Non-Grassmann mechanical model of the Dirac equation

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Abstract

We construct a new example of the spinning-particle model without Grassmann variables. The spin degrees of freedom are described on the base of an inner anti-de Sitter space. This produces both Γ^μ and $\Gamma^{\mu\nu}$-matrices in the course of quantization. Canonical quantization of the model implies the Dirac equation. We present the detailed analysis of both the Lagrangian and the Hamiltonian formulations of the model and obtain the general solution to the classical equations of motion. Comparing *Zitterbewegung* of the spatial coordinate with the evolution of spin, we ask on the possibility of space-time interpretation for the inner space of spin. We enumerate similarities between our analogous model of the Dirac equation and the two-body system subject to confining potential which admits only the elliptic orbits of the order of de Broglie wave-length. The Dirac equation dictates the perpendicularity of the elliptic orbits to the direction of center-of-mass motion.

1   Introduction

Although the true understanding of spin is achieved in the framework of quantum electrodynamics, a lot of efforts has been spent in attempts to construct the relativistic mechanical model of a spinning electron [2-19]. Just after the introduction of the electron spin as a new quantum number by Pauli [1], Uhlenbeck and Goudsmit

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suggested its naive interpretation in terms of the inner angular momentum [2]. The first papers devoted the semiclassical description of spin dynamics can be traced back up to Frenkel[1] [4] and Thomas [5]. Bargmann, Michel and Telegdi have demonstrated [7] that the models practically exactly reproduces the spin dynamics of polarized beams in uniform fields. However, the miracle is that the models based on these schemes do not produce the Dirac equation through the canonical quantization. One possible solution to the problem has been found by Berezin and Marinov [8, 9] in the framework of Grassmann mechanics. The problem here is that the Grassmann mechanics represents a rather formal mathematical construction. It leads to certain difficulties [8, 21] in attempts to use it for the description of spin effects on the semiclassical level, before the quantization. We also point out that there is no generalization of Grassmann mechanics on higher spins [10].

Hence it would be interesting to construct the non-Grassmann model of the Dirac equation. While the problem has a long history (see [2-7, 10-19, 23, 24, 26] and references therein), there appears to be no wholly satisfactory solution to date. Our believe on the existence of such a kind model is based on the following well-known observation. The Dirac spinor Ψ can be used to construct the four-dimensional current vector, $\bar{\Psi}\Gamma^\mu\Psi$, which is preserved for solutions to the Dirac equation, $\partial_\mu(\bar{\Psi}\Gamma^\mu\Psi) = 0$. Hence its null-component, $\Psi^\dagger\Psi \geq 0$, admits the probabilistic interpretation, and we expect that one-particle sector of the Dirac equation admits description in the framework of relativistic quantum mechanics (RQM). So we can look for the corresponding semiclassical model, which would lead to the Dirac equation in the course of canonical quantization.

However, it is well-known that adopting the RQM interpretation, we arrive at the rather strange and controversial picture. To remind this, we use the Dirac matrices α^i and β, to represent the Dirac equation in the form of the Schrödinger equation

$$i\hbar\partial_t\Psi = \hat{H}\Psi, \qquad \hat{H} = c\alpha^i\hat{p}_i + mc^2\beta. \qquad (1)$$

Then $\hat{H}$ may be interpreted as the Hamiltonian. If we pass from the Schrödinger to Heisenberg picture, time derivative of an operator is $i\hbar\dot{a} = [a, H]$. For the basic operators of the Dirac theory we obtain

$$\dot{x}_i = c\alpha_i, \qquad i\hbar\dot{\alpha}_i = 2(cp_i - H\alpha_i), \qquad \dot{p}_i = 0. \qquad (2)$$

[1]See a modern revision of Frenkel´s paper in [6].

Some immediate consequences of these equations are enumerated below.

- The wrong balance of the number of degrees of freedom. According to the first equation from (2), the operator $c\alpha^i$ represents velocity of the "center of charge" x^i [11]. Then the physical meaning of the operator p^i became rather obscure in both the semiclassical and the RQM framework.

  Various approaches to the problem has been considered in the literature. Schrödinger noticed [11] that besides the center of charge, x, in the Dirac theory we can construct the "center-of-mass" operator $\tilde{x}_i = x_i + \frac{1}{2}i\hbar c H^{-1}\alpha_i$ in such a way, that p^i turns out to be the mechanical momentum for $\tilde{x}$, $\dot{\tilde{x}} \sim p$ (various versions of this operator have been discussed in the works [13, 14, 15]). Following this way, Schrödinger assumed the naive interpretation of the Dirac electron as a kind of composed system (we return to this subject in Subsection 6.1). In contrast, Foldy and Wouthuysen [14] assumed that x^i does not correspond to an observable quantity.

- Free electron follows complicated trembling trajectory. The equations (2) can be solved, with the result for $x^i(t)$ being [11, 12]

$$x^i = a^i + bp^i t + c^i \exp(-\frac{2iH}{\hbar}t). \tag{3}$$

  The trajectory is a superposition of the rectilinear motion along a straight line, $a_i + bp_i t$, and the rapid oscillations with higher frequency $\frac{2H}{\hbar} \sim \frac{2mc^2}{\hbar}$. The oscillator motion is called *Zitterbewegung*. Schrödinger had compared spin with the angular momentum associated to the *Zitterbewegung*. He found [11] that they differ by the factor 2 and concluded that spin can not be identified with the *Zitterbewegung*. In our model, we would be able to construct the variables for which the identification turns out to be possible, see Subsection 6.1.

- Since the velocity operator $c\alpha^i$ has eigenvalues $\pm c$, we conclude that a measurement of a component of the velocity of a free electron is certain to lead to the result $\pm c$. Besides, since the operators α^i do not commute, components of velocity in different directions can not be instantaneously measured.

In view of this, one may assume that the Dirac equation does not admit the RQM interpretation. Other possibility might be that the basic operator $\hat{x}$ appeared in the Dirac equation do not correspond to the physically observable quantity. This possibility is supported by the seminal work of Foldy and Wouthuysen [14], where they have constructed (in a Lorentz non-covariant manner) the position operator with reasonable properties. In particular, it obeys the equation $\dot{\mathbf{X}}^{i} = \frac{p^{i}}{p^{0}}$. This leads to further complications, as the Dirac equation gives no evidence which of these two operators should be identified with the position of an electron. We return to the subject in Section 5, where we propose the Lorentz-covariant classical-mechanical analog for the Foldy-Wouthuysen operator.

To understand the controversial properties of the one-particle Dirac equation, it would be desirable to have at our disposal the spinning-particle model which leads to the Dirac equation in the course of canonical quantization. We construct and discuss the analogous model of such a kind in this work. It shows the same properties as those of the Dirac equation in the RQM interpretation. Analyzing the present model, we have been able to identify the origin of the problems, see Section 7. The modified model which turns out to be free of the problems mentioned above has been proposed in the recent work [23].

The work is organized as follows. The operators of the Dirac theory which are associated with spin are the Γ^{μ}-matrices as well as the Lorentz generators $\Gamma^{\mu\nu}$. Their commutators can be identified with the Poisson-brackets of angular momentum of five-dimensional space with the metric $\eta = (-, +, +, +, -)$ [23]. So the space can be taken as the underlying configuration space for the description of spin. As the number of independent angular-momentum components is less than dimension of the spin phase space, dynamics of spin should be restricted to an appropriate subspace which we call the spin surface[2] [23]. The corresponding configuration space turns out to be the anti-de Sitter space. To make the work self-consistent, we present the construction of the spin surface in Section 2.

Hamiltonian formulation of the model with spin sector of such a kind has been announced in [26]. In the Sections 3, 4 and 5

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[2]For the case of non-relativistic spin [24], the surface can be identified with the group manifold $SO(3)$. This has the natural structure of a fiber bundle with base S^2 and fiber $SO(2)$ The spin is identified with coordinates of the base. We are grateful to Prof. A. Nersessian for the comments on this subject.

we carry out the detailed analysis of Hamiltonian and Lagrangian formulations of the model and analyze it on the classical as well as on the quantum level. In the Section 6 the classical equations of motion are integrated and discussed in details. Section 7 is left for the conclusions.

2 Algebraic construction of relativistic spin surface

We start from the Dirac equation written in the manifestly-covariant form

$$(\hat{p}_\mu \Gamma^\mu + mc)\Psi(x^\mu) = 0, \tag{4}$$

where $\hat{p}_\mu = -i\hbar\partial_\mu$. We use the representation with hermitian Γ^0 and antihermitian Γ^i

$$\Gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{5}$$

then $[\Gamma^\mu, \Gamma^\nu]_+ = -2\eta^{\mu\nu}$, $\eta^{\mu\nu} = (-+++)$, and $\Gamma^0\Gamma^i$, Γ^0 are the Dirac matrices α^i, β [12]. We take the classical counterparts of the operators $\hat{x}^\mu$ and $\hat{p}_\mu = -i\hbar\partial_\mu$ in the standard way, which are x^μ, p^ν, with the Poisson brackets $\{x^\mu, p^\nu\} = \eta^{\mu\nu}$.

Let us look for the classical variables that could produce the Γ-matrices. According to the canonical quantization paradigm, the classical variables, say z^α, corresponding to the Hermitian operators $\hat{z}^\alpha$ should obey the quantization rule

$$[\hat{z}^\alpha, \hat{z}^\beta] = i\hbar \left. \{z^\alpha, z^\beta\} \right|_{z \to \hat{z}}. \tag{6}$$

In this equation, $[\,,\,]$ is the commutator of the operators and $\{\,,\,\}$ stands for the classical bracket[3]. To avoid the operator-ordering problems, we will consider only the sets of operators which form the Lie algebra, $[\hat{z}^\alpha, \hat{z}^\beta] = c^{\alpha\beta}{}_\gamma \hat{z}^\gamma$. So our first task is to study the algebra of Γ-matrices. We note that commutators of Γ^μ do not form closed Lie algebra, but produce $SO(1,3)$ Lorentz generators

$$[\Gamma^\mu, \Gamma^\nu] = -2i\Gamma^{\mu\nu}, \tag{7}$$

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[3]It is the Poisson (Dirac) bracket in a theory without (with) second-class constraints.

where $\Gamma^{\mu\nu} \equiv \frac{i}{2}(\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu)$. The set Γ^μ, $\Gamma^{\mu\nu}$ forms a closed algebra. Besides the commutator (7), we have

$$[\Gamma^{\mu\nu}, \Gamma^\alpha] = 2i(\eta^{\mu\alpha}\Gamma^\nu - \eta^{\nu\alpha}\Gamma^\mu),$$
$$[\Gamma^{\mu\nu}, \Gamma^{\alpha\beta}] = 2i(\eta^{\mu\alpha}\Gamma^{\nu\beta} - \eta^{\mu\beta}\Gamma^{\nu\alpha} - \eta^{\nu\alpha}\Gamma^{\mu\beta} + \eta^{\nu\beta}\Gamma^{\mu\alpha}). \qquad (8)$$

The algebra can be identified with $SO(2,3)$ Lorentz algebra with generators $\hat{J}^{AB}$:

$$[\hat{J}^{AB}, \hat{J}^{CD}] = 2i(\eta^{AC}\hat{J}^{BD} - \eta^{AD}\hat{J}^{BC} - \eta^{BC}\hat{J}^{AD} + \eta^{BD}\hat{J}^{AC}), \qquad (9)$$

assuming $\Gamma^\mu \equiv \hat{J}^{5\mu}$, $\Gamma^{\mu\nu} \equiv \hat{J}^{\mu\nu}$.

To reach the algebra starting from a classical-mechanics model, we introduce ten-dimensional "phase" space of the spin degrees of freedom, ω^A, π^B, equipped with the Poisson bracket $\{\omega^A, \pi^B\} = \eta^{AB}$. Consider the inner angular momentum

$$J^{AB} \equiv 2(\omega^A\pi^B - \omega^B\pi^A). \qquad (10)$$

Poisson brackets of these quantities form the algebra

$$\{J^{AB}, J^{CD}\}_{PB} = 2(\eta^{AC}J^{BD} - \eta^{AD}J^{BC} - \eta^{BC}J^{AD} + \eta^{BD}J^{AC}). \qquad (11)$$

Comparing (11) with (9) we conclude that the operators Γ^μ, $\Gamma^{\mu\nu}$ could be obtained by quantization of J^{AB}.

Since J^{AB} are the variables which we are interested in, we try to take them as coordinates of the space ω^A, π^B. The Jacobian of the transformation $(\omega^A, \pi^B) \to J^{AB}$ has rank equal seven[4]. So, only seven among ten functions $J^{AB}(\omega, \pi)$, $A < B$, are independent quantities. They can be separated as follows. By construction, the quantities (10) obey the identity $\epsilon^{\mu\nu\alpha\beta}J^5{}_\nu J_{\alpha\beta} = 0$, this can be solved as

$$J^{ij} = (J^{50})^{-1}(J^{5i}J^{0j} - J^{5j}J^{0i}). \qquad (12)$$

Hence we can take $J^{5\mu}$, J^{0i} as the independent variables. We could complete the set up to a base of the phase space (ω^A, π^B) adding three more coordinates, for instance ω^3, ω^5, π^5. Quantizing the complete set we obtain, besides the desired operators $\hat{J}^{5\mu}$, $\hat{J}^{0i}$, some

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[4]The rank has been computed using the program: Wolfram Mathematica 8.

extra operators $\hat{\omega}^3$, $\hat{\omega}^5$, $\hat{\pi}^5$. They are not present in the Dirac theory, and are not necessary for description of spin. So we need to reduce the dimension of our space from ten to seven imposing three constraints which we denote $T_a(\omega, \pi) = 0$, $a = 3, 4, 5$. We require the surface defined by the constraints be invariant under action of the angular-momentum generators, that is

$$\{T_a(\omega, \pi), J^{AB}\} = 0. \tag{13}$$

The only quadratic $SO(2,3)$-invariants which can be constructed from ω^A, π^B are $\omega^A\omega_A$, $\omega^A\pi_A$ and $\pi^A\pi_A$. So we restrict our model to live on the surface defined by the equations

$$T_3 \equiv \pi^A\pi_A + a_3 = 0; \tag{14}$$

$$T_4 \equiv \omega^A\omega_A + a_4 = 0, \quad T_5 \equiv \omega^A\pi_A = 0, \tag{15}$$

where a_3, a_4 are positive numbers[5]. It is called the spin surface.

The first equation from (15) states that the configuration space of the spin degrees of freedom is anti-de Sitter space.

