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On the relationship between Hamiltonian chaos and classical gravity

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Abstract

It is known that Hamiltonian equations of motion for low-dimensional chaotic systems are typically formulated using fractional derivatives. The evolution of such systems is governed by the fractional diffusion equation, which describes self-similar and non-Gaussian processes with strong intermittencies. We confirm, in this context, that the dynamics of a Brownian particle driven by space-time dependent fluctuations evolves towards Hamiltonian chaos and fractional diffusion. The corresponding motion of the particle has a time-dependent and nowhere vanishing acceleration. Invoking the equivalence principle of general relativity leads to the conclusion that fractional diffusion is locally equivalent to a transient gravitational field. It is shown that gravity becomes renormalizable as Newton's constant converges towards a dimensionless quantity.

1. Introduction

Fractional diffusion equations have emerged in recent years as a powerful tool for the analysis of stochastic processes and complex dynamics. In particular, fractional diffusion has been successfully linked to the study of Hamiltonian chaos in low-dimensional systems [1–4,10,11]. In this work we investigate an unexpected connection that may be established between Hamiltonian chaos and the classical theory of gravitation. The object of study is the Brownian motion of a free non-relativistic particle evolving in an environment that is random and space-time dependent. Despite its simplicity, this model offers a convenient benchmark for probing dissipative systems of higher complexity. [1]

Our three main findings are that (i) fluctuations are capable of migrating Brownian motion into Hamiltonian chaos, (ii) the Brownian particle moves as if subjected to a locally transient gravitational field and (iii) Newton's constant converges towards a dimensionless quantity as the dynamics makes the transition from fractional to the classical regime. The last finding opens the door for full renormalization of the theory, in manifest contrast with quantum gravity. The approach may be extended to include open dynamical systems and stochastic field models and may thus provide valuable insights into the long-standing issue of unification in field theory [25–28]. This is particularly attractive in light of the recently discovered decoherence mechanism responsible for the transition from quantum to classical behavior in systems strongly coupled to their environment [6,7].

It is instructive to point out that our conclusions are consistent with El Naschie's conjecture on the connection between gravitation and the Cantorian topology of space–time on or above the Planck scale ($M_{\mathrm{Pl}} \sim 10^{19}$ GeV)

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[1] We recall that, in general, there is a large spectrum of persistent fluctuations that may perturb the evolution of any dynamical system in a variety of physical settings. Examples include thermal fluctuations in statistical ensembles, Poincare resonances [21] and vacuum fluctuations in quantum physics [8].

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[12,22,24]. Our results are also relevant for theories concerned with the statistical nature of gravitational interaction in ultra-high energy physics. These models are based upon the prediction that the underlying structure of space–time undergoes large stochastic fluctuations as a result of short-distance gravitational effects [29,30].

The outline of the paper is as follows: Section 2 derives the relationship between the Langevin equation of Brownian motion and Hamiltonian chaos. A brief review of the classical Hamilton–Jacobi formalism is outlined in Section 3. The generalization of Hamilton–Jacobi equation to fractional diffusion is presented in Section 4. Section 5 establishes the explicit connection between Hamiltonian chaos and classical gravity. Renormalization is discussed in Section 6 and concluding remarks are presented in Section 7.

2. Noise driven dynamics and Hamiltonian chaos

It is well known that classical Langevin equation describes the transport of a non-relativistic Brownian particle moving in a dissipative and disordered environment [13]. Let m_0 denote the mass of the particle, γ the damping coefficient and $\eta(x,t)$ the stochastic force exerted on the particle. If there are no external potentials and the motion takes place in one dimension, the Langevin equation reads

$$m_0\ddot{x} + \gamma\dot{x} = \eta(x,t) \tag{1}$$

It is customary to assume that the stochastic force has a noise-like distribution characterized by a constant average and a shift-invariant correlation function

$$\begin{aligned} &\langle \eta(x,t) \rangle = \text{const.} \\ &\langle \eta(x,t)\eta(x',t') \rangle = D w_x(x-x') w_t(t-t') \end{aligned} \tag{2}$$

The fluctuation–dissipation theorem requires [13,14]

$$D \sim \gamma k T \tag{3}$$

where D are the diffusion coefficient and T the temperature. [2]

