Cooperative oligopoly games: a probabilistic approach

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Abstract

We analyze the core of a cooperative Cournot game. We assume that when contemplating a deviation, the members of a coalition assign positive probability over all possible coalition structures that the non-members can form. We show that when the number of firms in the market is sufficiently large then the core of the underlying cooperative game is non-empty. Moreover, we show that the core of our game is a subset of the γ core.

Keywords: Cooperative game with externalities; Cournot market; core bounded rationality

1 Introduction

The issue of cooperation among firms in oligopolistic markets constantly attracts the interest of economists. Among other avenues, economists analyze cooperation in the market by examining the non-emptiness of the core of an appropriately defined cooperative game. The core consists of all these allocations of total market profits that cannot be blocked by any coalition of firms. When the members of a coalition contemplate to block (or not) an allocation they need to calculate the worth of their coalition. In a market environment such a calculation is not a trivial task, though, as the coalition's worth depends on how the non-members would act. Namely, it depends on the coalition structure that the outsiders will form.

Different beliefs about the reaction of the outsiders lead to different notions of core. The α and β cores (Aumann 1959) are based on min-max behavior on

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behalf of the non-members; the γ core (Chander and Tulkens 1997) is based on the assumption that outsiders play individual best replies to the deviant coalition. Various authors applied these core notions to the study of oligopolistic markets. Rajan (1989) used the concept of γ core and showed that it is non-empty for a market with 4 firms. Currarini and Marini (1998) built a refinement of the γ core by assuming that the deviant coalition acts as a Stackelberg leader in the product market. Zhao (1999) showed that the α and β cores of oligopolistic markets are non-empty.

The seminal works of Bloch (1996) and Ray & Vohra (1999) go one step further, as in their approaches the reactions of the outsiders –and the resulting coalition structures– are deduced via equilibrium arguments. However, Sandholm et.al (1999) showed that for an n-player game the number of different coalition structures is $O(n^n)$ and $\omega(n^{\frac{n}{2}})$. Hence computing which coalition structure the outsiders form is, in general, a particularly difficult task (in fact the problem is NP-complete).

The last result gives the motivation of the current paper. We analyze the core of a Cournot oligopoly assuming that no group of firms has the computational ability to accurately deduce the coalition structure that other firms will form. Instead, when a coalition contemplates a deviation from the grand coalition it assumes that all possible partitions of the outsiders can arise with positive probability. As a first step, we assume that coalition structures are all equiprobable. By imposing a uniform distribution over outsiders' reactions, the current paper can be seen as offering a boundedly rational approach to the literature of cooperative games with externalities.

We derive the worth function of any coalition using the above scenario and we examine the core of the corresponding game. We focus in a market with linear demand and cost functions and product homogeneity. Our main result says that when the number of firms is sufficiently large then no coalition has incentive to break full cooperation; hence the core is non-empty. We also examine the relation of our core with the γ core and we show that the former core is a subset of the latter.

In what follows, we present the model in section 2 and in section 3 we present our results. Section 4 provides concluding remarks.

2 Model

We consider a market with the set $N = \{1, 2, ..., n\}$ of firms. Firms produce a homogeneous product facing the inverse demand function $P = max\{a - Q, 0\}$ where P is the market price, $Q = q_1 + q_2 + ... + q_n$ is the market quantity, q_l is the quantity of firm l, $l = 1, 2, ..., n$ and $a > 0$. Firm l produces with the cost function $C(q_l) = cq_l, l = 1, 2, ..., n$, where $c < a$.

Let $S \subseteq N$ denote a coalition with $|S| = s$ firms and let N/S denote the complementary set of S. The value (worth) of S is the sum of its members' profits. In

order to compute this value, the members of S need to predict how the members of N/S partition themselves into coalitions. The set N/S can partition into disjoint subsets in B_{n-s} possible ways, where B_{n-s} is Bell's $(n-s)^{th}$ number (Bell (1934)). The B_{n-s} different partitions define B_{n-s} different coalition structures that coalition S might face in the market. In this paper we incorporate the assumption of bounded computational abilities of agents with regard to which structure will form. One way to do this is by assuming that the members of coalition S treat all structures as equiprobable, with probability $\frac{1}{B_{n-s}}$.