In the Hamiltonian formulation, the equations $T_a = 0$ appear as the Dirac constraints. So, we classify them in accordance with their algebraic properties with respect to the Poisson bracket. The brackets read

$$\{T_3, T_4\} = -4T_5, \qquad \{T_3, T_5\} = -2T_3 + 2a_3,$$
$$\{T_4, T_5\} = 2T_4 - 2a_4. \tag{16}$$

Taking the combination

$$\tilde{T}_3 \equiv T_3 + \frac{a_3}{a_4}T_4, \tag{17}$$

we have the algebra

$$\{\tilde{T}_3, T_4\} = -4T_5, \qquad \{\tilde{T}_3, T_5\} = -2T_3 + 2\frac{a_3}{a_4}T_4,$$
$$\{T_4, T_5\} = -2a_4 + 2T_4, \tag{18}$$

that is Poisson bracket of $\tilde{T}_3$ with any constraint vanishes on the surface, while the Poisson brackets of constraints T_4, T_5 form an invertible matrix on the surface. In the Dirac terminology, the set

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[5]The positivity guarantees the causal dynamics of our particle, see Eq. (71).

$\tilde{T}_3$ is the first-class constraint, while T_4, T_5 is a pair of second-class constraints.

There are various reasons to take the functions T_a invariant.
A) Consistency of canonical quantization of a system with second-class constraints implies replacement the Poisson by the Dirac bracket, the latter is constructed with help of the constraints. For the momentum generators it reads $(i = 4, 5)$

$$\{J^{AB}, J^{CD}\}_{DB} = \{J^{AB}, J^{CD}\} - \{J^{AB}, T_i\}\{T_i, T_j\}^{-1}\{T_j, J^{CD}\}. \quad (19)$$

If the surface is invariant, the second term on the r. h. s. vanishes, and the Dirac bracket of J^{AB} coincides with the Poisson bracket, Eq. (11). So, as before, we have the desired algebra.
B) Presence the first-class constraint $\tilde{T}_3$ implies that we deal with a theory with local symmetry. Generators of the symmetry are proportional to the constraints [21, 27, 22]. Suppose that the first-class constraint is not invariant, that is we have $\{\tilde{T}_3, J^{AB}\}_{PB} \neq 0$. It should imply that the variables J^{AB} are not inert under the local symmetry, $\delta J^{AB} \sim \{\tilde{T}_3, J^{AB}\}_{PB} \neq 0$. Hence the non-invariant constraints would be responsible for gauging out some of the variables J^{AB}, which is not under our interest now.

Let us discuss convenient parametrization of the spin surface. The matrix $\frac{\partial(J^{5\mu}, J^{0i}, T_4, T_5, \omega^5)}{\partial(\omega^A, \pi^B)}$ has rank equal ten. So the quantities

$$J^{5\mu}, \ J^{0i}, \ T_4, \ T_5, \ \omega^5, \quad (20)$$

can be taken as coordinates of the space (ω^A, π^B). The equation $J^{AB} = 2(\omega^A \pi^B - \omega^B \pi^A)$ implies the identity

$$J^{AB}J_{AB} = 8[(\omega^A)^2(\pi^B)^2 - (\omega^A \pi_A)^2] =$$
$$8[(T_4 - a_4)(T_3 - a_3) - (T_5)^2], \quad (21)$$

then the constraint T_3 can be written in the coordinates (20) as follows:

$$T_3 = \frac{(J^{AB})^2 + 8(T_5)^2}{8(T_4 - a_4)} + a_3, \quad (22)$$

where J^{ij} are given by Eq. (12). Note that T_3 does not depend on ω^5. On the hyperplane $T_4 = T_5 = 0$ it reduces to

$$- 8a_4 T_3 = (J^{AB})^2 - 8a_3 a_4 = 0. \quad (23)$$

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Eq. (23) states that the value of $SO(2,3)$-Casimir operator $(J^{AB})^2$ is equal to $8a_3a_4$. In quantum theory, for the operators (9), (8) we have: $\hat{J}^{AB}\hat{J}_{AB} = 20\hbar^2$. So we can fix the product of our parameters as

$$a_3a_4 = \frac{5\hbar^2}{2}.\qquad(24)$$

As we have pointed out above, the function T_3 represents generator of local symmetry. The coordinate ω^5 is not inert under the symmetry, $\delta\omega^5 \sim \{T_3, \omega^5\} \neq 0$. Hence ω^5 is gauge non-invariant (hence non-observable) variable.

Summing up, we have restricted dynamics of spin on the surface (14), (15). If (20) are taken as coordinates of the phase space, the surface is the hyperplane $T_4 = T_5 = 0$ with the coordinates $J^{5\mu}, J^{0i}, \omega^5$ subject to the condition (23). Since ω^5 is gauge non-invariant coordinate, we can discard it. It implies that we can quantize $J^{5\mu}, J^{0i}$ instead of the initial variables ω^A, π^B.

Following the canonical quantization paradigm, the variables must be replaced by Hermitian operators[6] with commutators resembling the Poisson bracket

$$[\ ,\ ] = i\hbar \{\ ,\ \}|_{J\to\hat{J}}.\qquad(25)$$

Similarly to the case of Γ-matrices, brackets of the variables $J^{5\mu}$, J^{0i} do not form closed Lie algebra. The non closed brackets are

$$\{J^{5i}, J^{5j}\} = \{J^{0i}, J^{0j}\} = -2J^{ij},\qquad(26)$$

where J^{ij} is given by Eq. (12). Adding them to the initial variables, we obtain the set $J^{AB} = (J^{5\mu}, J^{0i}, J^{ij})$ which obeys the desired algebra (11).

According to Eqs. (9), (11) the quantization is achieved replacing the classical variables $J^{5\mu}, J^{\mu\nu}$ on Γ-matrices[7]. We assume that ω^A has a dimension of length, then J^{AB} has the dimension of the Planck's constant. Hence the quantization rule is

$$J^{5\mu} \to \hbar\Gamma^{\mu}, \quad J^{\mu\nu} \to \hbar\Gamma^{\mu\nu}.\qquad(27)$$

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[6]The matrices Γ^{μ}, $\Gamma^{\mu\nu}$ are Hermitian operators with respect to the scalar product $(\Psi_1, \Psi_2) = \Psi_1^{\dagger}\Gamma^0\Psi_2$.

[7]Replacing (12) by an operator $\hat{J}^{ij}(\Gamma^{\mu}, \Gamma^{0i})$ we arrange the operators Γ in such a way, that $\hat{J}^{ij}(\Gamma) = \Gamma^{ij}$.

This implies that the Dirac equation (4) can be produced by the constraint

$$T_2 \equiv p_\mu J^{5\mu} + mc\hbar = 0. \tag{28}$$

3 Hamiltonian formulation

Our next task is to formulate the variational problem for our spinning particle. As it has been discussed above, we need a theory which implies the constraints (14), (15) and (28). Since they are written on the phase space, it is natural to start from construction of an action functional in the Hamiltonian formalism.

Hamiltonian action of non-singular (that is non-constrained) system reads $\int d\tau [P\dot{Q} - H(Q, P)]$. When the theory is singular, Hamiltonian action acquires more complicated form [20, 21, 22]. To remind its structure, as well as to justify our choice of the action (38), (39), we outline the Hamiltonian formulation for singular Lagrangian theory of a special form. For our purposes it will be sufficient to discuss the Lagrangian

$$S = \int d\tau L(Q, \dot{Q}, e_a), \tag{29}$$

that is the configuration variables are divided on two groups, Q and e_a, where e_a enter into the action without derivatives. We also suppose that

$$\det \frac{\partial^2 L}{\partial \dot{Q} \partial \dot{Q}} \neq 0. \tag{30}$$

Following the standard prescription, we construct Hamiltonian formulation for the action (29). Canonical momenta are defined by

$$P = \frac{\partial L}{\partial \dot{Q}}, \tag{31}$$

$$\pi_{ea} = \frac{\partial L}{\partial \dot{e}_a} = 0. \tag{32}$$

The phase space $(Q, P; e_a, \pi_{ea})$ is equipped with canonical (nondegenerated) Poisson bracket. Nonvanishing brackets are $\{Q, P\}_{PB} = 1$, $\{e_a, \pi_{ea}\}_{PB} = 1$.

According to Eq. (32), in the theory there are the primary constraints $\pi_{ea} = 0$. Due to the condition (30), Eqs. (31) can be resolved with respect to $\dot{Q}$, $\dot{Q} = f(Q, P, e_a)$. Using these expressions, we construct complete Hamiltonian according to the standard rule

$$H(Q, P, e_a, \pi_{ea}, \lambda_{ea}) =$$
$$\left[P\dot{Q} + \pi_{ea}\dot{e}_a - L(Q, \dot{Q}, e_a) \right]\Big|_{\dot{Q}=f(Q,P,e),\pi_{ea}=0} + \lambda_{ea}\pi_{ea}, \qquad (33)$$

where $\lambda_{ea}(\tau)$ are the Lagrangian multipliers for the primary constraints (32). Given complete Hamiltonian and the Poisson brackets, the temporal evolution of any quantity A is given by the equation $\dot{A} = \{A, H\}_{PB}$. In particular, the basic variables obey the Hamiltonian equations

$$\dot{Q} = \{Q, H\}_{PB}, \quad \dot{P} = \{P, H\}_{PB}, \quad \dot{e}_a = \lambda_{ea}, \quad \dot{\pi}_{ea} = 0. \qquad (34)$$

The condition of preservation in time for the primary constraints generally implies the secondary constraints denoted by T_a

$$\dot{\pi}_{ea} = \{\pi_{ea}, H\}_{PB} = -\frac{\partial H}{\partial e_a} \equiv T_a = 0. \qquad (35)$$

Dynamics of the system can be equivalently obtained starting from the Hamiltonian action functional

$$S_H = \int d\tau \left[P\dot{Q} + \pi_{ea}\dot{e}_a - H(Q, P, e_a, \pi_{ea}, \lambda_{ea}) \right]. \qquad (36)$$

Performing variation of the action with respect to all the variables $(Q, P, e_a, \pi_{ea}, \lambda_{ea})$, we obtain the complete set of equations governing the dynamics, (32), (34), (35).

The procedure can be inverted. Given Hamiltonian action (36), we can restore the corresponding Lagrangian formulation. In the process, we can omit π_{ea} because e_a enter into the formulation without derivatives. Assuming that the Hamiltonian (33) is at most quadratic function of the momenta P, we write it in the form $H = \frac{1}{2}PG(Q, e)P + g(Q, e)$. We solve the Hamiltonian equations $\dot{Q} = GP$ with respect to P. The solution reads $P = \tilde{G}\dot{Q}$, where $\tilde{G}$ is the inverse matrix for G. We substitute these P back into Eq. (36), obtaining the Lagrangian action

$$S = \int d\tau \left[\frac{1}{2}\dot{Q}\tilde{G}\dot{Q} - g(Q, e) \right]. \qquad (37)$$

Applying the hamiltonization procedure to the action, we expect to arrive back at the Hamiltonian formulation (36). We do this for our model in Sect. 5.

We are interested in to construct the variational problem which implies the constraints (14), (15) and (28). The analysis made above suggests to take the following Hamiltonian action[8]

$$S_H = \int d\tau \left(p_\mu \dot{x}^\mu + \pi_A \dot{\omega}^A + \pi_{e_l} \dot{e}_l - H \right), \tag{38}$$

with the Hamiltonian constructed on the base[9] of constraints T_2, T_3 and T_4

$$H = \frac{e_l}{2} T_l + \lambda_{e_l} \pi_{e_l} =$$
$$\frac{e_2}{2}(p_\mu J^{5\mu} + mc\hbar) + \frac{e_3}{2}[(\pi^A)^2 + a_3] + \frac{e_4}{2}[(\omega^A)^2 + a_4] + \lambda_{e_l} \pi_{e_l}. \tag{39}$$

Variation of S_H with respect to momenta gives us the Hamiltonian equations for position variables,

$$\frac{\delta S_H}{\delta p_\mu} = 0 \quad \Rightarrow \quad \dot{x}^\mu = \frac{e_2}{2} J^{5\mu} = e_2(\omega^5 \pi^\mu - \pi^5 \omega^\mu), \tag{40}$$

$$\frac{\delta S_H}{\delta \pi_\mu} = 0 \quad \Rightarrow \quad \dot{\omega}^\mu = e_3 \pi^\mu + e_2 \omega^5 p^\mu, \tag{41}$$

$$\frac{\delta S_H}{\delta \pi_5} = 0 \quad \Rightarrow \quad \dot{\omega}^5 = e_3 \pi^5 + e_2 p_\mu \omega^\mu, \tag{42}$$

$$\frac{\delta S_H}{\delta \pi_{e_l}} = 0 \quad \Rightarrow \quad \dot{e}_l = \lambda_{e_l}. \tag{43}$$

Variation of S_H with respect to the position variables gives the equations for momenta,

$$\frac{\delta S_H}{\delta x^\mu} = 0 \quad \Rightarrow \quad \dot{p}_\mu = 0, \tag{44}$$

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[8]The Hamiltonian (39) is slightly different from those of the work [26]. It will lead to more simple Lagrangian. Both theories have the same physical sector.

[9]Peculiar property of the model is that the equation $T_5 = 0$ appears as the third-stage constraint, from the condition of preservation in time of the constraint T_4. So, it is not necessary to include T_5 into the Hamiltonian.

$$\frac{\delta S_H}{\delta \omega^\mu} = 0 \quad \Rightarrow \quad \dot{\pi}^\mu = e_2 \pi^5 p^\mu - e_4 \omega^\mu, \tag{45}$$

$$\frac{\delta S_H}{\delta \omega^5} = 0 \quad \Rightarrow \quad \dot{\pi}^5 = e_2 p_\mu \pi^\mu - e_4 \omega^5. \tag{46}$$

Besides we obtain the primary constraints $\pi_{e_l} = 0$, which appear from variation with respect to the Lagrange multipliers λ_{e_l}. At last, the variation of S_H with respect to e_l gives a part of the desired constraints of the theory,

$$\frac{\delta S_H}{\delta e_2} = 0 \quad \Rightarrow \quad p_\mu J^{5\mu} + mc\hbar = 0, \tag{47}$$

$$\frac{\delta S_H}{\delta e_3} = 0 \quad \Rightarrow \quad (\pi^A)^2 + a_3 = 0, \tag{48}$$

$$\frac{\delta S_H}{\delta e_4} = 0 \quad \Rightarrow \quad (\omega^A)^2 + a_4 = 0. \tag{49}$$

Preservation in time of the constraint $(\omega^A)^2 + a_4 = 0$ gives the constraint $T_5 = \omega^A \pi^A = 0$, which preservation, in turn, leads to the $e_4 - \frac{a_3}{a_4} e_3 = 0$. To see this, we use equations of motion (41), (42), (45) and (46),

$$[(\omega^A)^2 + a_4]^\cdot = 2\dot{\omega}^A \omega^A = 2e_3 \omega^A \pi^A = 0 \Rightarrow \omega^A \pi^A = 0; \tag{50}$$

$$(\omega^A \pi^A)^\cdot = e_3(\pi^A)^2 - e_4(\omega^A)^2 = 0 \Rightarrow e_4 - \frac{a_3}{a_4} e_3 = 0. \tag{51}$$

Time derivative of the constraint $e_4 - \frac{a_3}{a_4} e_3 = 0$ determines the Lagrangian multiplier λ_{e4}, $\lambda_{e4} = \frac{a_3}{a_4} \lambda_{e3}$. Preservation in time of the constraints $T_2 = 0$ and $T_3 = 0$ gives no new equations.

