On a question of L.A. Shemetkov concerning the intersection of $\mathfrak{F}$-maximal subgroups of finite groups

–Dedicated to Professor K.P. Shum on the occasion of his 70-th birthday

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Abstract

We investigate the influence of the intersection of the $\mathfrak{F}$-maximal subgroups on the structure of a finite group. In particular, answering a question of L.A Shemetkov we give conditions under which a hereditary saturated formation $\mathfrak{F}$ has a property that for any finite group G, the $\mathfrak{F}$-hypercentre of G coincides with the intersection of all $\mathfrak{F}$-maximal subgroups of G.

1 Introduction

Throughout this paper, all groups are finite. We use $\mathfrak{N}$, $\mathfrak{S}$ and $\mathfrak{U}$ to denote the classes of all nilpotent, of all soluble and of all supersoluble groups respectively.

Let $\mathfrak{X}$ be a class of groups. The symbol $\pi(\mathfrak{X})$ denotes the set of all primes p such that p divides $|G|$ for some $G \in \mathfrak{X}$. A chief factor H/K of a group G is called $\mathfrak{X}$-central in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{X}$ (see [1, p. 127-128]). A normal subgroup N of G is said to be $\mathfrak{X}$-*hypercentral in* G if either $N = 1$ or $N \neq 1$ and every chief factor of G below N is $\mathfrak{X}$-central in G. The symbol $Z_{\mathfrak{X}}(G)$ denotes the $\mathfrak{X}$-hypercentre of G, that is, the product of all normal $\mathfrak{X}$-hypercentral subgroups of G [2, p. 389]. If $1 \in \mathfrak{X}$ and G is a group, then we write $G^{\mathfrak{X}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{X}$. A group G is called *s-critical for* $\mathfrak{X}$ or simply $\mathfrak{X}$-*critical* if G is not in $\mathfrak{X}$ but all proper subgroups of G are in $\mathfrak{X}$ [2, p. 517]. A subgroup U of a group G is called $\mathfrak{X}$-*maximal* in G provided that (a) $U \in \mathfrak{X}$, and (b) if $U \leq V \leq G$ and $V \in \mathfrak{X}$, then $U = V$ [2, p. 288].

Some classes of $\mathfrak{X}$-maximal subgroups ($\mathfrak{X}$-projectors, $\mathfrak{X}$-injectors, $\mathfrak{X}$-covering subgroups and at al) have been studied by a large number of authors and they play an important role in the theory

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of soluble groups [2]. In this paper, we investigate the influence of the intersection of all $\mathfrak{X}$-maximal subgroups of a group G on the structure of G. We denote this intersection by $\mathrm{Int}_{\mathfrak{X}}(G)$.

In the paper [3], Baer proved that $\mathrm{Int}_{\mathfrak{N}}(G)$ coincides with the hypercentre $Z_{\infty}(G) = Z_{\mathfrak{N}}(G)$ of G. But in general, $Z_{\mathfrak{X}}(G) < \mathrm{Int}_{\mathfrak{X}}(G)$ even when $\mathfrak{X} = \mathfrak{U}$ and G is soluble (see Example 5.13 below).

L.A. Shemetkov asked in 1995 at the Gomel Algebraic Seminar the following question (the formulation of this question was also given in [4, p. 41]): *What are the non-empty hereditary saturated formations $\mathfrak{F}$ with the property that for each group G, the equality*

$$Int_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G) \qquad (*)$$

holds? Our main goal here is to give an answer to this question.

A class $\mathfrak{F}$ of groups is said to be a *formation* if either $\mathfrak{F} = \varnothing$ or $\mathfrak{F} \neq \varnothing$ and for any group G, each homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be: *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; *hereditary* if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$.

Let $\mathfrak{F}$ be a saturated formation with $\pi(\mathfrak{F}) \neq \varnothing$. Then for any $p \in \pi(\mathfrak{F})$ we write $\mathfrak{F}(p)$ to denote the intersection of all formations containing the set $\{G/O_{p',p}(G) \mid G \in \mathfrak{F}\}$, and let $F(p)$ denote the class of all groups G such that $G^{\mathfrak{F}(p)}$ is a p-group.

Remark. We will show (see Lemma 2.1 below) that the function f of the form

$$f : \mathbb{P} \to \{\text{group formations}\},$$

where $f(p) = F(p)$ for all $p \in \pi(\mathfrak{F})$, and $f(p) = \varnothing$ for all $p \notin \pi(\mathfrak{F})$, is the canonical local definition of $\mathfrak{F}$ (see p. 361 in [2]). Therefore, our notation $F(p)$ follows the terminology of [2, Chapter IV].

Definition. Let $\mathfrak{F}$ be a hereditary saturated formation with $\pi(\mathfrak{F}) \neq \varnothing$. We say that $\mathfrak{F}$ satisfies:

(1) The *boundary condition* if for any $p \in \pi(\mathfrak{F})$, $G \in \mathfrak{F}$ whenever G is an $F(p)$-critical group.

(2) The *boundary condition in the class of all soluble groups* if for any $p \in \pi(\mathfrak{F})$, $G \in \mathfrak{F}$ whenever G is a soluble $F(p)$-critical group.

If $\mathfrak{F}$ is a non-empty formation with $\pi(\mathfrak{F}) = \varnothing$, then $\mathfrak{F} = (1)$ is the class of all groups G with $|G| = 1$, and therefore for any group G we have $Z_{\mathfrak{F}}(G) = 1 = \mathrm{Int}_{\mathfrak{F}}(G)$. For the general case, we prove

Theorem A. *Let $\mathfrak{F}$ be a hereditary saturated formation with $\pi(\mathfrak{F}) \neq \varnothing$. Equality (*) holds for each group G if and only if $\mathfrak{F}$ satisfies the boundary condition.*

Theorem B. *Let $\mathfrak{F}$ be a hereditary saturated formation with $\pi(\mathfrak{F}) \neq \varnothing$. Equality (*) holds for each soluble group G if and only if $\mathfrak{F}$ satisfies the boundary condition in the class of all soluble groups.*

The proofs of Theorems A and B rely on the following general facts on the subgroup $\mathrm{Int}_{\mathfrak{F}}(G)$.

Theorem C. *Let $\mathfrak{F}$ be a non-empty hereditary saturated formation. Let H, E be subgroups of a group G, N a normal subgroup of G and $I = Int_{\mathfrak{F}}(G)$.*

(a) $\mathrm{Int}_{\mathfrak{F}}(H)N/N \leq \mathrm{Int}_{\mathfrak{F}}(HN/N)$.

(b) $\mathrm{Int}_{\mathfrak{F}}(H) \cap E \leq \mathrm{Int}_{\mathfrak{F}}(H \cap E)$.

(c) *If $H/H \cap I \in \mathfrak{F}$, then $H \in \mathfrak{F}$.*

(d) *If $H \in \mathfrak{F}$, then $IH \in \mathfrak{F}$*

(e) *If $N \leq I$, then $I/N = \mathrm{Int}_{\mathfrak{F}}(G/N)$.*

(f) $\mathrm{Int}_{\mathfrak{F}}(G/I) = 1$.

(g) *If every $\mathfrak{F}$-critical subgroup of G is soluble and $\psi_e(N) \leq I$, then $N \leq I$.*

(h) $Z_{\mathfrak{F}}(G) \leq I$.

It this theorem $\psi_e(N)$ denotes the subgroup of N generated by all its cyclic subgroups of prime order and order 4 [5].

We prove Theorems A, B and C in Section 3. In Section 4 it is shown that the formation of all nilpotent groups, the formation of all p-decomposable groups (for any prime p), and the formation of all groups G with $G' \leq F(G)$ satisfy the boundary condition, and that the formation of all soluble groups of nilpotent length at most r (for any fixed $r \in \mathbb{N}$) satisfies the boundary condition in the class of all soluble groups. We also consider here some classes of saturated formations which do not satisfy the boundary condition. Finally, in Section 5, some further applications of the subgroup $\mathrm{Int}_{\mathfrak{F}}(G)$ are discussed.

All unexplained notation and terminology are standard. The reader is referred to [2], [6] and [7] if necessary.

2 Preliminaries

The product $\mathfrak{M}\mathfrak{H}$ of the formations $\mathfrak{M}$ and $\mathfrak{H}$ is the class of all groups G such that $G^{\mathfrak{H}} \in \mathfrak{M}$. We use $\mathfrak{G}_\pi$ to denote the class of all π-groups. In particular, we write $\mathfrak{G}_p$ to denote the class of all p-groups if $\pi = \{p\}$, p is a prime. The product of any two formations is itself a formation [2, Chapter IV, Theorem 1.8]. Therefore, if $\mathfrak{F}$ is a saturated formation and if $p \in \pi(\mathfrak{F})$, then $F(p) = \mathfrak{G}_p\mathfrak{F}(p)$ is a formation.

