Random matrices and Riemann hypothesis

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Abstract

The curious connection between the spacings of the eigenvalues of random matrices and the corresponding spacings of the nontrivial zeros of the Riemann zeta function is analyzed on the basis of the geometric dynamical global program of Langlands whose fundamental structures are shifted quantized conjugacy class representatives of bilinear algebraic semigroups. The considered symmetry behind this phenomenology is the differential bilinear Galois semigroup shifting the product, right by left, of automorphism semigroups of cofunctions and functions on compact transcendental quanta.

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Contents

Introduction

It became more and more evident over the years that **there must exist a connection between the eigenvalues of random matrices and the nontrivial zeros of the Riemann zeta function $\boldsymbol{\zeta(s)}$** in such a way that the distribution of the nontrivial zeros of $\zeta(s)$ can be approached by techniques developed in random matrix theory.

And, as the distributions over ensembles of random matrices may be related to the statistical properties of waves in chaotic dynamical systems [Yau], **the Riemann dynamics should be chaotic**.

Studying numerically the nontrivial zeros of $\zeta(s)$, Odlyzko [Odl] found that the mean spacings between their imaginary part γ_j and the spacings between the eigenvalues of large unitary matrices correspond. This work was based on researches of Montgomery [Mon] who found that the pair correlation of the γ_j is equal to the Gaussian unitary ensemble (GUE) pair correlation.

In their famous paper [K-S], M. Katz and P. Sarnak put the big questions: "**Why is this so and what does it tell us about the nature (e.g. spectral) of the zeroes? Also, what is the symmetry behind this "GUE" law?**"

It is the purpose of this paper to propose a satisfying solution to these questions based **on the following considerations:**

1) **The fundamental structures behind this problem are those of the geometric-dynamical global program of Langlands, i.e. double symmetric towers of shifted conjugacy class representatives of bilinear algebraic semigroups over sets of increasing finite algebraic extensions of number fields**.

2) **The quantization of these conjugacy class representatives, which are abstract subsemivarieties [Har], by algebraic (resp. transcendental) quanta being equivalently: a) irreducible closed algebraic (resp. transcendental) subsets characterized by an extension degree $\boldsymbol{N}$** (which is the Artin conductor); **b) unitary irreducible representations of these semigroups**.

3) **The considered symmetry behind this phenomenology is the one of the differential bilinear Galois semigroup shifting the product, right by left, of automorphism semigroups of cofunctions and functions on compact transcendental quanta**.

4) **The Riemann dynamics then corresponds to the wave chaotic dynamics described by random matrix theory**.

The general proposed solution is organized in three parts (or chapters):

1) **Reminder of universal algebraic structures of the global program of Langlands [Pie2], [Lan].**

2) **Reminder of universal dynamical structures of the global program of Langlands [Pie3].**

3) **Responses to five questions allowing finding a solution to this problem**:

   a) **What is behind random matrices leading to the Gaussian orthogonal ensemble (GOE) as well as the Gaussian unitary ensemble (GUE)?**
   b) **What is behind the partition and correlation functions between eigenvalues of random matrices?**
   c) **What interpretation can we give to the local spacings between the eigenvalues of large random matrices?**
   d) **What interpretation can we give to the spacings between the nontrivial zeros of the Riemann zeta function $\zeta(s)$?**
   e) **What is the curious connection between c) and d)?**

Let us analyse more concretely these three parts.

In chapter 1, the bilinear global version of the Langlands program is recalled to be **based on:**

a) **bisemiobjects $(o_R \times o_L)$ composed of the products of right semiobjetcs o_R**, localized on the lower half space or on $I\!R_-$, **and of left symmetric semiobjects o_L**, localized on the upper half space or on $I\!R_+$.

b) **the product, right by left, of two symmetric towers of increasing algebraic or compact transcendental extensions composed of an increasing number of algebraic or transcendental compact quanta and defining a (bisemi)lattice of biquanta**, i.e. the product, right by left, of two symmetric (semi)lattices of algebraic or transcendental quanta.

c) **Abstract bisemivarieties $G^{2n}(F_{\overline{v}} \times F_v)$ over products, right by left, $(F_{\overline{v}} \times F_v)$ of sets of increasing compact transcendental extensions** (or archimedean completions) $F_v = \{F_{v_1}, \dots, F_{v_j}, \dots, F_{v_r}\}$ and $F_{\overline{v}} = \{F_{\overline{v}_1}, \dots, F_{\overline{v}_j}, \dots, F_{\overline{v}_r}\}$ in such a way that they are (functional) representation spaces of the **algebraic bilinear** semigroups $\mathbf{GL_{2n}}(\widetilde{\boldsymbol{F}}_{\overline{\boldsymbol{v}}}\times\widetilde{\boldsymbol{F}}_{\boldsymbol{v}})=\boldsymbol{T}^t_{\mathbf{2n}}(\widetilde{\boldsymbol{F}}_{\overline{\boldsymbol{v}}})\times\boldsymbol{T}_{\mathbf{2n}}(\widetilde{\boldsymbol{F}}_{\boldsymbol{v}})$ **being the $2n$-dimensional bilinear representations** of the products, right by left, $\mathrm{Gal}(\widetilde{F}_{\overline{v}}/k)\times\mathrm{Gal}(\widetilde{F}_v/k)$ **of global Weil semigroups**.

d) **The bilinear abstract parabolic semigroups $\boldsymbol{P}^{(\mathbf{2n})}(\boldsymbol{F}_{\overline{\boldsymbol{v}}^1}\times\boldsymbol{F}_{\boldsymbol{v}^1})$**, over products $(F_{\overline{v}^1}\times F_{v^1})$ of unitary transcendental pseudoramified extensions, in such a way that they **are unitary representation spaces of the algebraic bilinear semigroup of matrices $\mathbf{GL_{2n}}(\widetilde{\boldsymbol{F}}_{\overline{\boldsymbol{v}}}\times\widetilde{\boldsymbol{F}}_{\boldsymbol{v}})$**.

The **abstract bisemivarieties $\boldsymbol{G}^{(\mathbf{2n})}(\boldsymbol{F}_{\overline{\boldsymbol{v}}}\times\boldsymbol{F}_{\boldsymbol{v}})$**, covered by their algebraic equivalents $G^{(2n)}(\widetilde{F}_{\overline{v}}\times\widetilde{F}_v)$, **are separated bisemischemes or quasiprojectives bisemivarieties** [**Har**] and constitute the corner stone of the global program of Langlands since, by an isomorphism of toroidal compactification, $G^{(2n)}(\widetilde{F}_{\overline{v}}\times\widetilde{F}_v)$ (and $G^{(2n)}(F_{\overline{v}}\times F_v)$) is transformed into **the cuspidal representation $\boldsymbol{\Pi}(\mathbf{GL_{2n}}(\widetilde{\boldsymbol{F}}_{\overline{\boldsymbol{v}}}\times\widetilde{\boldsymbol{F}}_{\boldsymbol{v}}))$ of $\mathbf{GL_{2n}}(\widetilde{\boldsymbol{F}}_{\overline{\boldsymbol{v}}}\times\widetilde{\boldsymbol{F}}_{\boldsymbol{v}})$**.
Remark that, as abstract (bisemi)varieties are quantized, their associated (bisemi)schemes, referring to quasi-projective (bisemi)varieties, are also quantized.
The differentiable functional representation space $\mathbf{FREPSP}(\mathbf{GL_{2n}}(\boldsymbol{F}_{\overline{\boldsymbol{v}}}\times\boldsymbol{F}_{\boldsymbol{v}}))$ of the complete bilinear semigroup $\mathbf{GL_{2n}}(\boldsymbol{F}_{\overline{\boldsymbol{v}}}\times\boldsymbol{F}_{\boldsymbol{v}})$ is a bisemisheaf $(\widehat{\boldsymbol{M}}^{(\mathbf{2n})}_{\boldsymbol{v_R}}\otimes\widehat{\boldsymbol{M}}^{(\mathbf{2n})}_{\boldsymbol{v_L}})$ of differential bifunctions on the abstract bisemivariety $\boldsymbol{G}^{(\mathbf{2n})}(\boldsymbol{F}_{\overline{\boldsymbol{v}}}\times\boldsymbol{F}_{\boldsymbol{v}})$ which is given by the product, right by left, of a right semisheaf $\widehat{M}^{(2n)}_{v_R}$ of differentiable cofunctions on the increasing conjugacy class representatives of the abstract right semivariety $G^{(2n)}(F_{\overline{v}})\equiv T^{(2n)}(F_{\overline{v}})$ by a left semisheaf $\widehat{M}^{(2n)}_{v_L}$ of symmetric differentiable functions on the increasing conjugacy class representatives of the abstract left semivariety $G^{(2n)}(F_v)\equiv T^{(2n)}(F_v)$.

Chapter 2 deals with dynamical (geometric) $\mathbf{GL(2n)}$-bisemistructures of the global program of Langlands generated from the action of the elliptic bioperator $(\boldsymbol{D}^{(\mathbf{2k})}_{\boldsymbol{R}}\otimes\boldsymbol{D}^{(\mathbf{2k})}_{\boldsymbol{L}})$ (i.e. the product of a right linear differential elliptic operator $D^{(2k)}_R$ acting on $2k$ variables by its left equivalent $D^{(2k)}_L$, $k\le n$) **on the bisemisheaf $(\widehat{\boldsymbol{M}}^{(\mathbf{2n})}_{\boldsymbol{v_R}}\otimes\widehat{\boldsymbol{M}}^{(\mathbf{2n})}_{\boldsymbol{v_L}})$** according to:

$$D^{(2k)}_R\otimes D^{(2k)}_L:\quad \mathrm{FREPSP}(\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v))\longrightarrow \mathrm{FREPSP}(\mathrm{GL}_{2n[2k]}(F_{\overline{v}}\times I\!\!R)\times(F_v\times I\!\!R))$$

where $\mathrm{FREPSP}(\mathrm{GL}_{2n[2k]}(F_{\overline{v}}\times I\!\!R)\times(F_v\times I\!\!R))$ is the functional representation space of $\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v)$ fibered or shifted in $2k$ bilinear geometric dimensions with $F_v^{S_{I\!\!R}}=(F_v\times I\!\!R)$ the set of increasing left compact transcendental extensions fibered or shifted by real numbers.
$\mathbf{FREPSP}(\mathbf{GL_{2k}}(\boldsymbol{F}_{\overline{\boldsymbol{v}}}\times I\!\!R)\times(\boldsymbol{F}_{\boldsymbol{v}}\times I\!\!R))$ is isomorphic to the total bisemispace of the tangent bibundle $\mathbf{TAN}(\widehat{\boldsymbol{M}}^{(\mathbf{2k})}_{\boldsymbol{v_R}}\otimes\widehat{\boldsymbol{M}}^{(\mathbf{2k})}_{\boldsymbol{v_L}})$ to the bisemisheaf $(\widehat{\boldsymbol{M}}^{(\mathbf{2k})}_{\boldsymbol{v_R}}\otimes\widehat{\boldsymbol{M}}^{(\mathbf{2k})}_{\boldsymbol{v_L}})\subset$

$(\widehat{M}^{(2n)}_{v_R} \otimes \widehat{M}^{(2n)}_{v_L})$ **of which bilinear fibre** $\mathcal{F}^{(2k)}_{R\times L}(\mathrm{TAN}) = \mathrm{(F)REPSP}(\mathrm{GL}_{2k}(I\!R \times I\!R))$ **is the (functional) representation space of** $\mathrm{GL}_{2k}(I\!R\times I\!R)$ **corresponding to the action of the bioperator** $(D^{(2k)}_R)\otimes(D^{(2k)}_L)$.

It is then proved that **the bilinear semigroup of matrices** $\mathrm{GL}_r(I\!R^{(2k)}\times I\!R^{(2k)})$ **of "algebraic" order** r, associated with the bilinear fibre $\mathrm{GL}_{2k}(I\!R\times I\!R)$ and referring to the action of the bioperator $(D^{(2k)}_R\otimes D^{(2k)}_L)$ on the bisemisheaf $(\widehat{M}^{(2k)}_{v_R}\otimes\widehat{M}^{(2k)}_{v_L})$, **corresponds to the** $2k$**-dimensional "geometric" bilinear real representation of the product, right by left, of "differential" Galois (or global Weil) semigroups** $\mathrm{Aut}_k(\phi_R(I\!R))\times\mathrm{Aut}_k(\phi_L(I\!R))$, shifting or fibering the product, right by left, of automorphism semigroups $\mathrm{Aut}_k(\phi_R(F_{\overline{v}}))\times\mathrm{Aut}_k(\phi_L(F_v))$ of cofunctions $\phi_R(F_{\overline{v}})$ and functions $\phi_L(F_v)$ respectively on the compact transcendental real extensions $F_{\overline{v}}$ and F_v by $\mathrm{Aut}_k(\phi_R(F_{\overline{v}}\times I\!R))\times\mathrm{Aut}_k(\phi_L(F_v\times I\!R))$ **such that**

$$\begin{aligned}\mathrm{GL}_r(\phi_R(I\!R^{(2k)})\times\phi_L(I\!R^{(2k)}))\\ &= \mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(I\!R))\times\mathrm{Aut}_k(\phi_L(I\!R)))\;, \quad 1\le j\le r\;.\end{aligned}$$

Similarly, **the unitary parabolic bilinear semigroup** $P_r(I\!R^{(2k)}\times I\!R^{(2k)})\subset\mathrm{GL}_r(I\!R^{(2k)}\times I\!R^{(2k)})$, **referring to the action of the bioperator** $(D^{(2k)}_R\otimes D^{(2k)}_L)$ **on the unitary bisemisheaf** $(\widehat{M}^{(2k)}_{\overline{v}^1_R}\otimes\widehat{M}^{(2k)}_{\overline{v}^1_L})\subset(\widehat{M}^{(2k)}_{v_R}\otimes\widehat{M}^{(2k)}_{v_L})$, **corresponds to the** $2k$**-dimensional bilinear real representation of the product, right by left, of "differential" inertia Galois (or global Weil) semigroups** $\mathrm{Int}_k(\phi_R(I\!R))\times\mathrm{Int}_k(\phi_L(I\!R))$ **such that**

$$P_r(\phi_R(I\!R^{(2k)})\times\phi_L(I\!R^{(2k)})) = \mathrm{Rep}^{(2k)}(\mathrm{Int}_k(\phi_R(I\!R))\times\mathrm{Int}_k(\phi_L(I\!R)))\;.$$

Let $O_r(I\!R)$ denote the orthogonal group of (algebraic) order r with entries in $I\!R$ and let $U_r(\mathbb{C})$ denote the unitary group with entries in $\mathbb{C}$. Then, the orthogonal bilinear semigroup $O_r(I\!R^{(2k)}\times I\!R^{(2k)})$ corresponds to the real parabolic bilinear semigroup $P_r(I\!R^{(2k)}\times I\!R^{(2k)})$ and the unitary bilinear semigroup $U_r(\mathbb{C}^k\times\mathbb{C}^k)$ corresponds to the complex parabolic bilinear semigroup $P_r(\mathbb{C}^k\times\mathbb{C}^k)$.

Chapter 3 deals with the connection between large random matrices and the solution of the Riemann hypothesis by responding to the five above mentioned questions. From now on, **the geometric dimension** $2k$ **will be taken to be** 1.

1. **The first question "What is behind random matrices leading to GOE and GUE?" leads to the eigenvalue problem in the frame of the geometric dynamical program of Langlands** recalled in chapter 2.

The symmetric group at the origin of the bilinear global program of Langlands is the bilinear semigroup of automorphisms $\mathrm{Aut}_k(F_{\overline{v}})\times\mathrm{Aut}_k(F_v)$ (resp. Galois automorphisms $\mathrm{Gal}(\widetilde F_{\overline{v}}/k)\times\mathrm{Gal}(\widetilde F_v/k)$) **of compact transcendental (resp. algebraic) quanta generating a bisemilattice** of compact transcendental (resp. algebraic) quanta **while the symmetry group at the origin of the dynamical (geometric) bilinear global program of Langlands is the bilinear semigroup of shifted automorphisms** $\mathrm{Aut}_k(\phi_R((F_{\overline{v}}\times I\!\!R))\times\mathrm{Aut}_k(\phi_L((F_v\times I\!\!R))$ **of bifunctions on compact transcendental biquanta** generating a bisemilattice of compact transcendental quanta.

It then results that **the bilinear semigroup of matrices $\mathbf{GL}_{\boldsymbol{r}}(I\!\!R\times I\!\!R)$ constitutes the "$\boldsymbol{r}$-dimensional algebraic" representation of the bilinear differential Galois semigroup associated with the action of the differential bioperator $(\boldsymbol{D_R}\otimes\boldsymbol{D_L})$** on the bisemisheaf $(\widehat M^{(1)}_{v_R}\otimes\widehat M^{(1)}_{v_L})$.

Let

$$(D_R\otimes D_L)(\phi(G^{(1)}(F_{\overline{v}_j}\times F_{v_j}))=E_{R\times L}(j)(\phi(G^{(1)}(F_{\overline{v}_j}\times F_{v_j}))\;,\qquad 1\le j\le r\;,$$

be the eigenbivalue equation related to the bisemisheaf $(\widehat M^{(1)}_{v_R}\otimes\widehat M^{(1)}_{v_L})$. Then, we have that

(a) **the $\boldsymbol{j}$-th eigenbifunction $\boldsymbol{\phi(G^{(1)}(F_{\overline{v}_j}\times F_{v_j}))}$ on the $\boldsymbol{j}$ transcendental compact biquanta, being the $\boldsymbol{j}$-th bisection of $\boldsymbol{(\widehat M^{(1)}_{v_R}\otimes\widehat M^{(1)}_{v_L})}$, corresponds to the $\boldsymbol{j}$-th eigenbivalue $\boldsymbol{E_{R\times L}(j)}$ which is the shift of this bifunction and the shift of the global Hecke character** associated with this subbisemilattice.

(b) the eigenbivalues of the matrix of $\mathrm{GL}_r(I\!\!R\times I\!\!R)$, constituting a representation of the bilinear differential Galois semigroup associated with the biaction of $(D_R\otimes D_L)$, are the eigenbivalues of the above eigenbivalue equation.

2. **The second question "What is behind the partition and correlation functions between eigenvalues of random matrices?" concerns the distribution of eigenvalues [D-H] of random matrices** of the Gaussian unitary (GUE) and orthogonal ensemble (GOE) **having $\boldsymbol{r}$ quantum states and characterized by a Hamiltonian matrix of order $\boldsymbol{r}$** whose entries are Gaussian random variables [D-S].

In the bilinear context envisaged in this paper, we are mostly interested by **the $\boldsymbol{m}$-point correlation function for the bilinear Gaussian unitary (or or-**

thogonal) ensemble BCUE (resp. BCOE) given by:

$$R_{m_{r_{R\times L}}}(x_1^2,\dots,x_m^2)=\frac{r!}{(r-m)!}\int_{\mathbb{R}^{r-m}} P_{r_{R\times L}}(x_1^2,\dots,x_r^2)\;dx_{m+1}\dots dx_r$$

$$=\det(K_r(x_k,x_\ell)^m_{k,\ell=1})$$

with

$$K_r(x_k,x_\ell)=\sum_{i=0}^{r-1}\psi_i(x_k)\psi_i(x_\ell)$$

$$\text{and}\qquad \psi_i(x)=h^{-1/2}P_i(x)\;e^{-r[(TG^T\times TG)(\mathbb{R}\times\mathbb{R})]/2}\;.$$

$P_i(x)$ being an orthogonal polynomial of degree i corresponding to the weight function $e^{-r[(TG^T\times TG)(\mathbb{R}\times\mathbb{R})]/2}$ where $G(\mathbb{R}\times\mathbb{R})=TG^T(\mathbb{R})\times TG(\mathbb{R})$ is the bilinear Gauss decomposition of the matrix G of BCOE.

$R_{m_{r_{R\times L}}}(x_1^2,\dots,x_m^2)$ is the probability of finding a level around each of the bipoints (i.e. entries in G) $x_1^2,\dots,x_m^2$, the positions of the remaining levels being unobserved.

Let

$$K_r(x,x)=\sum_{i=0}^{r-1}\psi_i(x)\psi_i(x)$$

be the energy level density with $\psi_i(x)$ given above.

Then, we have found that:

(a) **the squares of the roots of the polynomial $P_i(x)$ correspond to the eigenbivalues of the product, right by left, $(U_{r_R}\times U_{r_L})$ of Hecke operators**.

(b) **the weight $e^{-r[(TG^T\times\mathbb{R})\times(TG\times\mathbb{R})]/2}$ is a measure of the eigenbivalues of the random matrix $G\in \mathrm{GL}_r(\mathbb{R}\times\mathbb{R})$ being a representation of the differential bilinear Galois semigroup**.

The orthogonal polynomials $P_i(x)$ satisfy the three term recurrent relation

$$\beta_{I+1}P_{i+1}(x)=(x-\alpha_i)P_i(x)-\beta_iP_{i-1}(x)$$

leading to a tower whose matricial form is

$$xP=JP+\beta_iP_i$$

where J is the Jacobi matrix and P a column vector of polynomials of increasing degrees.

It was then proven that **the $\boldsymbol{i}$ roots of $\boldsymbol{P_i(x)}$ are the eigenvalues of the Jacobi symmetric matrix being a representation** of the Hecke operator U_{r_i}.

We are then led to the conclusions:

(a) **The probabilistic interpretation of quantum (field) theories is related to the bilinear semigroup of automorphisms $\mathbf{Aut}_{\boldsymbol{k}}(\boldsymbol{F_{\overline{v}}}) \times \mathbf{Aut}_{\boldsymbol{k}}(\boldsymbol{F_v})$ of compact transcendental biquanta generating a bisemilattice of these**.

(b) **The $\boldsymbol{m}$-point correlation function for BCUE (or BCOE) $\boldsymbol{R_{m_r}(x_1^2,\dots,x_m^2)}$ constitutes a representation of the bilinear semigroup of automorphisms $\mathbf{Aut}_{\boldsymbol{k}}(\boldsymbol{\phi_R}(\boldsymbol{F_{\overline{v}}} \times I\!R)) \times \mathbf{Aut}_{\boldsymbol{k}}(\boldsymbol{\phi_L}(\boldsymbol{F_v} \times I\!R))$ of bifunctions on shifted compact transcendental biquanta**.

3. **The third question "What interpretation can we give to the local spacings [Gau] between the eigenvalues of large random matrices" depends on the dynamical global program of Langlands** developed in chapter 2 and summarized in the first question: it results from the r-dimensional algebraic representation of $\mathrm{GL}_r(I\!R \times I\!R)$ associated with the action of the differential bioperator $(D_R \otimes D_L)$ on the bisemisheaf $(\widehat{M}^{(1)}_{v_R} \otimes \widehat{M}^{(1)}_{v_L})$ and leading to the above mentioned eigenbivalue equation in such a way that:

   (a) **the consecutive spacings**

   $$\boldsymbol{\delta E_{R\times L}(j) = E_{R\times L}(j+1) - E_{R\times L}(j)\ , \qquad 1 \le j \le r \le \infty\ ,}$$

   between the eigenbivalues of the random matrix $\boldsymbol{G} \in \mathbf{GL}_{\boldsymbol{r}}(I\!R \times I\!R)$ are infinitesimal bigenerators of one biquantum of the Lie subbisemialgebra $\mathbf{gl}_1(\boldsymbol{F_{\overline{v}^1}} \times \boldsymbol{F_{v^1}})$ of the bilinear parabolic unitary semigroup $\boldsymbol{P_1(F_{\overline{v}^1} \times F_{v^1})} \in \mathbf{GL}_1(\boldsymbol{F_{\overline{v}}} \times \boldsymbol{F_v})$ and correspond to the energies of one free biquantum from subbisemilattices of $\boldsymbol{(j+1)}$ biquanta.

   (b) the k-th consecutive spacings

   $$\delta E^{(k)}_{R\times L}(j) = E_{R\times L}(j+k) - E_{R\times L}(j)$$

   between the eigenbivalues of the random matrix $G \in \mathrm{GL}_r(I\!R \times I\!R)$ are the infinitesimal bigenerators on k biquanta of the Lie subbisemialgebra $\mathrm{gl}_1(F_{\overline{v}^k} \times F_{v^k})$ of the bilinear k-th semigroup $\mathrm{gl}_1(F_{\overline{v}^k} \times F_{v^k})$ and correspond to the energies of k free biquanta from subbisemilattices of $(j+k)$ biquanta.

The consecutive spacings $\delta E_{R\times L}(j)$ between the eigenbivalues of the matrix G of $\mathrm{GL}_r(I\!\!R\times I\!\!R)$ decompose into:

$$\delta E_{R\times L}(j)=\delta EF_{R\times L}(j)+\delta EV_{R\times L}(j)$$

where **$\delta EF_{R\times L}(j)$ and $\delta EV_{R\times L}(j)$ denote respectively the fixed (or constant) and variable consecutive spacings between the r eigenbivalues of G**.

Then, we have that:

(a) the consecutive spacings

$$E^{BCUE}_{R\times L}(j+1)-E^{BCUE}_{R\times L}(j)=\delta EV^{BCUE}_{R\times L}(j)$$

between the eigenbivalues $E^{BCUE}_{R\times L}(j+1)$ and $E^{BCUE}_{R\times L}(j)$ of a unitary random matrix of $U_r(I\!\!\!C\times I\!\!\!C)$ (or $O_r(I\!\!R\times I\!\!R)$) are the variable (unitary) infinitesimal bigenerators on one biquantum on the envisaged Lie subbisemialgebra or the variable (unitary) energies $\delta E^{BCUE}_{R\times L}(j)$ of one biquantum in subbisemilattices of $(j+1)$ biquanta.

(b) **the k-th consecutive spacings**

$$\boldsymbol{E^{BCUE}_{R\times L}(j+k)-E^{BCUE}_{R\times L}(j)=\delta EV^{(k)BCUE}_{R\times L}(j)}$$

between the eigenbivalues of a unitary random matrix of $U_r(I\!\!\!C\times I\!\!\!C)$ (or $O_r(I\!\!R\times I\!\!R)$), are the variable energies $\delta E^{(k)BCUE}_{R\times L}(j)$ on k biquanta in subbisemilattices of $(j+k)$ biquanta.

4. **The fourth question: "What interpretation can we give to the spacings between the nontrivial zeros [Zag], [Pol], of the Riemann zeta function $\zeta(s)$?" depends on the solution of the Riemann hypothesis [Bom] proposed in [Pie7]** and briefly recalled now.