In the result, we have two pairs of second-class constraints, $e_4 - \frac{a_3}{a_4} e_3 = 0$, $\pi_{e_4} = 0$ and $T_4 = 0$, $T_5 = 0$, while $\pi_{e_2} = 0$, $\pi_{e_3} = 0$, $T_2 = 0$ and $\tilde{T}_3 = T_3 + \frac{a_3}{a_4} T_4 = 0$ represent first-class constraints. The first two of them are primary, and according to the general theory [20, 21, 22, 27] it indicates on invariance of S_H with respect to two-parameter group of local transformations.

One of them is the well-known reparametrization transformation, its infinitesimal form is

$$\delta Y = \alpha \dot{Y}, \quad \delta e_l = (\alpha e_l)^\cdot, \quad \delta \pi_{e_l} = 0, \quad \delta \lambda_{e_l} = (\delta e_l)^\cdot, \tag{52}$$

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where $Y = (x^\mu, p_\mu, \omega^A, \pi_B)$ and $\alpha = \alpha(\tau)$ is an arbitrary function. Variation of S_H with respect to (52) is equal to the total derivative

$$\delta S_H = \int d\tau [\alpha(p_\mu \dot{x}^\mu + \pi_A \omega^A - \frac{1}{2} e_l T_l)]^\cdot. \tag{53}$$

The other symmetry is given by[10]

$$\delta x^\mu = 0, \quad \delta p_\mu = 0, \tag{54}$$

$$\delta \omega^A = \epsilon e_3 \pi^A, \quad \delta \pi^A = -\epsilon e_3 \omega^A, \tag{55}$$

$$\delta e_2 = 0, \quad \delta e_3 = (\epsilon e_3)^\cdot, \quad \delta e_4 = (\epsilon e_4)^\cdot, \tag{56}$$

$$\delta \lambda_{e_l} = (\delta e_l)^\cdot, \quad \delta \pi_{e_l} = 0, \tag{57}$$

Variation of S_H is equal to a total derivative as well.

Comment. In the Berezin-Marinov model [8] the Dirac equation is implied by the supersymmetric gauge transformations. In our model the Dirac equation is associated with the ε-symmetry. So, it represents the bosonic analogue of BM supersymmetric transformations.

According to general theory [20, 21, 22], the local symmetries indicate on two-parametric degeneracy in solutions to equations of motion. Indeed, we note that $\lambda_{e_2}(\tau)$ and $\lambda_{e_3}(\tau)$ cannot be determined neither with the system of constraints nor with the dynamical equations. As a consequence (see Eq. (43)), the variables e_2 and e_3 turn out to be the arbitrary functions as well. Since they enter into the equations for x^μ, ω^A, and π_A, general solution for these variables contains, besides the arbitrary integration constants, the arbitrary functions e_a. The variables with ambiguous evolution have no physical meaning [21]. Hence our next task is to find candidates for observables, which are variables with unambiguous dynamics. Equivalently, we can look for the gauge-invariant variables.

The only unambiguous among the initial variables is p^μ, see Eq. (44). x^μ has one-parameter ambiguity due to e_2, while ω^A and π^B have two-parameter ambiguity due to e_2 and e_3. Inspection of equations of motion for ω^A and π^B allows us to construct more

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[10]It is Hamiltonian counterpart of the Lagrangian symmetry which will be discussed below, see Eqs. (88)-(90).

quantities with one-parameter ambiguity, they turn out to be the angular-momentum components

$$\dot{J}^{\mu\nu} = e_2(p^\mu J^{5\nu} - p^\nu J^{5\mu}), \tag{58}$$

$$\dot{J}^{5\mu} = e_2 p_\nu J^{\nu\mu}. \tag{59}$$

We also point out that J^{AB} is invariant under the ε-transformation (55). The constraints (14) and (15) determine the square of the angular-momentum tensor, see Eq. (23). Besides, they guarantee that $J^{5\mu}$ is the time-like vector

$$(J^{5\mu})^2 = -4(a_3(\omega^5)^2 + a_4(\pi^5)^2) < 0, \tag{60}$$

for positive values of a_3, a_4.

To proceed further, it is instructive to compare equations for x^μ, $J^{\mu\nu}$ and $J^{5\mu}$ with those for x^μ of spinless relativistic particle (see, for example, [28, 29]). If we use the parametric representation $x^\mu(\tau) = (ct(\tau), x^i(\tau))$ for the trajectory $x^i(t)$, the spinless particle can be described by the action

$$S = \int d\tau \left(\frac{1}{2e} \dot{x}^2 - \frac{e}{2} m^2 c^2 \right). \tag{61}$$

It implies the Hamiltonian equations

$$\dot{x}^\mu = e p^\mu, \qquad \dot{p}_\mu = 0, \tag{62}$$

as well as the constraint

$$p^2 + m^2 c^2 = 0. \tag{63}$$

The auxiliary variable e enters into general solution for $x^\mu(\tau)$ as an arbitrary function. The ambiguity reflects the freedom in the choice of parametrization for the particle trajectory

$$\begin{array}{ll} \tau \to \tau' = \tau + \alpha, & \\ x^\mu(\tau) \to x'^\mu(\tau') = x^\mu(\tau), \quad \text{then} & \begin{array}{l} \delta x^\mu = \alpha \dot{x}^\mu, \\ \delta e = (\alpha e)\dot{}. \end{array} \\ e(\tau) \to e'(\tau') = (1 + \dot{\alpha}) e(\tau). & \end{array} \tag{64}$$

The action (61) turns out to be invariant under the reparametrizations.

By construction, the expression for the physical trajectory $x^i(t)$ is obtained resolving the equation $x^0 = x^0(\tau)$ with respect of τ, $\tau = \tau(x^0)$, then $x^i(t) \equiv x^i(\tau(x^0))$. The last equality implies

$$\frac{dx^i}{dt} = c\frac{\dot{x}^i}{\dot{x}^0} = c\frac{p^i}{p^0}. \tag{65}$$

As it should be, the physical coordinate $x^i(t)$ has unambiguous evolution.

To see physical meaning of the constraint (63), we take square of Eq. (65) and use (63) to estimate the particle's speed

$$\left(\frac{dx^i}{dt}\right)^2 = c^2 \frac{(p^i)^2}{(p^i)^2 + m^2 c^2} \Rightarrow \left(\frac{dx^i}{dt}\right)^2 < c^2. \tag{66}$$

Hence the constraint (63) guarantees that the particle's speed can not exceed the speed of light.

The same result can be reproduced in the Lagrangian formulation. Indeed, variation of the action (61) with respect to e implies that x^μ is the time-like vector, $(\dot{x}^\mu)^2 = -e^2 m^2 c^2 < 0$. This also allows us to estimate the particle's speed

$$\left(\frac{dx^i}{dt}\right)^2 = c^2 \left(\frac{\dot{x}^i}{\dot{x}^0}\right)^2 = c^2 \left(1 - \frac{e^2 m^2 c^2}{(\dot{x}^0)^2}\right) \Rightarrow \left(\frac{dx^i}{dt}\right)^2 < c^2. \tag{67}$$

Let us return to the spinning particle. We note both x^μ and J^{AB} are invariant under ε-transformations (54), (55). So the ambiguity presented in equations of motion (40), (58) and (59) is due to the reparametrization symmetry (52). In accordance with this observation, we can assume that the functions $x^\mu(\tau)$, $J^{AB}(\tau)$ represent the physical variables $x^i(t)$ and $J^{AB}(t)$ in the parametric form. Then equations of motion for the physical variables read

$$\frac{dx^i}{dt} = c \frac{J^{5i}}{J^{50}}, \tag{68}$$

$$\frac{dJ^{\mu\nu}}{dt} = c \frac{\dot{J}^{\mu\nu}}{\dot{x}^0} = 2c \frac{p^\mu J^{5\nu} - p^\nu J^{5\mu}}{J^{50}}, \tag{69}$$

$$\frac{dJ^{5\mu}}{dt} = c \frac{\dot{J}^{5\mu}}{\dot{x}^0} = 2c \frac{p_\nu J^{\nu\mu}}{J^{50}}. \tag{70}$$

As it should be, they are unambiguous. General solution to these equations will be obtained in Sect. 6.

Although there is no the mass-shell constraint $p^2 + m^2 c^2 = 0$ in our model, our particle's speed cannot exceed the speed of light. To see this, we take square of Eq. (68) and use the fact that $J^{5\mu}$ is the time-like vector, see Eq. (60), to estimate the particle's speed

$$\left(\frac{dx^i}{dt}\right)^2 = c^2 \frac{(J^{5i})^2}{(J^{50})^2} = c^2 \frac{(J^{5i})^2}{(J^{5i})^2 + 4(a_3(\omega^5)^2 + a_4(\pi^5)^2)} \Rightarrow$$

$$\left(\frac{dx^i}{dt}\right)^2 < c^2. \tag{71}$$

Note that the spinning particle has causal dynamics for both positive and negative values of p^2.

4 Lagrangian formulation

In this section we reconstruct the configuration-space formulation of the theory (38). We obtain various equivalent forms of the Lagrangian action and analyze the Lagrangian equations. While this is much less systematic procedure as compare with the Dirac method, at the end we arrive at the essentially the same results as those of the previous section. In what follows, we suppress the four-dimensional indexes, for example, we write $x^\mu\omega_\mu \equiv (x\omega)$. We introduce also the following condensed notation: $P_a = (p_\mu, \pi_\nu, \pi_5)$, $Q^a = (x^\mu, \omega^\nu, \omega^5)$ and $a_2 \equiv mc\hbar$. Then the Hamiltonian (39) reads

$$H = \frac{1}{2}P_a G^{ab} P_b + \frac{1}{2}e_4(\omega^A)^2 + \frac{a_l}{2}e_l, \qquad l = 2, 3, 4, \tag{72}$$

where $G^{ab}(\omega^A, e_l)$ is a 9×9 non-singular matrix, $\det G = e_2^8 e_3(\omega^5)^6(\omega^A)^2$, which is schematically written as

$$G^{ab} = \begin{pmatrix} 0_{(4\times4)} & e_2\omega^5\eta^{\mu\nu}_{(4\times4)} & e_2\omega^\nu_{(4\times1)} \\ e_2\omega^5\eta^{\mu\nu}_{(4\times4)} & e_3\eta^{\mu\nu}_{(4\times4)} & 0_{(4\times1)} \\ e_2\omega^\mu_{(1\times4)} & 0_{(1\times4)} & -e_3{}_{(1\times1)} \end{pmatrix}. \tag{73}$$

The notation $0_{(4\times4)}$ indicates that the first block of the matrix G^{ab} is composed of the null 4×4-matrix, and so on.

To find the Lagrangian, we write Hamiltonian equations for the position variables Q^a, $\dot{Q}^a = G^{ab}P_b$, and resolve them with respect to P_a, $P_a = \tilde{G}_{ab}\dot{Q}^b$, where $\tilde{G}$ is the inverse matrix for G. We substitute these P_a back into the Hamiltonian action (38), which gives the desired Lagrangian

$$L = \frac{1}{2}\tilde{G}_{ab}\dot{Q}^a\dot{Q}^b - \frac{a_l}{2}e_l - \frac{1}{2}e_4(\omega^A)^2. \tag{74}$$

Hence the problem of restoring the Lagrangian formulation is re-

duced to obtaining the inverse matrix for G^{ab}. It reads

$$\tilde{G}_{ab} = \begin{pmatrix} \tilde{G}_{\mu\nu} & -\frac{e_2\omega^5}{e_3}\tilde{G}_{\mu\nu} & \frac{1}{e_2(\omega^A)^2}\omega_\nu \\ -\frac{e_2\omega^5}{e_3}\tilde{G}_{\mu\nu} & \frac{1}{e_3(\omega^A)^2}\omega_\mu\omega_\nu & -\frac{\omega^5}{e_3(\omega^A)^2}\omega_\nu \\ \frac{1}{e_2(\omega^A)^2}\omega_\mu & -\frac{\omega^5}{e_3(\omega^A)^2}\omega_\mu & \frac{(\omega^5)^2}{e_3(\omega^A)^2} \end{pmatrix}, \tag{75}$$

where

$$\tilde{G}_{\mu\nu} = -\frac{e_3}{(e_2\omega^5)^2}\left[\eta_{\mu\nu} - \frac{\omega_\mu\omega_\nu}{(\omega^A)^2}\right]. \tag{76}$$

It is invertible, with the inverse matrix being

$$G^{\mu\nu} \equiv -\frac{(e_2\omega^5)^2}{e_3}\left[\eta^{\mu\nu} - \frac{\omega^\mu\omega^\nu}{(\omega^5)^2}\right]. \tag{77}$$

We substitute these expressions into Eq. (74) and obtain the manifest form of our Lagrangian

$$L = -\frac{e_3}{2(e_2\omega^5)^2}\dot{x}^2 + \frac{1}{e_2\omega^5}(\dot{x}\dot{\omega}) + \frac{1}{2e_3(\omega^A)^2}\left(\omega_A\dot{\omega}^A - \frac{e_3}{e_2\omega^5}(\dot{x}\omega)\right)^2 - \frac{e_2}{2}a_2 - \frac{e_3}{2}a_3 - \frac{e_4}{2}[(\omega^A)^2 + a_4]. \tag{78}$$

It is manifestly Poincare-invariant. The variables ω^5, e_l, are scalars under the Poincare transformations. The remaining variables transform according to the rule

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \qquad \omega'^\mu = \Lambda^\mu{}_\nu \omega^\nu. \tag{79}$$

To discuss the local symmetries, it is convenient to rewrite L in an equivalent form by rearranging its terms as follows:

$$L = \frac{1}{2}\tilde{G}_{\mu\nu}Dx^\mu Dx^\nu + \frac{1}{2e_3}(\dot{\omega}^A)^2 - \frac{e_4}{2}(\omega^A)^2 - \frac{a_l}{2}e_l, \quad l = 2, 3, 4. \tag{80}$$

$\tilde{G}_{\mu\nu}$ is the upper-left block of the matrix $\tilde{G}_{ab}$, see Eq. (76). Besides, in Eq. (80) it has been denoted

$$Dx^\mu = \dot{x}^\mu - I^{5\mu}, \tag{81}$$

$$I^{5\mu} = \frac{e_2}{e_3}(\omega^5\dot{\omega}^\mu - \dot{\omega}^5\omega^\mu). \tag{82}$$

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The Lagrangian implies the following expressions for conjugate momenta

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \tilde{G}_{\mu\nu} D x^\nu = \tilde{G}_{\mu\nu}(\dot{x}^\nu - I^{5\nu}), \tag{83}$$

$$\pi_\mu = \frac{\partial L}{\partial \dot{\omega}^\mu} = \frac{1}{e_3}(\dot{\omega}_\mu - e_2 \omega^5 p_\mu), \tag{84}$$

$$\pi^5 = \frac{\partial L}{\partial \dot{\omega}_5} = \frac{1}{e_3}(\dot{\omega}^5 - e_2(\omega p)). \tag{85}$$

Local symmetries of the theory form the two-parameter group. The first is the reparametrization symmetry with the parameter $\alpha(\tau)$

$$\delta_\alpha x^\mu = \alpha \dot{x}^\mu, \qquad \delta_\alpha \omega^A = \alpha \dot{\omega}^A, \qquad \delta_\alpha e_l = (\alpha e_l)^\cdot, \tag{86}$$

The variations (86) imply

$$\delta \tilde{G}_{\mu\nu} = (\alpha \tilde{G}_{\mu\nu})^\cdot - 2\dot{\alpha}\tilde{G}_{\mu\nu}, \quad \delta D x^\mu = (\alpha D x^\mu)^\cdot,$$
$$\delta\left(\frac{1}{2e_3}(\dot{\omega}^A)^2\right) = \left(\alpha \frac{1}{2e_3}(\dot{\omega}^A)^2\right)^\cdot;$$
$$\delta(e_4[(\omega^A)^2 + a_4]) = (\alpha e_4[(\omega^A)^2 + a_4])^\cdot. \tag{87}$$

Using these equalities it is easy to verify that $\delta L = (\alpha L)^\cdot$.