A function $f : \mathbb{P} \rightarrow \{\text{group formations}\}$ is called a *formation function*. The symbol $LF(f)$ denotes the collection of all groups G such that either $G = 1$ or $G \neq 1$ and $G/C_G(H/K) \in f(p)$ for every chief factor H/K of G and every $p \in \pi(H/K)$. A formation function f is called *integrated* if $f(p) \subseteq LF(f)$ for all primes p, and *full* if $f(p) = \mathfrak{G}_p f(p)$ for all primes p. If for a formation $\mathfrak{F}$ we have $\mathfrak{F} = LF(f)$, then f is called a local definition of $\mathfrak{F}$. It is well known that $O_{p',p}(G) = \cap\{C_G(H/K) \mid H/K \text{ is a chief factor of } G \text{ and } p \in \pi(H/K)\}$. Therefore, $G \in \mathfrak{F} = LF(f)$ if and only if either $G = 1$ or $G \neq 1$ and $G/O_{p',p}(G) \in f(p)$ for all $p \in \pi(G)$.

Lemma 2.1. *Let $\mathfrak{F}$ be a non-empty saturated formation. Then $\mathfrak{F} = LF(f)$, where $f(p) =$*

$F(p) \subseteq \mathfrak{F}$ *for all* $p \in \pi(\mathfrak{F})$, *and* $f(p) = \varnothing$ *for all primes* $p \notin \pi(\mathfrak{F})$.

Proof. Define a function t as follows:

$$t(p) = \begin{cases} \mathfrak{F}(p), & \text{if } p \in \pi(\mathfrak{F}), \\ \varnothing, & \text{if } p \notin \pi(\mathfrak{F}) \end{cases}.$$

Let $\mathfrak{M} = LF(t)$. Then $\mathfrak{F} \subseteq \mathfrak{M}$. On the other hand, by the Gaschütz-Lubeseder-Schmid theorem [2, Chapter IV, Theorem 4.4], there is a formation function h such that $\mathfrak{F} = LF(h)$. Moreover, $t(p) \leq h(p)$ for all primes p and therefore $\mathfrak{M} \subseteq \mathfrak{F}$. Hence $\mathfrak{F} = \mathfrak{M} = LF(t)$. Now the assertion follows from Proposition 3.8 (a) in [2, Chapter IV].

From Theorem 17.14 in [1] we get

Lemma 2.2. *Let $\mathfrak{F}$ be a non-empty saturated formation. A chief factor H/K of a group G is $\mathfrak{F}$-central in G if and only if $G/C_G(H/K) \in F(p)$ for all primes $p \in \pi(H/K)$.*

In view of Lemma 2.1 and Proposition 3.16 in [2, IV] we have

Lemma 2.3. *Let $\mathfrak{F}$ be a hereditary saturated formation. Then for any prime $p \in \pi(\mathfrak{F})$, the formation $F(p)$ is hereditary.*

We shall need in our proofs a few facts about the $\mathfrak{F}$-hypercentre.

Lemma 2.4. *Let $\mathfrak{F}$ be a non-empty saturated formation. Let G be a group and $H \leq G$.*

(1) *If H is normal in G, then $Z_{\mathfrak{F}}(G)H/H \leq Z_{\mathfrak{F}}(G/H)$*

(2) *If $\mathfrak{F}$ is hereditary, then $Z_{\mathfrak{F}}(G) \cap H \leq Z_{\mathfrak{F}}(H)$.*

(3) *If $G/Z_{\mathfrak{F}}(G) \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Proof. (1) This follows from the G-isomorphism $Z_{\mathfrak{F}}(G)H/H \simeq Z_{\mathfrak{F}}(G)/Z_{\mathfrak{F}}(G) \cap H$ since for any two G-isomorphic chief factors H/K and T/L of G we have $(H/K) \rtimes (G/C_G(H/K)) \simeq (T/L) \rtimes (G/C_G(T/L))$.

(2) Let $1 = Z_0 < Z_1 < \ldots < Z_t = Z_{\mathfrak{F}}(G)$ be a chief series of G below $Z_{\mathfrak{F}}(G)$ and $C_i = C_G(Z_i/Z_{i-1})$. Let p be a prime divisor of $|Z_i \cap H/Z_{i-1} \cap H| = |Z_{i-1}(Z_i \cap H)/Z_{i-1}|$. Then p divides $|Z_i/Z_{i-1}|$, so $G/C_i \leq F(p)$ by Lemma 2.2. Hence by Lemma 2.3, $H/H \cap C_i \simeq C_iH/C_i \in F(p)$. But $H \cap C_i \leq C_H(Z_i \cap H/Z_{i-1} \cap H)$. Hence $H/C_H(Z_i \cap H/Z_{i-1} \cap H) \in F(p)$ for all primes p dividing $|Z_i \cap H/Z_{i-1} \cap H|$. Thus $Z_{\mathfrak{F}}(G) \cap H \leq Z_{\mathfrak{F}}(H)$ by Lemma 2.2 and [2, Chapter A, Theorem 3.2].

(3) This follows from Lemmas 2.1, 2.2 and the Jordan-Hölder theorem [2, Chapter A, Theorem 3.2].

The following lemma is a corollary of general results on f-hypercentral action (see [2, Chapter IV, Section 6]). For reader's convenience, we give a direct proof.

Lemma 2.5. *Let $\mathfrak{F}$ be a saturated formation. Let E be a normal p-subgroup of a group G. If $E \leq Z_{\mathfrak{F}}(G)$, then $G/C_G(E) \in F(p)$.*

Proof. Let $1 = E_0 < E_1 < \ldots < E_t = E$ be a chief series of G below E. Let $C_i = C_G(E_i/E_{i-1})$ and $C = C_1 \cap \ldots \cap C_t$. Then $C_G(E) \leq C$ and so $C/C_G(E)$ is a p-group by Corollary 3.3 in [8, Chapter 5]. On the other hand, by Lemma 2.2, $G/C_i \in F(p)$, so $G/C \in F(p)$. Hence $G/C_G(E) \in F(p) = \mathcal{G}_p F(p)$.

Lemma 2.6. *Let G be a group and p a prime such that $O_p(G) = 1$. If G has a unique minimal normal subgroup, then there exists a simple $\mathbb{F}_pG$-module which is faithful for G.*

Proof. Let C_p be a group of order p. Consider $A = C_p \wr G = K \rtimes G$, the regular wreath product of C_p with G, where K is the base group of A. Let

$$1 = K_0 < K_1 < \ldots < K_t = K, \qquad (*)$$

where K_i/K_{i-1} is a chief factor of A for all $i = 1, \ldots, t$. Let $C_i = C_A(K_i/K_{i-1})$, N a minimal normal subgroup of G and $C = C_1 \cap \ldots \cap C_t$. Suppose that $C_i \cap G \neq 1$ for all $i = 1, \ldots, t$. Then $N \leq C \cap G$. Hence N stabilizes Series (*), so N is a p-group by Corollary 3.3 in [8, Chapter 5], which implies $N \leq O_p(G)$. This contradiction shows that for some i we have $C_G(K_i/K_{i-1}) = 1$. The lemma is proved.

Lemma 2.7. Let $\mathcal{F}$ be a non-empty saturated formation.

(1) *If for some prime p we have $\mathcal{F} = \mathcal{G}_p\mathcal{F}$, then $F(p) = \mathcal{F}$.*

(2) *If $\mathcal{F} = \mathcal{N}\mathcal{H}$ for some non-empty formation $\mathcal{H}$, then $F(p) = \mathcal{G}_p\mathcal{H}$ for all primes p.*

Proof. (1) By Lemma 2.1, $F(p) \subseteq \mathcal{F}$, so we need only prove that $\mathcal{F} \subseteq F(p)$. Suppose that this is false and let A be a group of minimal order in $\mathcal{F} \setminus F(p)$. Then $A^{F(p)}$ is a unique minimal normal subgroup of A and $O_p(A) = 1$. By Lemma 2.6 there is a simple $\mathbb{F}_pA$-module P which is faithful for A. Then $G = P \rtimes A \in \mathcal{G}_p\mathcal{F} = \mathcal{F}$, so $A \simeq G/P = G/O_{p',p}(G) \in F(p)$, a contradiction. Thus $F(p) = \mathcal{F}$.

(2) The inclusion $F(p) \subseteq \mathcal{G}_p\mathcal{H}$ is evident. Suppose that $\mathcal{G}_p\mathcal{H} \not\subseteq F(p)$ and let A be a group of minimal order in $\mathcal{N}_p\mathcal{H} \setminus F(p)$. Then $A^{F(p)}$ is a unique minimal normal subgroup of A and $O_p(A) = 1$. Hence $A \in \mathcal{H}$ and there exists a simple $\mathbb{F}_pA$-module P which is faithful for A. Then $G = P \rtimes A \in \mathcal{G}_p\mathcal{H} \subseteq \mathcal{F}$, so $A \simeq G/P = G/O_{p',p}(G) \in F(p)$, a contradiction. The lemma is proved.