A convenient noise representation is provided by the delta-kicked model [15]. Under the most general circumstances, the function $\eta(x,t)$ may be factored as

$$\eta(x,t) = \zeta(x) \sum_{n=0}^{\infty} \delta(t - n\tau) \tag{4}$$

in which $\tau = 2\pi/\Omega$ stands for the period separating successive kicks and the space-dependent amplitude is considered a superposition of power terms

$$\zeta(x) = \sum_{m=0}^{\infty} a_m x^m \tag{5}$$

The sum of delta-kicks may be expanded in harmonics of Ω to obtain

$$\sum_{n=0}^{\infty} \delta(t - n\tau) = 1 + 2\sum_{n=1}^{\infty} \cos(n\Omega t) \tag{6}$$

In what follows we assume that, on a suitably chosen observation scale, the fundamental noise mode ($n = 1$) is predominant and the rest of harmonics cancel out by destructive interference. As a result, the following condition holds

$$\sum_{n=2}^{\infty} \sum_{m=0}^{\infty} a_m x^m \cos(n\Omega t) \to 0 \tag{7}$$

[2] It is important to emphasize that, according to the fluctuation–dissipation theorem, any system undergoing random perturbations must include damping as a mechanism for relaxation towards thermal equilibrium.

The Langevin model may thus be transformed into a set of coupled differential equations using the parameterization

$$\begin{aligned} y_1 &= \dot{x} \\ y_2 &= \dot{y}_1 = \frac{1 + 2\cos(y_3 t)}{m_0}\left(\sum_{m=0}^{\infty} a_m x^m\right) - \frac{\gamma}{m_0} y_1 \\ y_3 &= \Omega \end{aligned} \tag{8}$$

The system (8) resembles the evolution equations for the damped driven pendulum [16]. It has a three-dimensional phase space which is the minimum dimension required for the onset of chaos in solutions of differential equations. According to the KAM theory, the winding number

$$w = \sqrt{\frac{\gamma}{m_0 \Omega^2}} \tag{9}$$

controls the transition from unperturbed motion to weak and fully developed Hamiltonian chaos [16,17]. A manifest example of such a transition is driving with a time-dependent noise frequency $\Omega(t)$. The corresponding phase space has a rich topological structure characterized by a mixture of periodic orbits layered between chaotic islands. Fluctuations in the driving frequency generated over short time intervals lead to progressive instability and eventual breakup of KAM tori [17,23]. It is of interest to mention that the last torus destroyed by noise corresponds to the most irrational winding number, i.e. to the golden mean

$$\phi = \frac{\sqrt{5} - 1}{2} \tag{10}$$

which is a key concept of the E^{∞} theory (see [22] and included references).

3. Overview of the classical Hamilton–Jacobi formalism

It is instructive, at this point, to bridge the gap between the Langevin formalism previously outlined and the canonical approach of classical mechanics based on the Hamilton–Jacobi equation. Consider the previous example of a free non-relativistic particle of mass m_0 moving in one dimension from origin to (x, t). In the absence of any damping and disorder, its trajectory is given by

$$x(t) = \dot{x}t \tag{11}$$

The action $S(x, t)$ satisfies the Hamilton–Jacobi equation [5]

$$\frac{\partial S}{\partial t} + \frac{1}{2m_0}\left(\frac{\partial S}{\partial x}\right)^2 = 0 \tag{12}$$

and has the explicit form

$$S(x, t) = \frac{m_0}{2}\frac{x^2}{t} + S_0 \tag{13}$$

where S_0 is an arbitrary additive constant. Setting

$$\begin{aligned} -\frac{\partial S}{\partial t} &= \frac{p^2}{2m_0} = \text{const.} \\ p &= m_0 \dot{x} \end{aligned} \tag{14}$$

recovers the uniform motion expressed by (11).