The $1D$-pseudounramified simple global elliptic $\Gamma_{\widehat{M}^{(1)}_{v^T_{R\times L}}}$ -bisemimodule

$$\phi^{(1),(nr)}_{R\times L}(x)=\sum_n\left(\lambda^{(nr)}(n)\;e^{-2\pi inx}\otimes_D\lambda^{(nr)}(n)\;e^{+2\pi inx}\right)\;,\quad x\in I\!\!R\;,$$

where $\lambda^{(nr)}(n)$ is a global Hecke character,

can be interpreted as the sum of products, right by left, of semicircles of level "n" on n transcendental compact quanta and constitutes a cuspidal representation of the bilinear semigroup $\mathrm{GL}_2(F^{nr}_{\overline{v}}\times F^{nr}_{v})$.

Let $\zeta_R(s_-)$ and $\zeta_L(s_+)$, $s_-=\sigma-i\tau$ and $s_+=\sigma+i\tau$, be the two zeta functions defined respectively in the lower and upper half planes.

They are (distribution) inverse space functions, i.e. energy functions on the variables s_- and s_+ conjugate to the complex space variables $z^*\in\mathbb{C}$ and $z\in\mathbb{C}$ of the cusp forms $f_L(z)$ and $f_R(z^*)$ submitted respectively to the transform maps $f_L(z)\to\zeta_L(s_+)$ and $f_R(z^*)\to\zeta_R(s_-)$ [Pie8].

Then, the kernel $\mathrm{Ker}(H_{\phi_{R\times L}\to\zeta_{R\times L}})$ of the map:

$$H_{\phi_{R\times L}\to\zeta_{R\times L}}:\qquad 2\phi^{(1),(nr)}_{R\times L}(x)\quad\longrightarrow\quad \zeta_R(s_-)\otimes_D\zeta_L(s_+)$$

is the set of squares of the trivial zeros of $\zeta_R(s_-)$, $\zeta_L(s_+)$ and $\zeta(s)$, corresponding to the degeneracies of the products, right by left, of circles $2\lambda^{(nr)}(n)e^{2\pi inx}$ on $2n$ transcendental quanta.

By this way, **we have a one-to-one correspondence between the trivial zeros of $\zeta(s)$ and the degeneracies of circles belonging to symmetric towers of cuspidal conjugacy class representatives of bilinear complete algebraic semigroups**.

Then, it is proved that **the products of the pairs of the trivial zeros of the Riemann zeta functions $\zeta_R(s_-)$ and $\zeta_L(s_+)$ are mapped into the products of the corresponding pairs of the nontrivial zeros according to**:

$$\begin{aligned}\{D_{4n^2,i^2}\cdot\varepsilon_{4n^2}\}:\qquad &\{\det(\alpha_{4n^2})\}_n\longrightarrow\{\det(D_{4n^2,i^2}\cdot\varepsilon_{4n^2}\cdot\alpha_{4n^2})_{ss}\}_n\\ &\{(-2n)\times(-2n)\}_n\longrightarrow\{\lambda^{(nr)}_+(4n^2,i^2,E_{4n^2})\times\lambda^{(nr)}_-(4n^2,i^2,E_{4n^2})\}_n\end{aligned}$$

where:

- α_{4n^2} is the split Cartan subgroup element associated with the integer $2n$;
- D_{4n^2,i^2} is the coset representative of the Lie (bisemi)algebra of the decomposition bisemigroup acting on α_{4n^2,i^2};
- ε_{4n^2} is the infinitesimal bigenerator of the considered bisemialgebra.

Every root of this Lie bisemialgebra is determined by the eigenvalues

$$\boldsymbol{\lambda^{(nr)}_\pm(4n^2,i^2,E_{4n^2})=\frac{1\pm i\sqrt{16n^2\cdot E_{4n^2}-1}}{2}}$$

of $D_{4n^2,i^2}\cdot\varepsilon_{4n^2}\cdot\alpha_{4n^2}$ which are the nontrivial zeros of $\zeta(s)$ written compactly according to $\left(\frac{1}{2}+ij_j\right)$ and $\left(\frac{1}{2}-ij_j\right)$, $j\leftrightarrow n$.

Then, we have that:

1) **the consecutive spacings**

$$\boldsymbol{\delta_{\gamma_j} = \gamma_{j+1} - \gamma_j \ , \qquad j = 1, 2, \dots \ ,}$$

between the nontrivial zeros of $\boldsymbol{\zeta(s)}$ are equivalently:

a) **the infinitesimal generators on one quantum of the Lie subsemialgebra $\mathbf{gl}_1(\boldsymbol{F}^{(nr)}_{\boldsymbol{v}^1})$ (or $\mathbf{gl}_1(\boldsymbol{F}^{(nr)}_{\overline{\boldsymbol{v}}^1})$) where $\boldsymbol{F}^{(nr)}_{\boldsymbol{v}^1}$ is a pseudounramified compact transcendental extension** (see section 1.1) of the linear parabolic unitary semigroup $P_1(F^{(nr)}_{v^1}) \subset \mathrm{GL}_1(F^{(nr)}_{v}) \equiv T_1(F^{(nr)}_{v})$ (or $P_1(F^{(nr)}_{\overline{v}^1})$;

b) **the energies of one free quantum in subsemilattices of $\boldsymbol{(j+1)}$ quanta**.

2) The k-th consecutive spacings

$$\delta^{(k)}_j = \gamma_{j+k} - \gamma_j$$

between the nontrivial zeros of $\zeta(s)$ are equivalently:

a) the infinitesimal generators on k quanta of the Lie subsemialgebra $\mathrm{gl}_1(F^{(nr)}_{v^k})$ (or $\mathrm{gl}_1(F^{(nr)}_{\overline{v}^k})$) of the k-th semigroup $\mathrm{GL}_1(F^{(nr)}_{v^k})$ (or $\mathrm{GL}_1(F^{(nr)}_{\overline{v}^k})$);

b) the energies of k free quanta in subsemilattices of $(j+k)$ quanta.

The fifth question "What is the curious connection between the spacings of the nontrivial zeros of $\boldsymbol{\zeta(s)}$ and the corresponding spacings between the eigenvalues of random matrices?" [Ke-S] finds response in the following statements (propositions of chapter 3):

- The consecutive spacings

$$\delta\gamma_j = \gamma_{+1} - \gamma_j \ , \qquad j \in \mathbb{N} \, , \quad 1 \le j \le r < \infty \ ,$$

between the nontrivial zeros of the Riemann zeta function $\zeta(s)$

correspond to the consecutive spacings

$$\delta E^{(nr)}_{R,L}(j) = E^{(nr)}_{R,L}(j+1) - E^{(nr)}_{R,L}(j) \ ,$$

($_{R,L}$ means right of left),

between the square roots of the pseudounramified eigenbivalues of a (large) random matrix of $\mathrm{GL}_r(\mathbb{R} \times \mathbb{R})$ (or of $\mathrm{GL}_r(\mathbb{C} \times \mathbb{C})$)

and are equivalently:

1) the infinitesimal generators on one quantum of the Lie subsemialgebra $\mathrm{gl}_1(F^{(nr)}_{v^1})$ of the linear parabolic unitary semigroup $P_1(F^{(nr)}_{v^1})$;
2) the energies of one transcendental compact pseudounramified ($N=1$) quantum in subsemilattices of $(j+1)$ transcendental compact pseudounramified quanta.

- The set $\{\delta E^{(nr)}_{R,L}(j)\}^r_{j=1}$ of consecutive spacings between the square roots of the eigenbivalues of a random matrix of $\mathrm{GL}_r(I\!R\times I\!R)$ as well as the set $\{\delta\gamma_j\}^r_{j=1}$ of consecutive spacings between the nontrivial zeros of $\zeta(s)$ constitutes a representation of the differential inertia Galois semigroup associated with the action of the differential operator D_L or D_R.

- Let $\{\delta E_{R,L}(j)\}_j$ be the set of consecutive spacings between the square roots of the eigenbivalues of a random matrix of $\mathrm{GL}_r(I\!R\times I\!R)$ or between the eigenvalues of a random matrix of $\mathrm{GL}_r(I\!R)$.

  Then, there is a surjective map:

$$IM_{E\to\gamma}: \qquad \{\delta E_{R,L}\}_j \quad \longrightarrow \quad \{\delta\gamma_j\}_j$$

  of which kernel $\mathrm{Ker}[IM_{E\to\gamma}]$ is the set $\{\delta E_{R,L}(j)-\delta E^{(nr)}_{R,L}(j)\}_j$ of differences of consecutive spacings between the square roots of the pseudoramified and pseudounramified eigenbivalues of a random matrix of $\mathrm{GL}_r(I\!R\times I\!R)$, i.e. the energies of one compact transcendental pseudoramified ($N>2$) quantum in subsemilattices of $(j+1)$ transcendental pseudoramified quanta.

- Finally, **we can gather the results of this paper in the following proposition (see 3.33)**.

  Let $\boldsymbol{\delta\gamma^{(k)}_j=\gamma_{j+k}-\gamma_j}$ denote the $\boldsymbol{k}$-th consecutive spacings between the nontrivial zeros of $\boldsymbol{\zeta(s)}$.

  Let*:*

  – $\delta E^{(k)}_{R,L}(j)=E_{R,L}(j+k)-E_{R,L}(j)$;
  – $\delta E^{(nr),(k)}_{R,L}(j)=E^{(nr)}_{R,L}(j+k)-E^{(nr)}_{R,L}(j)$;
  – $\delta EV^{(k),(nr),BCOE}_{R,L}(j)=E^{(nr),BCOE}_{R,L}(j+k)-E^{(nr),BCOE}_{R,L}(j)$, $1\le j\le r$, $k\le j$,

  be the $\boldsymbol{k}$-th consecutive spacings between respectively*:*

  – *the pseudoramified eigenvalues of a random matrix of $GL_r(I\!R)$;*
  – *the pseudounramified eigenvalues of a random matrix of $GL_r(I\!R)$;*

– *the pseudounramified eigenvalues of a random unitary matrix of* $O_r(I\!R)$.

Then, we have*:*

1) $\boldsymbol{\delta\gamma_j^{(k)} = \delta E_{R,L}^{(nr),(k)}(j)}$ **which are equivalently***:*

   a) **the infinitesimal generators of** $\boldsymbol{k}$ **quanta of the Lie subsemialgebra** $\mathrm{gl}_1(\boldsymbol{F_{v^k}^{(nr)}})$ **of the linear** $\boldsymbol{k}$**-th semigroup** $\mathrm{GL}_1(\boldsymbol{F_{v^k}^{(nr)}}) \subset \mathrm{GL}_1(\boldsymbol{F_{v}^{(nr)}})$ *;*

   b) **the energies of** $\boldsymbol{k}$ **transcendental compact pseudounramified (** $\boldsymbol{N=1}$ **) quanta in subsemilattices in** $\boldsymbol{(j+k)}$ **transcendental pseudounramified quanta***;*

   c) **a representation of the differential Galois semigroup** *associated with the action of the differential operator* D_L *or* D_R *on a function on* k *transcendental pseudounramified quanta;*

2) *a surjective map:*

$$IM_{E\to\gamma}^{(k)} : \qquad \{\delta E_{R,L}^{(k)}(j)\}_j \longrightarrow \{\delta\gamma_j^{(k)}\}_j$$

   of which kernel is the set $\{\delta E_{R,L}^{(k)}(j) - \delta E_{R,L}^{nr,(k)}(j)\}_j$ *of difference of* k*-th consecutive spacings between the pseudoramified and pseudounramified eigenvalues of a random matrix of* $\mathrm{GL}_r(I\!R)$ *;*

3) *a bijective map:*

$$IM_{\gamma\to E_{BCOE}^{(nr)}}^{(k)} : \qquad \{\delta\gamma_j^{(k)}\}_j \longrightarrow \{\delta EV_{R,L}^{(k),(nr),BCOE}(j)\}_j$$

   where $\delta\gamma_j^{(k)}$ *denotes a* k*-th (variable) consecutive spacing verifying* $\delta\gamma_j^{(k)} = \delta E_{R,L}^{(k),(nr),BCOE}(j)$.

- It has been inferred for a long time that the nontrivial zeros of the Riemann zeta function are probably related to the eigenvalues of some wave dynamical system. The connection between these two fields at the light of the global program of Langlands can be summarised by the proposition 3.35:

  The squares of the nontrivial zeros $\boldsymbol{\gamma_j}$ **of the Riemann zeta function** $\boldsymbol{\zeta(s)}$ **are pseudounramified eigenbivalues of the eigenbivalue biwave operation:**

$$\boldsymbol{(D_R\otimes D_L)(\phi(G^{(1)}(F_{\overline{v}_j}^{(nr)}\times F_{v_j}^{(nr)}))) = \gamma_j^2(\phi(G^{(1)}(F_{\overline{v}_j}^{(nr)}\times F_{v_j}^{(nr)})))}$$

  of which eigenbifunctions are the sections of the bisemisheaf $\boldsymbol{(\widehat{M}_{v_R}^{(1)}\otimes \widehat{M}_{v_L}^{(1)})}$ **being interpreted as the internal stringfield of an elementary (bisemi)particle** [Pie8].

1 Universal algebraic structures of the global program of Langlands

1.1 Symmetric structures of the program of Langlands

In analogy with the fundamental theorem of the Galois theory which establishes a one-to-one correspondence between the set of closed intermediate fields of a given finite extension F of a number field k and the set of all closed normal subgroups of the Galois group $\mathrm{Aut}_k F$, we consider **a set of increasing finite algebraic extensions of *k*** in one-to-one correspondence with the corresponding Galois (sub)groups. The envisaged global program of Langlands is then constructed on n-dimensional representations of such Galois (sub)groups and is recalled in this chapter.
Since symmetric semiobjects have to be considered in any generality [Pie4], we take into account **the bilinear global version of the Langlands program** concerning the generation of general symmetric structures, i.e. **double symmetric towers of conjugacy class representatives of (bilinear) algebraic (semi)groups**.

1.2 Algebraic and transcendental symmetric extensions

Let then the set $\widetilde F$ of finite algebraic extensions of a number field k of characteristic 0 be a set of symmetric splitting fields composed of the left and right real symmetric splitting semifields $\widetilde F_L^+$ and $\widetilde F_R^+$ given respectively by the sets of positive and symmetric negative simple real roots.

Assume that the **set of all increasing left (resp. right) splitting semifields** [Wei]:

$$\begin{aligned} &\widetilde F_{v_1} \subset \cdots \subset \widetilde F_{v_{j,m_j}} \subset \cdots \subset \widetilde F_{v_{r,m_r}} \\ (\text{resp.}\quad &\widetilde F_{\overline v_1} \subset \cdots \subset \widetilde F_{\overline v_{j,m_j}} \subset \cdots \subset \widetilde F_{\overline v_{r,m_r}}\ ) \end{aligned}$$

is a set of increasing left (resp. right) real algebraic extensions characterized by degrees:

$$[\widetilde F_{v_j} : k] = [\widetilde F_{\overline v_j} : k] = * + j\ N\ , \qquad 1 \le j \le r < \infty\ ,$$

which are integers modulo N, where

- $*$ denotes an integer inferior to N ($*=0$ for the zero class);
- m_j labels the multiplicity of the envisaged extension.

These algebraic extensions are then said to be pseudoramified in contrast with the pseudounramified extensions $\{\widetilde F^{(nr)}_{v_{j,m_j}}\}_{j,m_j}$ (resp. $\{\widetilde F^{(nr)}_{\overline v_{j,m_j}}\}_{j,m_j}$) which are characterized by their global residue degrees f_{v_j} (resp. $f_{\overline v_j}$):

$$f_{v_j}=[\widetilde F^{(nr)}_{v_{j,m_j}}:k]=[\widetilde F^{(nr)}_{\overline v_{j,m_j}}:k]=j \qquad \text{(case } N=1)$$

The smallest left (resp. right) (pseudoramified) splitting (sub)semifield $\widetilde F_{v_1}$ (resp. $\widetilde F_{\overline v_1}$) characterized by an extension degree

$$[\widetilde F_{v_1}:k]=[\widetilde F_{\overline v_1}:k]=N \qquad \text{(case } j=1)$$

was interpreted as being a left (resp. right) algebraic quantum, i.e. an irreducible closed left (resp. right) algebraic subset.

According to the fundamental theorem of Galois, there could exist a one-to-one correspondence between the intermediate subsemifields

$$\begin{aligned} &\widetilde F_{v_r}\supseteq\cdots\subseteq\widetilde F_{v_j}\supseteq\cdots\subseteq\widetilde F_{v_1}\supseteq k\\ \text{(resp.}\quad &\widetilde F_{\overline v_r}\supseteq\cdots\supseteq\widetilde F_{\overline v_j}\supseteq\cdots\supseteq\widetilde F_{\overline v_1}\supseteq k\ ) \end{aligned}$$

and the corresponding closed subsemigroups of the Galois semigroup $\mathrm{Gal}(\widetilde F^+_L/k)$ (resp. $\mathrm{Gal}(\widetilde F^+_R/k)$):

$$\{1\}\subseteq\mathrm{Gal}(\widetilde F_{v_1}/k)\subseteq\cdots\subseteq\mathrm{Gal}(\widetilde F_{v_j}/\widetilde F_{v_{j-1}})\subseteq\cdots\subseteq\mathrm{Gal}(\widetilde F_r/\widetilde F_{r-1})\ .$$

But, the extension degree $[\widetilde F_{v_r}:k]=[\widetilde F_{\overline v_r}:k]$ would then be the product of the above intermediate algebraic extension degrees which is generally not verified by the given extension degree $[\widetilde F_{v_r}:k]=[\widetilde F_{\overline v_r}:k]=r\cdot N$ which belongs to the zero class of integers modulo N required by the searched one-to-one correspondence between the representations of the associated Galois (sub)groups and the corresponding cuspidal representations [Pie2]. However, every algebraic extension $\widetilde F_{v_j}$ (resp. $\widetilde F_{\overline v_j}$), $1\le j\le r\le\infty$, characterized by the degree $[\widetilde F_{v_j}:k]=[\widetilde F_{\overline v_j}:k]=j\cdot N$, can be decomposed according to the fundamental theorem of Galois.

Let then $\{\widetilde F_{v_{j,m_j}}\}_{j,m_j}$ (resp. $\{\widetilde F_{\overline v_{j,m_j}}\}_{j,m_j}$) be the set of increasing algebraic extensions characterized by degrees $\{[\widetilde F_{v_{j,m_j}}:k]=j\cdot N\}_{j,m_j}$ (resp. $\{[\widetilde F_{\overline v_{j,m_j}}:k]=j\cdot N\}_{j,m_j}$) and whose global Weil (or Galois) subgroups [Pie2] are $\mathrm{Gal}(\widetilde F_{v_{j,m_j}}/k)$ (resp. $\mathrm{Gal}(\widetilde F_{\overline v_{j,m_j}}/k)$).
By **an isomorphism of compactification**

$$c_{v_{j,m_j}}:\quad \widetilde F_{v_{j,m_j}}\longrightarrow F_{v_{j,m_j}} \qquad \text{(resp.}\quad c_{\overline v_{j,m_j}}:\quad \widetilde F_{\overline v_{j,m_j}}\longrightarrow F_{\overline v_{j,m_j}}\ )\ ,$$

each algebraic extensions $\widetilde{F}_{v_{j,m_j}}$ (resp. $\widetilde{F}_{\overline{v}_{j,m_j}}$) **is sent into its compact image** $F_{v_{j,m_j}}$ (resp. $F_{\overline{v}_{j,m_j}}$) **which is a closed compact subset of** $\mathbb{R}_+$ **(resp.** $\mathbb{R}_-$ **)**.
Each compact image $F_{v_{j,m_j}}$ (resp. $F_{\overline{v}_{j,m_j}}$) of the algebraic extension $\widetilde{F}_{v_{j,m_j}}$ (resp. $\widetilde{F}_{\overline{v}_{j,m_j}}$) **is thus a transcendental extension [Hun] of which transcendence degree** $\mathrm{tr}\cdot d\cdot F_{v_{j,m_j}}$ (resp. $\mathrm{tr}\cdot d\cdot F_{\overline{v}_{j,m_j}}$) is given by:

$$\mathrm{tr}\cdot d\cdot F_{v_{j,m_j}}=[\widetilde{F}_{v_{j,m_j}}:k]=j\cdot N \qquad (\text{resp.}\quad \mathrm{tr}\cdot d\cdot F_{\overline{v}_{j,m_j}}=[\widetilde{F}_{\overline{v}_{j,m_j}}:k]=j\cdot N\ ).$$

Every transcendental extension $F_{v_{j,m_j}}$ (resp. $F_{\overline{v}_{j,m_j}}$) is also an archimedean completion which can be viewed as resulting from the semigroup of automorphisms $\mathrm{Aut}_k(F_{v_{j,m_j}})$ (resp. $\mathrm{Aut}_k(F_{\overline{v}_{j,m_j}})$) which is a semigroup of reflections (or permutations) of a transcendental quantum $F_{v^1_{j,m_j}}\subset F_{v_{j,m_j}}$ (resp. $F_{\overline{v}^1_{j,m_j}}\subset F_{\overline{v}_{j,m_j}}$) characterized by the corresponding extension degree $[\widetilde{F}_{v^1_{j,m_j}}:k]=[\widetilde{F}_{\overline{v}^1_{j,m_j}}:k]=N$ [Pie5].

This implies that **the compact transcendental extension** $\boldsymbol{F_{v_{h,m_h}}}$ **(resp.** $\boldsymbol{F_{\overline{v}_{h,m_h}}}$ **), composed of** $\boldsymbol{h}$ **left (resp. right) transcendental quanta, of transcendental degree** $\boldsymbol{h\cdot N}$ **over** $\boldsymbol{k}$, is included into the transcendental extension $F_{v_{j,m_j}}$ (resp. $F_{\overline{v}_{j,m_j}}$), composed of j left (resp. right) compact transcendental quanta, in the sense of the fundamental theorem of Galois theory. Indeed, the transcendental quantum $F_{v^1_j}\subset F_{v_{j,m_j}}$ (resp. $F_{\overline{v}^1_j}\subset F_{\overline{v}_{j,m_j}}$) of the transcendental extension $F_{v_{j,m_j}}$ (resp. $F_{\overline{v}_{j,m_j}}$) is homeomorphic to the transcendental quantum $F_{v^1_h}\subset F_{v_h}$ (resp. $F_{\overline{v}^1_h}\subset F_{\overline{v}_{h,m_h}}$) of the transcendental extension $F_{v_{h,m_h}}$ (resp. $F_{\overline{v}_{h,m_h}}$) since, by Galois automorphism and isomorphisms of compactification, they are the compact images either of the nonunits of the algebraic quantum $\widetilde{F}_{v^1_j}$ (resp. $\widetilde{F}_{\overline{v}^1_j}$) of the algebraic extension $F_{v_{j,m_j}}$ (resp. $F_{\overline{v}_{j,m_j}}$) or of the nonunits of the algebraic quantum $F_{v^1_h}$ (resp. $F_{\overline{v}^1_h}$) of the algebraic extension $F_{v_{h,m_h}}$ (resp. $F_{\overline{v}_{h,m_h}}$).
Thus, in the case of transcendental extensions or compact archimedean completions, we have that:

1.3 Proposition

If F_{v_r} *(resp.* $F_{\overline{v}_r}$ *) is a finite dimensional compact transcendental extension of* k *,* **there is a one-to-one correspondence between the set of intermediate semifields**

$$\begin{gathered}\boldsymbol{F_{v_r}\supset\cdots\supset F_{v_j}\supset\cdots\supset F_{v_1}\supset k}\\ \textbf{(resp.}\quad\boldsymbol{F_{\overline{v}_r}\supset\cdots\supset F_{\overline{v}_j}\supset\cdots\supset F_{\overline{v}_1}\supset k}\ \textbf{)}\end{gathered}$$

and the set of corresponding subsemigroups of the semigroup of automorphisms $\mathrm{Aut}_k(F_{v_r})$ (resp. $\mathrm{Aut}_k(F_{\overline{v}_r})$):

$$
\begin{aligned}
&\{1\} \subseteq \mathrm{Aut}_k(F_{v_1}) \subset \cdots \subset \mathrm{Aut}_{F_{v_h}}(F_{v_j}) \subset \cdots \subset \mathrm{Aut}_{F_{v m}}(F_{v_r}) \\
\textit{(resp.}\quad &\{1\} \subseteq \mathrm{Aut}_k(F_{\overline{v}_1}) \subset \cdots \subset \mathrm{Aut}_{F_{\overline{v}_h}}(F_{\overline{v}_j}) \subset \cdots \subset \mathrm{Aut}_{F_{\overline{v} m}}(F_{\overline{v}_r})\ )
\end{aligned}
$$

in such a way that:

a) *the relative degree (or transcendental dimension) of two intermediate semifields must be an integer;*

b) F_{v_r} *(resp.* $F_{\overline{v}_r}$ *) is transcendental over every intermediate semifield* F_{v_j} *(resp.* $F_{\overline{v}_j}$ *) if their transcendence degrees or global residue degrees verify [Hun]:*

$$
\begin{aligned}
&\mathrm{tr}\cdot d\cdot F_{v_r}/k = (\mathrm{tr}\cdot d\cdot F_{v_r}/F_{v_j}) + (\mathrm{tr}\cdot d\cdot F_{v_j}/k) \\
\textit{(resp.}\quad &\mathrm{tr}\cdot d\cdot F_{\overline{v}_r}/k = (\mathrm{tr}\cdot d\cdot F_{\overline{v}_r}/F_{\overline{v}_j}) + (\mathrm{tr}\cdot d\cdot F_{\overline{v}_j}/k)\ ).
\end{aligned}
$$

Proof: In the Galois (i.e. algebraic) case, we would have that $\mathrm{Gal}(\widetilde{F}_{v_r}/\widetilde{F}_{v_j})$, corresponding to $\mathrm{Aut}_{F_{v_j}} F_{v_r}$ in the transcendental case, is normal in $\mathrm{Gal}(\widetilde{F}_{v_r}/k)$ and that

$$\mathrm{Gal}(\widetilde{F}_{v_j}/k) = \mathrm{Gal}(\widetilde{F}_{v_r}/k)\big/(\widetilde{F}_{v_r}/F_{v_j})$$

implying the relative degree $\left[\widetilde{F}_{v_r} : \widetilde{F}_{v_j}\right] = \frac{r}{j}$, being the relative index of the corresponding semigroup, is not an integer unless j divides r. ■

1.4 Bisemilattices of algebraic and transcendental quanta

According to section 1.1. and [Pie2], only symmetric semiobjects have to be considered in any generality in such a way that **a bisemiobject $O_R \times O_L$ is composed of the product of the right semiobject O_R, localized in the lower half space or on $I\!R_-$, and of the left symmetric semiobject O_L, localized in the upper half space or on $I\!R_+$.** So, instead of envisaging the set of left (resp. right) increasing transcendental (or algebraic) extensions:

$$F_v = \{F_{v_1}, \dots, F_{v_{j,m_j}}, \dots, F_{v_{r,m_r}}\} \qquad \text{(resp.} \quad F_{\overline{v}} = \{F_{\overline{v}_1}, \dots, F_{\overline{v}_{j,m_j}}, \dots, F_{\overline{v}_{r,m_r}}\}\ );$$

we shall take into account their (diagonal) product

$$F_{\overline{v}} \times F_v : \{F_{\overline{v}_1} \times F_{v_1}, \dots, F_{\overline{v}_{j,m_j}} \times F_{v_{j,m_j}}, \dots, F_{\overline{v}_{r,m_r}} \times F_{v_{r,m_r}}\}$$

in one-to-one correspondence with their semigroups of automorphisms:

$$\begin{aligned}\mathrm{Aut}_k(F_{\overline{v}}) \times \mathrm{Aut}_k(F_v) = \\ \{\mathrm{Aut}_k F_{\overline{v}_1} \times \mathrm{Aut}_k F_{v_1}, \dots, \mathrm{Aut}_k F_{\overline{v}_{j,m_j}} \times \mathrm{Aut}_k F_{v_{j,m_j}}, \dots, \\ \mathrm{Aut}_k F_{\overline{v}_{r,m_r}} \times \mathrm{Aut}_k F_{v_{r,m_r}}\}\ .\end{aligned}$$

Let

$$\begin{aligned}\mathrm{Gal}(\widetilde F_{\overline{v}}/k) \times \mathrm{Gal}(\widetilde F_v/k) = \\ \{\mathrm{Gal}(\widetilde F_{\overline{v}_1}/k) \times \mathrm{Gal}(\widetilde F_{v_1}/k), \dots, \{\mathrm{Gal}(\widetilde F_{\overline{v}_{j,m_j}}/k) \times \mathrm{Gal}(\widetilde F_{v_{j,m_j}}/k), \dots, \\ \{\mathrm{Gal}(\widetilde F_{\overline{v}_{r,m_r}}/k) \times \mathrm{Gal}(\widetilde F_{v_{r,m_r}}/k)\}\end{aligned}$$

denote the **products, right by left, of the Galois semigroups** of the sets of increasing algebraic extensions characterized by increasing degrees belonging to the zero class of integers modulo N: they are thus, **Weil global semigroups** [**Pie2**].
We thus have the isomosphism:

$$\mathrm{Gal}(\widetilde F_{\overline{v}}/k) \times \mathrm{Gal}(\widetilde F_v/k) \quad \xrightarrow{\sim} \quad \mathrm{Aut}_k(F_{\overline{v}}) \times \mathrm{Aut}_k(F_v)$$

associated with the isomorphism:

$$\widetilde F_{\overline{v}} \times \widetilde F_v \quad \xrightarrow{\sim} \quad F_{\overline{v}} \times F_v$$

between the product, right by left, of sets of symmetric algebraic and transcendental extensions (or archimedean completions), which consists in the **the product, right by left, of two symmetric towers of increasing algebraic or compact transcendental extensions composed of an increasing number of algebraic or transcendental compact quanta**: this defines a bi(semi)lattice of biquanta, i.e. the product, right by left, of two symmetric (semi)lattices of algebraic or transcendental quanta.