Lemma 2.8 [9, Chapter VI, Theorem 25.4]. *Let $\mathcal{F}$ be a saturated formation. Let G be a group whose $\mathcal{F}$-residual $G^{\mathcal{F}}$ is soluble. Suppose that every maximal subgroup of G not containing $G^{\mathcal{F}}$ belongs to $\mathcal{F}$.*

(a) *$P = G^{\mathcal{F}}$ is a p-group for some prime p and P is of exponent p or of exponent 4 (if P is a non-abelian 2-group).*

(b) *$P/\Phi(P)$ is a chief factor of G and $(P/\Phi(P)) \rtimes (G/C_G(P/\Phi(P))) \notin \mathcal{F}$.*

Let H and K be subgroups of a group G. If $HK = G$, then K is called a *supplement* of H in G. If, in addition, $HT \neq G$ for all proper subgroups T of K, then K is called a *minimal supplement* of H in G.

Lemma 2.9. *Let $\mathfrak{F}$ be a hereditary saturated formation. Let $N \le U \le G$, where N is a normal subgroup of a group G.*

(i) *If $G/N \in \mathfrak{F}$ and V is a minimal supplement of N in G, then $V \in \mathfrak{F}$.*

(ii) *If U/N is an $\mathfrak{F}$-maximal subgroup of G/N, then $U = U_0N$ for some $\mathfrak{F}$-maximal subgroup U_0 of G.*

(iii) *If V is an $\mathfrak{F}$-maximal subgroup of U, then $V = H \cap U$ for some $\mathfrak{F}$-maximal subgroup H of G.*

Proof. (i) It is clear that $V \cap N \le \Phi(V)$. Hence from $V/V \cap N \simeq VN/N = G/N \in \mathfrak{F}$ we have $V \in \mathfrak{F}$ since $\mathfrak{F}$ is saturated.

(ii) Let V be a minimal supplement of N in U. Then $V \in \mathfrak{F}$ by (i). Let U_0 be an $\mathfrak{F}$-maximal subgroup of G such that $V \le U_0$. Then $U_0N/N \simeq U_0/U_0 \cap N \in \mathfrak{F}$ and $U/N \le U_0N/N$. Hence $U = U_0N$.

(iii) Let H be an $\mathfrak{F}$-maximal subgroup of G such that $V \le H$. Then $V \le H \cap U \in \mathfrak{F}$ since $\mathfrak{F}$ is hereditary, which implies $V = H \cap U$.

Lemma 2.10. *Let $\mathfrak{F}$ be a saturated formation with $p \in \pi(\mathfrak{F})$. Suppose that G is a group of minimal order in the set of all $F(p)$-critical groups G with $G \notin \mathfrak{F}$. Then $O_p(G) = 1 = \Phi(G)$ and $G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G.*

Proof. Let N be a minimal normal subgroup of G. Then $G/N \in \mathfrak{F}$. Indeed, suppose that $G/N \notin \mathfrak{F}$. Since $F(p)$ is a formation, and $F(p) \subseteq \mathfrak{F}$ by Lemma 2.1, it follows that $G/N \notin F(p)$ and that every maximal subgroup of G/N belongs to $F(p)$. Thus G/N is an $F(p)$-critical group with $G/N \notin \mathfrak{F}$. But then $|G/N| < |G|$ contradicts the minimality of G. Hence $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is a saturated formation, $N = G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G and $\Phi(G) = 1$. Suppose that $N \le O_p(G)$ and let M be a maximal subgroup of G such that $G = NM$. Then $G/N \simeq M/N \cap M \in F(p) = \mathfrak{G}_pF(p)$, so $G \le F(p) \subseteq \mathfrak{F}$. This contradiction shows that $O_p(G) = 1$.

Lemma 2.11 [9, Chapter 1, Lemma 4.4]. *Let L be a normal subgroup of a group G such that $L \le \Phi(G)$. If G/L has a normal Hall π-subgroup, then does G.*

Lemma 2.12 [10, Lemma 2.2]. *Let $p \ne q$ be primes dividing the order of a group G, P a Sylow p-subgroup of G. If every maximal subgroup of P has a q-closed supplement in G, then G is q-closed.*

The following lemma is well known.

Lemma 2.13. *Let A and B be proper subgroups of G such that $G = AB$. Then $A^xB = G$ and $G \ne AA^x$ for all $x \in G$.*

3 Proofs of Theorems A, B and C

Proof of Theorem C (a) First we suppose that $H = G$. If U/N is an $\mathfrak{F}$-maximal subgroup of G/N, then for some $\mathfrak{F}$-maximal subgroup U_0 of G we have $U = U_0N$ by Lemma 2.9 (ii). Let $\text{Int}_{\mathfrak{F}}(G/N) = U_1/N \cap \ldots \cap U_t/N$, where U_i/N is an $\mathfrak{F}$-maximal subgroup of G/N for all $i = 1, \ldots, t$. Let V_i be an $\mathfrak{F}$-maximal subgroup of G such that $U_i = V_iN$. Then $I \leq V_1 \cap \ldots \cap V_t$, so $IN/N \leq \text{Int}_{\mathfrak{F}}(G/N)$.

Now let H be any subgroup of G. And let $f : H/H \cap N \to HN/N$ be the canonical isomorphism from $H/H \cap N$ onto HN/N. Then $f(\text{Int}_{\mathfrak{F}}(H/H \cap N)) = \text{Int}_{\mathfrak{F}}(HN/N)$ and $f(\text{Int}_{\mathfrak{F}}(H)(H \cap N)/(H \cap N)) = \text{Int}_{\mathfrak{F}}(H)N/N$. But from above we have $\text{Int}_{\mathfrak{F}}(H)(H \cap N)/(H \cap N) \leq \text{Int}_{\mathfrak{F}}(H/H \cap N)$. Hence $\text{Int}_{\mathfrak{F}}(H)N/N \leq \text{Int}_{\mathfrak{F}}(HN/N)$.

(b) If V is any $\mathfrak{F}$-maximal subgroup of H, then $V = H \cap U$ for some $\mathfrak{F}$-maximal subgroup U of G by Lemma 2.9 (iii). Thus there are $\mathfrak{F}$-maximal subgroups $U_1, \ldots, U_t$ of G such that $\text{Int}_{\mathfrak{F}}(H) = U_1 \cap \ldots \cap U_t \cap H$, hence $I \cap H \leq \text{Int}_{\mathfrak{F}}(H)$ and $\text{Int}_{\mathfrak{F}}(H) \cap E = \text{Int}_{\mathfrak{F}}(H) \cap (H \cap E) \leq \text{Int}_{\mathfrak{F}}(H \cap E)$.

(c) First we suppose that $H = G$. Let U be a minimal supplement of I in G. Then $U \in \mathfrak{F}$ by Lemma 2.9 (i). Let V be an $\mathfrak{F}$-maximal subgroup of G containing U. Then $G = IU =\leq V \in \mathfrak{F}$. Finally, in the general case we have $I \cap H \leq \text{Int}_{\mathfrak{F}}(H)$ by (b), so from $H/H \cap I \in \mathfrak{F}$ we deduce $H/\text{Int}_{\mathfrak{F}}(H) \in \mathfrak{F}$ and hence $H \in \mathfrak{F}$.

(d) Since $H \in \mathfrak{F}$, $HI/I \simeq H/H \cap I \in \mathfrak{F}$. By (b), $I \leq \text{Int}_{\mathfrak{F}}(HI)$. Hence $HI/\text{Int}_{\mathfrak{F}}(HI) \in \mathfrak{F}$. Thus $HI \in \mathfrak{F}$ by (c).

(e) In view of Lemma 2.9 (ii) it is enough to prove that if U is an $\mathfrak{F}$-maximal subgroup of G, then U/N is an $\mathfrak{F}$-maximal subgroup of G/N. Let $U/N \leq X/N$, where X/N is an $\mathfrak{F}$-maximal subgroup of G/N. By Lemma 2.9 (ii), $X = U_0N$ for some $\mathfrak{F}$-maximal subgroup U_0 of G. But $N \leq U_0$, so $U/N \leq U_0/N$ and hence $U = U_0$. Thus $U/N = X/N$.

(f) This follows from (e).