As it is known, the Hamilton–Jacobi equation may be converted to a second-order partial differential equation describing standard diffusion or wave propagation. To elaborate on this point we proceed by analogy with the path integral formalism of quantum mechanics [8,31,32]. The probability amplitude for a given space–time path $x(t)$ is given by

$$\rho[x(t)] = \rho_0 \exp\{\mathrm{i}S[x(t)]\} \tag{15}$$

Assuming that the technique of analytic continuation is applicable [9,32], (15) becomes

$$\rho[x(t)] = \rho_0 \exp\{-S_{\mathrm{E}}[x(t)]\} \tag{16}$$

where $S_E(\bullet)$ represents the Euclidean action. Taking into account that momentum is a constant of motion, or $\partial^2 S_E/\partial x^2 \to 0$, the Hamilton–Jacobi equation (12) assumes the form

$$\frac{\partial\rho}{\partial t} + \frac{1}{2m_0}\frac{\partial^2\rho}{\partial x^2} = 0 \tag{17}$$

For sufficiently small space–time paths the probability amplitude is proportional to the action, that is

$$\rho[\Delta x(t)] \approx \rho_0\{1 - S_E[\Delta x(t)]\} \tag{18}$$

We shall use relation (18) in the next section.

4. Generalization of Hamilton–Jacobi formalism to fractional diffusion

For the sake of clarity we briefly summarize results obtained so far. It was found in Section 2 that, if conditions required by KAM theory are met, path dependent fluctuations are capable of migrating the classical Brownian motion into Hamiltonian chaos. The adequate formulation of this noise-driven regime requires use of fractional space and time derivatives. Section 3 has pointed out that the canonical treatment of motion in classical mechanics is based upon the Hamilton–Jacobi equation. A natural question arises on how to properly apply the Hamilton–Jacobi formulation to Hamiltonian chaos. This is the object of the current section.

Let $P(x,t)$ represent the probability density function of finding the particle at location x at instant t. Fractional diffusion equation is defined by two critical exponents (α,β) corresponding to the space and time derivatives of $P(x,t)$ [1,2]. To simplify the presentation and without any loss of generality, we set below $m_0 = \frac{1}{2}$ in (17). Fractional diffusion of the Brownian particle then takes the form

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^\alpha P}{\partial|x|^\alpha} + \frac{t^{-\beta}}{\Gamma(1-\beta)}\delta(x) \tag{19}$$

for positive time intervals $t > 0$ and point-like source functions [1,2,4]. Particular cases include Levy transport ($\beta = 1$) and fractal Brownian motion ($0 < \beta < 1$, $\alpha = 2$). The probability density stays positive if the range of the two exponents is limited to the intervals below

$$\begin{aligned} &0 < \alpha \leqslant 2 \\ &0 < \beta \leqslant 1 \end{aligned} \tag{20}$$

To simplify the formalism we adopt below the hypothesis that the integral over all possible paths connecting the initial and final space-time points can be approximated by a single contribution arising from the most dominant path. Let Δ represent the linear extent of the particle motion. Following Section 3, we note that $P(x,t)$ is equivalent to

$$P(x,t) = \frac{1}{\Delta}\left(\frac{\rho[x(t)]}{\rho_0}\right)^2 \tag{21}$$

and satisfies the normalization condition

$$\int_{-\infty}^{\infty} P(x,t)\,\mathrm{d}x = 1 \tag{22}$$

For sufficiently small paths we have from (18)

$$P(\Delta x,t) \approx \frac{1}{\Delta}\{1 - 2S_E[\Delta x(t)]\} \tag{23}$$

which shows that, up to an additive constant and a scaling factor, the probability density function and Euclidean action are identical. Under these circumstances, the asymptotic solution of the fractional diffusion equation (19) reads [2]

$$S_E[\Delta x(t)] \approx \left(-\frac{\Delta}{2}\right)\left[\frac{1}{\pi}\frac{t^\beta}{|\Delta x|^{\alpha+1}}\frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\sin\frac{\pi\alpha}{2}\right] \tag{24}$$

for ultra-short time intervals obeying

$$(\Delta x)^{\alpha} \gg t^{\beta} \tag{25}$$

Dimensional analysis of (24) in light of normalization (22) leads to

$$[t]^{\beta} = [\Delta x]^{\alpha} \tag{26}$$

in which [•] stands for the unit of time and space.