Let j denote a coalition structure with j members (coalitions), $j = 1, 2, ..., n-s$. Observe that all coalition structures with j members induce the same profit for S (as firms are symmetric). The number of coalition structures with j members is k_j, where k_j is given by the coefficients of the Bell polynomials $B_{n-s,j}$ (the Stirling numbers of the second kind) which give the number of ways to partition a set of $n-s$ objects into j groups:

$$k_j = \left\{ \begin{matrix} n-s \\ j \end{matrix} \right\} = \frac{1}{j!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} (j-i)^{n-s} \tag{1}$$

What matters for S is the number of coalitions it faces in the market (if two coalition structures have the same number of firms they induce the same profit for S). So define the function

$$f_{n,s}(j) = \frac{k_j}{B_{n-s}}, \;\; j = 1, 2, ..., n-s \tag{2}$$

which is the probability that a coalition S with s members assigns to coalition structures with j coalitions, $j = 1, 2, ..., n-s$. With a slight abuse of notation, let q_i^j denote the quantity that coalition i chooses, $i = 1, 2, ..., j$, under structure j. Let also q_s denote the quantity of coalition S. The profit function that coalition S faces is then given by

$$\pi(S) = \sum_{j=1}^{n-s} f_{n,s}(j)(a - q_s - \sum_{i=1}^{j} q_i^j - c)q_s \tag{3}$$

Moreover, from the perspective of coalition S, the profit function of coalition i within structure j is

$$\pi_i^j = (a - q_s - \sum_{i=1}^{j} q_i^j - c)q_i^j, \;\; i = 1, 2, ..., j, \;\; j = 1, 2, ..., n-s$$

Hence the maximization problems to solve for are

$$max_{q_s} \pi(S)$$

$$max_{q_i^j} \pi_i^j, \;\; i = 1, 2, ..., j, \;\; j = 1, 2, ..., n-s$$

By symmetry, the solution of the above problems will involve $q_1^j = q_2^j = ... = q_j^j \equiv q^j$, $j = 1, 2, ..., n-s$. Define

$$F_{f_{n,s}} = \sum_{j=0}^{n-s} \frac{j \cdot f_{n,s}(j)}{j+1} \tag{4}$$

Then it is easy to show that

$$q_s = \frac{1 - F_{f_{n,s}}}{2 - F_{f_{n,s}}}(a - c) \tag{5}$$

and for $j = 1, 2, ..., n-s$,

$$q_i^j = q^j = \frac{a-c}{(j+1)(2 - F_{f_{n,s}})}, \quad i = 1, 2, ..., j \tag{6}$$

Using (5),(6) and (1) in (3), we obtain $v(S)$ as[1]

$$v(S) = \frac{(\alpha - c)^2}{B_{n-s}} \frac{1 - F_{f_{n,s}}}{(2 - F_{f_{n,s}})^2} \sum_{j=0}^{n-s} \frac{\left\{ {n-s \atop j} \right\}}{j+1} \tag{7}$$

Hence our game is the pair (N, v) where v is defined by (7).

3 Properties of the game

In this section we derive some important properties of the game (N, v).

Lemma 1 *Let $v^n(s)$ denote the value for coalition $S \neq \emptyset$ with $|S| = s$ in a game with n players. For every positive integer k we have that $v^n(s) = v^{n+k}(s+k)$.*

Proof. $v^n(s) = \frac{(\alpha-c)^2}{B_{n-s}} \cdot \frac{1-F_{f_{n,s}}}{(2-F_{f_{n,s}})^2} \cdot \sum_{j=0}^{n-s} \frac{\left\{ {n-s \atop j} \right\}}{j+1} =$

$$= \frac{(\alpha-c)^2}{B_{(n+k)-(s+k)}} \cdot \frac{1-F_{f_{n,s}}}{(2-F_{f_{n,s}})^2} \cdot \sum_{j=0}^{(n+k)-(s+k)} \frac{\left\{ {(n+k)-(s+k) \atop j} \right\}}{j+1} = v^{n+k}(s+k)$$

because from (4) $F_{f_{n+k,s+k}} = F_{f_{n,s}}$. ■

An almost immediate implication of Lemma 1 is the monotonicity of $v(S)$ in $|S| = s$.

Lemma 2 *For every S with $|S| = s \leq n$, $v^n(S)$ is strictly increasing in s.*

[1]Normally, we should sum from $j = 1$ up to $n - s$ but for convenience we start from $j = 0$ with the understanding that $f_{n,s}(0) = 0$.

Proof. We will use induction on the number of players n. For the base case, $n = 2$, we have to prove that $v^2(2) > v^2(1) > v^2(0)$. We have that $v^2(2) = \left(\frac{a-c}{2}\right)^2 > \left(\frac{a-c}{3}\right)^2 = v^2(1) > 0 = v^2(0)$, so we have the base case. Assume for the induction hypothesis that in a game with n players and for an arbitrary s, $1 < s \leq n$ we have that $v^n(s) > v^n(s-1)$. We will prove that $v^{n+1}(s) > v^{n+1}(s-1)$. But this is an immediate result of lemma 1 and the induction hypothesis since $v^{n+1}(s) = v^n(s-1) > v^n(s-2) = v^{n+1}(s-1)$. ■

We are now ready to state and prove the main result of this section.