(g) Suppose that this assertion is false and let G be a counterexample with $|G||N|$ minimal. Then there is an $\mathfrak{F}$-maximal subgroup U of G such that $N \nleq U$. Let $E = NU$. Then $E/N \simeq U/U \cap N \in \mathfrak{F}$. By (b), $\psi_e(N) \leq I \cap E \leq \text{Int}_{\mathfrak{F}}(E)$. Suppose that $E \neq G$. Then $N \leq \text{Int}_{\mathfrak{F}}(E)$ by the choice of (G, N), so $G/\text{Int}_{\mathfrak{F}}(E) \in \mathfrak{F}$. Hence $E \in \mathfrak{F}$ by (c), so $U = E$. Therefore $N \leq U$, a contradiction. Thus $E = G$. Let M be any maximal subgroup of G. We show that $M \in \mathfrak{F}$. Since $\psi_e(N \cap M) \leq \psi_e(N)$, $\psi_e(N \cap M) \leq I \cap M$. Hence $\psi_e(N \cap M) \leq \text{Int}_{\mathfrak{F}}(M)$ by (b). Therefore $N \cap M \leq \text{Int}_{\mathfrak{F}}(M)$ by the choice of (G, N). Note also that $M/M \cap N \in \mathfrak{F}$. Indeed, if $N \leq M$, then $M/N \leq G/N \in \mathfrak{F}$. On the other hand, if $N \nsubseteq M$, then $M/M \cap N \simeq NM/N = G/N \in \mathfrak{F}$ since $\mathfrak{F}$ is hereditary. Therefore $M \in \mathfrak{F}$ by (c). Hence $I = \Phi(G)$ and G is an $\mathfrak{F}$-critical group. Since $G/N \in \mathfrak{F}$, $\psi_e(G^{\mathfrak{F}}) \leq \psi_e(N) \leq I$. Thus for any $x \in G^{\mathfrak{F}} \setminus \Phi(G^{\mathfrak{F}})$ we have $x \in \psi_e(N) \leq I = \Phi(G)$ by Lemma 2.8. Therefore $G^{\mathfrak{F}} \leq \Phi(G)$, so $I = G \in \mathfrak{F}$, a contradiction. Hence we have (g).

(h) Let H be a subgroup of G such that $H \in \mathfrak{F}$. Then $HZ_{\mathfrak{F}}(G)/Z_{\mathfrak{F}}(G) \simeq H/H \cap Z_{\mathfrak{F}}(G) \in \mathfrak{F}$ and $Z_{\mathfrak{F}}(G) \leq Z_{\mathfrak{F}}(HZ_{\mathfrak{F}}(G))$ by Lemma 2.4 (2). Hence $HZ_{\mathfrak{F}}(G) \in \mathfrak{F}$ by Lemma 2.4 (3). Thus $Z_{\mathfrak{F}}(G) \leq I$.

Proof of Theorem A. First we suppose that $\mathfrak{F}$ satisfies the boundary condition. We shall show that for every group G we have $Z_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{F}}(G)$. Suppose that this is false and let G be a counterexample with minimal order. Let $Z = Z_{\mathfrak{F}}(G)$ and $I = \mathrm{Int}_{\mathfrak{F}}(G)$. Then $Z < I$ by Theorem C (h), so $I \neq 1$ and $G \notin \mathfrak{F}$. Let N be a minimal normal subgroup of G, L a minimal normal subgroup of G contained in I.

(1) *$IN/N \leq Z_{\mathfrak{F}}(G/N) = Int_{\mathfrak{F}}(G/N)$.*

Indeed, by Theorem C (a) we have $IN/N \leq \mathrm{Int}_{\mathfrak{F}}(G/N)$. On the other hand, by the choice of G, $\mathrm{Int}_{\mathfrak{F}}(G/N) = Z_{\mathfrak{F}}(G/N)$.

(2) *$L \nleq Z$.*

Suppose that $L \leq Z$. Then $Z/L = Z_{\mathfrak{F}}(G/L)$ and $I/L = \mathrm{Int}_{\mathfrak{F}}(G/L)$ by Theorem C (e). But by (1), $Z_{\mathfrak{F}}(G/L) = \mathrm{Int}_{\mathfrak{F}}(G/L)$. Hence $I/L = Z/L$, so $I = Z$, a contradiction.

(3) *If $L \leq M < G$, then $L \leq Z_{\mathfrak{F}}(M)$.*

Let V be any $\mathfrak{F}$-maximal subgroup of M. Then $V = H \cap M$ for some $\mathfrak{F}$-maximal subgroup H of G by Lemma 2.9 (iii). Hence L is contained in the intersection of all $\mathfrak{F}$-maximal subgroups of M. But $|M| < |G|$, so $\mathrm{Int}_{\mathfrak{F}}(M) = Z_{\mathfrak{F}}(M)$ by the choice of G. Hence $L \leq Z_{\mathfrak{F}}(M)$

(4) *$L = N$ is a unique minimal normal subgroup of G.*

Suppose that $L \neq N$. From Theorem C (a) and (1) we deduce that $NL/N \leq Z_{\mathfrak{F}}(G/N)$, so from the G-isomorphism $NL/N \simeq L$ we obtain $L \leq Z$, which contradicts (2).

(5) *$L \nleq \Phi(G)$.*

Suppose that $L \leq \Phi(G)$. Then L is a p-group for some prime p. Let $C = C_G(L)$. Let M be any maximal subgroup of G. Then $L \leq M$, so $L \leq Z_{\mathfrak{F}}(M)$ by (3). Hence $M/M \cap C \in F(p)$ by Lemma 2.5. If $C \nleq M$, then $G/C = CM/C \simeq M/M \cap C \in F(p)$, so $L \leq Z_{\mathfrak{F}}(G)$ by Lemma 2.2, contrary to (2). Hence $C \leq M$ for all maximal subgroups M of G, so C is nilpotent. Therefore in view of (4), C is a p-group since C is normal in G. Hence for every maximal subgroup M of G we have $M \in \mathfrak{G}_pF(p) = F(p)$. By Lemma 2.1, $F(p) \subseteq \mathfrak{F}$. Hence $G \not\subseteq F(p)$ and so G is an $F(p)$-critical group. But $\mathfrak{F}$ satisfies the boundary condition and so $G \in \mathfrak{F}$, a contradiction. Hence we have (5).

(6) *L is not abelian.*

Suppose that L is abelian. Then from (4) and (5) we deduce that $G = L \rtimes M$ for some maximal subgroup M of G and $C = C_G(L) = L$. Let E be a maximal subgroup of M, $V = LE$. Then by (3), $L \leq Z_{\mathfrak{F}}(V)$, so $E \simeq V/L = V/C_V(L) \in F(p)$ by Lemma 2.5. Hence $M \in \mathfrak{F}$ since $\mathfrak{F}$ satisfies the boundary condition. But $L \leq I$, so $G \in \mathfrak{F}$ by Theorem A (c), a contradiction.

The final contradiction for the sufficiency.

Let $p \in \pi(L)$. First we show that each maximal subgroup M of G containing L belongs to $F(p)$. By (3), $L \leq Z_{\mathfrak{F}}(M)$. Let

$$1 = L_0 < L_1 < \ldots < L_n = L \qquad (*)$$

be a chief series of M below L. Let $C_i = C_M(L_i/L_{i-1})$ and $C = C_1 \cap \ldots \cap C_n$. Since by Lemma 2.2, $M/C_i \in F(p)$ for all $i = 1, \ldots, n$, $M/C \in F(p)$. By (4), L is a unique minimal normal subgroup of

G and L is non-abelian by (6). Hence $C_G(L) = 1$, so for any minimal normal subgroup R of M we have $R \leq L$. Suppose that $C \neq 1$ and let R be a minimal normal subgroup of M contained in C. Then $R \leq L$ and $R \leq C_A(H/K)$ for each chief factor H/K of M by [2, Chapter A, Theorem 3.2]. Thus R is abelian and hence L is abelian. This contradiction shows that $C = 1$, so $M \in F(p)$.

Now let U be a minimal supplement of L in G, V a maximal subgroup of U. Then $LV \neq G$, so $LV \leq T$ for some maximal subgroup T of G. Hence $T \in F(p)$, so $V \in F(p)$ by Lemma 2.3. Therefore every maximal subgroup of U belongs to $F(p) \subseteq \mathfrak{F}$. Hence $U \in \mathfrak{F}$, so $G \in \mathfrak{F}$ by Theorem C (c). This contradiction completes the proof of the sufficiency.

Now suppose that the equality $Z_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ holds for each group G. We shall show that $\mathfrak{F}$ satisfies the boundary condition. Suppose that this is false. Then there is a prime $p \in \pi(\mathfrak{F})$ such that the set of all $F(p)$-critical groups A with $A \notin \mathfrak{F}$ is non-empty. Let us choose in this set a group G with minimal $|G|$. Then by Lemma 2.10, $G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G and $O_p(G) = 1 = \Phi(G)$. Hence by Lemma 2.6, there exists a a simple $\mathbb{F}_pG$-module P which is faithful for G. Let $A = P \rtimes G$ and M be any maximal subgroup of A. If $P \nleq M$, $M \simeq A/P \simeq G \notin \mathfrak{F}$. On the other hand, if $P \leq M$, $M = M \cap PG = P(M \cap G)$, where $M \cap G$ is a maximal subgroup of G. Hence $M \cap G \in F(p)$, so $M \in \mathfrak{G}_pF(p) = F(p) \subseteq \mathfrak{F}$ by Lemma 2.1. Therefore P is contained in the intersection of all $\mathfrak{F}$-maximal subgroups of A. Hence $P \leq Z_{\mathfrak{F}}(A)$ by our assumption about $\mathfrak{F}$, so $G \simeq A/P = A/C_A(P) \in F(p) \subseteq \mathfrak{F}$ by Lemma 2.5. This contradiction completes the proof of the result.