1.5 Abstract bisemivarieties

Let $B_{\widetilde F_v}$ (resp. $B_{\widetilde F_{\overline{v}}}$) be a left (resp. right) division semialgebra of real dimension $2n$ over the set $\widetilde F_v$ (resp. $\widetilde F_{\overline{v}}$) of increasing real pseudoramified extensions:
$B_{\widetilde F_v}$ (resp. $B_{\widetilde F_{\overline{v}}}$) is then isomorphic to the semialgebra of Borel upper (resp. lower) triangular matrices:

$$B_{\widetilde F_v} \simeq T_{2n}(\widetilde F_v) \qquad\qquad (\text{resp.}\quad B_{\widetilde F_{\overline{v}}} \simeq T^t_{2n}(\widetilde F_{\overline{v}}\ ))$$

allowing introducing the algebraic bilinear semigroup of matrices by:

$$B_{\widetilde F_{\overline v}}\otimes B_{\widetilde F_v}\simeq T^t_{2n}(\widetilde F_{\overline v})\times T_{2n}(\widetilde F_v)\simeq \mathrm{GL}_{2n}(\widetilde F_{\overline v}\times \widetilde F_v)$$

in such a way that its representation (bisemi)space is given by the tensor product $\widetilde M^{(2n)}_{v_R}\otimes \widetilde M^{(2n)}_{v_L}$ of a right $T^t_{2n}(\widetilde F_{\overline v})$-semimodule $\widetilde M^{(2n)}_{v_R}$, localized in the lower half space, by a left $T_{2n}(\widetilde F_v)$-semimodule $\widetilde M^{(2n)}_{v_L}$, localized in the upper half space.

The $\mathbf{GL_{2n}}(\widetilde{\boldsymbol F}_{\overline{\boldsymbol v}}\times\widetilde{\boldsymbol F}_{\boldsymbol v})$-bisemimodule $\widetilde{\boldsymbol M}^{(2n)}_{\boldsymbol{v_R}}\otimes\widetilde{\boldsymbol M}^{(2n)}_{\boldsymbol{v_L}}$ is an algebraic bilinear (affine) semigroup noted $\boldsymbol G^{(2n)}(\widetilde{\boldsymbol F}_{\overline{\boldsymbol v}}\times\widetilde{\boldsymbol F}_{\boldsymbol v})$ whose bilinear semigroup of Galois automorphisms is $\mathrm{GL}(\widehat M^{(2n)}_{\overline v_R}\otimes \widehat M^{(2n)}_{v_L})$.
$\mathrm{GL}(\widehat M^{(2n)}_{\overline v_R}\otimes \widehat M^{(2n)}_{v_L})$ constitutes the $2n$-dimensional equivalent of the product $\mathrm{Gal}(\widetilde F_{\overline v}/k)\times \mathrm{Gal}(\widetilde F_v/k)$ of Galois semigroups in such a way that $G^{(2n)}(\widetilde F_{\overline v}\times\widetilde F_v)$ becomes the $2n$-dimensional (irreducible) representation space $\mathrm{Irr\,Rep}^{(2n)}_{\mathrm{Gal}_{F^+_{R\times L}}}(\mathrm{Gal}(\widetilde F_{\overline v}/k)\times \mathrm{Gal}(\widetilde F_v/k))$ of $\mathrm{Gal}(\widetilde F_{\overline v}/k)\times \mathrm{Gal}(\widetilde F_v/k)$.

$$G^{(2n)}(\widetilde F_{\overline v}\times\widetilde F_v)=\mathrm{Irr\,Rep}^{(2n)}_{\mathrm{Gal}_{F^+_{R\times L}}}(\mathrm{Gal}(\widetilde F_{\overline v}/k)\times \mathrm{Gal}(\widetilde F_v/k))$$

implies the mononorphims:

$$\sigma_{\widetilde v_R}\times\sigma_{\widetilde v_L}:\qquad (\mathrm{Gal}(\widetilde F_{\overline v}/k)\times \mathrm{Gal}(\widetilde F_v/k))\quad\longrightarrow\quad \mathrm{GL}(\widehat M_{\overline v_R}\otimes \widehat M_{v_L})\approx G^{(2n)}(\widetilde F_{\overline v}\otimes\widetilde F_v)\ .$$

The isomorphism

$$\mathrm{Gal}(\widetilde F_{\overline v}/k)\times \mathrm{Gal}(\widetilde F_v/k)\quad\overset{\sim}{\longrightarrow}\quad \mathrm{Aut}_k(F_{\overline v})\times \mathrm{Aut}_k(F_v)$$

between the products, right by left, of Weil global semigroups and corresponding automorphism semigroups of transcendental extensions leads to the commutative diagram

$$\begin{tikzcd}
\mathrm{Gal}(\widetilde F_{\overline v}/k)\times \mathrm{Gal}(\widetilde F_v/k) \arrow[r, "\sigma_{\widetilde v_R}\times\sigma_{\widetilde v_L}"] \arrow[d, "\wr"] & G^{(2n)}(\widetilde F_{\overline v}\times\widetilde F_v) \arrow[d, "\wr"] \\
\mathrm{Aut}_k(F_{\overline v})\times \mathrm{Aut}_k(F_v) \arrow[r, "\sigma_{v_R}\times\sigma_{v_L}"] & G^{(2n)}(\widetilde F_{\overline v}\times\widetilde F_v)
\end{tikzcd}$$

where **the monomorphism $\boldsymbol\sigma_{\boldsymbol{v_R}}\times\boldsymbol\sigma_{\boldsymbol{v_L}}$ generates the abstract bisemivariety $\boldsymbol G^{(2n)}(\boldsymbol F_{\overline{\boldsymbol v}}\times\boldsymbol F_{\boldsymbol v})$ on the product of sets of symmetric compact transcendental extensions (or archimedean completions)**.
The abstract bisemivariety $G^{(2n)}(F_{\overline v}\times F_v)$ is covered by the algebraic bilinear (affine) semigroup $G^{(2n)}(\widetilde F_{\overline v}\times\widetilde F_v)$ and is thus a complete (locally) compact (algebraic) bilinear semigroup.

At every infinite biplace $\overline{v}_j \times v_j$ of $F_{\overline{v}} \times F_v$ corresponds a conjugacy class $g^{(2n)}_{v_{R\times L}}[j]$ of the abstract bisemivariety $G^{(2n)}(\widetilde F_{\overline{v}} \times \widetilde F_v)$.
The number of representatives of $g^{(2n)}_{v_{R\times L}}[j]$ corresponds to the number of equivalent extensions of $\widetilde F_{\overline{v}_j} \times \widetilde F_{v_j}$;

Let $\mathcal{O}_{\widetilde F_v}$ (resp. $\mathcal{O}_{\widetilde F_{\overline{v}}}$) be the maximal order of $\widetilde F_v$ (resp. $\widetilde F_{\overline{v}}$).
Then, $\Lambda_v = \mathcal{O}_{B_{\widetilde F_v}}$ (resp. $\Lambda_{\overline{v}} = \mathcal{O}_{B_{\widetilde F_{\overline{v}}}}$) in the division semialgebra $B_{\widetilde F_v}$ (resp. $B_{\widetilde F_{\overline{v}}}$) is a pseudo-ramified $\mathbb{Z}/N\ \mathbb{Z}$-lattice, in the left (resp. right) $B_{\widetilde F_v}$-semimodule $\widetilde M^{(2n)}_{v_L}$ (resp. $B_{\widetilde F_{\overline{v}}}$-semimodule $\widetilde M^{(2n)}_{v_R}$).
So, we have that

$$\Lambda_v \simeq T_{2n}(\mathcal{O}_{\widetilde F_v}) \simeq T_{2n}(\mathcal{O}_{F_v}) \qquad (\text{resp.}\quad \Lambda_{\overline{v}} \simeq T^t_{2n}(\mathcal{O}_{\widetilde F_{\overline{v}}}) \simeq T^t_{2n}(\mathcal{O}_{F_{\overline{v}}})\ )$$

leading to

$$\begin{aligned}\Lambda_{\overline{v}} \times \Lambda_v &= T^t_{2n}(\mathcal{O}_{F_{\overline{v}}}) \times T_{2n}(\mathcal{O}_{F_v})\\ &= \mathrm{GL}_{2n}(\mathcal{O}_{F_{\overline{v}}} \times \mathcal{O}_{F_v})\\ &= \mathrm{GL}_{2n}((\mathbb{Z}/N\ \mathbb{Z})^2)\ .\end{aligned}$$

Then, the representation space $\mathrm{Repsp}(\mathrm{GL}_{2n}(\mathbb{Z}/N\ \mathbb{Z})^2)$ of $\mathrm{GL}_{2n}((\mathbb{Z}/N\ \mathbb{Z})^2)$ decomposes according to:

$$\mathrm{Repsp}(\mathrm{GL}_{2n}(\mathbb{Z}/N\ \mathbb{Z})^2) = \bigoplus_j \bigoplus_{m_j} \left(\Lambda_{\overline{v}_{j,m_j}} \otimes \Lambda_{\overline{v}_{j,m_j}}\right)$$

where $\Lambda_{v_{j,m_j}}$ (resp. $\Lambda_{\overline{v}_{j,m_j}}$) is the (j,m_j)-th subsemilattice referring to the conjugacy class representative $g^{(2n)}_{v_L}[j,m_j] \in G^{(2n)}(F_v)$ (resp. $g^{(2n)}_{v_R}[j,m_j] \in G^{(2n)}(F_{\overline{v}})$).
The pseudo-ramified Hecke bisemialgebra $H_{R\times L}(2n)$ of all Hecke bioperators $T_R(2n;r) \otimes T_L(2n;r)$, **having a representation in the arithmetic subgroup of matrices $\mathrm{GL}_{2n}((\mathbb{Z}/N\ \mathbb{Z})^2)$, generates the endomorphisms of the $B_{F_{\overline{v}}} \times B_{F_v}$-bisemimodule $(M^{(2n)}_{v_R} \otimes M^{(2n)}_{v_L})$ decomposing it according to the bisubsemilattices $(\Lambda_{\overline{v}_{j,m_j}} \otimes \Lambda_{\overline{v}_{j,m_j}})$** [Pie3]:

$$M^{(2n)}_{v_R} \otimes M^{(2n)}_{v_L} = \bigoplus_{j,m_j} (M^{(2n)}_{\overline{v}_{j,m_j}} \otimes M^{(2n)}_{v_{j,m_j}})\ .$$

Let

$$\begin{aligned} F^{(nr)}_v &= \{F^{(nr)}_{v_1},\dots,F^{(nr)}_{v_{j,m_j}},\dots,F^{(nr)}_{v_{r,m_r}}\}\\ (\text{resp.}\quad F^{(nr)}_{\overline{v}} &= \{F^{(nr)}_{\overline{v}_1},\dots,F^{(nr)}_{\overline{v}_{j,m_j}},\dots,F^{(nr)}_{\overline{v}_{r,m_r}}\}\ )\end{aligned}$$

be the set of left (resp. right) increasing pseudounramified transcendental extensions homeomorphic to the corresponding pseudounramified algebraic extensions introduced in section 1.2.

Let $G^{(2n)}(F_{\overline{v}}^{(nr)}\times F_v^{(nr)})$ be the complete bilinear semigroup with entries in $(F_{\overline{v}}^{(nr)}\times F_v^{(nr)})$. Then, the kernel $\operatorname{Ker}(G^{(2n)}_{F\to F^{(nr)}})$ of the map:

$$G^{(2n)}_{F\to F^{(nr)}}: \qquad G^{(2n)}(F_{\overline{v}}\times F_v) \quad\longrightarrow\quad G^{(2n)}(F_{\overline{v}}^{(nr)}\times F_v^{(nr)})$$

is the smallest bilinear normal pseudoramified subgroup of $G^{(2n)}(F_{\overline{v}}\times F_v)$:

$$\operatorname{Ker}(G^{(2n)}_{F\to F^{(nr)}}) = P^{(2n)}(F_{\overline{v}^1}\times F_{v^1})\ ,$$

i.e. **the parabolic bilinear subsemigroup** over the product $F_{\overline{v}^1}\times F_{v^1}$ of sets

$$F_{\overline{v}^1}=\{F_{\overline{v}^1_1},\dots,F_{\overline{v}^1_{j,m_j}},\dots,F_{\overline{v}^1_{r,m_r}}\}\qquad\text{and}\qquad F_{v^1}=\{F_{v^1_1},\dots,F_{v^1_{j,m_j}},\dots,F_{v^1_{r,m_r}}\}$$

of “unitary” transcendental preudoramified extensions.

1.6 Proposition

The bilinear abstract parabolic semigroup $P^{(2n)}(F_{\overline{v}^1}\times F_{v^1})$ is the unitary (irreducible) representation space of the complete bilinear semigroup $\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v)$ of matrices.

Proof:

1) The abstract bisemivariety $G^{(2n)}(F_{\overline{v}}\times F_v)$, covered by the algebraic (affine) bisemivariety $G^{(2n)}(\widetilde{F}_{\overline{v}}\times \widetilde{F}_v)$, acts by conjugation on the bilinear parabolic subsemigroup $P^{(2n)}(F_{\overline{v}^1}\times F_{v^1})$ in such a way that the number of conjugates of $P^{(2n)}(F_{\overline{v}^1_j}\times F_{v^1_j})$ in the conjugacy class representative $G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}})\equiv g^{(2n)}_{v_{R\times L}}[j,m_j]\in G^{(2n)}(F_{\overline{v}}\times F_v)$ is the index
$$\left|G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}}) : P^{(2n)}(F_{\overline{v}^1_j}\times F_{v^1_j}\right| = j$$
of the normalizer $P^{(2n)}(F_{\overline{v}^1_j}\times F_{v^1_j})$ in $G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}})$.

2) Let $\operatorname{Out}(G^{(2n)}(F_{\overline{v}}\times F_v))=\operatorname{Aut}(G^{(2n)}(F_{\overline{v}}\times F_v))\big/\operatorname{Int}(G^{(2n)}(F_{\overline{v}}\times F_v))$ be the bisemigroup [Pie4] of transcendental automorphisms of the complete bilinear semigroup $G^{(2n)}(F_{\overline{v}}\times F_v)$ where $\operatorname{Int}(G^{(2n)}(F_{\overline{v}}\times F_v))$ is the bisemigroup of transcendental inner automorphisms.

   As, we have that:
$$\operatorname{Int}(G^{(2n)}(F_{\overline{v}}\times F_v))=\operatorname{Aut}(P^{(2n)}(F_{\overline{v}^1}\times F_{v^1}))$$
   and considering 1), it appears that $P^{(2n)}(F_{\overline{v}^1}\times F_{v^1})$ is the unitary representation bisemispace with respect to the abstract bisemivariety $G^{(2n)}(F_{\overline{v}}\times F_v)$. ■

1.7 Corollary

The rank of the bilinear parabolic semigroup $P^{(2n)}(F_{\overline{v}^1_j}\times F_{v^1_j})$ *restricted to the* j*-th conjugacy class is*

$$r_{P^{(2n)}(F_{\overline{v}^1_j}\times F_{v^1_j})}\simeq (m_j\cdot N)^n\times_{(D)}(m_j\cdot N)^n$$

and the rank of the conjugacy class representative $G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}})$ *is*

$$r_{G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}})}\simeq (j\cdot m_j\cdot N)^n\times_{(D)}(j\cdot m_j\cdot N)^n$$

where $\times_D$ *is the notation for a diagonal product [Pie4].*

Proof: As the bilinear parabolic semigroup $P^{(2n)}(F_{\overline{v}^1_j}\times F_{v^1_j})$ restricted to the j-th conjugacy class is the $2n$-dimensional representation space of the product, right by left, of inertia subgroups

$$I_{F_{\overline{v}_j}}\times I_{F_{v_j}}=[\mathrm{Gal}(F_{\overline{v}_j}/k)\times\mathrm{Gal}(F_{v_j}/k)]\Big/[\mathrm{Gal}(F^{(nr)}_{\overline{v}_j}/k)\times\mathrm{Gal}(F^{(nr)}_{v_j}/k)]$$

according to [Pie2] and as the order of $I_{F_{\overline{v}_j}}$ or $I_{F_{v_j}}$ is N, we have that

$$r_{P^{(2n)}(F_{\overline{v}^1_j}\times F_{v^1_j})}=(m_j\cdot N)^n\times_{(D)}(m_j\cdot N)^n$$

if it is taken into account that "m_j" real equivalent conjugacy class representatives $g^{(n)}_{v_{R,L}}[j]$ of dimension "n" cover one complex conjugacy class representative $g^{(2n)}_{\omega_{R,L}}(j)$ [Pie5] of dimension $2n$:

$$g^{(2n)}_{\omega_{R,L}}[j]=\{g^{(n)}_{v_{R,L}}[j,m_j]\}_{m_j}\ .$$

It is then immediate to see that the rank of $G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}})=g^{(2n)}_{R\times L}[j,m_j]$ is

$$r_{G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}})}\simeq (j\cdot m_j\cdot N)^n\times_{(D)}(j\cdot m_j\cdot N)^n \qquad\blacksquare$$

1.8 Corollary

1) *The number of transcendental quanta of the bilinear parabolic semigroup* $P^{(2n)}(F_{\overline{v}^1_j}\times F_{v^1_j})$, $\forall\ j\in I\!N$, *is*

$$n(P^{(2n)}(F_{\overline{v}^1_j}\times F_{v^1_j}))=\mathbb{1}^n\times_{(D)}\mathbb{1}^n\longrightarrow(m_j)^n\times_{(D)}(m_j)^n\ .$$

2) *The number of transcendental quanta of the conjugacy class representative* $G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}})$ *is*

$$n(G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}}))=(j\cdot m_j)^n\times_{(D)}(j\cdot m_j)^n\ .$$

Proof: This results from corollary 1.7 by taking into account that the degree of a transcendental quantum is N. According to section 1.5, we thus have **a bisemilattice of** $\boldsymbol{\sum\limits_j (j \times m_j)^n}$ **transcendental biquanta**. ■

1.9 Covering of complex abstract bisemivarieties by real abstract bisemivarieties

Let $M^{(2n)}_{\omega_R}\otimes M^{(2n)}_{\omega_L}$ denote a complex $\mathrm{GL}_n(F_{\overline{\omega}}\times F_\omega)$-bisemimodule being the representation space of the complete bilinear semigroup of matrices $\mathrm{GL}_n(F_{\overline{\omega}}\times F_\omega)$ of the product $(F_{\overline{\omega}}\times F_\omega)$, right by left, of complex transcendental extensions covered by their real equivalents $(F_{\overline{v}}\times F_v)$ as developed in [Pie2].

If each conjugacy class representative $G^{(2n)}(F_{\overline{\omega}_j}\times F_{\omega_j})$ of $G^{(2n)}(F_{\overline{\omega}}\times F_\omega)\equiv M^{(2n)}_{\omega_R}\otimes M^{(2n)}_{\omega_L}$ is unique in this j-th class, then $G^{(2n)}(F_{\overline{\omega}_j}\times F_{\omega_j})$ is covered by m_j real equivalent conjugacy class representatives $G^{(n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}})$ of $G^{(n)}(F_{\overline{v}}\times F_v)$ [Pie2].

So, **the complex bipoints of $\boldsymbol{G^{(2n)}(F_{\overline{\omega}}\times F_\omega)}$ are in one-to-one correspondence with the real bipoints of $\boldsymbol{G^{(2n)}(F_{\overline{v}}\times F_v)}$ and we have the inclusion:**

$$\boldsymbol{\frac{G^{(2n)}(F_{\overline{v}}\times F_v)}{G^{(n)}(F_{\overline{v}}\times F_v)}\simeq M^{(2n)}_{v_R}\otimes M^{(2n)}_{v_L}\hookrightarrow M^{(2n)}_{\omega_R}\otimes M^{(2n)}_{\omega_L}}$$

of the real abstract bisemivariety $\boldsymbol{M^{(2n)}_{v_R}\otimes M^{(2n)}_{v_L}}$ into the complex abstract bisemivariety $\boldsymbol{M^{(2n)}_{\omega_R}\otimes M^{(2n)}_{\omega_L}}$.

1.10 Corner stone of the global program of Langlands

The real abstract bisemivariety $M^{(2n)}_{v_R}\otimes M^{(2n)}_{v_L}\equiv G^{(2n)}(F_{\overline{v}}\times F_v)$, being the representation space of the bilinear algebraic semigroup $\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v)$ of matrices, constitutes the $2n$-dimension representation space of the product, right by left, $\mathrm{Gal}(\widetilde{F}_{\overline{v}}/k)\times\mathrm{Gal}(\widetilde{F}_v/k)$ of Galois semigroups according to section 1.5. So, we get the isomorphism $\mathrm{GL}(\widetilde{M}^{(2n)}_{v_R}\otimes\widetilde{M}^{(2n)}_{v_L})\simeq\mathrm{GL}_{2n}(\widetilde{F}_{\overline{v}}\times\widetilde{F}_v)$.

So, $G^{(2n)}(F_{\overline{v}}\times F_v)$ constitutes the corner stone of the real global correspondence of Langlands recalled in [Pie5] since, by an isomormosphism of toroidal compactification, $G^{(2n)}(F_{\overline{v}}\times F_v)$ is transformed into **the cuspidal representation $\boldsymbol{\Pi(\mathrm{GL}_{2n}(\widetilde{F}_{\overline{v}}\times\widetilde{F}_v))}$ of the algebraic bilinear semigroup of matrices $\boldsymbol{\mathrm{GL}_{2n}(\widetilde{F}_{\overline{v}}\times\widetilde{F}_v)}$**.

1.11 Reducible representations of $\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v)$

We have envisaged until now that the representation of the general bilinear semigroup $\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v)$ was irreducible. But, in consideration of the future developments of this

paper, it is useful to take into account its reducibility.
Let $2n = 2_1 + 2_2 + \cdots + 2_k + \cdots + 2_\ell$ be a partition of $2n$ labeling the reducible representation of $T_{2n}(F_v)$ (resp. $T^t_{2n}(F_{\overline{v}})$).
Then, we have that:

1) **the representation $\mathrm{Rep}(\mathrm{GL}_{2n}(F_{\overline{v}} \times F_v))$ of the general bilinear semigroup $\mathrm{GL}_{2n}(F_{\overline{v}} \times F_v) = T^t_{2n}(F_{\overline{v}}) \times T^t_{2n}(F_v)$ is orthogonally completely reducible** if it decomposes diagonally according to the direct sum of 2-dimensional irreducible representations $\mathrm{Rep}(\mathrm{GL}_{2_k}(F_{\overline{v}} \times F_v))$:

$$\mathrm{Rep}(\mathrm{GL}_{2n}(F_{\overline{v}} \times F_v)) = \boxplus_{2_k=2}^{2n} \mathrm{Rep}(\mathrm{GL}_{2_k}(F_{\overline{v}} \times F_v))$$

and it is nonorthogonally completely reducible if it decomposes diagonally according to the direct sum of irreducible 2-dimensional bilinear representations $\mathrm{Rep}(\mathrm{GL}_{2_k}(F_{\overline{v}} \times F_v))$ and offdiagonally according to the direct sum of irreducible mixed bilinear representations $\mathrm{Rep}(T^t_{2_k}(F_{\overline{v}}) \times T_{2_\ell}(F_v))$.