From equations (54)-(56), (84) and (85) we expect that the second symmetry would be

$$\delta_\epsilon x^\mu = 0, \tag{88}$$

$$\delta_\epsilon \omega^\mu = \epsilon \dot{\omega}^\mu - \epsilon e_2 \omega^5 p^\mu, \qquad \delta_\epsilon \omega^5 = \epsilon \dot{\omega}^5 - \epsilon e_2(\omega p), \tag{89}$$

$$\delta_\epsilon e_2 = 0, \quad \delta_\epsilon e_3 = (\epsilon e_3)^\cdot, \quad \delta_\epsilon e_4 = (\epsilon e_4)^\cdot. \tag{90}$$

To simplify the computations, we combine δ_ϵ and δ_α, $\delta = \delta_\epsilon + \delta_\alpha$, with $\alpha = -\epsilon$. The combination reads

$$\delta x^\mu = -\epsilon \dot{x}^\mu, \tag{91}$$

$$\delta \omega^\mu = -\epsilon e_2 \omega^5 p^\mu, \qquad \delta \omega^5 = -\epsilon e_2(\omega p), \tag{92}$$

$$\delta e_2 = -(\epsilon e_2)^\cdot, \quad \delta e_3 = \delta e_4 = 0. \tag{93}$$

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where p^μ is given by (83). These transformations imply

$$\delta(\omega^A)^2 = 0, \tag{94}$$

$$\delta Dx^\mu = -(\epsilon Dx^\mu)\dot{} - \epsilon e_2 \left[\omega^5 \left(\frac{\dot\omega^\mu}{e_3} \right)\dot{} - \right.$$
$$\left. \left(\frac{\dot\omega^5}{e_3} \right)\dot{} \omega^\mu \right] - \frac{e_2}{e_3} \delta(\omega^5 \dot\omega^\mu - \dot\omega^5 \omega^\mu), \tag{95}$$

$$\delta\tilde{G}_{\mu\nu} = 2 \left[\frac{(\epsilon e_2)\dot{}}{e_2} + \frac{\epsilon e_2(\omega p)}{\omega^5} \right] \tilde{G}_{\mu\nu} - \frac{\epsilon e_3}{e_2 \omega^5 (\omega^A)^2} (p_\mu \omega_\nu + p_\nu \omega_\mu), \tag{96}$$

then

$$(\delta\tilde{G}_{\mu\nu}) Dx^\mu Dx^\nu = 2 \frac{(\epsilon e_2)\dot{}}{e_2} (pDx). \tag{97}$$

Using these formulas variation of the action (80) under the transformations (91)-(93) reads, up to total-derivative terms

$$\delta L = \frac{1}{2} (\delta\tilde{G}_{\mu\nu}) Dx^\mu Dx^\nu + p_\mu \delta Dx^\mu - \left(\frac{\dot\omega^A}{e_3} \right)\dot{} \delta\omega^A =$$
$$\frac{(\epsilon e_2)\dot{}}{e_2} (pDx) -$$
$$p_\mu \left[(\epsilon Dx^\mu)\dot{} + \epsilon e_2 \left[\omega^5 \left(\frac{\dot\omega^\mu}{e_3} \right)\dot{} - \left(\frac{\dot\omega^5}{e_3} \right)\dot{} \omega^\mu \right] + \frac{e_2}{e_3} \delta(\omega^5 \dot\omega^\mu - \dot\omega^5 \omega^\mu) \right] +$$
$$\epsilon e_2 p_\mu \left[\omega^5 \left(\frac{\dot\omega^\mu}{e_3} \right)\dot{} - \left(\frac{\dot\omega^5}{e_3} \right)\dot{} \omega^\mu \right] =$$
$$\frac{(\epsilon e_2)\dot{}}{e_2} (pDx) + \epsilon(\dot{p}Dx) - \frac{e_2}{e_3} p_\mu \delta(\omega^5 \dot\omega^\mu - \dot\omega^5 \omega^\mu). \tag{98}$$

Compute the last term

$$-\frac{e_2}{e_3} p_\mu \delta(\omega^5 \dot\omega^\mu - \dot\omega^5 \omega^\mu) =$$
$$\frac{e_2}{e_3} \left(\epsilon e_2(\omega p)(\dot\omega p) + \omega^5(\epsilon e_2 \omega^5 p^\mu)\dot{} p_\mu - (\epsilon e_2(\omega p))\dot{}(\omega p) - \dot\omega^5 \epsilon e_2 \omega^5 p^2 \right) =$$
$$\frac{e_2}{e_3} \left(\epsilon e_2(\omega p))(\omega p)\dot{} - \epsilon e_2(\omega p)(\omega\dot{p}) + \omega^5 \epsilon e_2 \dot\omega^5 p^2 + (\omega^5)^2(\epsilon e_2)\dot{} p^2 + \right.$$
$$\left. (\omega^5)^2 \epsilon e_2(\dot{p}p) - (\epsilon e_2)\dot{}(\omega p)^2 - \epsilon e_2(\omega p)\dot{}(\omega p) - \dot\omega^5 \epsilon e_2 \omega^5 p^2 \right) =$$
$$\frac{e_2}{e_3} \left((\omega^5)^2 \epsilon e_2 \dot{p}(\eta - \frac{\omega\omega}{(\omega^5)^2})p + (\omega^5)^2(\epsilon e_2)\dot{} p(\eta - \frac{\omega\omega}{(\omega^5)^2})p \right) =$$

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$$-\epsilon(\dot{p}G\tilde{G}Dx) - \frac{(\epsilon e_2)^{\cdot}}{e_2}(pG\tilde{G}Dx) =$$
$$-\epsilon(\dot{p}Dx) - \frac{(\epsilon e_2)^{\cdot}}{e_2}(pDx). \tag{99}$$

So the last term in (98) cancel exactly the first two terms.

Let us present some other possible forms of the action (78).

The action implies the kinematic constraint $(\omega^A)^2 + a_4 = 0$ which is taken into account with help of the Lagrangian multiplier e_4. According to classical mechanics [30, 22], we can solve the constraint and substitute the result back into the action, thus obtaining

$$L = -\frac{e_3}{2(e_2\omega^5)^2}\left(\dot{x}^2 + \frac{1}{a_4}(\dot{x}\omega)^2\right) + \frac{1}{e_2\omega^5}(\dot{x}\dot{\omega}) - \frac{e_2}{2}a_2 - \frac{e_3}{2}a_3, \tag{100}$$

where now $\omega^5 \equiv \pm\sqrt{a_4 + \omega^2}$. Now, if we omit the term $\omega^A\dot{\omega}_A$ in (78), it acquires the form[11]

$$L = -\frac{e_3}{2(e_2\omega^5)^2}\left(\dot{x}^2 - \frac{1}{(\omega^A)^2}(\dot{x}\omega)^2\right) + \frac{1}{e_2\omega^5}(\dot{x}\dot{\omega}) -$$
$$\frac{e_2}{2}a_2 - \frac{e_3}{2}a_3 - \frac{e_4}{2}[(\omega^A)^2 + a_4] \equiv$$
$$\frac{1}{2}\tilde{G}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{e_2\omega^5}(\dot{x}\dot{\omega}) - \frac{e_2}{2}a_2 - \frac{e_3}{2}a_3 - \frac{e_4}{2}[(\omega^A)^2 + a_4] \tag{101}$$

It is equivalent to (78) since excluding the kinematic constraint we arrive at the expression (100). As compared with (78), the action (101) does not involve derivatives of ω^5.

The kinetic part of the actions presented above contains the cross-term $(\dot{x}\dot{\omega})$. Starting from the action (80), we can diagonalize its kinetic part in an appropriately chosen variables. To achieve this, we introduce the auxiliary variable $\sigma(\tau)$ and use the first-order trick [22] to replace

$$-\frac{(Dx\omega)^2}{(\omega^A)^2} \quad \text{by} \quad \sigma^2(\omega^A)^2 + 2\sigma(Dx\omega). \tag{102}$$

Then the first term in (80) reads

$$\frac{1}{2}\tilde{G}_{\mu\nu}Dx^\mu Dx^\nu = -\frac{e_3}{2(e_2\omega^5)^2}\left[(Dx)^2 - \frac{(Dx\omega)^2}{(\omega^A)^2}\right] =$$

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[11]The actions (100) and (101) can be further simplified by rescaling $e_2\omega^5 = \tilde{e}_2$.

$$-\frac{e_3}{2(e_2\omega^5)^2}\left[(Dx)^2 + 2\sigma(Dx\omega) + \sigma^2(\omega^A)^2\right] =$$
$$-\frac{e_3}{2(e_2\omega^5)^2}\left[(Dx^\mu + \sigma\omega^\mu)^2 - \sigma^2(\omega^5)^2\right]. \qquad (103)$$

Now the cross-terms contained in the expression $Dx^\mu + \sigma\omega^\mu$ can be diagonalized as follows

$$Dx^\mu + \sigma\omega^\mu \equiv \dot{\tilde{x}}^\mu + \tilde{\sigma}\omega^\mu. \qquad (104)$$

We have introduced the new variables (the variable $\tilde{x}^5$ will appear below)

$$\tilde{x}^\mu = x^\mu - \frac{e_2\omega^5}{e_3}\omega^\mu,$$
$$\tilde{x}^5 = 0 - \frac{e_2\omega^5}{e_3}\omega^5, \qquad (105)$$

$$\tilde{\sigma} = \sigma + \left(\frac{e_2\omega^5}{e_3}\right)^\cdot + \frac{e_2\dot{\omega}^5}{e_3}. \qquad (106)$$

Comment. Hamiltonian formulation of the theory with σ-variable implies two second-class constraints associated with this variable. The derivative-dependent transformation (106) represents an example of conversion of the second-class constraints, see [31] for the details.

In the new variables, the last term in (103) reads

$$- \sigma^2(\omega^5)^2 = -\left(\tilde{\sigma}\omega^5 - (\frac{e_2\omega^5}{e_3})\omega^5 - \frac{e_2\omega^5}{e_3}\dot{\omega}^5\right)^2 =$$
$$-\left(\tilde{\sigma}\omega^5 + \left(-\frac{e_2(\omega^5)^2}{e_3}\right)^\cdot\right)^2 = -\left(\dot{\tilde{x}}^5 + \tilde{\sigma}\omega^5\right)^2, \qquad (107)$$

where we have replaced the variable e_2 by $\tilde{x}^5$ of Eq. (105). $\tilde{x}^5$ together with $\tilde{x}^\mu$ form a five-dimensional space with the metric $\eta_{AB} = (-, +, +, +, -)$

Using the expressions (103)-(107) in Eq. (80), the latter acquires the form

$$L = -\frac{1}{2e_3}\left(\frac{\omega^5}{\tilde{x}^5}\right)^2\left(\dot{\tilde{x}}^A + \tilde{\sigma}\omega^A\right)^2 + \frac{1}{2e_3}(\dot{\omega}^A)^2 +$$

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$$\frac{e_3 \tilde{x}^5}{2(\omega^5)^2} a_2 - \frac{e_3}{2} a_3 - \frac{e_4}{2} [(\omega^A)^2 + a_4]. \qquad (108)$$

This almost five-dimensional form of the Lagrangian has been obtained also in [26]. If we use the first-order trick (102) to exclude the $\tilde{\sigma}$-variable, the Lagrangian reads

$$L = \frac{1}{2} G_{AB} \dot{\tilde{x}}^A \dot{\tilde{x}}^B + \frac{1}{2e_3} (\dot{\omega}^A)^2 + \frac{e_3 \tilde{x}^5}{2(\omega^5)^2} a_2 - \frac{e_3}{2} a_3 - \frac{e_4}{2} [(\omega^A)^2 + a_4], \quad (109)$$

where the five-dimensional "metric" is

$$G_{AB} = -\frac{1}{e_3} \left(\frac{\omega^5}{\tilde{x}^5} \right)^2 \left[\eta_{AB} - \frac{\omega_A \omega_B}{(\omega^C)^2} \right]. \qquad (110)$$

In contrast to $\tilde{G}_{\mu\nu}$, the matrix G_{AB} has null-vector ω^B, $G_{AB}\omega^B = 0$, hence it is not invertible.