Proof of Theorem B. See the proof of Theorem A.

4 Some classes of formations satisfying the boundary condition

Classes of soluble groups with limited nilpotent length. Following [2, Chapter VII, Definitions 6.9] we write $l(G)$ to denote the nilpotent length of the group G. Recall that $\mathfrak{N}^r$ is the product of r copies of $\mathfrak{N}$; $\mathfrak{N}^0$ is the class of groups of order 1 by definition. It is well known that $\mathfrak{N}^r$ is the class of all soluble groups G with $l(G) \leq r$. It is known also that $\mathfrak{N}^r$ is a hereditary saturated formation (see, for example, [2, p. 358]).

Proposition 4.1. *For any $r \in \mathbb{N}$, the formation $\mathfrak{N}^r$ satisfies the boundary condition in the class of all soluble groups. The formation $\mathfrak{N}$ satisfies the boundary condition.*

Proof. We proceed by induction on r. Let $\mathfrak{F} = \mathfrak{N}^r$, $\mathfrak{H} = \mathfrak{N}^{r-1}$. It is clear that $\mathfrak{F} = \mathfrak{N}\mathfrak{H}$, so $F(p) = \mathfrak{N}_p\mathfrak{H}$ for all primes p by Lemma 2.7 (2). If $r = 1$, then for any prime p we have $F(p) = \mathfrak{N}_p$, so $\mathfrak{F} = \mathfrak{N}$ satisfies the boundary condition.

Now suppose that $r > 1$. Assume that $\mathfrak{F}$ does not satisfy the boundary condition in the class of all soluble groups. Then there is a prime p such that the set of all soluble $F(p)$-critical groups A with $A \notin \mathfrak{F}$ is non-empty. Let G be a group of minimal order in this set. Then $O_p(G) = 1 = \Phi(G)$ and $R = G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G by Lemma 2.10. Hence G is a primitive group and R is a q-group for some prime $q \neq p$. Therefore $G = R \rtimes M$ for some maximal subgroup M of G and $R = C_G(R) = F(G)$ by Theorem 15.2 in [2, Chapter A].

Let M_1 be any maximal subgroup of M. Then $RM_1 \in F(p) = \mathfrak{N}_p\mathfrak{H}$. Since $R = C_G(R)$, $O_{q'}(RM_1) = 1$. Hence $O_{q',q}(RM_1) = O_q(RM_1)$ and $O_p(RM_1) = 1$. Therefore $RM_1 \in \mathfrak{H}$. Let $H(q) = \mathfrak{G}_q\mathfrak{H}(q)$, where $\mathfrak{H}(q)$ is the intersection of all formations containing the set $\{A/O_{p',\,p}(A) \mid A \in \mathfrak{H}\}$. Then by Lemma 2.7, $H(q) = \mathfrak{N}_q\mathfrak{N}^{r-2}$. Hence $M_1/M_1 \cap RO_q(M_1) \simeq RM_1/RO_q(M_1) = RM_1/O_q(RM_1) = RM_1/O_{q',q}(RM_1) \in \mathfrak{N}_q\mathfrak{N}^{r-2}$. Thus $M_1 \in \mathfrak{N}_q\mathfrak{N}^{r-2}$. Therefore every maximal subgroup of M belongs to $H(q)$. By induction, $\mathfrak{H} = \mathfrak{N}^{r-1}$ satisfies the boundary condition in the class of all soluble groups. Therefore $M \in \mathfrak{H}$, so $G = R \rtimes M \in \mathfrak{F} = \mathfrak{N}^r$. This contradiction completes the proof of the proposition.

We use $\mathbb{P}$ to denote the set of all primes.

Proposition 4.2. *Let $\{\pi_i \mid i \in I\}$ be a partition of $\mathbb{P}$, and $\mathfrak{F}$ the class of all groups G of the form $G = A_{i_1} \times \ldots \times A_{i_t}$, where A_{i_j} is a Hall π_{i_j}-subgroup of G, $i_1, \ldots, i_t \in I$. Then $\mathfrak{F}$ is a hereditary saturated formation satisfying the boundary condition.*

Proof. It is clear that the class $\mathfrak{F}$ is closed under taking subgroups, homomorphic images and direct products. Hence $\mathfrak{F}$ is a hereditary formation. Moreover, in view of Lemma 2.11 this formation $\mathfrak{F}$ is saturated. We show that for any prime $p \in \pi_i$, $F(p) = \mathfrak{G}_{\pi_i}$. Clearly $F(p) \subseteq \mathfrak{G}_{\pi_i}$. Suppose that the inverse inclusion is not true and let A be a group of minimal order in $\mathfrak{G}_{\pi_i} \setminus F(p)$. Then $A^{F(p)}$ is a unique minimal normal subgroup of A and $O_p(A) = 1$. Hence there is a simple $\mathbb{F}_pA$-module P which is faithful for A by Lemma 2.6. Then $G = P \rtimes A \in \mathfrak{G}_{\pi_i} \subseteq \mathfrak{F}$, so $A \simeq G/P = G/O_{p',p}(G) \in F(p)$. This contradiction shows that $F(p) = \mathfrak{G}_{\pi_i}$. Now let G be any $F(p)$-critical group. Then $|G| = q$ for some prime $q \notin \pi_i$ and so $G \in \mathfrak{F}$. Hence $\mathfrak{F}$ satisfies the boundary condition.

Proposition 4.3. *Let $\{\pi_i \mid i \in I\}$ be a partition of $\mathbb{P}$, and $\mathfrak{F}$ the class of all soluble groups G of the form $G = A_{i_1} \times \ldots \times A_{i_t}$, where A_{i_j} is a Hall π_{i_j}-subgroup of G, $i_1, \ldots, i_t \in I$. Then $\mathfrak{F}$ is a hereditary saturated formation satisfying the boundary condition in the class of all soluble groups.*

Proof. See the proof of Proposition 4.2.

Lattice formations. A subgroup H is said to be $\mathfrak{F}$-subnormal in a group G if either $H = G$ or there exists a chain of subgroups

$$H = H_0 < H_1 < \ldots < H_t = G$$

such that H_{i-1} is a maximal subgroup of H_i and $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$ for all $i = 1, \ldots, t$ [7, p. 236].

A formation $\mathfrak{F}$ is said to be a lattice formation (see [7, Section 6]) if the set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of the lattice of all subgroups in every group.

Proposition 4.4. *Every lattice formation $\mathfrak{F}$ with $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$ is a hereditary saturated formation satisfying the boundary condition in the class of all soluble groups.*

Proof. This follows from Proposition 4.3 and Corollary 6.3.16 in [7].

Proposition 4.5. *Let $\mathfrak{F}$ be the class of all groups with $G' \leq F(G)$. Then $\mathfrak{F}$ is a hereditary saturated formation satisfying the boundary condition.*

Proof. It is clear that $\mathfrak{F}$ is a hereditary formation and $\mathfrak{F}$ is saturated by Theorem 4.2 d) in [19, Chapter III]. Moreover, $\mathfrak{F} = \mathfrak{N}\mathcal{A}$, where $\mathcal{A}$ is the formation of all abelian groups. Hence by Lemma 2.7 (2), $F(p) = \mathfrak{G}_p\mathcal{A}$ for all primes p. Assume that $\mathfrak{F}$ does not satisfy the boundary condition. Then for some prime p, the set of all $F(p)$-critical groups A with $A \notin \mathfrak{F}$ is non-empty. Let G be a group of minimal order in this set. Then $O_p(G) = 1 = \Phi(G)$ and $L = G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G by Lemma 2.10. Hence G is a primitive group.

First we show that G is soluble. Suppose that this is false. Let $q \neq p$ be any prime divisor of $|G|$. Suppose that G is not q-nilpotent. Then G has a q-closed Schmidt subgroup $H = Q \rtimes R$ [19, Chapter IV, Satz 5.4], where Q is a Sylow q-subgroup of H, R is a cyclic Sylow r-subgroup of H. Since G is not soluble, $H \neq G$. Hence $H \leq M \in F(p)$ for some maximal subgroup M of G. Since $M \in \mathfrak{G}_p\mathcal{A}$, $M' \leq O_p(M)$ and hence $H' \leq Q \cap O_p(H) = 1$. Therefore H is abelian. This contradiction shows that G is q-nilpotent for all primes $q \neq p$, so $G^{\mathfrak{N}}$ is a Sylow p-subgroup of G. Hence G is soluble. Therefore $L = C_G(L) = F(G)$ is a q-group for some prime $q \neq p$ and $G = L \rtimes M$ for some maximal subgroup M of G by Theorem 15.2 in [2, Chapter A]. Let M_1 be any maximal subgroup of M. Then $LM_1 \in F(p)$, so LM_1 is abelian since $L = C_G(L)$. Hence $M_1 = 1$, so $G' = L$ is nilpotent. Therefore $G \in \mathfrak{F}$. This contradiction completes the proof of the result.