Indeed, taking into account the existence of cross products in the definition of bilinear semigroups [Pie4], we have that the representation of $\mathrm{GL}_{2n}(F_{\overline{v}} \times F_v)$ can be reduced to:

$$\begin{aligned}\mathrm{Rep}(\mathrm{GL}_{2n}(F_{\overline{v}} \times F_v)) &= \left(\boxplus_{2_k=2}^{2n} \mathrm{Rep}(T^t_{2_k}(F_{\overline{v}})) \otimes \boxplus_{2_\ell=2}^{2n} \mathrm{Rep}(T_{2_\ell}(F_v)) \right) \\ &= \boxplus_{2_k=2}^{2n} \mathrm{Rep}(\mathrm{GL}_{2_k}(F_{\overline{v}} \times F_v)) \boxplus_{2_k \neq 2\ell}^{2n} \mathrm{Rep}(T^t_{2_k}(F_{\overline{v}})) \times T_{2_\ell}(F_v))\end{aligned}$$

If the mixed bilinear representations $\boxplus_{2_k \neq 2\ell}^{2n} \mathrm{Rep}(T^t_{2_k}(F_{\overline{v}})) \times T_{2_\ell}(F_v)$ are equal to 0, then the above completely reducible nonorthogonal representation of $\mathrm{GL}_{2n}(F_{\overline{v}} \times F_v)$ reduces to the orthogonal case.

2 Universal dynamical structures of the global program of Langlands

2.1 Dynamical functional representation spaces of abstract bisemivarieties

In order to generate dynamical $\mathrm{GL}(2n)$-bisemistructures [Pie3] referring to the geometric program of Langlands, we have to take into account the differentiable functional representation space $\mathrm{FREPSP}(\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v))$ of the complete bilinear semigroup $\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v)$, i.e. **a bisemisheaf $\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L}$ of differentiable bifunctions on the abstract bisemivariety $G^{(2n)}(F_{\overline{v}}\times F_v)$** which is given by the product, right by left, of a right semisheaf $\widehat{M}^{(2n)}_{v_R}$ of differentiable (co)functions on the abstract right semivariety $G^{(2n)}(F_{\overline{v}})\equiv T^{(2n)}(F_{\overline{v}})$ by a left semisheaf $\widehat{M}^{(2n)}_{v_L}$ of symmetric differentiable functions on the abstract left semivariety $G^{(2n)}(F_v)\equiv T^{(2n)}(F_v)$.

This functional representation space $\mathrm{FREPSP}(\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v))$ of bilinear geometric dimension $2n$ splits into:

$$\mathrm{FREPSP}(\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v))=\mathrm{FREPSP}(\mathrm{GL}_{2k}(F_{\overline{v}}\times F_v))\oplus\mathrm{FREPSP}(\mathrm{GL}_{2n-2k}(F_{\overline{v}}\times F_v))$$

in such a way that $\mathrm{FREPSP}(\mathrm{GL}_{2k}(F_{\overline{v}}\times F_v))$, $k\le n$, be the functional representation space of bilinear geometric dimension $2k$ of the bilinear semigroup $\mathrm{GL}_{2k}(F_{\overline{v}}\times F_v)$ on which acts **the elliptic bioperator $D^{2k}_R\otimes D^{2k}_L$**, i.e. the product of a right linear differential elliptic operator D^{2k}_R acting on $2k$ variables by its left equivalent D^{2k}_L, by its biaction:

$$D^{2k}_R\otimes D^{2k}_L:\quad \mathrm{FREPSP}(\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v))\longrightarrow\mathrm{FREPSP}(\mathrm{GL}_{2n[2k]}(F_{\overline{v}}\times I\!\!R)\times(F_v\times I\!\!R))$$

where $\mathrm{FREPSP}(\mathrm{GL}_{2n[2k]}(F_{\overline{v}}\times I\!\!R)\times(F_v\times I\!\!R))$ is the functional representation space of $\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v)$ fibered or shifted into "$2k$" bilinear geometric dimensions.

$F^{S_{I\!\!R}}_v=(F_v\times I\!\!R)$ (resp. $F^{S_{I\!\!R}}_{\overline{v}}=(F_{\overline{v}}\times I\!\!R)$) denotes the set of increasing left (resp. right) transcendental extensions (or archimedean completions) fibered or shifted by real numbers

$$F_v\times I\!\!R=\{F^{S_{I\!\!R}}_{v_1},\dots,F^{S_{I\!\!R}}_{v_{j,m_j}},\dots,F^{S_{I\!\!R}}_{v_{r,m_r}}\}$$
$$(\text{resp.}\quad F_{\overline{v}}\times I\!\!R=\{F^{S_{I\!\!R}}_{\overline{v}_1},\dots,F^{S_{I\!\!R}}_{\overline{v}_{j,m_j}},\dots,F^{S_{I\!\!R}}_{\overline{v}_{r,m_r}}\}\ )$$

from their unshifted equivalents, i.e. fibre bundles with vertical fibres $I\!\!R$ and basis F_v (resp. $F_{\overline{v}}$).

Remark that the shifted transcendental extension $F^{S_{I\!\!R}}_{v_{j,m_j}}$ (resp. $F^{S_{I\!\!R}}_{\overline{v}_{j,m_j}}$) is composed of j left (resp. right) fibered or shifted transcendental quanta.

2.2 Bilinear fibre of tangent bibundle

According to chapter 3 of [Pie3], the shifted functional representation space $\mathrm{FREPSP}(\mathrm{GL}_{2n[2k]}(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R))$ decomposes into:

$$\begin{aligned}&\mathrm{FREPSP}(\mathrm{GL}_{2n[2k]}(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R))\\&\qquad\qquad=\mathrm{FREPSP}(\mathrm{GL}_{2k}(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R))\oplus\mathrm{FREPSP}(\mathrm{GL}_{2n-2k}(F_{\overline{v}}\times F_v))\end{aligned}$$

in such a way that $\mathrm{FREPSP}(\mathrm{GL}_{2k}(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R))$ is the **total bisemispace $(\boldsymbol{\Delta^{2k}_R}\times\boldsymbol{\Delta^{2k}_L})$ of the tangent bibundle** $\mathrm{TAN}(\widehat M^{(2n)}_{v_R}\otimes\widehat M^{(2n)}_{v_L})$ to the bisemisheaf $(\widehat M^{(2n)}_{v_R}\otimes\widehat M^{(2n)}_{v_L})\equiv\mathrm{FREPSP}(\mathrm{GL}_{2k}(F_{\overline{v}}\times F_v))$ and is isomorphic to the adjoint (functional) representation space of $\mathrm{GL}_{2k}(F_{\overline{v}}\times F_v)$ corresponding to the action of the bioperator $(D^{2k}_R\times D^{2k}_L)$ on $(\widehat M^{(2n)}_{v_R}\otimes\widehat M^{(2n)}_{v_L})$ which maps $\mathrm{TAN}(\widehat M^{(2n)}_{v_R}\otimes\widehat M^{(2n)}_{v_L})\simeq\mathrm{Ad(F)\,REPSP}(\mathrm{GL}_{2k}(F_{\overline{v}}\times F_v)$ into itself.
We have thus that:

$$\begin{aligned}\Delta^{2k}_R\times\Delta^{2k}_L&\simeq\mathrm{Ad(F)\,REPSP}(\mathrm{GL}_{2k}(F_{\overline{v}}\times F_v)\\&\simeq\mathrm{(F)REPSP}(\mathrm{GL}_{2k}(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R))\ ,\end{aligned}$$

where $\mathrm{(F)REPSP}(\mathrm{GL}_{2k}(I\!R\times I\!R))$ is the bilinear fibre $\mathcal{F}_{R\times L}(\mathrm{TAN})$ of $\mathrm{TAN}(\widehat M^{(2n)}_{v_R}\otimes\widehat M^{(2n)}_{v_L})$.

2.3 Symbol of the bioperator $(D^{2k}_R\times D^{2k}_L)$

Referring to the classical definition [A-S] of the symbol $\sigma(D)$ of a differential linear operator D dealing with the unit sphere bundles in the cotangent vector bundle $T^*(X)$ of the compact smooth manifold X, we introduce the **symbol $\boldsymbol{\sigma(D^{2k}_R\times D^{2k}_L)}$ of the differential bioperator $\boldsymbol{(D^{2k}_R\times D^{2k}_L)}$ by**:

$$\sigma(D^{2k}_R\times D^{2k}_L)\simeq\mathrm{REPSP}(P_{2k}(I\!R\times I\!R)_{|F_{\overline{v}^1}\times F_{v^1}}))\ ,$$

i.e. the unitary functional representation space of $\mathrm{GL}_{2k}((I\!R\times I\!R)_{|F_{\overline{v}}\times F_v})$ given by **the functional representation space of the fibering or shifting bilinear parabolic semigroup $\boldsymbol{P_{2k}(I\!R\times I\!R)_{|F_{\overline{v}^1}\times F_{v^1}}}$)**.
So, the differential bioperator $D^{2k}_R\times D^{2k}_L$ maps from the bisemisheaf

$$(\widehat M^{(2n)}_{v_R}\otimes\widehat M^{(2n)}_{v_L})=\mathrm{FREPSP}(\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v))$$

into the corresponding perverse bisemisheaf

$$(\widehat M^{(2n)}_{v_R}[2k]\otimes\widehat M^{(2n)}_{v_L}[2k])=\mathrm{FREPSP}(\mathrm{GL}_{2n[2k]}(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R))\ ,$$

shifted in $2k$ geometric dimensions, according to:

$$D_R^{2k}\times D_L^{2k}:\quad (\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L})\longrightarrow(\widehat{M}^{(2n)}_{v_R}[2k]\otimes\widehat{M}^{(2n)}_{v_L}[2k])\ ,$$

while $\sigma(D_R^{2k}\times D_L^{2k})$ maps the "unitary" bisemisheaf

$$\begin{aligned}(\widehat{M}^{(2n)}_{v^1_R}\otimes\widehat{M}^{(2n)}_{v^1_L}) &= \mathrm{FREPSP}(P_{2n}(F_{\overline{v}^1}\times F_{v^1}))\\ &\equiv \mathrm{FREPSP}(\mathrm{GL}_{2n}(F_{\overline{v}^1}\times F_{v^1}))\end{aligned}$$

into the corresponding "unitary" perverse bisemisheaf

$$\begin{aligned}(\widehat{M}^{(2n)}_{v^1_R}[2k]\otimes\widehat{M}^{(2n)}_{v^1_L}[2k]) &= \mathrm{FREPSP}(P_{2n[2k]}(F_{\overline{v}^1}\times I\!\!R)\times(F_{v^1}\times I\!\!R))\\ &\equiv \mathrm{FREPSP}(\mathrm{GL}_{2n[2k]}(F_{\overline{v}^1}\times I\!\!R)\times(F_{v^1}\times I\!\!R))\end{aligned}$$

according to:

$$\sigma(D_R^{2k}\times D_L^{2k}):\quad (\widehat{M}^{(2n)}_{v^1_R}\otimes\widehat{M}^{(2n)}_{v^1_L})\longrightarrow(\widehat{M}^{(2n)}_{v^1_R}[2k]\otimes\widehat{M}^{(2n)}_{v^1_L}[2k])\ .$$

2.4 Proposition

1. *The bilinear semigroup of matrices* $\mathrm{GL}_r(I\!\!R^{(2k)}\times I\!\!R^{(2k)})$ *with algebraic orders "r", referring to the biaction of the bioperator* $(D_R^{2k}\times D_L^{2k})$ *on the real bisemisheaf* $(\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L})$ *over the abstract real bisemivariety* $G^{(2k)}(F_{\overline{v}}\times F_v)\equiv \mathrm{REPSP}(\mathrm{GL}_{2k}(F_{\overline{v}}\times F_v)$*, corresponds to the* $2k$*-dimensional bilinear real representation of the product, right by left, of "differential" Galois (or global Weil) semigroups* $\mathrm{Aut}_k(\phi_R(I\!\!R))\times\mathrm{Aut}_k(\phi_L(I\!\!R))$ *fibering or "shifting" the product, right by left, of automorphism semigroups* $\mathrm{Aut}_k(\phi_R(F_{\overline{v}}))\times\mathrm{Aut}_k(\phi_L(F_v))$ *of cofunctions* $\phi_R(F_{\overline{v}})$ *and functions* $\phi_L(F_v)$ *respectively on the compact transcendental real extensions* $F_{\overline{v}}$ *and* F_v *by* $\mathrm{Aut}_k(\phi_R(F_{\overline{v}}\times I\!\!R))\times\mathrm{Aut}_k(\phi_L(F_v\times I\!\!R))$*.*

   So we have that:

$$\mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(I\!\!R))\times\mathrm{Aut}_k(\phi_L(I\!\!R)))=\mathrm{GL}_r(\phi_R(I\!\!R^{(2k)})\times\phi_L(I\!\!R^{(2k)}))\ .$$

2. *The unitary parabolic bilinear semigroup* $P_r(I\!\!R^{(2k)}\times I\!\!R^{(2k)})\subset\mathrm{GL}_r(I\!\!R^{(2k)}\times I\!\!R^{(2k)})$*, referring to the biaction of the bioperator on the unitary bisemisheaf* $(\widehat{M}^{(2n)}_{v^1_R}\otimes\widehat{M}^{(2n)}_{v^1_L})$ *over the unitary abstract bisemivariety* $P^{(2k)}(F_{\overline{v}^1}\times F_{v^1})=\mathrm{REPSP}(P_{2k}(F_{\overline{v}^1}\times F_{v^1}))$*, corresponds to the* $2k$*-dimensional bilinear representation of the product, right by left, of "differential" inertia Galois (or global Weil) semigroups* $\mathrm{Int}_k(\phi_R(I\!\!R))\times$

$\mathrm{Int}_k(\phi_L(I\!R))$ *fibering or shifting the product, right by left, of internal automorphism semigroups* $\mathrm{Int}_k(\phi_R(F_{\overline{v}^1}))\times\mathrm{Int}_k(\phi_L(F_{v^1}))$ *of cofunctions* $\phi_R(F_{\overline{v}^1})$ *and functions* $\phi_L(F_{v^1})$ *respectively on the unitary compact transcendental real extensions* $F_{\overline{v}^1}$ *and* F_{v^1} *by* $\mathrm{Int}_k(\phi_R(F_{\overline{v}^1}\times I\!R))\times\mathrm{Int}_k(\phi_L(F_{v^1}\times I\!R))$ *.*

So we have that:

$$\mathrm{Rep}^{(2k)}(\mathrm{Int}_k(\phi_R(I\!R))\times\mathrm{Int}_k(\phi_L(I\!R)))=P_r(\phi_R(I\!R^{(2k)})\times\phi_L(I\!R^{(2k)}))\ .$$

Proof:

1. **The abstract bisemivariety $\boldsymbol{G^{(2k)}(F_{\overline{v}}\times F_v)\equiv M^{(2k)}_{\overline{v}_R}\otimes M^{(2k)}_{v_L})}$ decomposes according to the increasing filtration of its conjugacy class representatives**:

$$\begin{aligned}G^{(2k)}(F_{\overline{v}_1}\times F_{v_1})\subset\cdots\subset G^{(2k)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}})\subset\cdots\subset G^{(2k)}(F_{\overline{v}_{r,m_r}}\times F_{v_{r,m_r}})\\ \equiv M^{(2k)}_{\overline{v}_1}\otimes M^{(2k)}_{v_1}\subset\cdots\subset M^{(2k)}_{\overline{v}_{j,m_j}}\otimes M^{(2k)}_{v_{j,m_j}}\cdots\subset M^{(2k)}_{\overline{v}_{r,m_r}}\otimes M^{(2k)}_{v_{r,m_r}}\ ,\\ 1\le j\le r\le\infty\end{aligned}$$

   in one-to-one correspondence with the filtration of the product $(F_{\overline{v}}\times F_v)$ of sets of archimedean transcendental extensions (see section 1.2).

   Similarly, the unitary abstract bisemivariety $P^{(2k)}(F_{\overline{v}^1}\times F_{v^1})\equiv M^{(2k)}_{\overline{v}^1_R}\otimes M^{(2k)}_{v^1_L})$, which is a bilinear parabolic subsemigroup, decomposes according to the increasing set of its representatives:

$$\begin{aligned}P^{(2k)}(F_{\overline{v}^1_1}\times F_{v^1_1})\subset\cdots\subset P^{(2k)}(F_{\overline{v}^1_{j,m_j}}\times F_{v^1_{j,m_j}})\subset\cdots\subset P^{(2k)}(F_{\overline{v}^1_{r,m_r}}\times F_{v^1_{r,m_r}})\\ \equiv M^{(2k)}_{\overline{v}^1_1}\otimes M^{(2k)}_{v^1_1}\subset\cdots\subset M^{(2k)}_{\overline{v}^1_{j,m_j}}\otimes M^{(2k)}_{v^1_{j,m_j}}\cdots\subset M^{(2k)}_{\overline{v}^1_{r,m_r}}\otimes M^{(2k)}_{v^1_{r,m_r}}\ ,\end{aligned}$$

   in one-to-one correspondence with the filtration of the product $(F_{\overline{v}^1}\times F_{v^1})$ of sets of unitary transcendental extensions.

2. Referring to the "infinite" general bilinear semigroup given by $\mathrm{GL}(F_{\overline{v}}\times F_v)=\varinjlim\mathrm{GL}_m(F_{\overline{v}}\times F_v)$ in such a way that $\mathrm{GL}_m(F_{\overline{v}}\times F_v)$ embeds in $\mathrm{GL}_{n+1}(F_{\overline{v}}\times F_v)$ according to the geometric dimension "m", **it is possible to introduce as in [Pie5] an "infinite quantum" general bilinear semigroup by**

$$\boldsymbol{\mathrm{GL}^{(Q)}(F^{(2k)}_{\overline{v}^1}\times F^{(2k)}_{v^1})=\lim_{j=1\to r}\mathrm{GL}^{(Q)}_j(F^{(2k)}_{\overline{v}^1}\times F^{(2k)}_{v^1})}$$

   in such a way that:

(a) $\mathrm{GL}_1^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)})$ is the parabolic, i.e. unitary, bilinear semigroup $\mathrm{GL}_{2k}(F_{\overline{v}^1} \times F_{v^1}) \equiv P_{2k}(F_{\overline{v}^1} \times F_{v^1})$.

(b) $\mathrm{GL}_j^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}) = \mathrm{GL}_{2k}(F_{\overline{v}_j} \times F_{v_j}) \simeq (F_{\overline{v}_j}^{(2k)} \times F_{v_j}^{(2k)})$.

(c) $\mathrm{GL}_j^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}) \subset \mathrm{GL}_{j+1}^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)})$ where the integer "j" denotes a global residue degree, i.e. an algebraic dimension, while the integer "$2k$" refers to a geometric dimension.

It was then proved in [Pie5] that the "infinite quantum" general bilinear semigroup $\mathrm{GL}^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)})$, defined with respect to the unitary parabolic bilinear semigroup $\mathrm{GL}_1^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}) = P_{2k}(F_{\overline{v}^1} \times F_{v^1})$, is the general bilinear semigroup $\mathrm{GL}_{2k}(F_{\overline{v}} \times F_v)$.

So, **the set $\{G^{(2k)}(F_{\overline{v}_{j,m_j}} \times F_{v_{j,m_j}})\}_{j,m_j}$ of conjugacy class representatives of the bisemivariety $G^{(2k)}(F_{\overline{v}} \times F_v)$**, generated from the bilinear semigroup of matrices $\mathrm{GL}_{2k}(F_{\overline{v}} \times F_v)$, **can be rewritten according to the set $\{G_{(Q)}^{(j,m_j)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)})\}_{j,m_j}$** of increasing conjugacy class representatives of the bisemivariety $G_{(Q)}^{(\overline{v}\times v)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}) = G^{(2k)}(F_{\overline{v}} \times F_v)$ generated from $\mathrm{GL}^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)})$.

And, thus, the filtration

$$G_{(Q)}^{(1)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}) \subset \cdots \subset G_{(Q)}^{(j,m_j)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}) \subset \cdots \subset G_{(Q)}^{(r,m_r)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)})$$

of conjugacy class representatives of $G^{(2k)}(F_{\overline{v}} \times F_v)$ is now written with reference to the increasing algebraic dimensions of the bilinear subsemigroups of matrices:

$$\mathrm{GL}_1^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}) \subset \cdots \subset \mathrm{GL}_{j,m_j}^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}) \subset \cdots \subset \mathrm{GL}_{r,m_r}^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}) .$$

3. Referring to section 2.3, we see that the set $\{\phi(G^{(2k)}(F_{\overline{v}_{j,m_j}} \times F_{v_{j,m_j}}))\}_{j,m_j}$ of conjugacy class representatives, or **bisections, of the bisemisheaf $(\widehat{M}_{v_R}^{(2k)} \otimes \widehat{M}_{v_L}^{(2k)}) = \mathrm{FREPSP}(\mathrm{GL}_{2k}(F_{\overline{v}} \times F_v))$**, which can be rewritten according to $\{\phi(G_{(Q)}^{(j,m_j)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}))\}_{j,m_j}$ with respect to $\mathrm{GL}^{(Q)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)})$, **are fibered or shifted, as tangent bibundles, under the action of the bioperator $(D_R^{2k} \otimes D_L^{2k})$ into**:

$$\begin{aligned}
D_R^{2k} \otimes D_L^{2k} :\ & (\widehat{M}_{v_R}^{(2n)} \otimes \widehat{M}_{v_L}^{(2n)}) \longrightarrow (\widehat{M}_{v_R}^{(2n)}[2k] \otimes \widehat{M}_{v_L}^{(2n)}[2k]) \\
& \{\phi(G_{(Q)}^{(j,m_j)}(F_{\overline{v}^1}^{(2k)} \times F_{v^1}^{(2k)}))\}_{j,m_j} \\
& \qquad \longrightarrow \{\phi(G_{(Q)}^{(j,m_j)}(F_{\overline{v}^1}^{(2k)} \times I\!R^{(2k)}) \times (F_{v^1}^{(2k)})) \times I\!R^{(2k)}\}_{j,m_j} .
\end{aligned}$$

Similarly, the set $\{\phi(P^{(2k)}(F_{\overline{v}^1_{j,m_j}}\times F_{v^1_{j,m_j}}))\}_{j,m_j}$ of bisections of the unitary bisemisheaf $(\widehat{M}^{(2k)}_{v^1_R}\otimes\widehat{M}^{(2k)}_{v^1_L})=\mathrm{FREPSP}(P_{2k}(F_{\overline{v}^1}\times F_{v^1}))$, which can be rewritten according to $\{\phi(P^{(j,m_j)}_{(Q)}(F^{(2k)}_{\overline{v}^1}\times F^{(2k)}_{v^1}))\}_{j,m_j}$ are fibered or shifted under the action of the bioperator $(D^{2k}_R\otimes D^{2k}_L)$ into:

$$\begin{aligned}D^{2k}_R\otimes D^{2k}_L:\ &(\widehat{M}^{(2n)}_{v^1_R}\otimes\widehat{M}^{(2n)}_{v^1_L})\longrightarrow(\widehat{M}^{(2n)}_{v^1_R}[2k]\otimes\widehat{M}^{(2n)}_{v^1_L}[2k])\\&\{\phi(P^{(j,m_j)}_{(Q)}(F^{(2k)}_{\overline{v}^1}\times F^{(2k)}_{v^1}))\}_{j,m_j}\\&\qquad\longrightarrow\{\phi(P^{(j,m_j)}_{(Q)}(F^{(2k)}_{\overline{v}^1}\times I\!R^{(2k)})\times(F^{(2k)}_{v^1}))\times I\!R^{(2k)}\}_{j,m_j}\ .\end{aligned}$$

4. Let the geometric dimension $2k$ be equal to $2k=1$. The tower of shifted real transcendental biextensions is:

$$\begin{aligned}(F_{\overline{v}_1}\times I\!R)\times(F_{v_1}\times I\!R)\subset\cdots\subset(F_{\overline{v}_{j,m_j}}\times I\!R)\times(F_{v_{j,m_j}}\times I\!R)\\\subset\cdots\subset(F_{\overline{v}_{r,m_r}}\times I\!R)\times(F_{v_{r,m_r}}\times I\!R)\ ,\end{aligned}$$

i.e. a tower of (real) shifted transcendental biquanta by vertical bifibres ($I\!R\times I\!R$).

They are the conjugacy class representatives generated under the action of representatives of $\mathrm{GL}_1(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R)$ and they can be rewritten as the filtration

$$\begin{aligned}G^{(1)}_{(Q)}((F_{\overline{v}^1}\times I\!R)\times(F_{v^1}\times I\!R)\subset\cdots\subset G^{(j,m_j)}_{(Q)}((F_{\overline{v}^1}\times I\!R)\times(F_{v^1}\times I\!R)\\\subset\cdots\subset G^{(r,m_r)}_{(Q)}((F_{\overline{v}^1}\times I\!R)\times(F_{v^1}\times I\!R))\end{aligned}$$

of the representatives of the bilinear subsemigroups of the (infinite) quantum general bilinear semigroup $\mathrm{GL}^{(Q)}((F_{\overline{v}^1}\times I\!R)\times(F_{v^1}\times I\!R))$.

The bilinear semigroup of automorphisms of these fibered or shifted transcendental extensions is:

$$\mathrm{Aut}_k(F_{\overline{v}}\times I\!R)\times\mathrm{Aut}_k(F_v\times I\!R)=\{\dots,\mathrm{Aut}_k(F_{\overline{v}_{j,m_j}}\times I\!R)\times\mathrm{Aut}_k(F_{v_{j,m_j}}\times I\!R),\dots\}\ .$$

So

$$[\mathrm{Aut}_k(F_{\overline{v}}\times I\!R)\times\mathrm{Aut}_k(F_v\times I\!R)]\big/[\mathrm{Aut}_k(F_{\overline{v}})\times\mathrm{Aut}_k(F_v)=\mathrm{Aut}_k(I\!R)_{|F_{\overline{v}}}\times\mathrm{Aut}_k(I\!R)_{|F_v}$$

corresponds to the bilinear semigroup of fibering or shifting automorphisms.

The functional representatives of $\mathrm{GL}^{(Q)}((F_{\overline{v}^1}\times I\!R)\times(F_{v^1}\times I\!R))$, or equivalently of $\mathrm{GL}_1(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R)$, are

$$F\ G^{(1)}_{(Q)}((F_{\overline{v}^1}\times I\!R)\times(F_{v^1}\times I\!R))\subset\cdots\subset F\ G^{(j,m_j)}_{(Q)}((F_{\overline{v}^1}\times I\!R)\times(F_{v^1}\times I\!R))\ ,$$

i.e. bifunctions on the birepresentatives $\{G^{(j,m_j)}_{(Q)}((F_{\overline{v}^1}\times I\!\!R)\times(F_{v^1}\times I\!\!R))\}_{j,m_j}$.