Proposition 1. *The game (N, v) has non-empty core for all $n \geq 11$.*

Proof. Since firms are identical, we can use Lemma 1 of Rajan (1989), according to which the core of a game is non empty if and only if for all $S : |S| = s \leq n$

$$\frac{v(n)}{n} \geq \frac{v(s)}{s} \tag{8}$$

It is easy to verify that inequality (8) does not hold for $3 \leq n \leq 10$, so for these values of n the core is empty.[2] We will prove the rest of the proposition using induction on n, $n \geq 11$.

Base: Table 1 in the Appendix establishes the base case ($n = 11$).

Induction hypothesis: For all $S : |S| = s \leq n$, $\frac{v^n(n)}{n} \geq \frac{v^n(s)}{s}$.

Induction step: We will show that for all $S : |S| = s \leq n+1$,

$$\frac{v^{n+1}(n+1)}{n+1} \geq \frac{v^{n+1}(s)}{s}$$

By Lemma 1 we have that $v^{n+1}(s) = v^{n+1}((s-1)+1) = v^n(s-1)$ and also that $v^{n+1}(n+1) = v^n(n)$. So we have to show that

$$\frac{v^n(n)}{n+1} \geq \frac{v^n(s-1)}{s} \tag{9}$$

From the Induction hypothesis we have

$$v^n(n) \geq \frac{n}{s-1} v^n(s-1)$$

and thus

$$(s-1)v^n(n) \geq n v^n(s-1) \tag{10}$$

Using Lemma 2,

$$v^n(n) > v^n(s-1) \tag{11}$$

[2]For $3 \leq n \leq 10$ and for all S with $|S| = 1$ it holds that $v^n(1) > \frac{v^n(n)}{n}$. See Table 2 in the Appendix.

Adding (10) and (11) gives $sv^n(n) \geq (n+1)v^n(s-1)$ which implies that (9) holds. So we have the proof for $n+1$ and thus the proposition. ■

Our result shows that for low values of n, the sign of $v^n(n)/n - v^n(S)/s$ can be negative for some s, whereas for large n the sign is always positive. To see why we need large n for core non-emptiness, we note that as n increases, both $v^n(n)/n$ and $v^n(S)/s$ decrease. However, the second term decreases faster than the first. Hence there would be an n above which the difference $v^n(n)/n - v^n(S)/s$ eventually becomes positive for all s.

3.1 Relation with the γ core

Let us now analyze the relation of our core with the γ core. Under the latter notion, the members of a deviant coalition believe that the outsiders play individual best replies. Below we show that the γ core is non-empty for our market and that our core is a subset of the γ core. Let $v_\gamma(S)$ denote the worth function of S under the scenario of γ-behavior of players. The pair (N, v_γ) will denote the game under the latter scenario; finally C_γ will denote the core of (N, v_γ) and C_f the core of our game.

Remark 1 $C_\gamma \neq \emptyset$

Proof. It is straightforward to show that $v_\gamma(S) = \dfrac{(a-c)^2}{(2+n-s)^2}$ and that $v_\gamma(N) = \dfrac{(a-c)^2}{4}$. Hence $\dfrac{v_\gamma(N)}{n} \geq \dfrac{v_\gamma(S)}{s}$ if and only if $sn^2 + (4s - 4 - 2s^2)n + s(4 + s^2 - 4s) \geq 0$ which holds. Hence the γ core is non-empty. ■

Lemma 3 $C_f \subset C_\gamma$

Proof. By Proposition 1, if $n \in \{3, 4, \cdots, 10\}$, we have $C_f = \emptyset \subset C_\gamma$. If $n \geq 11$, then $C_f \neq \emptyset$. In this case, it suffices to show that $v(S) > v_\gamma(S)$. To this end, let us give a useful representation of $v(S)$. The representation is based on harmonic numbers. Recall that the k-th harmonic number is defined as

$$h^k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} = \sum_{j=0}^{k-1} \frac{1}{1+j}$$

Let us now define

$$h_f^k = \sum_{j=0}^{k-1} \frac{f(j)}{1+j}$$

as the *k-th probabilistic harmonic number* where $f(.)$ is a probability distribution on $\{0, 1, 2, \cdots, k\}$. To make a connection between the above concept and our game, notice that we can write (7) as

$$v(S) = (a-c)^2 \frac{1 - F_{f_{n,s}}}{(2 - F_{f_{n,s}})^2} h_{f_{n,s}}^{n-s+1} \tag{12}$$

where

$$h_{f_{n,s}}^{n-s+1} = \sum_{j=0}^{n-s} \frac{f_{n,s}(j)}{1+j}$$

Notice next that

$$F_{f_{n,s}} = \sum_{j=0}^{n-s} \frac{j \cdot f_{n,s}(j)}{j+1} = \sum_{j=0}^{n-s} [1 - \frac{1}{j+1}] f_{n,s}(j) = 1 - \sum_{j=0}^{n-s} \frac{f_{n,s}(j)}{j+1}$$