A group G is called a p-decomposable if $G = P \times H$, where P is the Sylow p-subgroup of G.

Corollary 4.6. *Let $\mathfrak{F}$ be one of the following formations:*

(1) *the class of all nilpotent groups* (Baer [3]);

(2) *the class of all groups G with $G' \leq F(G)$;*

(3) *the class of all p-decomposable groups (p is a prime).*

Then for each group G, $Z_{\mathfrak{F}}(G) = Int_{\mathfrak{F}}(G)$.

Corollary 4.7. *Let $\mathfrak{F}$ be one of the following formations:*

(1) *the class of all soluble groups G with $l(G) \leq r$ ($r \in \mathbb{N}$)* (Sidorov [4]);

(2) *any lattice formation $\mathfrak{F}$ with $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$.*

Then for each soluble group G, $Z_{\mathfrak{F}}(G) = Int_{\mathfrak{F}}(G)$.

Some classes of formations not satisfying the boundary condition. We end this section with some examples of saturated formations which do not satisfy the boundary condition.

Lemma 4.8 *Let $\mathfrak{F}$ be any non-empty saturated formation. Suppose that for some prime p we have $F(p) = \mathfrak{F}$. Then $\mathfrak{F}$ does not satisfy the boundary condition.*

Proof. Indeed, in this case every $\mathfrak{F}$-critical group is also $F(p)$-critical.

Corollary 4.9 *Let p be a prime and $\mathfrak{F}$ is one of the following formations:*

(1) *the class of all p-soluble groups;*

(2) *the class of all p-supersoluble groups;*

(3) *the class of all p-nilpotent groups;*

(4) *the class of all soluble groups.*

Then $\mathcal{F}$ does not satisfy the boundary condition.

Proof. It is clear that for any prime $q \neq p$ we have $\mathcal{F} = \mathcal{G}_q\mathcal{F}$. Hence $F(q) = \mathcal{F}$ by Lemma 2.7 (1). Now we use Lemma 4.8.

5 Further applications

Based on the subgroup $\mathrm{Int}_{\mathcal{F}}(G)$ you can achieve the development of many well-known results. The observations in this section are partial illustrations to this.

A solubility criterion. It is clear that $\mathrm{Int}_{\mathcal{S}}(G)$ is the radical $R(G)$ of G, that is, the largest soluble normal subgroup of G.

Theorem 5.1. *Suppose that a group G has three subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime. If $A_i \cap A_j \leq R(A_i) \cap R(A_j)$ for all $i \neq j$, then G is soluble.*

Proof. Assume that this theorem is false and let G be a counterexample of minimal order. Fist we shall show that $A_i \cap A_j \neq 1$ for all $i \neq j$. Suppose, for example, that $A_1 \cap A_2 = 1$. Then A_1 and A_2 are Hall subgroups of G. Hence, for any prime p dividing $|G : A_3|$, p either divides $|G : A_1|$ or divides $|G : A_2|$. The contradiction shows that $|G : A_3| = 1$, that is, $G = A_3$. Therefore A_1, A_2 are contained in $R(G)$. It follows that $G = A_1A_2 = R(G)$, a contradiction. Therefore $A_i \cap A_j \neq 1$ for all $i \neq j$.

Now we prove that G/N is soluble for any abelian minimal normal subgroup N of G. Let $i \neq j$. Since N is abelian, N is a p-group for some prime p. Hence either $N \leq A_i$ or $N \leq A_j$. In the former case we have

$$A_i/N \cap A_jN/N = N(A_i \cap A_j)/N \leq N(R(A_i) \cap R(A_j))/N \leq R(A_i/N) \cap R(A_jN/N)$$

by Theorem C (a). Therefore the hypothesis holds for G/N and so G/N is soluble by the choice of G.

Finally, we shall prove that G has an abelian minimal normal subgroup. Since $A_1 \cap A_2 \neq 1$ and $A_1 \cap A_2 \leq R(A_2)$, for some minimal normal subgroup V of A_2 we have $V \leq R(A_2)$. Hence V is a p-group for some prime p. Then either p does not divide $|G : A_1|$ or p does not divide $|G : A_3|$. Assume that p does not divide $|G : A_1|$. Then for some $b \in A_2$, we have $V \leq A_1^b$. Hence $V = V^{b^{-1}} \leq A_1 \cap A_2 \leq R(A_2)$, which implies that $V^G = V^{A_2A_1} = V^{A_1} \leq A_1$. It follows that $E = V^G \cap A_2 \leq A_1 \cap A_2 \leq R(A_1)$ and E is normal in A_2. Hence $E^G = E^{A_2A_1} = E^{A_1} \leq A_1$. It follows that $E^G = E^{A_1} \leq (R(A_1))^{A_1} = R(A_1)$ and so E^G is soluble. This shows that G has an abelian minimal normal subgroup N and we have already proved that G/N is soluble, so G is soluble contrary to the choice of G. This contradiction completes the proof of the result.

Corollary 5.2 (Wielandt [11]). *If G has three soluble subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself soluble.*

Two characterizations of supersolubility.

Lemma 5.3. *Let N be a soluble normal subgroup of a group G, p a prime divisor of $|G|$ and P a Sylow p-subgroup of G. Suppose that $P \nleq N$ and that every maximal subgroup M of P has a supplement T in G such that $T \cap M \leq \mathrm{Int}_{\mathfrak{U}}(T)M_G$. Then every maximal subgroup V/N of NP/N has a supplement T/N in G/N such that*

$$(T/N) \cap (V/N) \leq \mathrm{Int}_{\mathfrak{U}}(T/N)(V/N)_{G/N}$$

.

Proof. We prove the lemma by induction on $|G|$. Let V/N be any maximal subgroup of NP/N and L a minimal normal subgroup of G contained in N. Then L is a q-group for some prime q. First suppose that $N = L$. If $q \neq p$, then $V = L \rtimes M$, for some maximal subgroup M of P. By hypothesis, there is a subgroup T such that $MT = G$ and $T \cap M \leq \mathrm{Int}_{\mathfrak{U}}(T)M_G$. Then $L \leq T$, $G/L = (V/L)(T/L)$ and

$$(V/L) \cap (T/L) = (LM/L) \cap (T/L) = L(M \cap T)/L \leq L\mathrm{Int}_{\mathfrak{U}}(T)M_G/L$$

$$= (LM_G/L)(L\mathrm{Int}_{\mathfrak{U}}(T)/L) \leq \mathrm{Int}_{\mathfrak{U}}(T/L)(V/L)_{G/L}$$

by Theorem C (a). If $q = p$, then V is a maximal subgroup of P and so for some supplement T of V in G we have $T \cap V \leq \mathrm{Int}_{\mathfrak{U}}(T)V_G$. Then $G/L = (V/L)(LT/L)$ and, as above, we deduce that

$$(V/L) \cap (TL/L) = L(V \cap T)/L \leq \mathrm{Int}_{\mathfrak{U}}(T)V_G L/L \leq \mathrm{Int}_{\mathfrak{U}}(TL/L)(V/L)_{G/L}$$

.

Finally, suppose that $L \neq N$. Obviously, the hypothesis holds for $(G/L, N/L)$. Hence, by induction, every maximal subgroup $(V/L)/(N/L)$ of $(PL/L)(N/L)/(N/L)$ has a supplement $(T/L)/(N/L)$ in $(G/L)/(N/L)$ such that

$$(T/L)/(N/L) \cap (V/L)/(N/L) \leq \mathrm{Int}_{\mathfrak{U}}((T/L)/(N/L))((V/L)/(N/L))_{(G/L)/(N/L)}.$$

Hence from the G-isomorphism $G/N \simeq (G/L)/(N/L)$, we obtain

$$(T/N) \cap (V/N) \leq \mathrm{Int}_{\mathfrak{U}}(T/N)(V/N)_{G/N}.$$

The lemma is proved.

Theorem 5.4. *A group G is supersoluble if and only if every maximal subgroup V of every Sylow subgroup of G has a supplement T in G such that $V \cap T \leq \mathrm{Int}_{\mathfrak{U}}(T)V_G$.*

Proof. We need only to prove "if" part. Suppose that it is false and let G be a counterexample of minimal order. The proof proceeds via the following steps.

(1) *If $V < P \leq E \leq G$, where P is a Sylow p-subgroup of G and V is a maximal subgroup of P, then V has a supplement T in E such that $T \cap V \leq \mathrm{Int}_{\mathfrak{U}}(T)V_E$.*

Indeed, let S be a supplement of V in G such that $S \cap V \leq \mathrm{Int}_{\mathfrak{U}}(S)V_G$. Then $T = S \cap E$ is a supplement of V in E and

$$V \cap T = V \cap S \cap E \leq \mathrm{Int}_{\mathfrak{U}}(S)V_G \cap E = (\mathrm{Int}_{\mathfrak{U}}(S) \cap E)V_G \leq \mathrm{Int}_{\mathfrak{U}}(S \cap E)V_E = \mathrm{Int}_{\mathfrak{U}}(T)V_E$$

by Theorem C (b).