And, thus,

$$\mathrm{Aut}_k(\phi_R(I\!\!R))\times\mathrm{Aut}_k(\phi_L(I\!\!R))=\{\dots,\mathrm{Aut}_k(\phi_{j,m_j}(I\!\!R))\times\mathrm{Aut}_k(\phi_{j,m_j}(I\!\!R)),\dots\}_{j,m_j}\ ,$$

where $\phi_{j,m_j}(I\!\!R)$ is the (j,m_j)-th function over $I\!\!R$ acting on the function $\phi_{j,m_j}(F_{v_{j,m_j}})$ over the transcendental extension $F_{v_{j,m_j}}$ or over the conjugacy class representative of $\mathrm{GL}_1(F_{v_{j,m_j}})$.

So, $\mathbf{Aut}_{\boldsymbol{k}}(\boldsymbol{\phi_R}(I\!\!R))\times\mathbf{Aut}_{\boldsymbol{k}}(\boldsymbol{\phi_L}(I\!\!R))$ **is the bilinear differential Galois semigroup where** $\mathbf{Aut}_{\boldsymbol{k}}(\boldsymbol{\phi_R}(I\!\!R))$ **is the set of linear differential Galois (semi)subgroups [Car], [And], acting on the set of sections of the considered** $\mathbf{1}\boldsymbol{D}$**-differential equation and** $\mathbf{Aut}_{\boldsymbol{k}}(\boldsymbol{\phi_L}(I\!\!R))$ **is the set of linear differential Galois (semi)subgroups acting on the symmetric set of sections**.

Referring to the fundamental theorem of Galois theory, we see that **the bilinear differential Galois semigroup** $\mathbf{Aut}_{\boldsymbol{k}}(\boldsymbol{\phi_R}(I\!\!R))\times\mathbf{Aut}_{\boldsymbol{k}}(\boldsymbol{\phi_L}(I\!\!R))$ **corresponds to the upper bilinear differential Galois semigroup** $\mathrm{Aut}_k(\phi_R(I\!\!R))_{|F_{\overline{v}_{r,m_r}}}\times\mathrm{Aut}_k(\phi_L(I\!\!R))_{|F_{v_{r,m_r}}}$ with respect to the upper algebraic dimension "r".

Similarly, the bilinear semigroup of unitary, i.e. internal, shifting automorphisms:

$$\mathrm{Int}_k(I\!\!R)_{|F_{\overline{v}^1}}\times\mathrm{Int}_k(I\!\!R)_{|F_{v^1}}=[\mathrm{Int}_k(F_{\overline{v}^1}\times I\!\!R)\times\mathrm{Int}_k(F_{v^1}\times I\!\!R)]\big/[\mathrm{Int}_k(F_{\overline{v}^1})\times\mathrm{Int}_k(F_{v^1})$$

has for differential Galois bilinear semigroup $\mathrm{Int}_k(\psi_R(I\!\!R))\times\mathrm{Int}_k(\psi_L(I\!\!R))$ where $\mathbf{Int}_{\boldsymbol{k}}(\boldsymbol{\psi_L}(I\!\!R))$ **is the unitary linear differential Galois (semi)group**, i.e. the inertia linear differential (semi)group, **acting on a "unitary section" of the envisaged** $\mathbf{1}\boldsymbol{D}$**-differential equation, and** $\mathbf{Int}_{\boldsymbol{k}}(\boldsymbol{\psi_R}(I\!\!R))$ **is the inertia linear differential (semi)group acting on the symmetric section**.

5. If the geometric dimension is $2k$, then we have a tower of fibered or shifted real conjugacy class representatives

$$G^{(2k)}((F_{\overline{v}_1}\times I\!\!R)\times(F_{v_1}\times I\!\!R))\subset\dots\subset G^{(2k)}((F_{\overline{v}_{j,m_j}}\times I\!\!R)\times(F_{v_{j,m_j}}\times I\!\!R))\subset\dots$$

generated under the (bi)action of $\mathrm{GL}_{2k}(F_{\overline{v}}\times F_v)$ or, equivalently, a tower of class representatives

$$\begin{aligned}&G^{(1)}_{(Q)}((F^{(2k)}_{\overline{v}^1}\times I\!\!R^{(2k)})\times(F^{(2k)}_{v^1}\times I\!\!R^{(2k)}))\\&\qquad\subset\dots\subset G^{(j,m_j)}_{(Q)}((F^{(2k)}_{\overline{v}^1}\times I\!\!R^{(2k)})\times(F^{(2k)}_{v^1}\times I\!\!R^{(2k)}))\subset\dots\end{aligned}$$

of the fibered or shifted bisemivariety $G^{(\overline{v}\times v)}_{(Q)}((F^{(2k)}_{\overline{v}^1}\times I\!R^{(2k)})\times(F^{(2k)}_{v^1}\times I\!R^{(2k)}))$ generated from $\mathrm{GL}^{(Q)}((F^{(2k)}_{\overline{v}^1}\times I\!R^{(2k)})\times(F^{(2k)}_{v^1}\times I\!R^{(2k)}))$.

Referring to 4), we see that the $2k$-dimensional representation of $(\mathrm{Aut}_k(\phi_R(F_{\overline{v}}\times I\!R))\times \mathrm{Aut}_k(\phi_L(F_v\times I\!R)))$ given by $\mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(F_{\overline{v}}\times I\!R))\times \mathrm{Aut}_k(\phi_L(F_v\times I\!R)))$ decomposes into:

$$\begin{aligned}&\mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(F_{\overline{v}}\times I\!R))\times \mathrm{Aut}_k(\phi_L(F_v\times I\!R)))\\ &\qquad\qquad = \mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(F_{\overline{v}}))\times \mathrm{Aut}_k(\phi_L(F_v)))\\ &\qquad\qquad\qquad\qquad \times \mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(I\!R))\times \mathrm{Aut}_k(\phi_L(I\!R)))\end{aligned}$$

which generates (and is isomorphic to) the fibered or shifted bisemivariety $G^{(2k)}(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R)$, rewritten according to $G^{(\overline{v}\times v)}_{(Q)}((F^{(2k)}_{\overline{v}^1}\times I\!R^{(2k)})\times(F^{(2k)}_{v^1}\times I\!R^{(2k)})$.

$\mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(I\!R))\times \mathrm{Aut}_k(\phi_L(I\!R)))$ is thus the $2k$-dimensional representation of the differential bilinear Galois semigroup $(\mathrm{Aut}_k(\phi_R(I\!R))\times \mathrm{Aut}_k(\phi_L(I\!R)))$ and is isomorphic to the shifting bisemivariety $G^{(2k)}(\phi_R(I\!R)\times\phi_L(I\!R))$ or $G^{(\overline{v}\times v)}_{(Q)}(\phi_R(I\!R^{(2k)})\times \phi_L(I\!R^{(2k)}))$.

So, we have **a tower of $2k$-dimensional representations of the differential bilinear Galois subsemigroups**

$$\begin{aligned}&\mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(I\!R)_{|F_{\overline{v}_1}})\times \mathrm{Aut}_k(\phi_L(I\!R)_{|F_{v_1}}))\\ &\qquad\subset\cdots\subset \mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(I\!R)_{|F_{\overline{v}_{j,m_j}}})\times \mathrm{Aut}_k(\phi_L(I\!R)_{|F_{v_{j,m_j}}}))\\ &\qquad\qquad\subset\cdots\subset \mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(I\!R)_{|F_{\overline{v}_{r,m_r}}})\times \mathrm{Aut}_k(\phi_L(I\!R)_{|F_{v_{r,m_r}}}))\end{aligned}$$

which are respectively in one-to-one correspondence with the bilinear semigroups:

$$\begin{aligned}&\mathrm{GL}_1(\phi_R(I\!R^{(2k)})\times\phi_L(I\!R^{(2k)})) = \mathrm{GL}_{2k}(\phi_R(I\!R)\times\phi_L(I\!R))_{|F_{\overline{v}_1}\times F_{v_1}}\\ &\quad\subset\cdots\subset \mathrm{GL}_j(\phi_R(I\!R^{(2k)})\times\phi_L(I\!R^{(2k)})) = \mathrm{GL}_{2k}(\phi_R(I\!R)\times\phi_L(I\!R))_{|F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}}}\\ &\quad\subset\cdots\subset \mathrm{GL}_r(\phi_R(I\!R^{(2k)})\times\phi_L(I\!R^{(2k)})) = \mathrm{GL}_{2k}(\phi_R(I\!R)\times\phi_L(I\!R))_{|F_{\overline{v}_{r,m_r}}\times F_{v_{r,m_r}}}\ .\end{aligned}$$

So, we get the thesis by considering the upper algebraic dimension "r":

$$\mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\phi_R(I\!R)_{|F_{\overline{v}_r}})\times \mathrm{Aut}_k(\phi_L(I\!R)_{|F_{v_r}})) = \mathrm{GL}_r(\phi_R(I\!R^{(2k)})\times\phi_L(I\!R^{(2k)}))$$

where $\phi_R(I\!R^{(2k)})\simeq I\!R^{(2k)}$ and $\phi_L(I\!R^{(2k)})\simeq I\!R^{(2k)}$ are generally constant functions.

The unitary case referring to the $2k$-dimensional representation of the bilinear inertia semigroup $\mathrm{Int}_k(\phi_R(\mathbb{R}))\times\mathrm{Int}_k(\phi_L(\mathbb{R}))$ can be reached similarly, i.e.

$$\begin{aligned}\mathrm{Rep}^{(2k)}(\mathrm{Int}_k(\phi_R(\mathbb{R})_{|F_{\overline{v}^1_r}})\times\mathrm{Int}_k(\phi_L(\mathbb{R})_{|F_{v^1_r}}))\\ &= P_r(\phi_R(\mathbb{R}^{(2k)})\times\phi_L(\mathbb{R}^{(2k)}))\\ &= \mathrm{GL}_r(\phi_R(\mathbb{R}^{(2k)})\times\phi_L(\mathbb{R}^{(2k)}))_{|F_{\overline{v}^1}\times F_{v^1}}\end{aligned}$$

■

2.5 Corollary

The complex bilinear representation of differential Galois (semi)groups can be found similarly as it was done for the real case.

Let $F_\omega=\{F_{\omega_1},\dots,F_{\omega_{j,m_j}},\dots,F_{\omega_{r,m_r}}\}$ *(resp.* $F_{\overline{\omega}}=\{F_{\overline{\omega}_1},\dots,F_{\overline{\omega}_{j,m_j}},\dots,F_{\overline{\omega}_{r,m_r}}\}$ *) be the set of complex transcendental extensions (or infinite complex archimedean completions) covered by its real equivalent* F_v *(resp.* $F_{\overline{v}}$ *).*

Then, **the complex bilinear semigroup** *of matrices* $\mathrm{GL}_r(\mathbb{C}^k\times\mathbb{C}^k)$*, of algebraic order "r", referring to the action of the bioperator* $(D^{2k}_R\otimes D^{2k}_L)$ *on the complex bisemisheaf* $(\widehat{M}^{(2k)}_{\overline{\omega}_R}\otimes\widehat{M}^{(2k)}_{\omega_L})$ *on the abstract complex bisemivariety* $G^{(2k)}(F_{\overline{\omega}}\times F_\omega)\equiv\mathrm{REPSP}(\mathrm{GL}_k(F_{\overline{\omega}}\times F_\omega))$*,* **corresponds to the k-dimensional complex bilinear representation of the product, right by left, of "differential" Galois (or global Weil) semigroups $(\mathrm{Aut}_k(\psi_R(\mathbb{C}))\times\mathrm{Aut}_k(\psi_L(\mathbb{C})))$ shifting the product, right by left, of automorphism semigroups $(\mathrm{Aut}_k(\psi_R(F_{\overline{\omega}}))\times\mathrm{Aut}_k(\psi_L(F_\omega)))$ of cofunctions $\psi_R(F_{\overline{\omega}})$ and $\psi_L(F_\omega)$ by $(\mathrm{Aut}_k(\psi_R(F_{\overline{\omega}}\times\mathbb{C}))\times\mathrm{Aut}_k(\psi_L(F_\omega)\times\mathbb{C}))$**.

So, we have that:

$$\mathrm{GL}_r(\phi_R(\mathbb{C}^k)\times\phi_L(\mathbb{C}^k))=\mathrm{Rep}^{(2k)}(\mathrm{Aut}_k(\psi_R(\mathbb{C})_{|F_{\overline{\omega}_r}}\times\mathrm{Aut}_k(\psi_L(\mathbb{C})_{|F_{\omega_R}})))\ .$$

And, in the unitary case, we have:

$$P_r(\psi_R(\mathbb{C}^k)\times\psi_L(\mathbb{C}^k))=\mathrm{Rep}^{(2k)}(\mathrm{Int}_k(\psi_R(\mathbb{C})_{|F_{\overline{\omega}^1_r}})\times\mathrm{Int}_k(\psi_L(\mathbb{C})_{|F_{\omega^1_R}}))$$

where:

- $P_r(\dots)$ *is the bilinear parabolic subsemigroup;*
- $\mathrm{Rep}^{(2k)}(\dots)$ *is the* k*-dimensional complex representation of* $(\dots)$.

2.6 Corollary

Let $O_r(\mathbb{R})$ denote the orthogonal group of algebraic order r with entries in the reals $\mathbb{R}$ and let $U_r(\mathbb{C})$ denote the unitary group of algebraic order r in the complexes $\mathbb{C}$.
Then, the orthogonal bilinear semigroup $O_r(\mathbb{R}^{(2k)}\times\mathbb{R}^{(2k)})$ corresponds to the real parabolic bilinear semigroup $P_r(\mathbb{R}^{(2k)}\times\mathbb{R}^{(2k)})$ and the unitary bilinear semigroup $U_r(\mathbb{C}^k\times\mathbb{C}^k)$ corresponds to the complex parabolic bilinear semigroup $P_r(\mathbb{C}^k\times\mathbb{C}^k)$.

Proof: This results from the definition of a bilinear semigroup recalled in section 1.5 and from proposition 2.4 and corollary 2.5.
We have more particularly that:

$$O_r(\mathbb{R}^{(2k)}\times\mathbb{R}^{(2k)})=\mathrm{Rep}^{(2k)}(\mathrm{Int}_k(\phi_R(\mathbb{R}))_{|F_{\overline{v}^1_r}}\times\mathrm{Int}_k(\phi_L(\mathbb{R}))_{|F_{v^1_r}})$$

and that

$$U_r(\mathbb{C}^k\times\mathbb{C}^k)=\mathrm{Rep}^{(2k)}(\mathrm{Int}_k(\psi_R(\mathbb{C}))_{|F_{\overline{\omega}^1_r}}\times\mathrm{Int}_k(\psi_L(\mathbb{C}))_{|F_{\omega^1_r}})\ . \qquad\blacksquare$$

2.7 Corollary

In the one-dimensional geometric case, i.e. when $2k=1$, we have that

1) **the orthogonal bilinear semigroup of algebraic order $\boldsymbol{r}$**

$$\boldsymbol{O_r(\mathbb{R}\times\mathbb{R})=P_r(\mathbb{R}\times\mathbb{R})}$$

corresponds to the product, right by left, of differential inertia Galois semigroups $\boldsymbol{(\mathrm{Int}_k(\phi_R(\mathbb{R}))_{|F_{\overline{v}^1_r}}\times\mathrm{Int}_k(\phi_L(\mathbb{R}))_{|F_{v^1_r}})}$ *shifting the product, right by left,*
$(\mathrm{Int}_k(\phi_R(F_{\overline{v}^1_r}))\times\mathrm{Int}_k(\phi_L(F_{v^1_r})))$ *of internal automorphism semigroups of cofunctions $\phi_R(F_{\overline{v}^1_r})$ and functions $\phi_L(F_{v^1_r})$ respectively on the unitary transcendental lower and upper real extensions $F_{\overline{v}^1_r}$ and $F_{v^1_r}$.*

2) **the unitary bilinear semigroup of algebraic order $\boldsymbol{r}$**

$$\boldsymbol{U_r(\mathbb{C}\times\mathbb{C})=P_r(\mathbb{C}\times\mathbb{C})}$$

corresponds to the product, right by left, of differential inertia Galois semigroups $\boldsymbol{(\mathrm{Int}_k(\psi_R(\mathbb{C}))_{|F_{\overline{\omega}^1_r}}\times\mathrm{Int}_k(\psi_L(\mathbb{C}))_{|F_{\omega^1_r}})}$ *shifting the product, right by left, $(\mathrm{Int}_k(\psi_R(F_{\overline{\omega}^1_r}))\times\mathrm{Int}_k(\psi_L(F_{\omega^1_r})))$ of internal automorphism semigroups of cofunctions $\psi_R(F_{\overline{\omega}^1_r})$ and functions $\psi_L(F_{\omega^1_r})$ respectively on the unitary complex transcendental upper extensions $F_{\overline{\omega}^1_r}$ and $F_{\omega^1_r}$.*

Proof:

1) $(\mathrm{Int}_k(\phi_R(F_{\overline{v}^1_r}))\times\mathrm{Int}_k(\phi_L(F_{v^1_r})))$ corresponds to the bilinear semigroup of internal automorphisms of bifunctions on "unitary" biquanta in a bisection $\phi_R(F_{\overline{v}_r})\times\phi_L(F_{v^1_r})$ at "r" biquanta $(F_{\overline{v}_r}\times F_{v_r})$ while $(\mathrm{Int}_k(\phi_R(F_{\overline{v}^1_r}\times I\!\!R))\times\mathrm{Int}_k(\phi_L(F_{v^1_r}\times I\!\!R)))$ corresponds to the bilinear semigroup of shifted internal automorphisms (under the action of a bioperator $(D_R\otimes D_L)$) of bifunctions on shifted "unitary" biquanta $((F_{\overline{v}^1_r}\times I\!\!R)\times(F_{v^1_r}\times I\!\!R))$ in a bisection $\phi_R(F_{\overline{v}_r}\times I\!\!R)\times\phi_L(F_{v_r}\times I\!\!R)$ at "r" shifted biquanta $(F_{\overline{v}_r}\times I\!\!R)\times(F_{v_r}\times I\!\!R)$.

2) The complex unitary case referring to $U_r(\mathbb{C}\times\mathbb{C})$ can be handled similarly as the real unitary case referring to $O_r(I\!\!R\times I\!\!R)$ by taking into account that a complex bisection $\psi_R(F_{\overline{\omega}_r})\times\psi_L(F_{\omega_r})$ is covered by real bisections $\{\phi_R(F_{\overline{v}_{r,m_r}})\times\phi_L(F_{v_{r,m_r}})\}_{m_r}$ as developed in section 1.9. ∎

2.8 Bilinear Hilbert semispaces and Von Neumann bisemialgebras

Let $(\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L})$ be the bisemisheaf of differentiable bifunctions on the abstract bisemivariety $G^{(2n)}(F_{\overline{v}}\times F_v)$.

The set

$$\{\phi^{(2n)}_{j,m_j}(g^{(2n)}_{v_{R\times L}}[j,m_j])\}_{j,m_j}=\{\phi^{(2n)}_{j,m_j}(g^{(2n)}_{v_R}[j,m_j])\otimes\phi^{(2n)}_{j,m_j}(g^{(2n)}_{v_L}[j,m_j])\}_{j,m_j}$$

of differentiable bifunctions, i.e. bisections of $(\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L})$,

forms an increasing filtration with respect to the algebraic dimension "j":

$$\phi^{(2n)}_1(g^{(2n)}_{v_{R\times L}}[1])\subset\cdots\subset\phi^{(2n)}_{j,m_j}(g^{(2n)}_{v_{R\times L}}[j,m_j])\subset\cdots\subset\phi^{(2n)}_{r,m_r}(g^{(2n)}_{v_{R\times L}}[r,m_r])\;,\qquad j\le r\le\infty\;,$$

on the corresponding filtration of conjugacy class representatives of the abstract bisemivariety $G^{(2n)}(F_{\overline{v}}\times F_v)$:

$$g^{(2n)}_{v_{R\times L}}[1]\subset\cdots\subset g^{(2n)}_{v_{R\times L}}[j,m_j]\subset\cdots\subset g^{(2n)}_{v_{R\times L}}[r,m_r]\;.$$

This bisemisheaf $(\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L})$ is transformed into **an extended internal left bilinear Hilbert semispace** $\boldsymbol{H}^+_{(\widehat{\boldsymbol{M}}^{(2n)}_{v_R}\otimes\widehat{\boldsymbol{M}}^{(2n)}_{v_L})}$ by taking into account

1) a map [Pie6]:

$$B_L\circ p_L:\qquad \widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L}\quad\longrightarrow\quad\widehat{M}^{(2n)}_{v_{L_R}}\otimes\widehat{M}^{(2n)}_{v_L})\equiv H^+_{(\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L})}$$

where:

- p_L is a projective linear map projecting the right semisheaf $\widehat{M}^{(2n)}_{v_R}$ onto the left semisheaf $\widehat{M}^{(2n)}_{v_L}$;
- B_L is a bijective linear isometric map from the projected right semisheaf $\widehat{M}^{(2n)}_{v_{L(P)_R}}$ to $\widehat{M}^{(2n)}_{v_{L_R}}$ mapping each covariant element of $\widehat{M}^{(2n)}_{v_{L(P)_R}}$ into a contravariant element.

2) an internal bilinear form defined from $H^+_{(\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L})}$ into $\mathbb{C}$ for every bisection $\phi^{(2n)}_{j,m_j}(g^{(2n)}_{v_{R\times L}}[j,m_j])$ by:

$$(\phi^{(2n)}_{j,m_j}(g^{(2n)}_{v_{L_R}}[j,m_j],\phi^{(2n)}_{j,m_j}(g^{(2n)}_{v_L}[j,m_j]))) \longrightarrow \mathbb{C}\ .$$

This bilinear Hilbert semispace $H^+_{(\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L})}$, noted in condensed form $H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}$, is a natural representation (bisemi)space for the bialgebra of elliptic bioperators $(D^{2k}_R\otimes D^{2k}_L)$ as noticed in [Pie6].

A bisemialgebra of von Neumann $\mathbb{M}_{R\times L}(H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}})$ in $H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}$ is an involutive subbisemialgebra of the bisemialgebra $(\mathcal{L}^B_R\otimes\mathcal{L}^B_L)(H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}})$ of bounded bioperators having a closed norm topology.

Due to the structure of the bisemisheaf $(\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L})$, the bilinear Hilbert semispace $H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}$ is "solvable" in the sense that we have a tower of embedded bilinear Hilbert subsemispaces

$$H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}(1)\subset\cdots\subset H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}(j)\subset\cdots\subset H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}(r)$$

where $H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}$ is given by the set $\{\phi^{(2n)}_{j,m_j}(g^{(2n)}_{v_{L_R}}[j,m_j])\otimes\phi^{(2n)}_{j,m_j}(g^{(2n)}_{v_L}[j,m_j])\}_{j,m_j}$ of bisections. But, we can also construct a tower of direct sums of embedded extended bilinear Hilbert subsemispaces:

$$H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}\{1\}\subset\cdots\subset H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}\{j\}\subset\cdots\subset H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}\{r\}$$

where $H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}\{j\}$ is defined by:

$$H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}\{j\}=\bigoplus_{\nu=1}^{j}H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}(\nu)\ .$$

2.9 Random bioperators

Let $(D_R^{2k}\otimes D_L^{2k})$ be the differential (elliptic) bioperator acting on the set $\{\phi(G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}}))\}_{j,m_j}$ (also written $\{\phi^{(2n)}_{j,m_j}(g^{(2n)}_{v_{R\times L}}[j,m_j])\}_{j,m_j}$) of differentiable bifunctions or bisections of the bisemisheaf $(\widehat M^{2n}_{v_R}\otimes\widehat M^{2n}_{v_L})$ according to:

$$\begin{aligned}D_R^{2k}\otimes D_L^{2k}:(\widehat M^{(2n)}_{v_R}\otimes\widehat M^{(2n)}_{v_L})&\longrightarrow(\widehat M^{(2n)}_{v_R}[2k]\otimes\widehat M^{(2n)}_{v_L}[2k])\\ \{\phi_{j,m_j}(G^{(2n)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}}))\}_{j,m_j}&\\ &\longrightarrow\{\phi_{j,m_j}(G^{(2n)}_{[2k]}((F_{\overline{v}_{j,m_j}}\times I\!R)\times(F_{v_{j,m_j}}\times I\!R)))\}_{j,m_j}\end{aligned}$$

where $\phi_{j,m_j}(G^{(2n)}_{[2k]}((F_{\overline{v}_{j,m_j}}\times I\!R)\times(F_{v_{j,m_j}}\times I\!R)))$ is the bifunction on the (j,m_j)-th conjugacy class representative $G^{(2n)}_{[2k]}((F_{\overline{v}_{j,m_j}}\times I\!R)\times(F_{v_{j,m_j}}\times I\!R))$ fibered or shifted in $2k$ bilinear geometric dimensions.

The bioperator $(D_R^{2k}\otimes D_L^{2k})$ is a random bioperator in the sense that, for every bifunction

$$\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}}))=\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{\overline{v}_{j,m_j}}))\times\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v_{j,m_j}}))$$

belonging to the bilinear Hilbert semispace $H^+_{\widehat M^{(2n)}_{v_{R\times L}}}$ the bilinear form

$$(D_R^{2k}(\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{\overline{v}_{j,m_j}}))),(D_L^{2k}(\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v_{j,m_j}})))))$$

is measurable.

The random bioperator $(D_R^{2k}\otimes D_L^{2k})$ acting on $H^+_{\widehat M^{(2n)}_{v_{R\times L}}}$ is a set $\{(D_R^{2k}(j,m_j)\otimes D_L^{2k}(j,m_j))\}_{j,m_j}$ of bioperators acting on the bisemisheaf $\widehat M^{(2n)}_{v_{R\times L}}$.