In the rest of this Section, we analyze the Euler-Lagrange equations which implies the action (80). They read

$$\frac{\delta S}{\delta e_2} = 0 \Rightarrow \tilde{G}_{\mu\nu} Dx^\mu \dot{x}^\nu + \frac{1}{2} mc\hbar e_2 = 0, \qquad (111)$$

$$\frac{\delta S}{\delta e_3} = 0 \Rightarrow \tilde{G}_{\mu\nu} Dx^\mu (\dot{x}^\nu + I^{5\nu}) - \frac{1}{e_3} (\dot{\omega}^A)^2 - a_3 e_3 = 0, \qquad (112)$$

$$\frac{\delta S}{\delta e_4} = 0 \Rightarrow (\omega^A)^2 + a_4 = 0, \quad \text{then} \quad \omega^A \dot{\omega}_A = 0, \qquad (113)$$

$$\frac{\delta S}{\delta x^\mu} = 0 \Rightarrow [\tilde{G}_{\mu\nu} Dx^\nu]^\cdot = 0, \qquad (114)$$

$$\frac{\delta S}{\delta \omega^\mu} = 0 \Rightarrow \left[\frac{\dot{\omega}_\mu}{e_3} - \frac{e_2}{e_3} \omega^5 \tilde{G}_{\mu\nu} Dx^\nu \right]^\cdot +$$
$$\left[\frac{(\omega Dx)}{(\omega^A)^2} - \frac{e_2 \dot{\omega}^5}{e_3} \right] \tilde{G}_{\mu\nu} Dx^\nu + e_4 \omega_\mu = 0, \qquad (115)$$

$$\frac{\delta S}{\delta \omega^5} = 0 \Rightarrow \left[\frac{\dot{\omega}^5}{e_3} - \frac{e_2}{e_3} \omega^\mu \tilde{G}_{\mu\nu} Dx^\nu \right]^\cdot - \frac{1}{\omega^5} \tilde{G}_{\mu\nu} Dx^\mu Dx^\nu +$$

$$\frac{e_3}{e_2^2 \omega^5}\left[\frac{(\omega Dx)}{(\omega^A)^2}\right]^2 - \frac{e_2}{e_3}\dot{\omega}^\mu \tilde{G}_{\mu\nu}Dx^\nu + e_4\omega^5 = 0. \qquad (116)$$

The equations (111)-(113) do not contain the second-order derivative and thus represent the Lagrangian constraints.

Equation (114) can be immediately integrated out

$$\tilde{G}_{\mu\nu}Dx^\nu = p_\mu = \text{const.} \qquad (117)$$

Contracting it with $G^{\alpha\mu}$, this can be written in the form

$$Dx^\mu = G^{\mu\nu}p_\nu, \quad \text{or} \quad \dot{x}^\mu = I^{5\mu} + G^{\mu\nu}p_\nu, \quad \text{or} \quad \dot{x}^\mu = \frac{e_2}{2}\tilde{J}^{5\mu}, \quad (118)$$

if we introduce the notation

$$\tilde{J}^{5\mu} = \frac{2}{e_2}(I^{5\mu} + G^{\mu\nu}p_\nu). \qquad (119)$$

Equation (118) can be compared with Eq. (40). The quantity $\tilde{J}^{5\mu}$ represents the Lagrangian counterpart of angular-momentum components $J^{5\mu}$, see Eq. (10).

Using (117), the Lagrangian constraint (111) reads $(p\dot{x}) + \frac{1}{2}mc\hbar e_2 = 0$. Replacing $\dot{x}$ with help of Eq. (118) we arrive at the classical analogy of the Dirac equation

$$p_\mu \tilde{J}^{5\mu} + mc\hbar = 0. \qquad (120)$$

The Lagrangian constraint (112) can be rewritten in two different forms. First, using the expression $Dx = \dot{x} - I^5$, we obtain

$$\tilde{G}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - a_3 e_3 - \tilde{G}_{\mu\nu}I^{5\mu}I^{5\nu} - \frac{1}{e_3}(\dot{\omega}^A)^2 = 0. \qquad (121)$$

The constraint (113) implies the identity

$$\tilde{G}_{\mu\nu}I^{5\mu}I^{5\nu} + \frac{1}{e_3}(\dot{\omega}^A)^2 = 0, \qquad (122)$$

which can be verified by direct computations. Using it in (121), we represent the Lagrangian constraint in the form

$$\tilde{G}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - a_3 e_3 = 0. \qquad (123)$$

Using the manifest form of $\tilde{G}_{\mu\nu}$, we separate $\dot{x}^2$ as follows

$$(\dot{x}^\mu)^2 = -\left[\frac{(\omega\dot{x})^2}{a_4} + (e_2\omega^5)^2 a_3\right] < 0. \qquad (124)$$

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Since $\dot{x}^\mu$ is the time-like four-vector, the particle's speed can not exceed the speed of light. Thus the constraint (112) guarantees causal propagation of the particle.

Second, using Eqs. (111) and (117) we can exclude x^μ from (112). Then it reads

$$(pI^5) = \frac{1}{e_3}(\dot{\omega}^A)^2 + \frac{1}{2}mc\hbar e_2 + a_3 e_3. \tag{125}$$

Below we present it in a more transparent form, see Eq. (134).

To simplify the equations (115) and (116) for the variables ω^A, we observe two consequences of Eq. (118).

Using (117), (118) as well as the manifest form of $G^{\mu\nu}$, we write $\tilde{G}_{\mu\nu}Dx^\mu Dx^\nu = p_\mu Dx^\mu = p_\mu G^{\mu\nu}p_\nu = -\frac{e_2^2(\omega^5)^2}{e_3}[p^2 - \frac{(\omega p)^2}{(\omega^5)^2}]$, hence

$$\tilde{G}_{\mu\nu}Dx^\mu Dx^\nu = -\frac{e_2^2}{e_3}[(\omega^5)^2 p^2 - (\omega p)^2]. \tag{126}$$

Contracting (117) with ω^μ, we obtain the following expression for (ωDx)

$$\frac{(\omega Dx)}{(\omega^A)^2} = \frac{e_2^2}{e_3}(\omega p). \tag{127}$$

These equalities together with the equation (117) allow us to exclude x^μ from equations (115) and (116)

$$\left[\frac{\dot{\omega}_\mu}{e_3} - \frac{e_2}{e_3}\omega^5 p_\mu\right]^\cdot - e_2\left[\frac{\dot{\omega}^5}{e_3} - \frac{e_2}{e_3}(\omega p)\right]p_\mu + e_4\omega_\mu = 0, \tag{128}$$

$$\left[\frac{\dot{\omega}^5}{e_3} - \frac{e_2}{e_3}(\omega p)\right]^\cdot - e_2\left[\frac{\dot{\omega}_\mu}{e_3} - \frac{e_2}{e_3}\omega^5 p_\mu\right]p^\mu + e_4\omega^5 = 0. \tag{129}$$

If we introduce the notation[12]

$$\pi_\mu \equiv \frac{\dot{\omega}_\mu}{e_3} - \frac{e_2}{e_3}\omega^5 p_\mu, \qquad \pi^5 \equiv \frac{\dot{\omega}^5}{e_3} - \frac{e_2}{e_3}(\omega p), \tag{130}$$

these equations acquire the form

$$\dot{\pi}^\mu = e_2\pi^5 p^\mu - e_4\omega^\mu, \tag{131}$$

[12]In the next section we confirm that the quantities p^μ and π^A are just the canonical momenta of the Hamiltonian formulation.

$$\dot{\pi}^5 = e_2(\pi p) - e_4\omega^5. \tag{132}$$

Under the condition (113), the quantities π^A obey the relation

$$\pi^A\omega_A = 0, \tag{133}$$

which can be verified by direct computation. Besides, $e_3\pi^A\pi_A$ turns out to be proportional to $-(pI^5) + \frac{1}{e_3}(\dot{\omega}^A)^2 + \frac{1}{2}mc\hbar$. Taking into account (125), we obtain

$$\pi^A\pi_A + a_3 = 0. \tag{134}$$

Thus the Lagrangian constraint (112) is equivalent to (134).

Summing up, in the notation (119) and (130) the Lagrangian equations (111), (112), (114)-(116) can be presented in an equivalent form as follows

$$p_\mu\tilde{J}^{5\mu} + mc\hbar = 0. \tag{135}$$

$$\pi^A\pi_A + a_3 = 0. \tag{136}$$

$$\dot{x}^\mu = \frac{e_2}{2}\tilde{J}^{5\mu}, \tag{137}$$

$$\dot{\pi}^\mu = e_2\pi^5 p^\mu - e_4\omega^\mu, \tag{138}$$

$$\dot{\pi}^5 = e_2(\pi p) - e_4\omega^5. \tag{139}$$

As it should be, they coincide with the Hamiltonian equations (40), (45)-(50).

5   Canonical analysis and quantization

Our Lagrangian action (80) does not involve derivatives of the variables e_a, hence it represents an example of singular Lagrangian theory. The Hamiltonian formulation in this case is obtained according to the Dirac procedure for hamiltonization of a constrained system. We do it here, with the aim to confirm that the Lagrangian (80) and the Hamiltonian (38) variational problems are actually equivalent.

For this aim, the most convenient form of the action turns out to be those written in Eq. (80). In this form, equations for conjugate momenta

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \tilde{G}_{\mu\nu} D x^\nu = \tilde{G}_{\mu\nu}(\dot{x}^\nu - I^{5\nu}), \qquad (140)$$

$$\pi_\mu = \frac{\partial L}{\partial \dot{\omega}^\mu} = \frac{\dot{\omega}_\mu}{e_3} - \frac{e_2}{e_3}\omega^5 \tilde{G}_{\mu\nu} D x^\nu = \frac{\dot{\omega}_\mu}{e_3} - \frac{e_2}{e_3}\omega^5 p_\mu, \qquad (141)$$

$$\pi^5 = \frac{\partial L}{\partial \dot{\omega}_5} = \frac{\dot{\omega}^5}{e_3} - \frac{e_2}{e_3}\omega^\mu \tilde{G}_{\mu\nu} D x^\nu = \frac{\dot{\omega}^5}{e_3} - \frac{e_2}{e_3}(\omega p), \qquad (142)$$

can be immediately resolved with respect to velocities as follows:

$$\dot{\omega}^\mu = e_3 \pi^\mu + e_2 \omega^5 p^\mu, \qquad (143)$$

$$\dot{\omega}^5 = e_3 \pi^5 + e_2(\omega p), \qquad (144)$$

$$\dot{x}^\mu = G^{\mu\nu} p_\nu + I^{5\mu} = e_2(\omega^5 \pi^\mu - \omega^\mu \pi^5) = \frac{e_2}{2} J^{5\mu}. \qquad (145)$$

The second equality in (145) follows with use of equations (143) and (144). Conjugate momenta for the variables e_l turn out to be the primary constraints

$$\pi_{e_l} = \frac{\partial L}{\partial \dot{e}_l} = 0; \qquad l = 2, 3, 4. \qquad (146)$$

Using the equalities (143)-(146) we exclude velocities from the expression $p_\mu \dot{x}^\mu + \pi_A \dot{\omega}^A + \pi_{el} \dot{e}_l - L + \lambda_{el} \pi_{el}$, and obtain the complete Hamiltonian

$$H = \frac{e_2}{2}(p_\mu J^{5\mu} + mc\hbar) + \frac{e_3}{2}[(\pi^A)^2 + a_3] + \frac{e_4}{2}[(\omega^A)^2 + a_4] + \lambda_{e_l} \pi_{e_l}(147)$$

As expected, it coincides with Eq. (39). The procedure of revealing the higher-stage constraints have been described in Sect. 3. We have obtained the following chains of constraints

$$\pi_{e2} = 0 \;\Rightarrow\; T_2 \equiv p_\mu J^{5\mu} + mc\hbar = 0. \qquad (148)$$

$$\left.\begin{array}{l} \pi_{e3} = 0, \;\Rightarrow\; T_3 \equiv (\pi^A)^2 + a_3 = 0 \\ \pi_{e4} = 0, \;\Rightarrow\; T_4 \equiv (\omega^A)^2 + a_4 = 0 \end{array}\right\} \Rightarrow T_5 \equiv \omega^A \pi_A = 0, \Rightarrow$$

$$e_4 - \frac{a_3}{a_4} e_3 = 0, \;\Rightarrow\; \lambda_{e4} = \frac{a_3}{a_4}\lambda_{e3}. \qquad (149)$$

The constraints T_2 and $\tilde{T}_3 = T_3 + \frac{a_3}{a_4}T_4$ form the first-class subset.

If we use the Dirac bracket to take into account the constraints (149), the Hamiltonian can be presented in terms of ε-invariant variables. First, we take into account the second-class pairs $e_4 - \frac{a_3}{a_4} e_3$, π_{e_4} and T_4, T_5. Denoting the constraints by K_l, the corresponding Dirac bracket of two phase-space functions f and g is

$$\{f, g\}_1 = \{f, g\} - \{f, K_l\}\{K_l, K_m\}^{-1}\{K_m, g\} = \{f, g\} +$$
$$\frac{1}{2(\omega^A)^2}\{f, (\omega^A)^2\}\{\omega^A \pi_A, g\} - \frac{1}{2(\omega^A)^2}\{f, \omega^A \pi_A\}\{(\omega^A)^2, g\} +$$
$$\{f, e_4 - \frac{a_3}{a_4} e_3\}\{\pi_{e4}, g\} - \{f, \pi_{e4}\}\{e_4 - \frac{a_3}{a_4} e_3, g\}, \tag{150}$$

where $\{,\}$ stands for the Poisson brackets. For the basic variables it reads

$$\{\omega^A, \pi^B\}_1 = \eta^{AB} - \frac{1}{(\omega^A)^2} \omega^A \omega^B, \quad \{\omega^A, \omega^B\}_1 = 0, \tag{151}$$

$$\{\pi^A, \pi^B\}_1 = \frac{1}{(\omega^A)^2}(\omega^B \pi^A - \omega^A \pi^B) = -\frac{1}{2(\omega^A)^2} J^{AB}. \tag{152}$$

Second, we impose the gauge conditions $e_3 = \text{const}$, $\omega^5 = \text{const}$ for the first-class constraints π_{e3} and T_3, and construct the Dirac bracket for this set of second-class functions

$$\{f, g\}_{DB} = \{f, g\}_1 + \{f, \omega^5\}_1 \Delta \{\pi^A \pi_A, g\}_1 - \{f, \pi^A \pi_A\}_1 \Delta \{\omega^5, g\}_1 +$$
$$\{f, e_3\}_1 \{\pi_{e3}, g\}_1 - \{f, \pi_{e3}\}_1 \{e_3, g\}_1, \tag{153}$$

where

$$\Delta = \frac{(\omega^A)^2}{2[(\omega^A)^2 \pi^5 - (\omega^A \pi_A)\omega^5]}. \tag{154}$$

It implies

$$\{\omega^A, \pi^B\}_{DB} = G^{AB} - \frac{J^{AC}\omega_C G^{5B}}{J^{5C}\omega_C}, \quad \{\omega^A, \omega^B\}_{DB} = 0. \tag{155}$$

$$\{\pi^A, \pi^B\}_{DB} = -\frac{J^{AB}}{2(\omega^C)^2} + \frac{G^{5A}J^{BC}\pi_C - (A \leftrightarrow B)}{J^{5C}\omega_C}. \tag{156}$$

We have denoted (see also Eq. (110))

$$G^{AB} = \left[\eta^{AB} - \frac{\omega^A \omega^B}{(\omega^C)^2}\right]. \tag{157}$$

These formulas acquire more simple form on the constraint surface

$$\{\omega^A, \pi^B\}_{DB} = G^{AB} - \frac{\pi^A G^{5B}}{\pi^5}, \quad \{\omega^A, \omega^B\}_{DB} = 0. \qquad (158)$$

$$\{\pi^A, \pi^B\}_{DB} = \frac{J^{AB}}{2a_4} - \frac{a_3}{a_4 \pi^5} \eta^{5[A} \omega^{B]}. \qquad (159)$$

The Dirac bracket of the basic variables is deformed as compared with the Poisson one. In contrast, brackets of J^{AB} keep their initial form. Indeed, the constraints T_a are $SO(2,3)$-invariants, $\{T_a, J^{AB}\} = 0$. On this reason, if we compute the Dirac bracket of the spin-tensor components, all its extra-terms vanish, $\{J^{AB}, J^{CD}\}_{DB}$ $= \{J^{AB}, J^{CD}\}$. Thus the transition from Poisson to the Dirac bracket does not modify the initial spin-tensor algebra (11).