(2) *G/N is supersoluble, for every abelian minimal normal subgroup N of G.*

By Lemma 5.3, the hypothesis is true for G/N. Hence G/N is supersoluble by the choice of G.

(3) *G is soluble.*

In view of (2), it is enough to prove that G has a non-identity soluble normal subgroup. Suppose that this is false. Then for every maximal subgroup V of any Sylow subgroup of G we have $V_G = 1$. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. If $|P| = p$, G has a normal p-complement E by [19, Chapter IV, Theorem 2.8]. On the other hand, by (1), the hypothesis holds for E. Hence E is supersoluble, which implies the solubility of G. Hence $|P| > p$. If $V \leq \mathrm{Int}_{\mathfrak{U}}(G)$ for some maximal subgroup V of P, then $\mathrm{Int}_{\mathfrak{U}}(G) \neq 1$ and so G has a non-identity soluble normal subgroup. Therefore every maximal subgroup V of P has a supplement T in G such that $T \neq G$ and $T \cap V \leq V_G\mathrm{Int}_{\mathfrak{U}}(T) = \mathrm{Int}_{\mathfrak{U}}(T)$. We claim that T is supersoluble. If $T \cap V = 1$, then $|T_p| = p$, for a Sylow p-subgroup T_p of T. Hence T supersoluble by (1) and the choice of G. Now assume that for some maximal subgroup V of P we have $1 \neq T \cap V \leq \mathrm{Int}_{\mathfrak{U}}(T)$. Since $|P \cap T : V \cap T| = |V(P \cap T) : V| = |P : V| = p$, the order of a Sylow p-subgroup of $T/\mathrm{Int}_{\mathfrak{U}}(T)$ divides p. Hence the hypothesis holds for $T/\mathrm{Int}_{\mathfrak{U}}(T)$ by (1) and Lemma 5.3. But since $T \neq G$, $T/\mathrm{Int}_{\mathfrak{U}}(T)$ is supersoluble by the choice of G. It follows that T is supersoluble by Theorem C (c). Therefore, our claim holds. This shows that every maximal subgroup of P has a supersoluble supplement in G. By Lemma 2.12, we see that G has a normal Sylow q-subgroup for some prime q dividing $|G|$. This contradiction completes the proof of (3)

(4) *$G = N \rtimes M$, where $N = C_G(N) = O_p(G)$ is a unique minimal normal subgroup of G (p is a prime), M is a supersoluble maximal subgroup of G with p divides $|M|$ and $|N| > p$.*

Let N be a minimal normal subgroup of G. Since the class of all supersoluble groups is a saturated formation, from (2) and (3) we deduce that N is a unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Hence G is a primitive group , so $N = C_G(N) = O_p(G) = F(G)$ for some prime p by Theorem 15.2 in [2, Chapter A]. Let M be a maximal subgroup of G such that $G = N \rtimes M$. Then M is supersoluble by (2). It is also clear that $|N| > p$. Suppose that N is a Sylow subgroup of G and let V be a maximal subgroup of N. Then $V_G = 1$, so V has a supplement T in G such that $VT = G$ and $T \cap V \leq \mathrm{Int}_{\mathfrak{U}}(T)$. But since $T \cap N$ is normal in G, the minimality of N implies that either $T = G$ or $T \cap V = 1$. In the former case, we have $1 \neq V \leq \mathrm{Int}_{\mathfrak{U}}(G)$ and so $N \leq \mathrm{Int}_{\mathfrak{U}}(G)$, which implies that G is supersoluble by Theorem C (c). In the second case, $|T \cap N| = p$, where $T \cap N$ is normal in G. Hence $N = N \cap T$ is a group with $|N| = p$. This contradiction shows that $p||M|$. Therefore (4) holds.

(5) $\pi(G) = \{p, q\}$, *where* $p < q$.

Suppose that $|\pi(G)| > 2$. Let $q \neq p$ be a prime divisor of $|G|$, Q a Sylow q-subgroup of G and P a Sylow p-subgroup of G. Since G is soluble, we may assume that Q and P are members of some Sylow system of G and so $E = PQ$ is a proper subgroup of G. By (1), the hypothesis holds for E. Hence E is supersoluble by the choice of G. If $q > p$, then Q is normal in E, which contradicts $C_G(N) = N$. Hence $p > q$ for any prime $q \neq p$ dividing $|G|$. Since G/N is supersoluble, a Sylow p-subgroup W of G/N is normal in G/N. Hence $W \leq O_p(G/N)$. By (4), $W \neq 1$. But $O_p(G/N) = O_p(G/C_G(N)) = 1$ (see [22, Appendix, Corollary 6.4]). This contradiction shows that $|\pi(G)| = 2$. In the above proof, we also see that $p > q$ is impossible. Therefore (5) holds.

Final contradiction. Let P_1 be a Sylow p-subgroup of M, V a maximal subgroup of a Sylow p-subgroup P of G containing P_1 and M_q a Sylow q-subgroup of M. Then $N \not\leq V$ and so $V_G = 1$. By hypothesis, V has a supplement T in G such that $V \cap T \leq \mathrm{Int}_{\mathfrak{U}}(T)$. If $T = G$, then $1 \neq V \leq \mathrm{Int}_{\mathfrak{U}}(G)$. Hence $N \leq \mathrm{Int}_{\mathfrak{U}}(G)$ since N is the only minimal normal subgroup of G. It follow from (2) that G is supersoluble by Theorem C (c), a contradiction. Hence $T \neq G$. In this case, as in the proof of (3), one can show that T is supersoluble. Hence a Sylow q-subgroup T_q of T is normal in T by (5). But T_q is a Sylow subgroup of G. Hence $T_q = (M_q)^x$ for some $x \in G$. Since $q > p$ and M is supersoluble, $M = N_G(M_q)$. Hence $T \leq N_G(T_q) = N_G(M_q^x) = (N_G(M_q))^x = M^x$. But then $G = VT = VM^x = VM$ by Lemma 2.13. It follows that $|G| = |V||M|/|V \cap M| \leq |V||M|/|P_1| < |M||N| = |G|$. This contradiction completes the proof of the result.

Note that if H is a group of G and H either is normal in G, has a complement in G, or has a supplement E in G with $E \in \mathfrak{F}$, then H has a supplement T in G such that $V \cap T \leq \mathrm{Int}_{\mathfrak{U}}(T)V_G$. Hence from Theorem 5.4 we get the following

Corollary 5.5 (Srinivasan [12]). *If the maximal subgroups of the Sylow subgroups of G are normal in G, then G is supersoluble.*

Corollary 5.6 (Ballester-Bolinches and Guo [13]). *A group G is supersoluble if every maximal subgroup of every Sylow subgroup of G has a complement in G.*

Corollary 5.7 (Guo, Shum and Skiba [14]). *A group G is supersoluble if and only if every maximal subgroup of every Sylow subgroup of G has a supersoluble supplement in G.*

In view of Theorem C (h) for every group G we have $Z_{\mathfrak{F}}(G) \leq \mathrm{Int}_{\mathfrak{F}}(G)$. Hence from Theorem 5.4 we also get

Corollary 5.8 (Guo and Skiba [15]). *A group G is supersoluble if and only if every maximal subgroup V of every Sylow subgroup of G either is normal or has a supplement T in G such that $V \cap T \le Z_{\mathfrak{U}}(T)$.*

It is well known that if every minimal subgroup of a group G is normal in G, then the commutator subgroup G' of G is 2-closed (Gaschütz [19, IV, Theorem 5.7]). On the other hand, if G is a group of odd order and every minimal subgroup of G is normal in G, then G is supersoluble (Buckley [16]). The following theorem covers both these observations.

Theorem 5.9. *A group G is $2'$-supersoluble if and only if every minimal subgroup L of G of odd order is contained in the intersection of all maximal $2'$-supersoluble subgroups of G.*

Proof. Let $\mathfrak{F}$ be the class of all $2'$-supersoluble groups and $I = \mathrm{Int}_{\mathfrak{F}}(G)$ the intersection of all maximal $2'$-supersoluble subgroups of G. It is well known that the class $\mathfrak{F}$ is a hereditary saturated formation (see [19, Chapter VI, Satz 8.6]). Assume that every minimal subgroup L of G of odd order is contained in I. We shall prove that G is $2'$-supersoluble. Assume that this is false and let G be a counterexample of minimal order .

The hypothesis holds for every subgroup of G by Theorem C (b). Hence every maximal subgroup of G is $2'$-supersoluble by the choice of G. Therefore every maximal subgroup of G is soluble.