2.10 Towers of embedded von Neumann subbisemialgebras

Referring to the tower of embedded bilinear Hilbert subsemispaces associated with the bisemisheaf $\widehat M^{(2n)}_{v_{R\times L}}$ and to the definition of a bisemialgebra of von Neumann $\mathbb{M}_{R\times L}(H^+_{\widehat M^{(2n)}_{v_{R\times L}}})$ given in section 2.8, it appears that there exists a **tower of embedded von Neumann subbisemialgebras**:

$$\mathbb{M}_{R\times L}(H^+_{\widehat M^{(2n)}_{v_{R\times L}}}(1))\subset\cdots\subset\mathbb{M}_{R\times L}(H^+_{\widehat M^{(2n)}_{v_{R\times L}}}(j))\subset\cdots\subset\mathbb{M}_{R\times L}(H^+_{\widehat M^{(2n)}_{v_{R\times L}}}(r))$$

according to the algebraic dimensions $1\le j\le r$, as well as a **tower of sums of embedded von Neumann subbisemialgebras**:

$$\mathbb{M}_{R\times L}(H^+_{\widehat M^{(2n)}_{v_{R\times L}}}\{1\})\subset\cdots\subset\mathbb{M}_{R\times L}(H^+_{\widehat M^{(2n)}_{v_{R\times L}}}\{j\})\subset\cdots\subset\mathbb{M}_{R\times L}(H^+_{\widehat M^{(2n)}_{v_{R\times L}}}\{r\})$$

where

$$\mathbb{M}_{R\times L}(H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}\{j\}) = \overset{j}{\underset{\nu=1}{\oplus}} \mathbb{M}_{R\times L}(H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}(\nu)) \; .$$

The bisemisheaf $(\widehat{M}^{2n}_{v_R}\otimes\widehat{M}^{2n}_{v_L})$ gives rise to the extended internal left bilinear Hilbert semispace $H^+_{\widehat{M}^{2n}_{v_R}\otimes\widehat{M}^{2n}_{v_L}}$ according to section 2.8.

Similarly, the diagonal bisemisheaf $(\widehat{M}^{2n}_{v_R}\otimes_D\widehat{M}^{2n}_{v_L})$, whose bisections are diagonal bisections

$$\phi_{j,m_j}(G^{(2n)}(F_{\overline{v}_{j,m_j}}\times_D F_{v_{j,m_j}})) = \phi_{j,m_j}(G^{(2n)}(F_{\overline{v}_{j,m_j}}))\times_D \phi_{j,m_j}(G^{(2n)}(F_{v_{j,m_j}}))$$

characterized by a diagonal bilinear basis (the offdiagonal bilinear basis elements being null) [Pie6], gives rise to the **diagonal internal left bilinear Hilbert semispace** $\mathcal{H}^+_{\widehat{M}^{2n}_{v_R}\otimes_D\widehat{M}^{2n}_{v_L}}$ by taking into account a $(B_L\circ p_L)$ map and the existence of an internal diagonal bilinear form, i.e. a scalar product, as in section 2.8.

2.11 Proposition

Let $\mathbb{M}_{R\times L}(H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}})$ be the bisemialgebra of von Neumann on the extended internal left bilinear Hilbert semispace $H^+_{\widehat{M}^{2n}_{v_R}\otimes\widehat{M}^{2n}_{v_L}}$ and let $\mathbb{M}_{R\times L}(\mathcal{H}^+_{\widehat{M}^{2n}_{v_R}\otimes_D\widehat{M}^{2n}_{v_L}})$ be the von Neumann bisemialgebra on the diagonal internal left bilinear Hilbert semispace $\mathcal{H}^+_{\widehat{M}^{2n}_{v_R}\otimes_D\widehat{M}^{2n}_{v_L}}$.

Then, **the discrete spectrum $\boldsymbol{\Sigma(D^{2k}_R\otimes D^{2k}_L)}$ of the bioperator $\boldsymbol{(D^{2k}_R\otimes D^{2k}_L)}$ is obtained by the composition of morphisms***:*

$$\begin{array}{llccc}
i_{\{j\}^D_{R\times L}}\circ i_{\{j\}_{R\times L}} & : & \mathbb{M}_{R\times L}(H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}) & \longrightarrow & \left[\mathbb{M}_{R\times L}(\mathcal{H}_{\widehat{M}^{(2n)}_{v_R}[2k]\otimes_D\mathcal{H}_{\widehat{M}^{(2n)}_{v_L}[2k]}})\right]_j \\
 & & (D^{2k}_R\otimes D^{2k}_L) & \longrightarrow & \Sigma(D^{2k}_R\otimes D^{2k}_L)
\end{array}$$

where $i_{\{j\}_{R\times L}}$ and $i_{\{j\}^D_{R\times L}}$ are given by:

$$\begin{array}{lll}
i_{\{j\}_{R\times L}}: & \mathbb{M}_{R\times L}(H^+_{\widehat{M}^{(2n)}_{v_{R\times L}}}) & \longrightarrow \left[\mathbb{M}_{R\times L}(H^+_{\widehat{M}^{(2n)}_{v_{R\times L}[2k]}}\{j\})\right]_j \\
i_{\{j\}^D_{R\times L}}: & \left[\mathbb{M}_{R\times L}(H^+_{\widehat{M}^{(2n)}_{v_{R\times L}[2k]}}\{j\})\right]_j & \longrightarrow \left[\mathbb{M}_{R\times L}(\mathcal{H}^+_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{j\})\right]_j \; .
\end{array}$$

$\left[\mathbb{M}_{R\times L}(\mathcal{H}^+_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{j\})\right]_j$ *is the increasing tower, over the running algebraic index "j", of sums of von Neumann subbisemialgebras:*

$$\mathbb{M}_{R\times L}(\mathcal{H}^+_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{j\}) = \overset{j}{\underset{\nu=1}{\oplus}} \mathbb{M}_{R\times L}(\mathcal{H}^+_{(\widehat{M}^{(2n)}_{v_R[2k]}\otimes\widehat{M}^{(2n)}_{v_L[2k]})}(\nu)$$

over respectively the tower:

$$\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{1\}\subset\cdots\subset\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{j\}\subset\cdots\subset\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{r\}$$

of sums of diagonal internal left bilinear Hilbert subsemispaces

$$\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{j\})=\bigoplus_{\nu=1}^{j}(\mathcal{H}^{+}_{(\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]})}(\nu))$$

shifted in $(2k)$ *bilinear geometric dimensions.*

Proof: The morphism $i_{\{j\}_{R\times L}}$

$$i_{\{j\}_{R\times L}}:\qquad \mathbb{M}_{R\times L}(H^{+}_{\widehat{M}^{(2n)}_{v_{R\times L}}})\longrightarrow\left[\mathbb{M}_{R\times L}(H^{+}_{\widehat{M}^{(2n)}_{v_{R\times L}[2k]}}\{j\})\right]_j$$

is in fact implicit depending on the decomposition of the bisemisheaf $(\widehat{M}^{(2n)}_{v_R}\otimes\widehat{M}^{(2n)}_{v_L})$ into bisections on the conjugacy class representatives $\{g^{(2n)}_{v_{R\times L}}[j,m_j]\}_{j,m_j}$ of the abstract bisemivariety $G^{(2n)}(F_{\overline{v}}\times F_v)$.

$\mathbb{M}_{R\times L}(\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_{R\times L}[2k]}}\{j\})$ is the subbisemialgebra of von Neumann on the **shifted** extended internal left bilinear Hilbert subsemispace $\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_{R\times L}[2k]}}\{j\}$.

The morphism

$$i_{\{j\}^D_{R\times L}}:\quad\left[\mathbb{M}_{R\times L}(H^{+}_{\widehat{M}^{(2n)}_{v_{R\times L}[2k]}}\{j\})\right]_j\longrightarrow\left[\mathbb{M}_{R\times L}(\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{j\})\right]_j$$

sends each von Neumann subbisemialgebra

$$\mathbb{M}_{R\times L}(H^{+}_{\widehat{M}^{(2n)}_{v_{R\times L}[2k]}})\{j\}=\bigoplus_{\nu=1}^{j}\mathbb{M}_{R\times L}(H^{+}_{\widehat{M}^{(2n)}_{v_{R\times L}[2k]}}(\nu))$$

on the shifted extended internal left bilinear Hilbert subsemispace

$$H^{+}_{\widehat{M}^{(2n)}_{v_{R\times L}[2k]}}\{j\}=\bigoplus_{\nu=1}^{j}H^{+}_{\widehat{M}^{(2n)}_{v_{R\times L}[2k]}}(\nu)$$

onto the corresponding von Neumann diagonal subbisemialgebra

$$\mathbb{M}_{R\times L}(\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{j\})=\bigoplus_{\nu=1}^{j}\mathbb{M}_{R\times L}(\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{j\})$$

on the diagonal internal left bilinear Hilbert subsemispace:

$$\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{j\}=\bigoplus_{\nu=1}^{j}\mathcal{H}^{+}_{\widehat{M}^{(2n)}_{v_R[2k]}\otimes_D\widehat{M}^{(2n)}_{v_L[2k]}}\{\nu\}$$

shifted in $(2k)$ bilinear geometric dimensions. ∎

2.12 Cuspidal representations of the global program of Langlands

The differential bioperator $(D^{2k}_R\otimes D^{2k}_L)$ maps the bisemisheaf $(\widehat M^{(2n)}_{v_R}\otimes\widehat M^{(2n)}_{v_L})$ into the corresponding perverse bisemisheaf $(\widehat M^{(2n)}_{v_R}[2k]\otimes\widehat M^{(2n)}_{v_L}[2k])$ according to:

$$(D^{2k}_R\otimes D^{2k}_L):\qquad \widehat M^{(2n)}_{v_R}\otimes\widehat M^{(2n)}_{v_L}\quad\longrightarrow\quad \widehat M^{(2n)}_{v_R}[2k]\otimes\widehat M^{(2n)}_{v_L}[2k]$$

in such a way that $\widehat M^{(2n)}_{v_R}[2k]\otimes\widehat M^{(2n)}_{v_L}[2k]$ decomposes into a tower of fibered or shifted bisections or bifunctions $\{\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{\overline v_{j,m_j}}))\times\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v_{j,m_j}}))\}$ (see section 2.10).

On the other hand, referring to the global program of Langlands [Pie2], **there is a one-to-one correspondence between the bisemisheaf $\boldsymbol{\widehat M^{(2n)}_{v_R}\otimes\widehat M^{(2n)}_{v_L}}$ over the abstract bisemivariety $\boldsymbol{G^{(2n)}(F_{\overline v}\times F_v)}$ and its cuspidal counterpart** $(\widehat M^{(2n)}_{v^T_R}\otimes\widehat M^{(2n)}_{v^T_L})$ on the toroidal abstract bisemivariety $G^{(2n)}(F^T_{\overline v}\times F^T_v)$ over the sets $F^T_v=\{F^T_{v_1},\dots,F^T_{v_{j,m_j}},\dots,F^T_{v,m_r}\}$ and $F^T_{\overline v}=\{F^T_{\overline v_1},\dots,F^T_{\overline v_{j,m_j}},\dots,F^T_{\overline v_{r,m_r}}\}$ of toroidal real archimedean completions or transcendental extensions.
The toroidal bisemisheaf has for bisections the bifunctions

$$\phi_{j,m_j}(G^{(2n)}(F^T_{\overline v_{j,m_j}}))\times\phi_{j,m_j}(G^{(2n)}(F^T_{v_{j,m_j}}))=\big(\lambda(2n,j,m_j)\;e^{-2\pi ijx}\big)\times\big(\lambda(2n,j,m_j)\;e^{+2\pi ijx}\big)$$

where:

- $\vec x=\sum\limits_{c=1}^{2n}x_c\;\vec e_c\,,\ x\in I\!R^{2n}\,;$
- $\lambda^2(2n,j,m_j)=\prod\limits_{c=1}^{2n}\lambda^2_c(2n,j,m_j)$ is a product of eigenbivalues $\lambda^2_c(2n,j,m_j)$ of the Hecke bioperator $(T_R(2n;r)\otimes T_L(2n;r))$ whose representation is $\mathrm{GL}_{2n}(\mathcal O_{F^T_{\overline v}}\times\mathcal O_{F^T_v})$ referring to section 1.5.

Let $\Gamma_{\widehat M^{(2n)}_{v^T_{R\times L}}}=\{\phi_{j,m_j}(G^{(2n)}(F^T_{\overline v_{j,m_j}}))\times\phi_{j,m_j}(G^{(2n)}(F^T_{v_{j,m_j}}))\}_{j,m_j}$ denote the set of increasing bisections of the bisemisheaf $\widehat M^{(2n)}_{v^T_R}\otimes\widehat M^{(2n)}_{v^T_L}$.

Then, **a global elliptic $\boldsymbol{(\Gamma_{\widehat M^{(2n)}_{v^T_{R\times L}}})}$-bisemimodule** $\phi^{(2n)}_{R\times L}(x)$, referring to the bihomomorphism

$$\phi^{(2n)}_{R\times L}(x):\qquad \Gamma_{\widehat M^{(2n)}_{v^T_{R\times L}}}\quad\longrightarrow\quad \mathrm{End}(\Gamma_{\widehat M^{(2n)}_{v^T_{R\times L}}})\,,$$

is given by:

$$\phi^{(2n)}_{R\times L}(x)=\sum_j\sum_{m_j}\big(\lambda(2n,j,m_j)\;e^{-2\pi ijx}\big)\times\sum_j\sum_{m_j}\big(\lambda(2n,j,m_j)\;e^{+2\pi ijx}\big)$$

in such a way that $\phi^{(2n)}_{R\times L}(x)$ **constitutes a real cuspidal representation of bilinear geometric dimension $\boldsymbol{2n}$, of the product, right by left, of Weil global semigroups $\mathbf{Gal}(\widetilde{\boldsymbol{F}}_{\overline{\boldsymbol{v}}}/\boldsymbol{k})\times\mathbf{Gal}(\widetilde{\boldsymbol{F}}_{\boldsymbol{v}}/\boldsymbol{k})$** according to the global program of Langlands.
Remark that $\phi^{(2n)}_{R\times L}(x)$ covers the corresponding "complex" cuspidal representation [Pie2].

2.13 Proposition

The global elliptic $(\boldsymbol{\Gamma}_{\widehat{\boldsymbol{M}}^{(2n)}_{\boldsymbol{v}^T_{R\times L}}})$-bisemimodule $\phi^{(2n)}_{R\times L}(x)$ *is the functional representation space* $\mathrm{FREPSP}(\mathrm{GL}_{2n}(F^T_{\overline{v}}\times F^T_v))$ *of the bilinear semigroup* $\mathrm{GL}_{2n}(F^T_{\overline{v}}\times F^T_v)$ *, over the product, right by left, of toroidal real archimedean completions* $F^T_{\overline{v}}$ *and* F^T_v *, under the (bi)action of the (bi)monomorphisms:*

$$\sigma_{v^T_R}\times\sigma_{v^T_L}:\qquad \mathrm{Aut}_k(F^T_{\overline{v}})\times\mathrm{Aut}_k(F^T_v)\quad\longrightarrow\quad G^{(2n)}(F^T_{\overline{v}}\times F^T_v)\ ,$$

where $\mathrm{Aut}_k(F^T_{\overline{v}}\times\mathrm{Aut}_k(F^T_v))$ *is the bilinear semigroup of automorphisms of toroidal transcendental extensions associated with a* $1D$ *-bisemilattice of transcendental biquanta.*
Proof: This results from the bimonomorphism

$$\sigma_{v_R}\times\sigma_{v_L}:\qquad \mathrm{Aut}_k(F_{\overline{v}})\times\mathrm{Aut}_k(F_v)\quad\longrightarrow\quad G^{(2n)}(F_{\overline{v}}\times F_v)$$

introduced in section 1.5 and generating the abstract bisemivariety $G^{(2n)}(F_{\overline{v}}\times F_v)$ from $\mathrm{GL}_{2n}(F_{\overline{v}}\times F_v)$ as well as from the definition of the global elliptic $\Gamma_{\widehat{M}^{(2n)}_{v^T_{R\times L}}}$ -bisemimodule $\phi^{(2n)}_{R\times L}(x)$ given in section 2.12. ■

2.14 Proposition

The shifted global elliptic bisemimodule $\mathrm{ELLIP}_{R\times L}(2n[2k],r)$ *resulting from the action*

$$(D^{2k}_R\otimes D^{2k}_L):\qquad \mathrm{ELLIP}_{R\times L}(2n,r)\quad\longrightarrow\quad \mathrm{ELLIP}_{R\times L}(2n[2k],r)$$

of the bioperator $(D^{2k}_R\otimes D^{2k}_L)$ *on the global elliptic* $\Gamma_{\widehat{M}^{(2n)}_{v^T_{R\times L}}}$ *-bisemimodule* $\phi^{(2n)}_{R\times L}(x)$ *, noted here* $\mathrm{ELLIP}_{R\times L}(2n,r)$ *, and generated under the (bi)monomorphism:*

$$\begin{aligned}\sigma_{v^T_R\otimes I\!R}\times\sigma_{v^T_L\otimes I\!R}:\qquad &\mathrm{Aut}_k(F^T_{\overline{v}}\times I\!R)\times\mathrm{Aut}_k(F^T_v\times I\!R)\\ &\qquad\longrightarrow\quad G^{(2n)}((F^T_{\overline{v}}\times I\!R)\times(F^T_v\times I\!R))\end{aligned}$$

gives rise to **the eigenbivalue equation:**

$$(\boldsymbol{D^{2k}_R}\otimes\boldsymbol{D^{2k}_L})(\mathbf{ELLIP}_{\boldsymbol{R\times L}}(\boldsymbol{2n,r}))=\boldsymbol{E_{2k_{R\times L}}(2n,j)}(\mathbf{ELLIP}_{\boldsymbol{R\times L}}(\boldsymbol{2n,r}))\ ,\ \boldsymbol{1\le j\le r}\ ,$$

where the eigenbivalues $\boldsymbol{E_{2k_{R\times L}}(2n,j)}$ are shifts in $\boldsymbol{2k}$ real dimensions of the global Hecke bicharacters $\boldsymbol{\lambda(2n,j,m_j)}$ associated with the subbisemilattices *characterized by the global residue degrees j .*

Proof: Similarly as in proposition 2.13, the (bi)map:

$$\sigma_{v_R^T\otimes I\!R}\times\sigma_{v_L^T\otimes I\!R} : \qquad \mathrm{Aut}_k(F^T_{\overline{v}}\times I\!R)\times\mathrm{Aut}_k(F^T_v\times I\!R) \quad \longrightarrow \quad G^{(2n)}((F^T_{\overline{v}}\times I\!R)\times(F^T_v\times I\!R))\ ,$$

where $\mathrm{Aut}_k(F^T_{\overline{v}}\otimes I\!R)\times\mathrm{Aut}_k(F^T_v\otimes I\!R)$ is the bilinear semigroup of automorphisms of fibered or shifted toroidal transcendental extensions, is responsible for the generation of the toroidal abstract fibered or shifted bisemivariety $G^{(2n)}((F^T_{\overline{v}}\times I\!R)\times(F^T_v\times I\!R))$, referring to section 2.1, of which functional representation space is the shifted global elliptic bisemimodule $\mathrm{ELLIP}_{R\times L}(2n[2k],r)$ obtained from the global elliptic bisemimodule $\mathrm{ELLIP}_{R\times L}(2n,r)$ under the action of the bioperator $(D^{2k}_R\otimes D^{2k}_L)$ giving rise to the eigenbivalue equation:

$$(D^{2k}_R\otimes D^{2k}_L)(\mathrm{ELLIP}_{R\times L}(2n,r))=E^{2k}_{R\times L}(2n,j)(\mathrm{ELLIP}_{R\times L}(2n,r))\ ,\qquad \forall\ j\ ,\quad 1\le j\le r\ .$$

The functional representation space $\mathrm{FREPSP}(\mathrm{GL}_{2n}(F^T_{\overline{v}}\times I\!R)\times(F^T_v\times I\!R))$ of the bilinear semigroup of matrices $\mathrm{GL}_{2n}((F^T_{\overline{v}}\times I\!R)\times(F^T_v\times I\!R))$ responsible for the generation of the abstract bisemivariety $G^{2n}((F^T_{\overline{v}}\times I\!R)\times(F^T_v\times I\!R))$ is given by the set of embedded bisemifunctions:

$$\begin{aligned}\phi_1(G^{(2n)}(F^T_{\overline{v}_1}\times I\!R))\times\phi_1(G^{(2n)}(F^T_{v_1}\times I\!R))&\\ \subset\cdots\subset\phi_{j,m_j}(G^{(2n)}(F^T_{\overline{v}_{j,m_j}}\times I\!R))\times\phi_{j,m_j}(G^{(2n)}(F^T_{v_{j,m_j}}\times I\!R))&\\ \subset\cdots\subset\phi_{r,m_r}(G^{(2n)}(F^T_{\overline{v}_{r,m_r}}\times I\!R))\times\phi_{r,m_r}(G^{(2n)}(F^T_{v_{r,m_r}}\times I\!R))&\end{aligned}$$

introduced in section 5.12. of [Pie3].
Each bisemifunction $\phi_{j,m_j}(G^{(2n)}(F^T_{\overline{v}_{j,m_j}}\times I\!R))\times\phi_{j,m_j}(G^{(2n)}(F^T_{v_{j,m_j}}\times I\!R))$ is the product, right by left, $T^{(2n)}_R([2k],(j,m_j))\times T^{(2n)}_L([2k],(j,m_j))$ of a $2n$-dimensional real semitorus $T^{(2n)}_R([2k],(j,m_j))$ shifted in $2k$ real dimensions and localized in the lower half space by its symmetric equivalent $T^{(2n)}_L([2k],(j,m_j))$ localized in the upper half space.
They have for analytic development:

$$T^{(2n)}_L([2k],(j,m_j))\simeq E_{2k_L}(2n,j,m_j))\ \lambda(2n,j,m_j)\ e^{2\pi ijx}$$

$$(\text{resp.}\quad T^{(2n)}_R([2k],(j,m_j))\simeq E_{2k_R}(2n,j,m_j))\ \lambda(2n,j,m_j)\ e^{-2\pi ijx}\ )$$

referring to section 2.12, where $E_{2k}(2n,j,m_j)$ is the shift in $2k$ real dimensions of the global Hecke character $\lambda(2n,j,m_j)$ being also a product of eigenvalues of the Hecke operator as described in section 2.12.

On the other hand, referring to proposition 2.11, the toroidal spectral representation of $(D_R^{2k} \otimes D_L^{2k})$ is given by the set of r-tuples:

$$\mathrm{ellip}_{R\times L}(2n,1) \subset \cdots \subset \mathrm{ellip}_{R\times L}(2n,j) \subset \cdots \subset \mathrm{ellip}_{R\times L}(2n,r)$$

where $\mathrm{ellip}_{R\times L}(2n,j)$ is given by

$$\mathrm{ellip}_{R\times L}(2n,j) = (\lambda(2n,j,m_j)\ e^{-2\pi ijx}) \times (\lambda(2n,j,m_j)\ e^{+2\pi ijx})$$

and to which corresponds the set of increasing eigenbivalues

$$E_{2k_{R\times L}}(2n,1) \subset \cdots \subset E_{2k_{R\times L}}(2n,j) \subset \cdots \subset E_{2k_{R\times L}}(2n,r)$$

where $E_{2k_{R\times L}}(2n,j)$ is the shift in $2k$ real dimensions of the Hecke bicharacter $\lambda^2(2k,j,m_j) \subset \lambda^2(2n,j,m_j)$ taking into account that

$$\lambda^2(2n,j,m_j) = \prod_{c=1}^{2k} \lambda_c^2(2k,j,m_j) \times \prod_{d=2n-2k}^{2n} \lambda_d^2(2n,j,m_j)$$

with $$\lambda^2(2k,j,m_j) = \prod_{c=1}^{2k} \lambda_c^2(2k,j,m_j)$$

and with $$\lambda^2(2n-2k,j,m_j) = \prod_{d=2n-2k}^{2n} \lambda_d^2(2n,j,m_j)$$ ■

3 Large random matrices and Riemann zeta function

3.1 Five questions to find a solution to this problem

Chapters 1 and 2 have been devoted to the mathematical tools necessary to clarify the conceptual framework behind the random matrices and, more particularly, the closed numerical connection between the spacings of the nontrivial zeros of the Riemann zeta function and the spacing of the eigenvalues of typical large random matrices.
It will be shown in this chapter that **the symmetry behind the Gaussian unitary ensemble (GUE) is the symmetric (bisemi)group of "Galois" automorphisms fibered or "shifted" algebraic and transcendental (bi)quanta.**
The constant reference to the global program of Langlands in chapter 1 **and to the geometric-shifted global program of Langlands, as well as to von Neumann (bisemi)algebras**, in chapter 2, **is thus not fortuitous**.
In order to find a solution to this problem, it will be answered in this chapter to **the five following questions**:

1) **What is behind random matrices leading to GOE (Gaussian orthogonal ensemble), as well as GUE (Gaussian unitary ensemble)?**

2) **What is behind the partition and correlation function(s) between eigenvalues of random matrices?**

3) **What interpretation can we give to the local spacings between the eigenvalues of large random matrices?**

4) **What interpretation can we give to the spacings between the nontrivial zeros of $\zeta(s)$?**

5) **What is the curious connection between 3) and 4)?**

But, first, we would like to outline that **the geometric dimension** envisaged in this chapter **will be one real, and, possibly, one complex**, i.e. that $n = 1$, in the sense of the reducible global program of Langlands developed in [Pie2]. Thus, only curves, covering possibly surfaces, will be considered in order to meet the conditions of question 4).

3.2 The first question "What is behind random matrices leading to GOE and GUE?"

This question reflects the importance of large random matrices in the present development of mathematics and of physics.

These random matrices lead to the eigenvalue problem in the frame of the geometric shifted global program of Langlands, recalled in section 1.11 and in chapter 2, in order to find a response to question 3).