5. Fractional diffusion as locally transient non-inertial motion

We may naturally associate the following Hamilton–Jacobi equation to the Euclidean action (24) [18]:

$$E_{\mathrm{fr}} = -\frac{\partial^{\beta} S_{\mathrm{E}}}{\partial t^{\beta}} = \frac{m_0}{2}\left[\frac{\partial^{\beta}(\Delta x)}{\partial t^{\beta}}\right]^2 \tag{27}$$

where E_{fr} is the energy transported by the fractional diffusion process and $\frac{\partial^{\beta}(\Delta x)}{\partial t^{\beta}}$ generalizes the ordinary velocity corresponding to $\beta = 1$. Hence

$$v_{\mathrm{fr}} = \frac{\partial^{\beta}(\Delta x)}{\partial t^{\beta}} = \sqrt{2}\left[\Delta \frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi|\Delta x|^{1+\alpha}}\right]^{1/2} \tag{28}$$

Following the rules of fractional differentiation [20], the generalized acceleration may be obtained from (28) as

$$a_{\mathrm{fr}} = \frac{\partial^{\beta} v_{\mathrm{fr}}}{\partial t^{\beta}} = v_{\mathrm{fr}} \frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{29}$$

This expression indicates that the free Brownian particle undergoes a space-time dependent non-inertial motion for $t < \infty$. The fractional acceleration vanishes in the limit $\beta = 1$ as $\Gamma(0) \to \infty$. According to the equivalence principle of general relativity, a non-inertial frame of reference is locally identical to a gravitational field. We conclude that, under the assumption that the equivalence principle holds for non-smooth trajectories, the statistical transport of the free Brownian particle is locally equivalent to the action of a transient gravitational field. The next section attempts to show that this field may be described by a renormalizable theory.

6. Dimensional analysis and renormalization

In the relativistic theory of gravitation Newton's constant carries a negative mass dimension. Power expanding the metric around the Lorentz solution leads to a non-polynomial action in this constant [19,32] (see Appendix A). As a result, quantum gravity theories founded on general relativity are considered non-renormalizable. The object of this section is to evaluate the impact of critical exponents (α, β) on renormalizability from arguments based on dimensional analysis.

(26)–(28) may be used to determine the dimensions of energy, fractional velocity and mass starting from the scalar nature of the Euclidean action. We find, respectively

$$\begin{aligned} [E_{\mathrm{fr}}] &= t^{-\beta} \\ [v_{\mathrm{fr}}] &= [\Delta x][t]^{-\beta} = [\Delta x]^{1-\alpha} \\ [m_0] &= \frac{[E_{fr}]}{[v_{\mathrm{fr}}]^2} = [\Delta x]^{\alpha-2} \end{aligned} \tag{30}$$

In order to include Newton's constant in these considerations it is necessary to write down a fundamental field equation. The most straightforward choice is the Poisson equation of classical field theory. Let Φ_{fr} and G_{fr} represent the gravitational potential and coupling constant induced by fractional diffusion. The natural generalization of Poisson's equation in 1 + 1 space–time is

$$\frac{\partial^{2\alpha}\Phi_{\mathrm{fr}}}{\partial(\Delta x)^{2\alpha}} = 4\pi G_{\mathrm{fr}}\rho \tag{31}$$

where ρ is the equivalent source of Φ_{fr}, expressed in units of mass per unit of length. [3] Since the standard Poisson equation is recovered in the limit $\alpha = 1$, it makes sense to change the upper bound in (20) such that $0 < \alpha \leqslant 1$. The solution of (31) for a uniform source and subject to the boundary condition

$$\Phi_{fr}(\Delta x = 0) = \Phi_0 \tag{32}$$

is supplied by [20]

$$\Phi_{fr}(\Delta x) = 4\pi G_{fr}\rho \frac{(\Delta x)^{2\alpha}}{\Gamma(2\alpha + 1)} + \Phi_0 \tag{33}$$

In general relativity the gravitational potential is dimensionless ([5] and Appendix A) which can be expressed as

$$[\Phi] = [m_0]^0 \tag{34}$$

In the framework provided by fractional diffusion this constraint may be relaxed to a less restrictive requirement, that is

$$[\Phi_{fr}] = [m_0]^{\gamma(\alpha)} \tag{35}$$

where $\gamma(\alpha)$ represents an α-dependent exponent obeying

$$\lim_{\alpha \to 1} \gamma(\alpha) = 0 \tag{36}$$

It is apparent that condition (36) does not uniquely determine the explicit form of $\gamma(\alpha)$. For example, two choices from the infinite span of possible solutions are

$$\begin{aligned} \gamma(\alpha) &= 1 - \alpha^2 \\ \gamma(\alpha) &= |\ln \alpha| \end{aligned} \tag{37}$$