By construction, the Dirac bracket of a constraint with any function vanishes identically. So, when we are dealing with the Dirac bracket, the constraints can be used in all the computations as strong equations. In particular, we can omit them in the expression (147). It gives the ε-invariant Hamiltonian

$$H_1 = \frac{e_2}{2}(p_\mu J^{5\mu} + mc\hbar) + \lambda_{e_2} \pi_{e_2}. \qquad (160)$$

The Hamiltonian equations (40)-(44), (58), (59) can now be obtained according the rule $\dot{A} = \{A, H_1\}_{DB}$ instead of $\dot{A} = \{A, H\}$.

As we have shown in Sect. 2, it is consistent to describe spin-sector of the model by the ε-invariant variables J^{AB} instead of the initial coordinates ω^A, π^B. Canonical quantization of the model is achieved replacing the variables x^μ, p^ν and J^{AB} by operators which obey the rule $[\,,\,] = i\hbar\{\,,\,\}$, where the nonvanishing classical brackets are

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu} \qquad (161)$$

$$\{J^{AB}, J^{CD}\} = 2(\eta^{AC}J^{BD} - \eta^{AD}J^{BC} - \eta^{BC}J^{AD} + \eta^{BD}J^{AC}). \ (162)$$

The operators

$$\begin{aligned}
\hat{x}^\mu &= x^\mu, & \hat{p}_\mu &= -i\hbar\partial_\mu, \\
\hat{J}^{5\mu} &= \hbar\Gamma^\mu, & \hat{J}^{\mu\nu} &= \hbar\Gamma^{\mu\nu},
\end{aligned} \qquad (163)$$

obey the necessary commutators. They act on the space of Dirac spinors $\Psi(x^\mu)$. The only constraint which has not been yet taken into account is $T_2 = p_\mu J^{5\mu} + mc\hbar = 0$. Since this is the first-class function, it is consistent to impose the corresponding operator on a state vector. This gives the Dirac equation, $(\Gamma^\mu \hat{p}_\mu + mc)\Psi = 0$.

The classical equations of motion (69) and (70) imply that the center-of-charge coordinate x^i experiences a complicated trembling motion in the theory without interaction (see also the next section). Besides the operator $\hat{x}$, some other versions for the position operator in the Dirac theory have been suggested and discussed in the literature [13, 14, 15]. In our model we can construct the following "center-of-mass" variable

$$\tilde{x}^\mu = x^\mu + \frac{1}{2p^2} J^{\mu\nu} p_\nu. \tag{164}$$

It obeys the equation

$$\dot{\tilde{x}}^\mu = \tilde{e} p^\mu; \quad \text{where} \quad \tilde{e} \equiv -\frac{e_2 mc\hbar}{2p^2}, \tag{165}$$

Note also that p^μ represents the mechanical momentum of the $\tilde{x}^\mu$-particle. The reparametrization-invariant quantity $\tilde{x}^i(t)$ moves along the straight line, $\frac{d\tilde{x}^i}{dt} = \frac{cp^i}{p^0}$. We propose the variable $\tilde{x}^\mu$ as the Lorentz-covariant analog of the Foldy-Wouthuysen operator.

Using components of the spin-tensor, we can construct the Pauli-Lubanski vector $S^\mu = \epsilon^{\mu\nu\alpha\beta} p_\nu J_{\alpha\beta}$. It has no precession in the free theory, $\dot{S}^\mu = 0$, and corresponds to the Bargmann-Michel-Telegdi spin-vector [7].

In the center-of-charge instantaneous rest frame, $J^{50} = \text{const}$, $J^{5i} = 0$, it reduces to $S^\mu = (0, S^i = \epsilon^{ijk} p_j J_{0k})$. The only J_{0k}-part of the angular-momentum tensor survives in this frame.

In the center of mass frame, $p^\mu = (p^0, 0, 0, 0)$, $\vec{S}$ is proportional to the three-dimensional rotation generator, $S^\mu = (0, \frac{1}{2} p^0 \epsilon^{ijk} J_{jk})$.

The equation (164) represents the phase-space transformation which is not the canonical one. As a consequence, the theory became non-commutative. Computing the Poisson brackets, we obtain

$$\{\tilde{x}^\mu, \tilde{x}^\nu\} = -\frac{1}{2p^2}\left(J^{\mu\nu} + \frac{1}{p^2} p^{[\mu} J^{\nu]\alpha} p_\alpha\right), \quad \{\tilde{x}^\mu, p^\nu\} = \eta^{\mu\nu}, \tag{166}$$

$$\{\tilde{x}^\mu, J^{5\nu}\} = \frac{1}{p^2}(\eta^{\mu\nu} J^{5\alpha} p_\alpha - J^{5\mu} p^\nu). \tag{167}$$

$$\{\tilde{x}^{\mu}, J^{\alpha\beta}\} = \frac{1}{p^2}(J^{\mu[\alpha}p^{\beta]} - \eta^{\mu[\alpha}J^{\beta]\gamma}p_{\gamma}), \qquad (168)$$

We also present the brackets with the Pauli-Lubanski vector

$$\{S^{\mu}, S^{\nu}\} = 2p^2 J^{\mu\nu} - 2p^{[\mu}J^{\nu]\alpha}p_{\alpha}. \qquad (169)$$

$$\{\tilde{x}^{\mu}, S^{\nu}\} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\rho\sigma} - \frac{1}{p^2}\epsilon^{\mu\nu\gamma\sigma}p_{\gamma}J_{\sigma}{}^{\alpha}p_{\alpha}, \qquad (170)$$

$$\{S^{\mu}, J^{5\alpha}\} = 2\epsilon^{\mu\alpha\sigma\nu}J^5{}_{\sigma}p_{\nu}, \quad \{S^{\mu}, J^{\alpha\beta}\} = 2p_{\nu}\epsilon^{\mu\nu\sigma[\alpha}J^{\beta]}{}_{\sigma}. \qquad (171)$$

We point out that the second term in Eq. (164) has the structure typical for non-commutative extensions of the usual mechanics, see [32, 33, 34, 35].

6 Solution to the classical equations of motion

We are interested in to solve equations of motion for the gauge-invariant variables $x^i(t)$, $p^{\mu}(t)$ and $J^{AB}(t)$, they are[13]

$$\frac{dx^i}{dt} = c\frac{J^{5i}}{J^{50}}, \qquad (172)$$

$$\frac{dJ^{5\mu}}{dt} = 2cp_{\nu}\frac{J^{\nu\mu}}{J^{50}}, \qquad (173)$$

$$\frac{dJ^{\mu\nu}}{dt} = 2c\frac{p^{\mu}J^{5\nu} - p^{\nu}J^{5\mu}}{J^{50}}, \qquad (174)$$

where p^{μ} = const. Besides, the spin-tensor J^{AB} and the time-like vector $J^{5\mu}$ obey the constraints

$$p_{\mu}J^{5\mu} + mc\hbar = 0, \qquad (175)$$

$$J^{AB}J_{AB} = 8a_3a_4. \qquad (176)$$

The differential equations written above are not polynomial. To avoid this difficulty, we remind that they were obtained from the

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[13]Solution to equations of motion for the initial variables ω^A and π^B as well as the subsequent construction on this base the physical variables $x^i(t)$, and $J^{AB}(t)$ are given in the Appendix.

equations for coordinates presented in parametric form

$$\dot{x}^\mu = \frac{e_2}{2} J^{5\mu}, \tag{177}$$

$$\dot{J}^{5\mu} = e_2 p_\nu J^{\nu\mu}, \tag{178}$$

$$\dot{J}^{\mu\nu} = e_2(p^\mu J^{5\nu} - p^\nu J^{5\mu}), \tag{179}$$

eliminating the ambiguity due to $e_2(\tau)$. The latter equations are polynomial, so their analysis represents more simple task. In the process, we can conventionally fix $e_2(\tau)$, since according to (172)-(176), the physical coordinates we are interested in do not depend on a particular choice of e_2. We take $e_2 = \text{const}$. After integrating the equations (177)-(179), we exclude τ from the resulting expressions thus obtaining the general solution to (172)-(174).

Let us start from Eq. (178). We compute its derivative and use the equations (179) and (175), it gives the closed equation for $J^{5\mu}$

$$\ddot{J}^{5\mu} - e_2^2 p^2 J^{5\mu} = e_2^2 mc\hbar p^\mu, \tag{180}$$

whose solution depends on the sign of the constant p^2. We discuss the two possibilities separately.

6.1 Spinning-particle with $p^2 < 0$, helical motions

In this case, the general solution to Eq. (180) is given by

$$J^{5\mu} = \frac{mc\hbar}{|p|^2} p^\mu + A^\mu \cos(\omega\tau) + B^\mu \sin(\omega\tau), \tag{181}$$

Where $|p| = \sqrt{-p^2}$, $\omega = e_2|p|$, and A^μ, B^μ are the integration constants. According to Eq. (175), they obey the restrictions

$$p_\mu A^\mu = 0, \qquad p_\mu B^\mu = 0. \tag{182}$$

We substitute Eq. (181) into (177). It gives closed equation for x^μ which can be integrated

$$x^\mu(\tau) = X^\mu + e_2 \frac{mc\hbar}{2|p|^2} p^\mu \tau + \frac{1}{2|p|} A^\mu \sin(\omega\tau) - \frac{1}{2|p|} B^\mu \cos(\omega\tau). \tag{183}$$

By construction, only the monotonic functions $x^0(\tau)$ are physically admissible. This implies

$$A^0 = B^0 = 0. \tag{184}$$

We also fix the initial instant to be zero, $X^0 = 0$. Hence

$$x^0 = ct = \frac{e_2 mc\hbar}{2|p|^2} p^0 \tau \Rightarrow \tau = \frac{2|p|^2}{e_2 m\hbar p^0} t. \qquad (185)$$

Similarly, if we substitute (181) into the equation $\dot{J}^{0i} = e_2(p^0 J^{5i} - p^i J^{50})$, it can be integrated as well

$$J^{0i} = \Sigma^{0i} + \frac{1}{|p|} p^0 A^i \sin(\omega\tau) - \frac{1}{|p|} p^0 B^i \cos(\omega\tau), \qquad (186)$$

Σ^{0i} is the integration constant. As we know, the remaining J's are not independent, and can be computed according the equation

$$J^{ij} = (J^{50})^{-1}(J^{5i} J^{0j} - J^{5j} J^{0i}). \qquad (187)$$

It reads

$$J^{ij} = \frac{p^{[i}\Sigma^{0j]}}{p^0} + \frac{|p|B^{[i}A^{j]}}{mc\hbar} + \left[\frac{1}{|p|}p^{[i}A^{j]} + \frac{|p|^2}{mc\hbar p^0}B^{[i}\Sigma^{0j]}\right]\sin(\omega\tau)$$
$$+ \left[-\frac{1}{|p|}p^{[i}B^{j]} + \frac{|p|^2}{mc\hbar p^0}A^{[i}\Sigma^{0j]}\right]\cos(\omega\tau)(188)$$

We have started our computations from the equation (180) which is a consequence of (178). To select the subset of solutions which obeys the initial system (177)-(179), we substitute (181), (186) and (188) into the equation (178). It is satisfied if

$$\Sigma^{0i} = 0. \qquad (189)$$

The next step is to satisfy the constraints (175), (176). The first one have been already taken into account, it implies (182). The constraint (176) leads to the following restriction:

$$\left(\frac{mc\hbar}{|p|}\right)^2 - (\vec{A}^2 + \vec{B}^2) + \left(\frac{|p|}{mc\hbar}\right)^2 (\vec{A} \times \vec{B})^2 = 4a_3 a_4. \qquad (190)$$

Besides, $J^{5\mu}$ turns out to be the time-like vector if

$$[\vec{A}\cos(\omega\tau) + \vec{B}\sin(\omega\tau)]^2 < \left(\frac{mc\hbar}{|p|}\right)^2, \quad \text{for any} \quad \tau. \qquad (191)$$

It implies

$$\vec{A}^2 < \left(\frac{mc\hbar}{|p|}\right)^2, \qquad \vec{B}^2 < \left(\frac{mc\hbar}{|p|}\right)^2. \qquad (192)$$

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The last step is to exclude the parameter τ from the expressions obtained. Using Eq. (185), we obtain the general solution to the equations (172)-(174)

$$x^i(t) = X^i + c\frac{p^i}{p^0}t + \frac{1}{2|p|}(A^i \sin(\tilde{\omega}t) - B^i \cos(\tilde{\omega}t)), \qquad (193)$$

$$J^{0i} = \frac{p^0}{|p|}(A^i \sin(\tilde{\omega}\tau) - B^i \cos(\tilde{\omega}\tau)), \qquad (194)$$

$$J^{5i}(t) = \frac{mc\hbar}{|p|^2}p^i + A^i \cos(\tilde{\omega}t) + B^i \sin(\tilde{\omega}t), \qquad (195)$$

$$J^{50}(t) = \frac{mc\hbar p^0}{|p|^2}, \qquad (196)$$

$$J^{ij}(t) = \frac{|p|}{mc\hbar}B^{[i}A^{j]} + \frac{1}{|p|}p^{[i}A^{j]}\sin(\tilde{\omega}t) - \frac{1}{|p|}p^{[i}B^{j]}\cos(\tilde{\omega}t), \quad (197)$$

where the frequency is

$$\tilde{\omega} = \frac{2|p|^3}{m\hbar p^0}, \qquad (198)$$

and the integration constants obey the restrictions (190), (191) and

$$\vec{p}\vec{A} = 0, \quad \vec{p}\vec{B} = 0. \qquad (199)$$

Let us discuss dynamics of the coordinate $x^i(t)$.