3.3 Bilinear differential Galois semigroup

The symmetry group behind or at the origin of the bilinear global program of Langlands is **the bilinear semigroup of automorphisms $\mathrm{Aut}_k(F_{\overline{v}})\times\mathrm{Aut}_k(F_v)$ (resp. Galois automorphisms $\mathrm{Gal}(F_{\overline{v}}/k)\times\mathrm{Gal}(F_v/k)$) of compact transcendental (resp. algebraic) quanta generating a bisemilattice of compact transcendental (resp. algebraic) quanta** referring to section 1.4.
Similarly, the symmetry group at the origin of the geometric bilinear global program of Langlands is the bilinear semigroup of fibered or shifted automorphisms $\mathrm{Aut}_k(F_{\overline{v}}\times I\!R)\times\mathrm{Aut}_k(F_v\times I\!R)$ of compact transcendental quanta generating a bisemilattice of compact shifted transcendental quanta according to proposition 2.4.
Referring to this same proposition, the one-dimensional shifted functional representation space $\mathrm{FREPSP}(\mathrm{GL}_1(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R))$ of $\mathrm{GL}_1(F_{\overline{v}}\times I\!R)\times(F_v\times I\!R)$ is given by the shifted bisemisheaf $\widehat{M}^{(1)}_{v_R}[1]\otimes\widehat{M}^{(1)}_{v_L}[1])$ whose set $\Gamma(\widehat{M}^{(1)}_{v_R}[1]\otimes\widehat{M}^{(1)}_{v_L}[1])$ of bisections is the set $\{\phi G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times I\!R)\times(F_{v^1}\times I\!R))\}_{j,m_j}$ of fibered or shifted differentiable bifunctions obtained from the set $\{\phi(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))\}_{j,m_j}$ of differentiable bifunctions under the action of the elliptic bioperator $D_R\otimes D_L$, $2k=1$.
The set

$$\{\phi(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times I\!R)\times(F_{v^1}\times I\!R))\}_{j,m_j}=\{\phi_R(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times I\!R))\otimes\phi_L(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times I\!R))\}_{j,m_j}$$

of differentiable bifunctions, being equivalent to the set $\{\phi(G^{(1)}(F_{\overline{v}_{j,m_j}}\times I\!R)\times(F_{v_{j,m_j}}\times I\!R))\}_{j,m_j}$ of differentiable bifunctions, is isomorphic to the set $\{\phi(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))\times\phi(G^{(j,m_j)}_{(Q)}(I\!R\times I\!R))\}_{j,m_j}$ of bifunctions in such a way that $\boldsymbol{\{\phi(G^{(j,m_j)}_{(Q)}(I\!R\times I\!R))\}^{r}_{j,m_j=1}}$ **is the fibering or shifting functional representation bisemispace obtained under(and being isomorphic to) the biaction of the bilinear differential Galois semigroup**

$$\begin{aligned}\mathrm{Aut}_k(\phi_R(I\!R))\times\mathrm{Aut}_k(\phi_L(I\!R))&\simeq\{\mathrm{Aut}_k(\phi_R(I\!R))_{|F_{\overline{v}_{j,m_j}}}\times\mathrm{Aut}_k(\phi_L(I\!R))_{|F_{v_{j,m_j}}}\}^{r}_{j=1}\\&\simeq\mathrm{GL}_r(\phi_R(I\!R)\otimes\phi_L(I\!R))\\&\simeq\mathrm{GL}_r(I\!R\times I\!R)\end{aligned}$$

according to proposition 2.4.

Thus, the bilinear semigroup $\mathrm{GL}_r(I\!R\times I\!R)$ of matrices constitutes the representation of the bilinear differential Galois semigroup associated with the action of the differential bioperator $(D_R\otimes D_L)$ on the bisemisheaf $(\widehat{M}^{(1)}_{v_R}\otimes\widehat{M}^{(1)}_{v_L})$ whose bisections are the set $\{\phi(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))\}^r_{j=1}=\{\phi(G^{(1)}(F_{\overline{v}_{j,m_j}}\times F_{v_{j,m_j}}))\}_{j,m_j}$ of differentiable bifunctions.
Let us set

$$\phi_j(G_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))=\sum_{j=1}^{j}\phi(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))$$

used in the following proposition.

3.4 Proposition

If

$$\phi_r(G_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))=\sum_{j=1}^{r}\phi(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))\ ,$$

let

$$(D_R\otimes D_L)(\phi_r(G_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))=E_{R\times L}(j)\ \phi_R(G_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))\ ,\qquad 1\le j\le r\ ,$$

be the eigenbivalue equation related to the bisemisheaf $(\widehat{M}^{(1)}_{v_R}\otimes\widehat{M}^{(1)}_{v_L})$ *and associated with the tower of shifted differentiable bifunctions* $\{\phi(G^{(j,m_j)}_{(Q)}((F_{\overline{v}^1}\times I\!R)\times(F_{v^1}\times I\!R)))\}^r_{j=1}$.
Then, **the eigenbivalues of the matrix of $\mathbf{GL}_r(I\!R\times I\!R)$, constituting a representation of the bilinear differential Galois semigroup associated with the biaction of $(D_R\otimes D_L)$, are the eigenbivalues $\{E_{R\times L}(j)\}^r_{j=1}$ of the above eigenbivalue equation**.

Proof:

1) $\mathrm{GL}_r(I\!R\times I\!R)$, being:

   1) the bilinear fibre $\mathcal{F}_{R\times L}(\mathrm{TAN})$ of the tangent bibundle $\mathrm{TAN}(\widetilde{M}^{(1)}_{v_R}\otimes\widetilde{M}^{(1)}_{v_L})\simeq$ Ad(F) REPSP($\mathrm{GL}_1(F_{\overline{v}}\times F_v)$ according to section 2.2,

   2) a representation of the bilinear differential Galois semigroup associated with the biaction of $(D_R\otimes D_L)$,

   constitutes a representation of the bioperator $(D_R\otimes D_L)$ because it generates endomorphisms of $\mathrm{TAN}(\widetilde{M}^{(1)}_{v_R}\otimes\widetilde{M}^{(1)}_{v_L})$.

2) Referring to propositions 2.11 and 2.14, we see that **the set of differentiable bifunctions** $\{\phi(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))\}_j$ constituting the r-bituple

$$\langle\phi(G^{(1)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1})),\dots,\phi(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1})),\dots,\phi(G^{(r,m_r)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))\rangle$$

is the spectral representation of $(\boldsymbol{D_R} \otimes \boldsymbol{D_L})$ (and the basis) **associated with the** $\boldsymbol{r}$**-tuple of eigenbivalues**:

$$\langle E_{R\times L}(1), \dots, E_{R\times L}(j), \dots, E_{R\times L}(r) \rangle \ . \qquad \blacksquare$$

3.5 Corollary

As the j*-th eigenbifunction* $\phi(G^{(j,m_j)}_{(Q)}(F_{\overline{v}^1} \times F_{v^1}))$ *is a bifunction on "*j*" transcendental compact biquanta, the corresponding eigenbivalue* $E_{R\times L}(j)$ *will be the shift of this bifunction corresponding to the biaction of the bioperator* $(D_R \otimes D_L)$*.*

Proof: We refer to section 2.10 and proposition 2.11 concerning;

1) the bisemialgebra of von Neumann $\mathbb{M}_{R\times L}(H^+_{\widehat{M}^{(1)}_{v_{R\times L}}})$ on the tower of embedded bilinear Hilbert semispaces associated with the bisemisheaf $\widehat{M}^{(1)}_{v_{R\times L}} \equiv \widehat{M}^{(1)}_{v_R} \otimes \widehat{M}^{(1)}_{v_L}$ and

2) the discrete spectrum $\Sigma(D_R \otimes D_L)$ of the differential bioperator $(D_R \otimes D_L)$. $\blacksquare$

3.6 Bilinear Gaussian orthogonal (resp. unitary) ensemble BGOE (resp. BGUE)

The **Gaussian orthogonal (resp. unitary) ensemble GOE (resp. GUE) is defined in the space of real symmetric (resp. hermitian) matrices by two requirements** [**Meh**]:

a) the ensemble is invariant under every transformation:

$$H \longrightarrow W^T \ H \ W \qquad\qquad (\text{resp.} \quad H \longrightarrow U^{-1} \ H \ U \ )$$

where:

- W (resp. U) is any real orthogonal (resp. unitary) matrix;
- H is a real symmetric (resp. hermitian) matrix, generally related to the hamiltonian matrix invariant (resp. not invariant) under time reversal.

b) the various elements H_{ij} are statistically independent.

In its simplified bilinear version, the hamiltonian H corresponds to the differential (random) bioperator $(D_R\otimes D_L)$ acting on the bisemisheaf $\widehat M^{(1)}_{v_{R\times L}}$ and belonging to the bisemialgebra of von Neumann $\mathbb{M}_{R\times L}(H^+_{\widehat M^{(1)}_{v_{R\times L}}})$ according to sections 2.8 to 2.11 and [Pie1].
Let then:

$$D_R\otimes D_L : \qquad \widehat M^{(1)}_{v_R}\otimes \widehat M^{(1)}_{v_L} \quad\longrightarrow\quad \widehat M^{(1)}_{v_R}[1]\otimes \widehat M^{(1)}_{v_L}[1]$$

be the biaction of $(D_R\otimes D_L)$ on the bisemisheaf $(\widehat M^{(1)}_{v_R}\otimes \widehat M^{(1)}_{v_L})$ generating the perverse bisemisheaf $(\widehat M^{(1)}_{v_R}[1]\otimes \widehat M^{(1)}_{v_L}[1])$ whose bisections $\{\phi_{j,m_j}(G^{(1)}_{[1]}(F_{\overline v_{j,m_j}}\times I\!\!R)\times(F_{v_{j,m_j}}\times I\!\!R))\}_{j,m_j}$ are differentiable bifunctions on the (j,m_j)-th conjugacy class representatives of the shifted or fibered bilinear semigroup $G^{(1)}((F_{\overline v}\times I\!\!R)\times(F_v\times I\!\!R))$.
Referring to [Pie1], **the perverse bisemisheaf $(\widehat M^{(1)}_{v_R}[1]\otimes \widehat M^{(1)}_{v_L}[1])$ is the operator-valued stringfield of an elementary (bisemi)particle**.
According to section 3.3, **the bilinear semigroup of matrices $\mathrm{GL}_r(I\!\!R\times I\!\!R)$ constitutes the representation of the bilinear differential Galois semigroup associated with the biaction of $(D_R\otimes D_L)$ on $(\widehat M^{(1)}_{v_R}\otimes \widehat M^{(1)}_{v_L})$**.
And, thus, **$\mathrm{GL}_r(I\!\!R\times I\!\!R)$ (and $O_r(I\!\!R\times I\!\!R)$), or its complex equivalent $\mathrm{GL}_r(\mathbb{C}\times\mathbb{C})$ (and $U_r(I\!\!R\times I\!\!R)$), is the new bilinear Gaussian real (orthogonal) (resp. complex ("unitary")) ensemble labeled BGOE (resp. BGUE) corresponding to GOE (resp. GUE)**.

3.7 Mixed higher bilinear KK-theory

To be complete, the deformations of random matrices have to be envisaged in the light of **the new interpretation of homotopy-cohomotopy** [Pie5] **viewed as deformations of Galois representations** in the context of mixed higher bilinear KK-theory related to the Langlands dynamical bilinear global program [Pie5].

Referring to proposition 2.4, we see that the $1D$-geometric infinite quantum general bilinear semigroup $\mathrm{GL}^{(Q)}(F_{\overline v^1}\times F_{v^1})$ is defined by:

$$\mathrm{GL}^{(Q)}(F_{\overline v^1}\times F_{v^1}) = \lim_{j=1\to r}\mathrm{GL}^{(Q)}_j(F_{\overline v^1}\times F_{v^1})$$

in such a way that $\mathrm{GL}^{(Q)}_1(F_{\overline v^1}\times F_{v^1})$ is the parabolic, i.e. unitary bilinear semigroup $P_1(F_{\overline v^1}\times F_{v^1})$, and that its shifted equivalent $\mathrm{GL}^{(Q)}((F_{\overline v^1}\times I\!\!R)\times(F_{v^1}\times I\!\!R))$ is given by

$$\mathrm{GL}^{(Q)}((F_{\overline v^1}\times I\!\!R)\times(F_{v^1}\times I\!\!R)) = \lim_{j=1\to r}\mathrm{GL}^{(Q)}_j((F_{\overline v^1}\times I\!\!R)\times(F_{v^1}\times I\!\!R))$$

leading to a filtration

$$G^{(1)}_{(Q)}((F_{\overline v^1}\times I\!\!R)\times(F_{v^1}\times I\!\!R))\subset\cdots\subset G^{(r,m_r)}_{(Q)}((F_{\overline v^1}\times I\!\!R)\times(F_{v^1}\times I\!\!R))$$

of its representatives.

So, **the bilinear version of the algebraic $\boldsymbol{K}$-theory restricted to $\boldsymbol{1D}$-geometric dimension is given by**:

$$K^1(G^{(1)}(F_{\overline{v}^1}\times F_{v^1}))=\Pi_1(\mathrm{BGL}^{(Q)}(F_{\overline{v}^1}\times F_{v^1})^+)$$

where the quantum classifying bisemispace $\mathrm{BGL}^{(Q)}(F_{\overline{v}^1}\times F_{v^1})$ is the base bisemispace of all equivalence classes of deformations of the Galois representations of $\mathrm{GL}^{(Q)}(\widetilde F_{\overline{v}^1}\times \widetilde F_{v^1})$ given by the kernels $\mathrm{GL}^{(Q)}(\delta F_{\overline{v^1+\ell}}\times \delta F_{v^1+\ell})$ of the maps:

$$\mathrm{GD}_\ell^{(Q)}:\qquad \mathrm{EGL}^{(Q)}(F_{\overline{v^1+\ell}}\times F_{v^1+\ell})\quad\longrightarrow\quad \mathrm{BGL}^{(Q)}(F_{\overline{v}^1}\times F_{v^1})$$

where $\mathrm{GD}_\ell^{(Q)}$ is a universal principal $\mathrm{GL}^{(Q)}(F_{\overline{v}^1}\times F_{v^1})$-bibundle.
Referring to chapter 3 of [Pie5], it is easy to see that

$$\mathrm{BGL}^{(Q)}(F_{\overline{v}^1}\times F_{v^1})\equiv \mathrm{GL}_1(F_{\overline{v}}\times F_v)$$

and that the maps $\mathrm{GD}_\ell^{(Q)}$ become

$$\mathrm{GD}_\ell^{(1)}:\qquad \mathrm{GL}_1(F_{\overline{v+\ell}}\times F_{v+\ell})\quad\longrightarrow\quad \mathrm{GL}_1(F_{\overline{v}}\times F_v)$$

having the same interpretation as $\mathrm{GD}_\ell^{(Q)}$.

In order to recall the **bilinear version of the mixed higher $\boldsymbol{KK}$-theory of Quillen adapted to the Langlands dynamical global program in $\boldsymbol{1D}$-geometric dimension** [Pie5], we have to take into account:

a) the bisemisheaf $(\widehat M^{(1)}_{v_R}\otimes \widehat M^{(1)}_{v_L})$, noted here $\mathrm{FG}^{(1)}(F_{\overline{v}}\times F_v)$ and being the functional representation space of $\mathrm{GL}_1(F_{\overline{v}}\times F_v)$;

b) the "plus" classifying bisemisheaf $\mathrm{BFGL}^{(Q)}(I\!\!R\times I\!\!R)^+=\mathrm{BFGL}_1(I\!\!R\times I\!\!R)^+$, being the base bisemisheaf of all equivalence classes of one-dimensional inverse deformations of the Galois differential representation of $\mathrm{FGL}_1(I\!\!R\times I\!\!R)$ due to the action of the bioperator $(D_R\otimes D_L)$ on $\mathrm{FG}^{(1)}(F_{\overline{v}}\times F_v)$, and corresponding to one-dimensional deformations of the Galois representation of $\mathrm{GL}_1(F_{\overline{v}}\times F_v)$ given by the kernel $\{\mathrm{GL}_1(\delta F_{\overline{v+\ell}}\times \delta F_{v+\ell})\}_\ell$ of $\mathrm{GD}_\ell^{(1)}$.

The higher (algebraic) $\boldsymbol{KK}$-theory is then given by:

$$\begin{aligned}&\boldsymbol{K_1}(\mathbf{FG^{(1)}}(I\!\!R\times I\!\!R))\times \boldsymbol{K^1}(\mathbf{FG^{(1)}}(\boldsymbol{F_{\overline{v}}}\times\boldsymbol{F_v}))\\&\qquad\qquad=\boldsymbol{\Pi^1}(\mathbf{BFGL_1}(I\!\!R\times I\!\!R)^+)\times\boldsymbol{\Pi_1}(\mathbf{BFGL_1}(\boldsymbol{F_{\overline{v}}}\times\boldsymbol{F_v})^+)\end{aligned}$$

in such a way that the bilinear contracting K-theory $K_1(\mathrm{FG}^{(1)}(I\!R\times I\!R))$ responsible for a differentiable biaction acts on the K-theory $K^1(\mathrm{FG}^{(1)}(F_{\overline{v}}\times F_v))$ of the bisemisheaf $\mathrm{FG}^{(1)}(F_{\overline{v}}\times F_v)$ in one-to-one correspondence with the biaction of the cohomotopy bisemigroup $\Pi^1(\mathrm{BFGL}_1(I\!R\times I\!R)^+)$ of the "plus" classifying bisemisheaf $\mathrm{BFGL}_1(I\!R\times I\!R)^+)$.

3.8 Proposition

The deformation

$$\mathrm{GD}_\ell^{(1)}: \qquad \mathrm{GL}_1(F_{\overline{v+\ell}}\times F_{v+\ell}) \quad\longrightarrow\quad \mathrm{GL}_1(F_{\overline{v}}\times F_v)$$

induces the following deformation:

$$\mathrm{GL}_{r+\delta r_\ell\to r}: \qquad \mathrm{GL}_{r+\delta r_\ell}(I\!R\times I\!R) \quad\longrightarrow\quad \mathrm{GL}_r(I\!R\times I\!R)$$

on random matrices $\mathrm{GL}_r(I\!R\times I\!R)$.

Proof: Indeed, the matrix of $\mathrm{GL}_{r+\delta r_\ell}(I\!R\times I\!R)$ of order $(r+\ell)$ constitutes a representation of the deformed bioperator or $((D_R+\delta D_R)\times(D_L+\delta D_L))$ acting on the deformed bifunction $\phi_{r+\delta r_\ell}(G_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))$ and has for spectral representation the $(r+\delta r_\ell)$-bituple:

$$\langle \phi(G^{(1+\ell)}_{(Q)}(F_{\overline{v^1+\ell}}\times F_{v^1+\ell})),\dots,\phi(G^{(r+\ell,m_r+\ell)}_{(Q)}(F_{\overline{v^1+\ell}}\times F_{v^1+\ell}))\rangle$$

decomposing into the sum of the r-bituple and the δr_ℓ-bituple according to:

$$\begin{aligned}
\langle \phi(G^{(1+\ell)}_{(Q)}&(F_{\overline{v^1+\ell}}\times F_{v^1+\ell})),\dots,\phi(G^{(r+\ell,m_r+\ell)}_{(Q)}(F_{\overline{v^1+\ell}}\times F_{v^1+\ell}))\rangle\\
&= \langle \phi(G^{(1)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1})),\dots,\phi(G^{(r,m_r)}_{(Q)}(F_{\overline{v}^1}\times F_{v^1}))\rangle\\
&\qquad + \langle \phi(G^{(\ell)}_{(Q)}(F_{\overline{v}^1_\ell}\times F_{v^1_\ell})),\dots,\phi(G^{(\ell)}_{(Q)}(F_{\overline{v}^1_\ell}\times F_{v^1_\ell}))\rangle\,.
\end{aligned}$$

The deformation $\mathrm{GL}_{r+\delta r_\ell}(I\!R\times I\!R)$ of the random matrix of $\mathrm{GL}_r(I\!R\times I\!R)$, constituting a deformation of the Galois differential representation $\mathrm{GL}_r(I\!R\times I\!R)$, corresponds to the cohomotopy bisemigroup $\Pi^1(\mathrm{BFGL}_1(I\!R\times I\!R)^+)$ according to section 3.7. ∎

3.9 The second question

The second question "What is behind the partition and correlation functions between eigenvalues of random matrices?" concerns the distribution of eigenvalues of random matrices. It will be seen that this problem **is based on the (bisemi)group of "Galois" automorphisms of shifted transcendental compact (bi)quanta which leads to a reevaluation of the probabilistic interpretation in quantum theories.**

3.10 Distribution of eigenvalues of GUE ensembles

Wigner introduced the idea of **statistical mechanics of nuclei based on a Gaussian ensemble (GUE) having "r" quantum states and characterized by a Hamiltonian symmetric matrix of order "r" whose entries are Gaussian random variables and to which a Gaussian statistical weight is associated** [Wig].

Unsatisfied by the impossibility of defining a uniform probability distribution on an infinite range, **F. Dyson introduced the circular unitary (resp. orthogonal) ensemble CUE (resp. COE)** [Dys] in such a way that the Hamiltonian is now described by a unitary matrix of order "r" whose eigenvalues are complex numbers $\exp(i\theta_j)$, $1 \le j \le r$, distributed around the unit circle.

This circular unitary (resp. orthogonal) ensemble corresponds to the Riemann symmetric space $U(r)/O(r)$ which "lives" in $\mathrm{GL}_r(\mathbb{C})/U(r)$ [Mez].

Let then $M = (M_{ij})_{i,j=1}^r$ be a random hermitian matrix to which is assigned the probability distribution

$$d\mu_r{}^{\mathrm{GUE}}(M) = \frac{1}{Z_r^{\mathrm{GUE}}}\ e^{-r\,\mathrm{tr}\,M^2}\ dM$$

where:

- $\mathrm{tr}\,M^2 = \sum\limits_{i,j=1}^{r} M_{ji}M_{ij} = \sum\limits_{i=1}^{r} M_{ii}^2 + 2\sum\limits_{i>j}|M_{ij}|^2$;
- $\mu_r^{\mathrm{GUE}}(dM) = \frac{1}{Z_r^{\mathrm{GUE}}} \prod\limits_{i=1}^{r}(e^{-rM_{ii}^2}) \prod\limits_{i>1}(e^{-2r|M_{ij}|^2})\ dM$.

.

The distribution of eigenvalues of M with respect to the ensemble μ_r^{GUE} is [Meh], [A-VM], [B-Z]:

$$d\mu_r^{\mathrm{GUE}}(\lambda) = \frac{1}{Z_r^{\mathrm{GUE}}}\prod_{i>j}(\lambda_i-\lambda_j)^2\prod_{i=1}^{r}e^{-r\lambda_i^2}\ d\lambda$$

where the partition function [Rue] of GUE is given by:

$$\begin{aligned} Z_r^{\mathrm{GUE}} &= \int \prod_{i>j}(\lambda_i-\lambda_j)^2\prod_{i=1}^{r}e^{-r\lambda_i^2}\ d\lambda_i \\ &= \frac{(2r)^{r/2}}{(2r)^{r^2/2}}\prod_{i=1}^{r}\ i!\ . \end{aligned}$$

3.11 m-point correlation function for GUE and Jacobi matrix

The joint probability density function for the eigenvalues of matrices from a Gaussian orthogonal, unitary or symplectic ensemble is given by:

$$P_{r\beta}(x_1,\ldots,x_r) = c_{r\beta}\exp\left(-\frac{\beta}{2}\sum_{i=1}^{r} x_i^2\right)\prod_{i>j}(x_i-x_j)^\beta\ , \qquad -\infty < x_i < +\infty\ ,$$

where $\beta = 1, 2$ or 4 according as the ensemble is orthogonal, unitary or symplectic and $c_{r2} = \dfrac{1}{Z_r^{\text{GUE}}}$.

The m-point correlation function for the Gaussian unitary ensemble is defined by [Dys]:

$$R_{mr}(x_1,\ldots,x_m) = \frac{r!}{(r-m)!}\int_{\mathbb{R}^{r-m}} P_r(x_1,\ldots,x_r)\ dx_{m+1}\ \ldots\ dx_r$$

which is the probability density of finding a level around each of the points (i.e. entries of M) $x_1,\ldots,x_m$, the positions of the remaining levels being unobserved.
The Dyson determinantal formulas for correlation functions is [Meh], [Ble], [B-H]:

$$R_{mr}(x_1,\ldots,x_m) = \det(K_r(x_k,x_\ell))_{k,\ell=1}^m$$

where $K_r(x_k,x_\ell)$ is given by:

$$K_r(x_k,x_\ell) = \sum_{i=0}^{r-1}\psi_i(x_k)\ \psi_i(x_\ell) \qquad\qquad \text{with} \quad \psi_i(x) = h_i^{-\frac{1}{2}}P_i(x)\ e^{-rM^2(x)/2}$$

where $P_i(x)$ is an orthogonal polynomial or degree i corresponding to the weight function $e^{-rM^2(x)/2}$ and verifying:

$$\int_{-\infty}^{+\infty} P_i(x)\ P_j(x)\ e^{-rM^2(x)}\ dx \simeq \delta_{ij}\ h_i\ .$$

These orthogonal polynomials, being sometimes Hermite polynomials, **satisfy the three term recurrent relation**:

$$\beta_{i+1}\ P_{i+1}(x) = (x-\alpha_i)\ P_i(x) - \beta_i\ P_{i-1}(x)$$

or

$$x\ P_i = \beta_i\ P_{i-1} + \alpha_i\ P_i + \beta_{i+1}\ P_{i+1}\ , \qquad \beta_i = \frac{h_i}{h_i-1}\ , \quad \alpha_i = (x\ P_i, P_i)\ .$$

If we set $P_{-1}(x)=0$, we get the tower:

$$\begin{aligned}
x\ P_0 &= \alpha_0\ P_0 + \beta_1\ P_1\ ,\\
x\ P_1 &= \beta_1\ P_0 + \alpha_1\ P_1 + \beta_2\ P_2\ ,\\
x\ P_2 &= \quad 0 \quad + \beta_2\ P_1 + \alpha_2\ P_2 + \beta_3\ P_3 + 0\ ,\\
&\vdots\\
x\ P_{i-1} &= \qquad 0 \qquad + \beta_{i-1}\ P_{i-2} + \alpha_{i-1}\ P_{i-1} + \beta_i\ P_i\ ,
\end{aligned}$$

which, in matricial form, is:

$$x\ \mathcal{P} = J\ \mathcal{P} + \beta_i\ P_i\ .$$

The matrix J is symmetric and is the Jacobi matrix such that the i roots of $P_i(x)$ verifying

$$P_i(x_j)=0\ , \qquad 1\leq j\leq i\ ,$$

lead to

$$x_j[\mathcal{P}(x_j)] = J\ [P(x_j)]\ .$$

The $\boldsymbol{i}$ roots of $\boldsymbol{P_i(x)}$ are then the eigenvalues of the Jacobi matrix $\boldsymbol{J}$.