As it is shown below, we use this redundancy to control the mass dimension of G_{fr}.
Since

$$[\rho] = \frac{[m_0]}{[\Delta x]} \tag{38}$$

we obtain from (30), (33), (35) and (38)

$$[G_{fr}] = [m_0]^{\gamma(\alpha) - \frac{3(\alpha-1)}{\alpha-2}} \tag{39}$$

Demanding a positive or vanishing mass dimension in (39) amounts to

$$\gamma(\alpha) \geqslant \frac{3(\alpha - 1)}{\alpha - 2} \tag{40}$$

which further restricts the space of acceptable functions $\gamma(\alpha)$.

Using (36) it is seen that condition (40) is automatically satisfied for $\alpha \to 1$, that is, when the dynamics makes the transition from fractional to the classical regime.

It is instructive to consider the particular choice $\gamma(\alpha) = |\ln \alpha|$. Condition (40) leads to

$$\alpha \leqslant 0.28683 \tag{41}$$

7. Concluding remarks

We have reported the close connection between Hamiltonian chaos and fractional diffusion, on the one hand, and classical theory of gravitation on the other. It was found that fractional diffusion enables Newton's constant to converge towards a dimensionless quantity and creates the necessary framework for renormalization. The approach is built upon the Hamilton–Jacobi formalism and may be thus extrapolated to a larger class of field theories. Our work complements

[3] A similar analysis may be carried out in $3+1$ space–time. It involves a lengthy derivation and it is not included here.

similar studies linking classical gravity to space–time fluctuations, as well as several papers on unification via fractal topology [24–28].

Appendix A

For ease of reading we briefly review in this Appendix A some key points regarding the renormalization topic of quantum gravity and related theories. Additional details may be found in [8,31,32].

The potential generated by a point mass m at a distance R in Newtonian gravitation is given by

$$\varphi = -G\frac{m}{R} \tag{A.1}$$

Let $g_{\mu\nu}$ denote the components of the metric tensor ($\mu, \nu = 0, 1, 2, 3$). The potential is a dimensionless quantity related to the magnitude of g_{00}, the temporal component of the metric tensor, via

$$g_{00} = 1 + 2\varphi \tag{A.2}$$

Because g_{00} and φ are both dimensionless and since, in natural units, distance is measured as reciprocal of mass

$$[R] = [m]^{-1} \tag{A.3}$$

it follows from (A.1) that Newton's constant has a -2 mass dimension, that is

$$[G] = [m]^{-2} \tag{A.4}$$

The negative mass dimension carried by G makes gravity non-renormalizable due to the following argument: the probability amplitude for graviton-graviton scattering at a given energy E may be computed using the series expansion

$$\text{amplitude} \sim 1 + GE^2 + (GE^2)^2 + \cdots \tag{A.5}$$

where different orders correspond to various Feynman diagrams. The series (A.5) is manifestly divergent and the resulting scattering amplitude lacks physical meaning.

A similar argument may be brought up in conjunction with any attempt to quantize gravity by power expanding the metric tensor $g_{\mu\nu}$ around the Euclidean metric $g^{(0)}_{\mu\nu}$ (where $g^{(0)}_{\mu\nu}$ is referred to as the Lorentz solution)

$$g_{\mu\nu}(x) = g^{(0)}_{\mu\nu} + \sqrt{G}h_{\mu\nu} \tag{A.6}$$

In the above, the metric deviations $h_{\mu\nu}$ are associated with the graviton field. Each term of the series contains derivatives and an ever-increasing number of $h_{\mu\nu}$ fields and powers of G. The action series assumes the generic form

$$S \sim \frac{1}{16\pi G}\int d^4x[\partial h\partial h + h\partial h\partial h + h^2\partial h\partial h + \cdots] \tag{A.7}$$

where space–time indexes μ, ν have been omitted for simplicity. The action expansion is not considered polynomial due to the very existence of a non-scalar Newton constant.

Dimensional analysis indicates that a renormalizable theory must be characterized by a coupling constant having a positive or vanishing mass dimension. Quantum electrodynamics, the electroweak model and quantum chromodynamics are examples of renormalizable theories because the fine-structure constant and gauge couplings g_1, g_2 and g_3 are dimensionless.

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