First we show that G is soluble. Assume that this is false. Then $G = G'$, and if $F = F(G)$, then $F = \Phi(G)$, G/F is a simple non-abelian group and every proper normal subgroup of G is contained in F. Hence $I = F$. It is clear that every maximal subgroup of G/F is soluble and hence by [17], G/F is isomorphic to one of the following groups: $PSL_2(p)$ (where $p > 3$ is a prime such that $p^2 + 1 \equiv 0(5)$), $PSL_2(3^p)$ (where p is an odd prime), $PSL_2(2^p)$ (where p is a prime), $PSL_3(3)$, a Suzuki group $Sz(2^p)$ (where p is an odd prime).

Let r be the largest prime dividing $|G/F|$ and G_r a Sylow r-subgroup of G. Then $r > 3$ by Burnside's p^aq^b-theorem. Let p be any odd prime dividing $|G/F|$ and C_p a subgroup of G of order p. Then $C_p \le I = F$. Suppose that $p < r$ and let P be a Sylow p-subgroup of F. We show that $E = P{G_r}^x$ is nilpotent for all $x \in G$. Suppose that this is false and let H be a Schmidt subgroup of E, that is, an $\mathfrak{N}$-critical group. Since G is not soluble, $E \ne G$ and hence H is supersoluble. Therefore ${G_r}^x$ is normal in $H = P \rtimes {G_r}^x$ since $p < r$, so H is nilpotent. This contradiction shows that $P{G_r}^x$ is nilpotent. Hence $\langle (G_r)^G \rangle = G \le C_G(P)$. Thus $P \le Z(G)$ and $P \le \Phi(G)$ since $F = \Phi(G)$. Let V be a Hall p'-subgroup of F. Then $PV/V \le Z(G/V)$ and $PV/V \le \Phi(G/V)$. Hence p divides $|M(G/F)|$, where $M(G/F)$ is the Schur multiplicator of G/F. Since $p > 2$, it follows that $p = 3$, $\pi(|M(G/F)|) \subseteq \{2, 3\}$ and 5 divides $|G/F|$ (see [18, Chapter 4]). Let G_3 be a Sylow 3-subgroup of G and R the Sylow 5-subgroup of $F(G)$. Since $V = RG_3$ is soluble, $V \ne G$ and so V is supersoluble. Hence for any chief factor H/K of V below R we deduce that $|V/C_V(H/K)|$ divides 4. Therefore $C_V(H/K) = V$, so $R \le Z_\infty(V)$ and hence V is nilpotent. Thus $R \le Z(G)$, which implies that 5 divides $|M(G/F)|$, a contradiction. Therefore G is soluble. But G is an $\mathfrak{F}$-critical group. Hence by Lemma 2.8, $G^{\mathfrak{F}}$ is a p-group for some odd prime p and $\psi_e(G^{\mathfrak{F}}) \le I$. Thus $G \in \mathfrak{F}$ by Theorem C (c)(g). This contradiction completes the proof of the result.

A nilpotency criterion. In the following theorem, $c(G)$ denotes the nilpotent class of the nilpotent group G.

Theorem 5.10. *Suppose that G has three subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime. Suppose that $A_i \cap A_j \leq Z_n(Int_{\mathfrak{N}}(A_i)) \cap Z_n(Int_{\mathfrak{N}}(A_j))$ for all $i \neq j$. Then G is nilpotent and $c(G) \leq n$.*

Proof. Let p be any prime dividing $|G|$. By hypothesis, there exists $i \neq j$ such that $p \nmid |G : A_i|$ and $p \nmid |G : A_j|$. Hence $p \nmid |G : A_i \cap A_j|$ and so G has a Sylow p-subgroup P such that $P \leq A_i \cap A_j \leq Z_n(\mathrm{Int}_{\mathfrak{N}}(A_i)) \cap Z_n(\mathrm{Int}_{\mathfrak{N}}(A_j))$. Since $\mathrm{Int}_{\mathfrak{N}}(A_i)$ is nilpotent, P is a characteristic subgroup of $\mathrm{Int}_{\mathfrak{N}}(A_i)$. On the other hand, $\mathrm{Int}_{\mathfrak{N}}(A_i)$ is characteristic in A_i. Hence P is normal in A_i. Similarly, we have $A_j \leq N_{A_j}(P)$. Therefore $G = A_iA_j \leq N_G(P)$. Thus G is nilpotent and $c(P) \leq n$ for all Sylow subgroups P of G, which implies $c(G) \leq n$.

The theorem is proved.

Corollary 5.11 (Kegel [20]). *If G has three nilpotent subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself nilpotent.*

Corollary 5.12 (Doerk [21]). *If G has three abelian subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself abelian.*

A question of Agrawal. Recall that a subgroup H of a group G is said to be S-quasinormal in G if $HP = PH$ for all Sylow subgroups P of G. The hyper-generalized-center $genz^*(G)$ of G coincides with the largest term of the chain of subgroups

$$1 = Q_0 \leq Q_1 \leq \ldots \leq Q_t \leq \cdots$$

where $Q_i(G)/Q_{i-1}(G)$ is the subgroup of $G/Q_{i-1}(G)$ generated by the set of all cyclic S-quasinormal subgroups of $G/Q_{i-1}(G)$ (see [22, page 22]). In the paper [23], Agrawal proved that $genz^*(G)$ is contained in every maximal supersoluble subgroup of the group G and posed the following question: *Does there exist a group G with $genz^*(G) \neq Int_{\mathfrak{U}}(G)$?* (see [23, page 19] or [22, page 22])

The following example gives a positive answer to this question and shows that there are soluble groups G with $\mathrm{Int}_{\mathfrak{U}}(G) \neq Z_{\mathfrak{U}}(G)$.

Example 5.13. Let C_p be a group of prime order p with $|\pi(\mathrm{Aut}(C_p))| > 1$. Let R and L be Hall subgroups of $\mathrm{Aut}(C_p)$ such that $\mathrm{Aut}(C_p) = R \times L$ and for any $r \in \pi(R)$ and $q \in \pi(L)$ we have $r < q$. Let $G = (C_p \rtimes R) \wr L = K \rtimes L$ be the regular wreath product of $C_p \rtimes R$ with L, where K is the base group of G. Let $P = C_p{}^{\natural}$ (we use here the terminology in [2, Chapter A]). Then by Proposition 18.5 in [2, Chapter A], G is a primitive group and P is a unique minimal normal subgroup of G. Hence $P = F(G) = C_G(P)$. Moreover, by Lemma 18.2 in [2, Chapter A], $G = P \rtimes M$, where $M \simeq U = R \wr L = D \rtimes L$, where D is the base group of U. It is clear that D is a Hall abelian subgroup of U and L is a cyclic subgroup of U such that for any $r \in \pi(D)$ and $q \in \pi(L)$ we have

$r < q$. Moreover, since $|\mathrm{Aut}(Cp)| = p-1$, D and L are groups of exponent dividing $p-1$. First we show that every supersoluble subgroup W of U is nilpotent. Suppose that this is false and let H be a Schmidt subgroup of W. Then $1 < D \cap H < H$, where $D \cap H$ is a Hall normal subgroup of H. By [19, Chapter IV, Satz 5.4], there are primes r and q such that $H = H_r \rtimes H_q$, where H_r is a Sylow r-subgroup of H, H_q is a cyclic Sylow q-subgroup of H. Hence $D \cap H = H_r$. Since $H \leq W$, H is supersoluble and hence $r > q$. But $Q \simeq H/D \cap H \simeq HD/D$ is isomorphic with some subgroup of L, so $r < q$. This contradiction shows that W is nilpotent.

Now we shall show that $P \leq \mathrm{Int}_{\mathfrak{U}}(G)$. Let V be any supersoluble subgroup of G and W a Hall p'-subgroup of V. Then $PV = PW$. It is clear that M is a Hall p'-subgroup of G, so for some $x \in G$ we have $W \leq M^x \simeq U^x$. Hence W is nilpotent since W is a subgroup of the supersoluble group V. It is clear that the Sylow subgroups of W are abelian, hence W is an abelian group of exponent dividing $p-1$. Hence PV is supersoluble by [22, Chapter 1, Theorem 1.9]. Therefore $P \leq \mathrm{Int}_{\mathfrak{U}}(G)$.

Finally, we show that $genz^*(G) = 1$. Indeed, suppose that $genz^*(G) \neq 1$. Then G has a non-identity cyclic S-quasinormal subgroup, say V. The subgroup V is subnormal in G by [22, Chapter 1, Corollary 6.3]. Therefore $V \leq F(G) = P$ by [2, Chapter A, Theorem 8.8]. Moreover, if Q is a Sylow q-subgroup of G, where $q \neq p$, then V is subnormal in VQ and so $Q \leq N_G(V)$. Hence V is normal in G and therefore $V = P$ is cyclic. But then $|P| = p = |{C_p}^{\natural}|$, a contradiction. Thus $genz^*(G) = 1 = Z_{\mathfrak{U}}(G)$.

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Among other results, the paper contains a solution of one open question of Agrawal (this question may be also found on page 22 in [M. Weinstein, Between Nilpotent and Solvable, Polygonal Publishing House, 1982]). So I a

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