Using the equations (199) and (191), we confirm once again its causal motion

$$\left(\frac{d\vec{x}}{dt}\right)^2 = c^2\frac{\vec{p}^2}{(p^0)^2} + \frac{|p|^4}{(m\hbar p^0)^2}(\vec{A}\cos(\tilde{\omega}t) + \vec{B}\sin(\tilde{\omega}t))^2 <$$
$$c^2\frac{\vec{p}^2}{(p^0)^2} + \frac{|p|^4}{(m\hbar p^0)^2}\frac{(mc\hbar)^2}{|p|^2} = c^2. \qquad (200)$$

The curve (193) is a helix which can be considered as a superposition of the rectilinear motion

$$\tilde{x}^i(t) = X^i + c\frac{p^i}{p^0}t, \qquad (201)$$

and the oscillatory motion

$$z^i(t) = \frac{1}{2|p|}(A^i \sin(\tilde{\omega}t) - B^i \cos(\tilde{\omega}t)), \qquad (202)$$

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the latter is the classical-mechanical analog of *Zitterbewegung*.

Both the conjugate momentum and the Dirac equation acquire certain interpretation in this picture. According to Eq. (199), the *Zitterbewegung* oscillations occur on the plane perpendicular to $\vec{p}$. This is the Dirac equation that dictates the perpendicularity.

On this plane, the trajectory is an ellipse. To see this, we take the coordinate system with the origin at $\tilde{x}^i$ and with the axis x_1 and x_2 on the plane of the vectors $\vec{A}$ and $\vec{B}$ in such a way, that the x_1-axis has the direction of the vector $\vec{B}$. Then $\vec{B} = (B^1, 0)$ and $\vec{A} = (A^1, A^2)$. In this coordinate system the parametric equations of the *Zitterbewegung* reads

$$x_1 = \frac{1}{2|p|}(A^1 \sin(\tilde{\omega}t) - B^1 \cos(\tilde{\omega}t)),$$
$$x_2 = \frac{1}{2|p|}A^2 \sin(\tilde{\omega}t). \tag{203}$$

To obtain its trajectory, we ask whether the points of the curve (203) can be identified with those of a second-order line, $C_{ij}x_ix_j = 1$. Short computation shows that the line is given by

$$C = \frac{4|p|^2}{(B^1)^2}\begin{pmatrix} 1 & -\frac{A^1}{A^2} \\ -\frac{A^1}{A^2} & \frac{(A^1)^2+(B^1)^2}{(A^2)^2} \end{pmatrix}. \tag{204}$$

Since $\det C = \left(\frac{4|p|^2}{A^2B^1}\right)^2 > 0$, the line represents an ellipse. Denoting its semi-axis as a and b, we can write

$$\frac{1}{a^2} + \frac{1}{b^2} = \text{Tr } C, \quad \frac{1}{a^2b^2} = \det C, \quad \text{then} \quad a^2 + b^2 = \frac{\text{Tr } C}{\det C}. \tag{205}$$

The last equation together with (192) allow us to estimate the size of the ellipse as follows:

$$a^2 + b^2 = \frac{\vec{A}^2 + \vec{B}^2}{4|p|^2} < \frac{1}{2}\left(\frac{mc\hbar}{|p|^2}\right)^2. \tag{206}$$

As we have seen above, canonical quantization of our model in the coordinates x, p and J leads to the Dirac equation. Now we look for the coordinates which may have a reasonable interpretation in the classical theory.

Let us compute the total number of physical degrees of freedom in the theory. Omitting the auxiliary variables and the corresponding constraints, we have 18 phase-space variables x^μ, p_μ, ω^A, π_A subject to the constraints (14), (15), (28). Taking into account that each second-class constraint rules out one variable, whereas each first-class constraint rules out two variables, the number of physical degrees of freedom is $18 - (2 + 2 \times 2) = 12$. Note that this is equal to the number of degrees of freedom of the two-body problem.

Further, we note that the *Zitterbewegung* (202) coincides with the evolution of the coordinate $\frac{J^{0i}}{2p^0}$ of the inner spin-space, see Eq. (194). Let us choose

$$\tilde{x}^i(t) = x^i - \frac{J^{0i}}{2p^0}, \qquad \tilde{p}^i = \frac{p^i}{p^0}; \tag{207}$$

$$J^i = \frac{J^{0i}}{2p^0}, \qquad V^i = \frac{J^{5i}}{J^{50}} - \frac{p^i}{p^0}. \tag{208}$$

as the new coordinates in the problem[14]. The spatial coordinates obey the equations

$$\frac{d\tilde{x}^i}{dt} = c\tilde{p}^i, \qquad \frac{dJ^i}{dt} = cV^i. \tag{209}$$

and, according to Eqs. (193)-(196), the general solution is

$$\tilde{x}^i = X^i + c\frac{p^i}{p^0}t, \tag{210}$$

$$J^i(t) = \frac{1}{2|p|}(A^i \sin(\tilde{\omega}t) - B^i \cos(\tilde{\omega}t)). \tag{211}$$

As we have seen, the trajectory of J^i is an ellipse on the plane perpendicular to $\vec{p}$. Thus, behavior of the coordinates $\tilde{x}^i$ and J^i is similar to those of the center-of-mass and the relative position of a two-body system subjected to a central field.

The Dirac electron obeys the mass-shell condition $(p^\mu)^2 = -m^2c^2$. Let us consider the subset of solutions in our model with this value of p^μ. Then for the two-particle system with a slowly moving center-of-mass $\tilde{x}^i$, we have $p^0 \sim |p| = mc$. Then the frequency (198) of the

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[14]We stress that we have performed a transformation on the phase-space.

relative position J^i approaches to the Compton frequency, $\tilde{\omega} \sim \frac{2mc^2}{\hbar}$. Besides, the size of elliptic orbit turns out to be of the order of de Broglie wave-length, $\sqrt{a^2 + b^2} < \frac{\hbar}{\sqrt{2}mc}$.

It would be interesting to quantize the model in the coordinates (207), (208) and to compare the results with those of Dirac theory in the Foldy-Wouthuysen representation.

6.2 Spinning-particle with $p^2 > 0$, hyperbolic motions

The general solution to (180) is now given by

$$J^{5\mu}(\tau) = -\frac{mc\hbar}{|p|^2}p^\mu + A^\mu e^{\omega\tau} + B^\mu e^{-\omega\tau}, \qquad (212)$$

where $|p| = \sqrt{p^2}$, $\omega = e_2|p|$. The constraint (175) implies $p_\mu A^\mu = 0$, $p_\mu B^\mu = 0$. With this $J^{5\mu}(\tau)$ at hands, we immediately integrate equations for x^μ and $J^{\mu\nu}$

$$x^\mu(\tau) = X^\mu - \frac{e_2 mc\hbar}{2|p|^2}p^\mu\tau + \frac{1}{2|p|}A^\mu e^{\omega\tau} -$$
$$- \frac{1}{2|p|}B^\mu e^{-\omega\tau}, \qquad (213)$$

$$J^{\mu\nu}(\tau) = \Sigma^{\mu\nu} + \frac{1}{|p|}(p^\mu A^\nu - p^\nu A^\mu)e^{\omega\tau} -$$
$$- \frac{1}{|p|}(p^\mu B^\nu - p^\nu B^\mu)e^{-\omega\tau}, \qquad (214)$$

where A^μ, B^μ, X^μ and Σ^{0i} are the integration constants. The components J^{ij} have been found with help of (187), it implies the following expression for Σ^{ij} in terms of the integration constants

$$\Sigma^{ij} = \frac{2|p|}{mc\hbar}(A^i B^j - A^j B^i). \qquad (215)$$

We have started our computations from the equation (180) which is a consequence of (178). To select the subset of solutions which obeys the initial system (177)-(179), we substitute (212)-(215) into the equation (178). This determines Σ^{0i} in terms of other integration constants

$$\Sigma^{0i} = \frac{2|p|}{mc\hbar}(A^0 B^i - B^0 A^i), \qquad (216)$$

as well as implies the restriction

$$A^0(\vec{p}\vec{B}) = B^0(\vec{p}\vec{A}). \tag{217}$$

The equations (215) and (216) can be unified into the expression

$$\Sigma^{\mu\nu} = \frac{2|p|}{mc\hbar}A^{[\mu}B^{\nu]}. \tag{218}$$

The next step is to satisfy the constraints (175), (176). They imply the following restrictions on the integration constants

$$p_\mu A^\mu = 0, \qquad p_\mu B^\mu = 0, \quad \text{then} \quad p_\mu \Sigma^{\mu\nu} = 0. \tag{219}$$

$$\left(\frac{mc\hbar}{2|p|}\right)^2 + AB - \left(\frac{|p|}{mc\hbar}\right)^2 [(\vec{A}\times\vec{B})^2 - (A^0\vec{B} - B^0\vec{A})^2] = a_3 a_4 \tag{220}$$

In obtaining the last equation we have used the restrictions (219). Besides, $J^{5\mu}$ turns out to be the time-like vector if

$$\left(\frac{mc\hbar}{|p|}\right)^2 + A^2 e^{2\omega\tau} + 2AB + B^2 e^{-2\omega\tau} < 0, \quad \text{for any } \tau. \tag{221}$$

Since $e^{2\omega\tau}$ increases, this inequality implies, in particular

$$A_\mu A^\mu < 0, \qquad B_\mu B^\mu < 0. \tag{222}$$

The solution (213) describes a self-accelerated motion. For the present case, we can not exclude the parameter τ analytically. For sufficiently big values of τ, we can neglect all the terms in the expression for x^μ as compared with the third term, then

$$x^0(\tau) \approx \frac{1}{2|p|}A^0 e^{\omega\tau} \Rightarrow e^{\omega\tau} = \frac{2|p|c}{A^0}t. \tag{223}$$

It gives the following asymptotic for $x^i(t)$

$$x^i(t) \approx X^i + c\frac{A^i}{A^0}t, \quad \text{when} \quad t \to +\infty. \tag{224}$$

Since A^μ is the time-like vector, see Eq. (222), we have $|\vec{v}| < c$. The hyperbolic motions are presented also in the Frenkel theory of an electron, see [36].

7 Conclusion

In this paper we have constructed the non-Grassmann mechanical model which implies the Dirac equation. To formulate a variational problem, we have identified the spin variables with coordinates of the base of a seven-dimensional fiber bundle embedded as a surface into ten-dimensional phase space equipped with the canonical Poisson bracket $\{\omega^A, \pi^B\} = \eta^{AB}$ (the configuration space ω^A turns out to be the anti-de Sitter space, $\omega^A \omega_A + a_4 = 0$). By construction, the values of Casimir operators of $SO(2,3)$-group are fixed within the classical theory (in the dynamical realization this is guaranteed by the first-class constraint $T_3 = 0$, see Eqs. (14) and (23)).

The surface can be covered by the coordinates $J^{5\mu}, J^{0i}, \omega^5$, adjusted with the structure of fiber bundle, then $J^{5\mu}, J^{0i}$ represents coordinates of the base, while ω^5 parameterizes the fiber. In accordance with this, we can discard ω^5 (in the dynamical realization ω^5 is affected by the local transformations, hence it is not observable quantity). In the result, only the coordinates of the base, $J^{5\mu}, J^{0i}$, are relevant to description of spin. Canonical quantization of the coordinates produces both Γ^μ and $\Gamma^{\mu\nu}$-matrices, see Eqs. (25)-(27).

It would be interesting to describe the underlying geometry of the fiber bundle (as it has been mentioned above, for the non-relativistic spin this is $SO(3)$ fiber bundle).

Besides the geometric constraints (14) and (15), which determines the spin surface, our model implies the dynamical first-class constraint (28). Being imposed on a state-vector, it implies the Dirac equation. Although there is no the mass-shell constraint $p^2 + m^2c^2 = 0$ in our model, our particle's speed cannot exceed the speed of light. This is due to the geometric constraints which imply the time-like character of the four-vector $J^{5\mu}$, see Eq. (60). In turn, this implies the causal dynamics of the coordinate x^μ, see Eq. (40).

Analyzing the general solution (193)-(??) to the classical equations of motion (172)-(176), we have constructed the coordinates (207), (208) which strongly resemble the two-body problem. If we assume space-time interpretation of the coordinates $\tilde{x}^i$ and J^i, they can be identified with the center-of-mass and the relative position of a two-body system subjected to a central field. The Dirac equation dictates the perpendicularity of the *Zitterbewegung*-plane to the

direction of the center-of-mass motion.

Dynamics of the subset of solutions with $(p^\mu)^2 = -m^2c^2$ is in correspondence with the dynamics of mean values of the corresponding operators in the Dirac theory. In particular, for the two-particle system with a slowly moving center-of-mass $\tilde{x}^i$, the frequency (198) of the relative position J^i approaches to the Compton frequency, $\tilde{\omega} \sim \frac{2mc^2}{\hbar}$. Besides, the size of elliptic orbit turns out to be of the order of de Broglie wave-length, $\sqrt{a^2 + b^2} \sim \frac{\hbar}{mc}$.

Our model shows the same undesirable properties as those of Dirac equation in the RQM interpretation. We finish with a brief comment on a modification which solves the problems. We recall that the Dirac equation (4) implies the Klein-Gordon one. In contrast, in classical mechanics the corresponding constraint (28) does not imply the mass-shell constraint $p^2 + m^2c^2 = 0$. So, the model presented here is not yet in complete correspondence with the Dirac theory. The semiclassical model that produces both constraints has been discussed in the recent work [23]. The extra first-class constraint implies that we are dealing with a theory with one more local symmetry, with the constraint being a generator of the symmetry [21, 22, 27, 37]. This leads to a completely different picture of the classical dynamics. The variable x^μ is not inert under the extra symmetry. Being gauge non-invariant, x^μ turns out to be an unobservable quantity. The variable $\tilde{x}^\mu$ of Eq. (164) is gauge invariant and should be identified with the position of the particle. Because p^μ is a mechanical momentum for $\tilde{x}^\mu$, the particle's speed cannot exceed the speed of light. In the absence of interaction it moves along the straight line. Hence the modified model is free of *Zitterbewegung.*

Acknowledgments

Appendix

In Section 6 we have solved equations of motion for the variables $J^{AB}(\tau)$ and $x^\mu(\tau)$. Here we present solution to equations of motion for the initial variables ω^A, π^B and x^μ. We shall restrict ourselves to the case $p^2 < 0$. We keep the notation $p^2 \equiv -|p|^2$. As discussed before, the equations for $J^{AB}(t)$ and $x^i(t)$ do not depend on the auxiliary variables $e_{2,3}$. So, we are free to select them as we want. We take $e_{2,3} = 1$ and define $a \equiv \frac{a_3}{a_4}$. The equations of motion for ω^A and π^B then read

$$\dot{\omega}^\mu = \pi^\mu + \omega^5 p^\mu, \quad \dot{\pi}^\mu = -a\omega^\mu + \pi^5 p^\mu; \tag{A.1}$$

$$\dot{\omega}^5 = \pi^5 + p\omega, \quad \dot{\pi}^5 = -a\omega^5 + p\pi. \tag{A.2}$$

Let us discuss one possible way to decouple the system above. The contraction of the equations for ω^μ and π^μ with p_μ gives

$$(p\omega)\dot{} = p\pi - |p|^2\omega^5, \quad (p\pi)\dot{} = -ap\omega - |p|^2\pi^5. \tag{A.3}$$