Hence

$$F_{f_{n,s}} = 1 - h_{f_{n,s}}^{n-s+1} \tag{13}$$

Then combining (12) and (13) we get

$$v(S) = \frac{(h_{f_{n,s}}^{n-s+1})^2}{(1 + h_{f_{n,s}}^{n-s+1})^2} (a-c)^2 \tag{14}$$

A similar representation can be given for $v_\gamma(S)$ as well. Let h_γ^{n-s+1} denote the probabilistic harmonic number associated with (N, v_γ). Then

$$h_\gamma^{n-s+1} = \frac{1}{n-s+1}$$

and thus

$$v_\gamma(S) = \frac{(h_\gamma^{n-s+1})^2}{(1 + h_\gamma^{n-s+1})^2} (a-c)^2$$

Hence in order to show that $v(S) > v_\gamma(S)$ it suffices to show that $h_f^{n-s+1} > h_\gamma^{n-s+1}$. The last inequality always holds as h_f^{n-s+1} is a weighted average of the list of numbers $(1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n-s+1})$ all of which are no less than h_γ^{n-s+1}. ■

The γ core is based on the worst scenario for the members of the deviant coalition S: all outsiders in N/S remain separate entities. Under our scenario, the singleton coalition structure is one only of the structures that the members of S take into account. Other, more favorable structures, occur with positive probability. Hence, under our framework, deviations from the grand coalition are "easier", which explains the relation between the two cores.

Let us conclude this section noticing that representations of the form (14) hold for any probability distribution that a coalition assigns over the set of coalition structures. Consider the distributions $g_{n,s}(j)$ and $z_{n,s}(j)$. Let (N, v_g) and (N, v_z) denote the corresponding games and let C_g and C_z denote the cores of the two games. We have the following

Corollary 1 *Consider two probability distributions $g_{n,s}(j)$ and $z_{n,s}(j)$ such that $h_{g_{n,s}}^{n-s+1} > h_{z_{n,s}}^{n-s+1}$, for all s. If $C_g \neq \emptyset$ then $C_z \neq \emptyset$ as well.*

Proof. Since $h_{g_{n,s}}^{n-s+1} > h_{z_{n,s}}^{n-s+1}$ then $v_g(S) > v_z(S)$. Let $x \in C_g$. Then $\sum_{i \in S} x_i \geq v_g(S) > v_z(S)$ for any S. Hence $x \in C_z$ and $C_z \neq \emptyset$. ■

Corollary 1 is useful in allowing for a operational comparison of the cores of two (or more) different games. If we know that the core of one of the two games is non-empty, Corollary 1 gives us a (convenient) sufficient condition for the non-emptiness of the core of the other: we simply need to compare two numbers, i.e., the probabilistic harmonic numbers induced by the corresponding distributions.

4 Conclusions

This paper has incorporated elements of bounded rationality into the study of cooperative oligopoly games with externalities. When a coalition contemplates to not cooperate with the rest of the players it assumes that all possible coalition structures can form with positive probability. Whenever the number of firms is sufficiently large, then the core of the game is non-empty. Furthermore, it is a subset of the γ core.

We assumed that all coalition structures occur according to the probability distribution $f_{n,s}$. Other probability schemes can produce core non-emptiness as well. Consider distributions under which the probabilities assigned to coalition structures with relatively many coalitions are higher compared to $f_{n,s}$. Clearly, under these distributions, the core will be non-empty more often, i.e., for more values of n. The reason is that these distributions penalize the structures that are more favorable for a deviant coalition (i.e, structures with few coalitions) and give more weight to less favorable structure (i.e., structures with many coalitions).

Finally, let us mention a few extensions of the current work. The analysis of oligopolistic markets with more general demand and cost functions and/or other modes of competition (e.g., product differentiation, price competition) are natural future directions. Further, the application of the current framework to other economic environments (e.g, environmental agreements, etc.) or to abstract cooperative games with externalities is of special interest.

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Appendix

s	$\frac{v(s)}{(a-c)^2}$
1	0.0226
2	0.0252
3	0.0285
4	0.0326
5	0.0378
6	0.0446
7	0.0539
8	0.0672
9	0.0865
10	0.1111
11	0.25

Table 1: values $v(s)$ of coalition $S : |S| = s$ in a game with $n = 11$ players.

n	$\frac{v(\{i\})}{(a-c)^2}$
3	0.0865
4	0.0672
5	0.0539
6	0.0446
7	0.0378
8	0.0326
9	0.0285
10	0.0252

Table 2: values $v(\{i\})$, $n \in \{3, 4, \cdots, 10\}$