It is suggestive to look for combinations of variables that decouple the system formed by (A.2) and (A.3), for the variables ω^5, π^5, $p\omega$ and $p\pi$. We point out that once the equations for ω^5, π^5 are solved, one can promptly write the solutions for ω^μ and π^μ since they obey a closed second order differential equation

$$\ddot{\omega}^\mu + a\omega^\mu = p^\mu(\pi^5 + \dot{\omega}^5), \tag{A.4}$$

$$\ddot{\pi}^\mu + a\pi^\mu = p^\mu(\dot{\pi}^5 - a\omega^5). \tag{A.5}$$

The system (A.2)-(A.3) may be solved in a sequence of steps. First we define

$$\pi^5_\pm = \pi^5 \pm \frac{p\pi}{|p|}, \tag{A.6}$$

$$\omega^5_\pm = \omega^5 \pm \frac{p\omega}{|p|}. \tag{A.7}$$

The second order equations for the variables above are closed for the pairs (ω^5_+, π^5_-) and (ω^5_-, π^5_+). In fact,

$$\ddot{\omega}^5_+ = -2|p|\pi^5_- - (|p|^2 + a)\omega^5_+, \tag{A.8}$$

$$\ddot{\pi}^5_- = -(|p|^2 + a)\pi^5_- - 2a|p|\omega^5_+, \tag{A.9}$$

and

$$\ddot{\omega}_-^5 = 2|p|\pi_+^5 - (|p|^2 + a)\omega_-^5, \qquad (A.10)$$

$$\ddot{\pi}_+^5 = -(|p|^2 + a)\pi_+^5 - 2a|p|\omega_-^5. \qquad (A.11)$$

The second step for solving the system (A.2)-(A.3) is given by the rotations,

$$z_+ = \sqrt{a}\omega_+^5 + \pi_-^5, \qquad (A.12)$$

$$z_- = \sqrt{a}\omega_+^5 - \pi_-^5. \qquad (A.13)$$

for (A.8)-(A.9) and

$$y_+ = -\sqrt{a}\omega_-^5 + \pi_+^5, \qquad (A.14)$$

$$y_- = -\sqrt{a}\omega_-^5 - \pi_+^5, \qquad (A.15)$$

for (A.10)-(A.11). Both systems are decoupled. The equations for $z_\pm$ and $y_\pm$ are

$$\ddot{z}_\pm = -\omega_\pm^2 z_\pm, \qquad (A.16)$$

$$\ddot{y}_\pm = -\omega_\pm^2 y_\pm, \qquad (A.17)$$

where $\omega_\pm = |p| \pm \sqrt{a}$. The general solution for $z_\pm$ and $y_\pm$ is given by

$$z_+ = A\cos(\omega_+\tau) + B\sin(\omega_+\tau), \qquad (A.18)$$

$$z_- = C\cos(\omega_-\tau) + D\sin(\omega_-\tau), \qquad (A.19)$$

$$y_+ = E\cos(\omega_+\tau) + F\sin(\omega_+\tau), \qquad (A.20)$$

$$y_- = G\cos(\omega_-\tau) + H\sin(\omega_-\tau), \qquad (A.21)$$

where capital letters are constants of integration. Inverting the expressions (A.12)-(A.15) and substituting (A.18)-(A.21) provides the solution for $\omega_\pm^5$, $\pi_\pm^5$. In turn, we invert (A.6)-(A.7) to obtain the solutions

$$\omega^5 = \frac{1}{4\sqrt{a}}[(A - E)\cos(\omega_+\tau) + (B - F)\sin(\omega_+\tau) +$$
$$(C - G)\cos(\omega_-\tau) + (D - H)\sin(\omega_-\tau)], \qquad (A.22)$$

$$\pi^5 = \frac{1}{4}[(A + E)\cos(\omega_+\tau) + (B + F)\sin(\omega_+\tau) -$$
$$-(C + G)\cos(\omega_-\tau) - (D + H)\sin(\omega_-\tau)], \qquad (A.23)$$

$$p\omega = \frac{|p|}{4\sqrt{a}}[(A + E)\cos(\omega_+\tau) + (B + F)\sin(\omega_+\tau) +$$

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$$+(C+G)\cos(\omega_-\tau)+(D+H)\sin(\omega_-\tau)], \quad \text{(A.24)}$$

$$p\pi = \frac{|p|}{4}[-(A-E)\cos(\omega_+\tau)+-(B-F)\sin(\omega_+\tau)+$$
$$(C-G)\cos(\omega_-\tau)+(D-H)\sin(\omega_-\tau)] \quad \text{(A.25)}$$

In order to solve the initial system of equations, we have taken one more time derivative: the variables $y_\pm$, $z_\pm$ obey second order differential equations. So, it is necessary to check the consistency of the solutions (A.22)-(A.25) substituting them in (A.2). lhs and rhs must coincide identically, which implies the restrictions

$$A + E = B - F, \quad \text{(A.26)}$$
$$E - A = B + F. \quad \text{(A.27)}$$

It follows that $E = B$, $F = -A$, $G = D$ and $H = -C$. We are now able to solve the equations (A.4), (A.5) for ω^μ and π^μ. Their general solution is given by

$$\omega^\mu(\tau) = K^\mu \cos(\sqrt{a}\tau) + L^\mu \sin(\sqrt{a}\tau)-$$
$$-\frac{(A+B)}{4|p|\sqrt{a}}p^\mu \cos(\omega_+\tau) + \frac{(A-B)}{4|p|\sqrt{a}}p^\mu \sin(\omega_+\tau)-$$
$$-\frac{(C+D)}{4|p|\sqrt{a}}p^\mu \cos(\omega_-\tau) + \frac{(C-D)}{4|p|\sqrt{a}}p^\mu \sin(\omega_-\tau) \quad \text{(A.28)}$$

$$\pi^\mu(\tau) = M^\mu \sin(\sqrt{a}\tau) + N^\mu \cos(\sqrt{a}\tau)-$$
$$-\frac{(A+B)}{4|p|\sqrt{a}}p^\mu \sin(\omega_+\tau) + \frac{(A-B)}{4|p|\sqrt{a}}p^\mu \cos(\omega_+\tau)-$$
$$-\frac{(C+D)}{4|p|\sqrt{a}}p^\mu \sin(\omega_-\tau) - \frac{(C-D)}{4|p|\sqrt{a}}p^\mu \cos(\omega_-\tau) \quad \text{(A.29)}$$

Substituting the expressions above in the initial system to check its consistency leads to

$$N^\mu = -\sqrt{a}K^\mu, \ M^\mu = \sqrt{a}L^\mu, \quad \text{(A.30)}$$
$$p_\mu K^\mu = p_\mu L^\mu = 0. \quad \text{(A.31)}$$

Redefining the constants of integration

$$\frac{A+B}{4} \to A, \qquad \frac{A-B}{4} \to B, \quad \text{(A.32)}$$

$$\frac{C+D}{4} \to C, \qquad \frac{C-D}{4} \to D, \quad \text{(A.33)}$$

we are left with the solutions

$$\omega^5(\tau) = \frac{1}{\sqrt{a}}(A\sin(\omega_+\tau) + B\cos(\omega_+\tau) + C\sin(\omega_-\tau) +$$
$$+ D\cos(\omega_-\tau)), \quad \text{(A.34)}$$

$$\pi^5(\tau) = A\cos(\omega_+\tau) - B\sin(\omega_+\tau) - C\cos(\omega_-\tau) +$$
$$+ D\sin(\omega_-\tau), \quad \text{(A.35)}$$

$$\omega^\mu(\tau) = K^\mu\cos(\sqrt{a}\tau) + L^\mu\sin(\sqrt{a}\tau) -$$
$$- \frac{A}{|p|\sqrt{a}}p^\mu\cos(\omega_+\tau) + \frac{B}{|p|\sqrt{a}}p^\mu\sin(\omega_+\tau) -$$
$$- \frac{C}{|p|\sqrt{a}}p^\mu\cos(\omega_-\tau) + \frac{D}{|p|\sqrt{a}}p^\mu\sin(\omega_-\tau) \quad \text{(A.36)}$$

$$\pi^\mu(\tau) = \sqrt{a}L^\mu\cos(\sqrt{a}\tau) - \sqrt{a}K^\mu\sin(\sqrt{a}\tau) +$$
$$+ \frac{A}{|p|}p^\mu\sin(\omega_+\tau) + \frac{B}{|p|}p^\mu\cos(\omega_+\tau) -$$
$$- \frac{C}{|p|}p^\mu\sin(\omega_-\tau) - \frac{D}{|p|}p^\mu\cos(\omega_-\tau). \quad \text{(A.37)}$$

Let us see what restrictions the constraints of the theory impose over the constants of integration. $T_3 = (\pi^A)^2 + a_3 = 0$ implies

$$- a_3 = aL^2\cos^2(\sqrt{a}\tau) + aK^2\sin^2(\sqrt{a}\tau) -$$
$$- 2aKL\sin(\sqrt{a}\tau)\cos(\sqrt{a}\tau) - (A^2 + B^2 + C^2 + D^2) +$$
$$+ 2\cos(2\sqrt{a}\tau)(AC + BD) + 2\sin(2\sqrt{a}\tau)(AD - BC). \quad \text{(A.38)}$$

The constraint $T_4 = (\omega^A)^2 + a_4 = 0$ leads to (we remind that $a = a_3/a_4$)

$$- a_3 = aL^2\sin^2(\sqrt{a}\tau) + aK^2\cos^2(\sqrt{a}\tau) +$$
$$+ 2aKL\sin(\sqrt{a}\tau)\cos(\sqrt{a}\tau) - (A^2 + B^2 + C^2 + D^2) -$$
$$- 2\cos(2\sqrt{a}\tau)(AC + BD) - 2\sin(2\sqrt{a}\tau)(AD - BC). \quad \text{(A.39)}$$

Adding the expressions (A.38) and (A.39), we arrive at

$$- 2a_3 = a(K^2 + L^2) - 2(A^2 + B^2 + C^2 + D^2). \quad \text{(A.40)}$$

$T_5 = \omega^A\pi^A = 0$ gives

$$0 = \cos(2\sqrt{a}\tau)\left[KL + \frac{2}{a}(BC - AD)\right] +$$

$$+\sin(2\sqrt{a}\tau)\left[\frac{1}{2}(L^2 - K^2) + \frac{2}{a}(AC + BD)\right], \qquad \text{(A.41)}$$

which must be zero throughout all the time, then

$$aKL = 2(AD - BC), \qquad \text{(A.42)}$$
$$a(K^2 - L^2) = 4(AC + BD). \qquad \text{(A.43)}$$

Now let us construct J^{AB} and x^μ. By definition,

$$J^{5\mu} = 2(\omega^5\pi^\mu - \pi^5\omega^\mu).$$

Thus

$$J^{5\mu} = \frac{2(A^2 + B^2 - C^2 - D^2)}{|p|\sqrt{a}}p^\mu +$$
$$+2|p|\alpha^\mu\sin(|p|\tau) + 2|p|\beta^\mu\cos(|p|\tau) \qquad \text{(A.44)}$$

where

$$\alpha^\mu = \frac{1}{|p|}[(A + C)L^\mu + (B - D)K^\mu], \qquad \text{(A.45)}$$

$$\beta^\mu = \frac{1}{|p|}[(B + D)L^\mu + (C - A)K^\mu]. \qquad \text{(A.46)}$$

To write $J^{5\mu}$ we have used the solutions (A.34)-(A.37). There is one more constraint to be taken into account: $T_2 = p_\mu J^{5\mu} + mc\hbar = 0$. We have

$$A^2 + B^2 - C^2 - D^2 = \frac{mc\hbar\sqrt{a}}{2|p|}. \qquad \text{(A.47)}$$

$J^{\mu\nu}$ can be found in the same way

$$\begin{aligned}J^{\mu\nu} &= 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu)\\ &= \Sigma_1^{\mu\nu} + \Sigma_2^{\mu\nu}\sin(|p|\tau) + \Sigma_3^{\mu\nu}\cos(|p|\tau), \qquad \text{(A.48)}\end{aligned}$$

where

$$\Sigma_1^{\mu\nu} = 2\sqrt{a}(K^\mu L^\nu - K^\nu L^\mu), \qquad \text{(A.49)}$$

$$\Sigma_2^{\mu\nu} = \frac{2}{|p|}[(A - C)(K^\mu p^\nu - K^\nu p^\mu) - (B + D)(L^\mu p^\nu - L^\nu p^\mu)], \text{(A.50)}$$

$$\Sigma_3^{\mu\nu} = \frac{2}{|p|}[(A + C)(L^\mu p^\nu - L^\nu p^\mu) + (B + D)(K^\mu p^\nu - K^\nu p^\mu)], \text{(A.51)}$$

Since $\dot{x}^\mu = \frac{1}{2}J^{5\mu}$, one has

$$x^\mu(\tau) = X^\mu + \frac{mc\hbar}{2|p|^2}p^\mu\tau - \alpha^\mu \cos(|p|\tau) + \beta^\mu \sin(|p|\tau). \quad (A.52)$$

We have already used (A.47). The physical trajectory $x^i = x^i(t)$ can be found with the same prescription as before: x^0 must be a monotonic function of τ, then we set $K^0 = L^0 = 0$. So, we have (for simplicity, we take X^0)

$$x^0 = ct = \frac{mc\hbar}{2|p|^2}p^0\tau \Rightarrow \tau = \frac{2|p|^2}{m\hbar p^0}. \quad (A.53)$$

Hence,

$$x^i(t) = X^i + c\frac{p^i}{p^0}t - \alpha^i \cos(\tilde{\omega}t) + \beta^i \sin(\tilde{\omega}t), \quad (A.54)$$

where $\tilde{\omega} = \frac{2|p|^3}{m\hbar p^0}$. We point out that the *Zitterbewegung* takes place in a plane orthogonal to the vector p^i once the restrictions (A.31) are reduced to $\vec{p}\cdot\vec{K} = \vec{p}\cdot\vec{L} = 0$, leading to $\vec{p}\cdot\vec{\alpha} = \vec{p}\cdot\vec{\beta} = 0$. One can also shows that the solution (A.44) for $J^{5\mu}(\tau)$ obeys the identity

$$(J^{5\mu}(\tau))^2 + 4[a_3(\omega^5(\tau))^2 + a_4(\pi^5(\tau))^2] \equiv 0, \quad (A.55)$$

that is $J^{5\mu}$ turns out to be the time-like vector.

In this appendix we have solved the equations of motion for the initial variables (x^μ, ω^A, π^B) and confirmed the results obtained in Section 6: the variables J^{AB} and x^μ experience *Zitterbewegung* with the angular frequency $\tilde{\omega} = \frac{2|p|^3}{m\hbar p^0}$.

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