3.12 Joint probability density function for the eigenvalues of matrices from BGUE and BGOE

In the bilinear case, the random matrix corresponding to M is

$$G = TG^T\times TG\in \mathrm{GL}_r(I\!\!R\times I\!\!R) \qquad (\text{or } \in \mathrm{GL}_r(I\!\!\!C\times I\!\!\!C)\ )$$

where $TG\in T_r(I\!\!R)$ (resp. $TG^T\in T_r^t(I\!\!R)$) is an upper (resp. lower) triangular matrix of order r with entries in $I\!\!R$ (or $I\!\!\!C$).
The BGUE (or BGOE) probability distribution corresponding to G is:

$$d_{\mu_r}^{\mathrm{BGUE}}(G) = \frac{1}{Z_r^{\mathrm{BGUE}}}\ e^{-rTr(TG^T\times TG)}\ dG$$

leading to

$$\mu_r^{\mathrm{BGUE}}(G) = \frac{1}{Z_r^{\mathrm{BGUE}}}\ \prod_{i=1}^{r} e^{-rG_{ii}^2} \prod_{i>j} e^{-2r|G_{ij}|^2}\ .$$

The joint probability density function for the eigenvalues of matrices from a

Gaussian bilinear orthogonal or unitary ensemble is thus:

$$P_{r_{R\times L}}(x_1,\dots,x_r)=c_r\exp\left(-\sum_{i=1}^{r} r x_i^2\right)\prod_{i>j}(x_i-x_j)^2$$

$$=c_r\prod_{i=1}^{r} e^{-rx_i^2}\prod_{i>j}^{r}(x_i-x_j)^2$$

which corresponds to the distribution of eigenvalues $d_{\mu_r}^{\mathrm{GUE}}(\lambda)$ of a random matrix M with respect to GUE, the eigenvalues $(x_1,\dots,x_r)$ being in fact eigenbivalues $(x_1^2,\dots,x_r^2)$.
The $\boldsymbol{m}$-point correlation function for the bilinear Gaussian ("unitary") (or real ("orthogonal")) ensemble BGUE (or BGOE):

$$R_{mr_{R\times L}}(x_1^2,\dots,x_m^2)=\frac{r!}{(r-m)!}\int_{\mathbb{R}^{r-m}}P_{r_{R\times L}}(x_1^2,\dots,x_r^2)\,dx_{m+1}\,\dots dx_r$$

then corresponds to the m-point correlation function for GUE $R_{mr}(x_1,\dots,x_m)$ developed in section 3.1.1.
$R_{mr_{R\times L}}(x_1^2,\dots,x_m^2)$ is thus also given by:

$$R_{mr_{R\times L}}(x_1^2,\dots,x_m^2)=\det(K_r(x_k,x_\ell)_{k,\ell=1}^{m})$$

with:

- $K_r(x_k,x_\ell)=\sum\limits_{i=0}^{r-1}\psi_i(x_k)\ \psi_i(x_\ell)$

  and

- $\psi_i(x)=h^{-\frac{1}{2}}P_i(x)\ e^{-r[(TG^T\times TG)(\mathbb{R}\times\mathbb{R})]/2}$.

As in section 3.11, $P_i(x)$ is an orthogonal polynomial of degree i associated with the weight function $e^{-r[(TG^T\times TG)(\mathbb{R}\times\mathbb{R})]/2}$ where $TG^T(\mathbb{R})\times TG(\mathbb{R})\equiv G(\mathbb{R}\times\mathbb{R})$ is the bilinear Gauss decomposition of the matrix G.
These orthogonal polynomials $P_i(x)$ also satisfy the three term recurrent relation leading to the Jacobi matrix.

3.13 Proposition

Let

$$K_r(x,x)=\sum_{i=0}^{r-1}\psi_i(x)\ \psi_i(x)$$

be the energy level density with $\psi_i(x)$ given by:

$$\psi_i(x)=h^{-\frac{1}{2}}P_i(x)\ e^{-r[(TG^T\times TG)(\mathbb{R}\times\mathbb{R})]/2}\ .$$

Then, we have that:

1) **the squares of the roots (x_j) of the polynomial $P_i(x)$ correspond to the eigenbivalues of the product, right by left, $(U_{r_R} \times U_{r_L})$ of the Hecke operators***;*

2) **the weight $e^{-r[(TG^T \times TG)(\mathbb{R}\times\mathbb{R})]}$ is a measure on the eigenbivalues of the matrix $G \in \mathrm{GL}_r(\mathbb{R}\times\mathbb{R})$ being a representation of the differential bilinear Galois semigroup***.*

Proof:

1) According to section 3.11, the roots of the orthogonal polynomial $P_i(x)$ are the eigenvalues of the Jacobi matrix J.

   On the other hand, the set $\{P_j(x)\}_{j=1}^{i}$ of orthogonal polynomials envisaged in the tower of the three term recurrent relations leading to the matricial form

$$x\ P = J\ P + \beta_i\ P_i$$

   is in one-to-one correspondence with the set $\{\phi(G^{(j)}_{(G)}(F_{\overline{v}^1} \times F_{v^1}))\}_{j=1}^{i}$ of differentiable eigenbifunctions, being the spectral representation of the bioperator $(D_R \otimes D_L)$ according to proposition 3.4 and constituting a representation of the bilinear semigroup of automorphisms $\mathrm{Aut}_k(F_{\overline{v}}) \times \mathrm{Aut}_k(F_v)$ of compact transcendental biquanta generating a bisemilattice according to section 3.3.

   This bisemilattice results from the action of Hecke bioperators $(U_{i_R} \otimes U_{i_L})$ generating the endomorphisms of the bisemisheaf $(\widehat{M}^1_{\overline{v}_R} \times \widehat{M}^1_{v_L})$ referring to sections 1.5 and 3.6.

   Consequently, **the Jacobi matrix J must be a representation of the Hecke operator U_{i_L}**, and, thus, the square of the roots of the polynomial $P_i(x)$ correspond to the eigenbivalues of the Hecke bioperator $(U_{i_R} \otimes U_{i_L})$ and are global Hecke (extended) bicharacters referring to [Pie2].

2) The weight $e^{-r[(TG^T \times TG)(\mathbb{R}\times\mathbb{R})]/2}$ is thus a representation of the differential bilinear Galois semigroup $\mathrm{Aut}_k(\phi_R(\mathbb{R})) \times \mathrm{Aut}_k(\phi_L(\mathbb{R}))$ shifting the product, right by left, of automorphism semigroups $\mathrm{Aut}_k(\phi_R(F_{\overline{v}})) \times \mathrm{Aut}_k(\phi_L(F_v))$ of cofunctions and functions on compact transcendental quanta according to proposition 2.4 and developments of N. Katz [Kat].

   This representation of the bilinear differential Galois semigroup is then associated with the biaction of the differential bioperator $(D_R \otimes D_L)$ of which eigenbivalues are shifts of global Hecke (extended) bicharacters referring to section 3.4 and proposition 3.5. ■

3.14 Corollary

The Jacobi matrix J is a representation of the Hecke operator.

Proof: This results directly from proposition 3.13. ■

3.15 Proposition

The probabilistic interpretation of quantum (field) theories is thus related to the bilinear semigroup of automorphisms $\mathrm{Aut}_k(F_{\overline{v}})\times\mathrm{Aut}_k(F_v)$ *(resp.* $\mathrm{Aut}_k(\widetilde{F}_{\overline{v}})\times\mathrm{Aut}_k(\widetilde{F}_v)$ *) of compact transcendental (resp. algebraic) biquanta generating a bisemilattice of these.*

Proof: The probabilistic interpretation in QFT is given by the function density

$$P_r(x,x)=\sum_{i=0}^{r-1}P_i(x)\ P_i(x)$$

where:

- $P_i(x)$ is an orthogonal polynomial of degree i;
- $P_r(x,x)\ dx\in K_r(x,x)$ being the (wave) function density gives the probability of finding a (bisemi)particle in a volume element $(x,x+dx)$.

As $P_i(x)\ P_i(x)$ or more exactly $P_i(-x)\ P_i(x)$, $x\in I\!R_+$, constitutes a representation of the bilinear semigroup of automorphisms $\mathrm{Aut}_k(F_{\overline{v}})\times\mathrm{Aut}_k(F_v)$ according to sections 1.5 to 2.12, we get the thesis. ■

3.16 Proposition

The m-point correlation function for BGUE (or BGOE) $R_{mr}(x_1^2,\dots,x_m^2)$ constitutes a representation of a the bilinear semigroup of automorphisms $\mathrm{Aut}_k(\phi_R(F_{\overline{v}}\times I\!R))\times\mathrm{Aut}_k(\phi_L(F_v\times I\!R))$ *of fibered or shifted compact transcendental biquanta:*

$$\mathrm{Rep}_{\mathrm{Aut}_k(\phi_{R\times L}(F_{\overline{v}}\times I\!R))}:\qquad \mathrm{Aut}_k(\phi_R(F_{\overline{v}}\times I\!R))\times\mathrm{Aut}_k(\phi_L(F_v\times I\!R))\quad\longrightarrow\quad R_{mr}(x_1,\dots,x_m)\ .$$

Proof: We refer to proposition 2.4 and section 3.12 showing that, in

$$K_r(x_k,x_\ell)=\sum_{i=0}^{r-1}\psi_i(x_k)\ \psi_i(x_\ell)$$

with

$$\psi_i(x)=h^{-\frac{1}{2}}\ P_i(x)\ \exp(-r[TG^T\times TG)(I\!R\times I\!R)]/2)\ ,$$

a) the polynomials $P_i(x)$ constitute a representation of the linear semigroup of automorphisms $\mathrm{Aut}_k(F_v)$ of compact transcendental quanta;

b) the weight $\exp(-r[TG^T\times TG)(I\!R\times I\!R)])$ is a representation of the differential bilinear Galois semigroup $\mathrm{Aut}_k(\phi_R(I\!R))\times\mathrm{Aut}_k(\phi_L(I\!R))$;

c) the products $\psi_i(x_k)\ \psi_i(x_\ell)$ constitute a representation of the bilinear semigroup of automorphisms $\mathrm{Aut}_k(\phi_R(F_{\overline{v}}\times I\!R))\times\mathrm{Aut}_k(\phi_L(F_v\times I\!R))$ of shifted compact transcendental biquanta. ∎

3.17 The third question "What interpretation can we give to the local spacings between the eigenvalues of large random matrices?"

This question [A-B-K], [Joh] is a direct consequence of the responses given to the two first questions and shows the central importance of the biquanta in this field as proved in the following sections.

3.18 Proposition

1) The consecutive spacings

$$\delta E_{R\times L}(j)=E_{R\times L}(j+1)-E_{R\times L}(j)\ ,\qquad 1\le j\le r<\infty\ ,$$

between the eigenbivalues of the random matrix G of $\mathrm{GL}_r(I\!R\times I\!R)$ are the infinitesimal bigenerators on one biquantum of the Lie subbisemialgebra $\mathrm{gl}_1(F_{\overline{v}^1}\times F_{v^1})$ of the bilinear parabolic unitary semigroup $P_1(F_{\overline{v}^1}\times F_{v^1})\subset\mathrm{GL}_1(F_{\overline{v}}\times F_v)$.

2) The k-th consecutive spacings

$$\delta E^{(k)}_{R\times L}(j)=E_{R\times L}(j+k)-E_{R\times L}(j)\ ,\qquad k\le j\ ,$$

between the eigenbivalues of the random matrix G of $\mathrm{GL}_r(I\!R\times I\!R)$ are the infinitesimal bigenerators on k biquanta of the Lie subbisemialgebra $\mathrm{gl}_1(F_{\overline{v}^k}\times F_{v^k})$ of the *bilinear k-th semigroup $\mathrm{GL}_1(F_{\overline{v}^k} \times F_{v^k}) \subset \mathrm{GL}_1(F_{\overline{v}} \times F_v)$, where F_{v^k} (and $F_{\overline{v}^k}$) denotes the set of transcendental subextensions characterized by a transcendence degree*

$$\mathrm{tr} \cdot d \cdot F_{v^k} = k \cdot N$$

referring to k biquanta included in transcendental extensions having higher or equal transcendence degree.

Proof: Referring to proposition 3.4 and corollary 3.5, it appears that the set $\{E_{R\times L}(j)\}_{j=1}^r$ of eigenvalues of a random matrix G of $\mathrm{GL}_r(I\!R \times I\!R)$, being shifts of eigenbifunctions of the eigenbivalue equation

$$(D_R \otimes D_L)(\phi_r(G_{(Q)}(F_{\overline{v}^1} \times F_{v^1}))) = E_{R\times L}(\phi_r(G_{(Q)}(F_{\overline{v}^1} \times F_{v^1})))$$

is the set of infinitesimal bigenerators on j biquanta of the Lie bisemialgebra $\mathrm{gl}_1(F_{\overline{v}} \times F_v)$ of the Lie bilinear semigroup $\mathrm{GL}_1(F_{\overline{v}} \times F_v)$.
And, thus, the consecutive spacings:

$$\delta E_{R\times L}(j) = E_{R\times L}(j+1) - E_{R\times L}(j) \; ,$$

between the eigenbivalues of a random matrix G of $\mathrm{GL}_r(I\!R \times I\!R)$ are the infinitesimal bigenerators on the biquantum on the Lie subbisemialgebra $\mathrm{gl}_1(F_{\overline{v}^1} \times F_{v^1})$ of the bilinear parabolic unitary semigroup $P_1(F_{\overline{v}^1} \times F_{v^1})$ on the product, right by left, of sets $F_{\overline{v}^1}$ and F_{v^1} of transcendental subextensions characterized by a transcendence degree $\mathrm{tr} \cdot d \cdot F_{v^1} = N$.
Similarly, the k-th consecutive spacings

$$\delta E^{(k)}_{R\times L}(j) = E_{R\times L}(j+k) - E_{R\times L}(j)$$

refer to the infinitesimal bigenerators on k biquanta of the Lie subbisemiagebra $\mathrm{gl}_1(F_{\overline{v}^k} \times F_{v^k})$ of the Lie bilinear k-th semigroup $\mathrm{GL}_1(F_{\overline{v}^k} \times F_{v^k}) \subset \mathrm{GL}_1(F_{\overline{v}} \times F_v)$. ■

3.19 Corollary

1) *The consecutive spacings*

$$\delta E_{R\times L}(j) = E_{R\times L}(j+1) - E_{R\times L}(j) \; , \qquad 1 \le j \le r < \infty \; ,$$

between the eigenbivalues of the random matrix G of $\mathrm{GL}_r(I\!R \times I\!R)$ correspond to the energies of one free biquantum from subbisemilattices of $(j+1)$ biquanta.

2) The k-th consecutive spacings

$$\delta E^{(k)}_{R\times L}(j)=E_{R\times L}(j+k)-E_{R\times L}(j)\ , \qquad k\le j\ ,$$

between the eigenbivalues of the random matrix G of $\mathrm{GL}_r(I\!R\times I\!R)$ correspond to the energies of k free biquanta from subbisemilattices of $(j+k)$ biquanta.

Proof: As the infinitesimal bigenerators $E_{R\times L}(j)$ of the Lie bisemiagebra $\mathrm{gl}_1(F_{\overline{v}}\times F_v)$ are shifts of eigenbifunctions on j transcendental biquanta according to section 3.3, they correspond to the energies of these j transcendental biquanta and thus to j transcendental biquanta of energy. ∎

3.20 Lemma

Let $\delta E_{R\times L}(j)$ denote the consecutive spacings between the eigenbivalues of the matrix G of $\mathrm{GL}_r(I\!R\times I\!R)$.
Then, $\delta E_{R\times L}(j)$ decomposes into:

$$\delta E_{R\times L}(j)=\delta EF_{R\times L}(j)+\delta EV_{R\times L}(j)$$

where $\delta EF_{R\times L}(j)$ and $\delta EV_{R\times L}(j)$ denote respectively the fixed (or constant) and variable consecutive spacings between the r eigenbivalues of the matrix G of $\mathrm{GL}_r(I\!R\times I\!R)$.

Proof: There are surjective maps

$$\delta E_{\to v}(j):\qquad \delta E_{R\times L}(j)\quad\longrightarrow\quad \delta EV_{R\times L}(j)$$

between the consecutive spacings $\delta E_{R\times L}(j)$, referring to the infinitesimal generators of Lie subbisemialgebras or energies of one biquantum in a lattice of $(j+1)$ biquanta, and the respective "variable" consecutive spacing $\delta EV_{R\times L}(j)$ in such a way that their kernels are the fixed consecutive spacings $\delta EF_{R\times L}(j)$ being equal for every integer j, $1\le j\le r<\infty$. Consequently, the energy $\delta E_{R\times L}(j)$ of one biquantum in a lattice of $(j+1)$ biquanta decomposes into a fixed part common to all considered bisections and into a variable part differing from one bisection to another. ∎

3.21 Proposition

1) The consecutive spacings

$$\delta EV^{\mathrm{BCUE}}_{R\times L}(j)=E^{\mathrm{BCUE}}_{R\times L}(j+1)-E^{\mathrm{BCUE}}_{R\times L}(j)\ , \qquad 1\le j\le r\ ,$$

between the eigenbivalues $E^{\rm BCUE}_{R\times L}(j+1)$ and $E^{\rm BCUE}_{R\times L}(j)$ of the unitary random matrix $u_r(\mathbb{C}\times\mathbb{C})\in U_r(\mathbb{C}\times\mathbb{C})$ (or $o_r(\mathbb{R}\times\mathbb{R})\in O_r(\mathbb{R}\times\mathbb{R})$) are the variable (unitary) infinitesimal bigenerators on one biquantum of the envisaged Lie subbisemialgebra or variable (unitary) energies $\delta EV^{\rm BCUE}_{R\times L}(j)$ of one biquantum in subbisemilattices of $(j+1)$ biquanta.

2) The k-th consecutive spacings

$$\delta EV^{(k){\rm BCUE}}_{R\times L}(j)=E^{\rm BCUE}_{R\times L}(j+k)-E^{\rm BCUE}_{R\times L}(j)\;, \qquad k\le j\;,$$

between the eigenbivalues of the unitary random matrix $u_r(\mathbb{C}\times\mathbb{C})$ (or $o_r(\mathbb{R}\times\mathbb{R})$) are the variable (unitary) infinitesimal bigenerators on k biquanta of the envisaged Lie subbisemialgebra or variable (unitary) energies $\delta EV^{(k){\rm BCUE}}_{R\times L}(j)$ on k biquanta in subbisemilattices of $(j+k)$ biquanta.

Proof: Similarly as it was developed in proposition 3.18, the set $\{E^{\rm BCUE}_{R\times L}(j)\}^r_{j=1}$ of eigenbivalues of $u_r(\mathbb{C}\times\mathbb{C})$, being shifts of eigenbifunctions of the eigenbivalue equation:

$$(D_R\otimes D_L)(\phi_r(P_{(Q)}(F_{\overline{v}^1}\times F_{v^1})))=E^{\rm BCUE}_{R\times L}(j)(\phi_r(P_{(Q)}(F_{\overline{v}^1}\times F_{v^1})))$$

is the set of unitary infinitesimal bigenerators of the Lie bisemialgebra $\mathrm{gl}_1(F_{\overline{v}^1}\times F_{v^1}))$ of the Lie bilinear parabolic semigroup $P_1(F_{\overline{v}^1}\times F_{v^1})$.
Indeed, the bilinear semigroup of matrices $U_r(\mathbb{C}\times\mathbb{C})$ (or $O_r(\mathbb{R}\times\mathbb{R})$), of the bilinear circular unitary (or orthogonal) ensemble BCUE (resp. BCOE), constitutes the representation of the unitary bilinear differential Galois semigroup associated with the biaction of the differential bioperator $(D_R\otimes D_L)$ on the unitary bisemisheaf $(\widehat{M}^{(1)}_{v^1_R}\otimes\widehat{M}^{(1)}_{v^1_R})\subset(\widehat{M}^{(1)}_{v_R}\otimes\widehat{M}^{(1)}_{v_R})$ according to proposition 2.4 to corollary 2.7.
And, thus, the consecutive spacings

$$E^{\rm BCUE}_{R\times L}(j+1)-E^{\rm BGUE}_{R\times L}(j)=\delta EV^{\rm BCUE}_{R\times L}(j)$$

between the eigenbivalues of $U_r(\mathbb{C}\times\mathbb{C})$ (or $O_r(\mathbb{R}\times\mathbb{R})$) are the variable (unitary) infinitesimal bigenerators of Lie subbisemialgebras $\mathrm{gl}_1(F_{\overline{v}^1}\times F_{v^1})$ or (unitary) variable energies $\delta EV^{\rm BCUE}_{R\times L}(j)$ of one biquantum in sublattices of $(j+1)$ biquanta.
The case of k-th consecutive spacings $\delta EV^{(k){\rm BCUE}}_{R\times L}(j)$ can be handled similarly by taking into account the content of proposition 3.18.
Remark finally that

$$\delta EV_{R\times L}(j)=\delta EV^{\rm BCUE}_{R\times L}(j)\equiv\delta E^{\rm BCUE}_{R\times L}(j)\;. \qquad\blacksquare$$

3.22 The fourth question "What interpretation can we give to the spacings between the nontrivial zeros of the zeta function $\zeta(s)$?"

This question depends on the solution of the Riemann hypothesis proposed in [Pie7] and is briefly recalled in the following sections [Edw], [Rie], [Tit].

3.23 Cuspidal representation given by global elliptic bisemimodule

Let $S_L(2,N=1)$ (resp. $S_R(2,N=1)$) denote the (semi)algebra of cusp forms $f_L(z)$ (resp. $f_R(z)$) of weight 2 and level $N=1$ holomorphic in the upper (resp. lower) half plane as developed in [Pie7]. $f_L(z)$ (resp. $f_R(z^*)$), expanded in Fourier series according to:

$$f_L(z)=\sum_n a_n\, q^n\ ,\ q=e^{2\pi iz}\ ,\ z\in\mathbb{C}\ ,\qquad (\text{resp.}\quad f_R(z^*)=\sum_n a_n\, q^{*n}\ ,\ q^*=e^{-2\pi iz^*}\ )$$

is the functional representation space of $G^{(2)}(F^T_\omega)\equiv T_2(F^T_\omega)$ (resp. $G^{(2)}(F^T_{\overline{\omega}})\equiv T^t_2(F^T_{\overline{\omega}})$) where F^T_ω (resp. $F^T_{\overline{\omega}}$) is the set of increasing toroidal complex transcendental extensions referring to section 1.9.

Then, we have that:

$$\begin{aligned}
f_R(z^*)\times f_L(z) &= \text{FREPSP}(\text{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega)\\
&= G^{(2)}(F^T_{\overline{\omega}}\times F^T_\omega)\\
&= M^{(2)}_{\omega^T_R}\otimes M^{(2)}_{\omega^T_L}
\end{aligned}$$

is a cusp biform of weight 2 and level 1.

On the other hand, we can consider the map:

$$\mathcal{M}_{f\to\zeta_L}:\quad f_L(z)\longrightarrow\zeta_L(s_+)\qquad(\text{resp.}\quad \mathcal{M}_{f\to\zeta_R}:\quad f_R(z^*)\longrightarrow\zeta_R(s_-)\ )$$

of the cusp form $f_L(z)$ (resp. $f_R(z^*)$) into the corresponding zeta function $\zeta_L(s_+)$ (resp. $\zeta_R(s_-)$) in such a way that $s_+=\sigma+i\tau$ (resp. $s_-=\sigma-i\tau$) be an "energy" variable conjugate to the spatial variable z (resp. z^*).

Referring to section 2.12 introducing a $2n$-dimensional global elliptic $(\Gamma_{\widehat{M}^{(2n)}_{v^T_{R\times L}}})$-bisemimodule $\phi^{(2n)}_{R\times L}(x)$, we see that it can be reduced to **a $1D$-pseudounramified simple global elliptic $(\Gamma_{\widehat{M}^{(2n)}_{v^T_{R\times L}}})$-bisemimodule**:

$$\phi^{(1)}_{R\times L}(x)=\sum_n(\lambda^{(nr)}(n)\ e^{-2\pi inx}\otimes_D\lambda^{(nr)}(n)\ e^{+2\pi inx})\ ,\qquad x\in\mathbb{R}\ ,$$

if:

a) pseudounramification is concerned, i.e. the conductor $N=1$;

b) simplicity is supposed, i.e. the multiplicity m_n is equal to one on each level "n".

This global elliptic $(\Gamma_{\widehat M^{(1)}_{v^T_{R\times L}}})$-bisemimodule

$$\phi^{(1),nr}_{R\times L}(x)=\phi^{(1),nr}_{R}(x)\otimes_D\phi^{(1),nr}_{L}(x)\qquad(\otimes_D:\text{ diagonal tensor product})$$

can be interpreted geometrically as **the sum of products, right by left, of semicircles of level "n" on n transcendental compact quanta** in such a way that $\phi^{(1),nr}_{R\times L}(x)$ be the cuspidal automorphic representation of the complete bilinear semigroup $\mathrm{GL}_2(F^{(nr)}_{\overline v}\times F^{(nr)}_{v})$ according to sections 1.5 and 2.12 and cover the weight 2 cusp biform $f_R(z^*)\otimes f_L(z)$.
On the other hand, as we are dealing with bisemiobjects, the zeta functions $\zeta_R(s_-)$ and $\zeta_L(s_+)$, defined respectively on the lower and upper half planes, are considered.

3.24 Proposition

Let

$$H_{\phi_{R\times L}\to\zeta_{R\times L}}:\qquad 2\phi^{(1),nr}_{R}(x)\otimes_D\phi^{(1),nr}_{L}(x)\quad\longrightarrow\quad\zeta_R(s_-)\otimes_D\zeta_L(s_+)$$

be the map between the double $1D$-pseudounramified simple global elliptic bisemimodule $\phi^{(1),nr}_{R\times L}(x)$ and the product, right by left, of zeta functions given by:

$$\zeta_R(s_-)\otimes_D\zeta_L(s_+)=\sum_n\left(n^{-s_-}\otimes_D n^{-s_+}\right)\ .$$

Then, **the kernel $\mathrm{Ker}(H_{\phi_{R\times L}\to\zeta_{R\times L}})$ of the map $H_{\phi_{R\times L}\to\zeta_{R\times L}}$ is the set of squares of trivial zeros of $\zeta_R(s_-)$ and $\zeta_L(s_+)$**.

Proof: The kernel $\mathrm{Ker}(H_{\phi_{R\times L}\to\zeta_{R\times L}})$ of $H_{\phi_{R\times L}\to\zeta_{R\times L}}$ maps into the set of "trivial" zeros of $\zeta_R(s_-)$ and $\zeta_L(s_+)$ which are the negative integers -2, -4, ...
Consequently, **this kernel must be the set of bipoints**:

$$\begin{aligned}\{\sigma_{n_R}\times\sigma_{n_L}&=2\lambda^{(nr)}(n)(e^{-2\pi inx}\mid x=0)\times 2\lambda^{(nr)}(n)(e^{-2\pi inx}\mid x=0)\}\\&=4(\lambda^{(nr)}(n^2))^2\ \}\\&=4f^2_{v_n}=4n^2\ \}\end{aligned}$$

where $(\lambda^{(nr)}(n^2))^2$ is the square of the global residue degree $f_{v_n}=n$ as proved in [Pie7],

in such a way that the left (resp. right) point $\boldsymbol{\sigma_{n_L} = 2\lambda^{(nr)}(n)\ e^{2\pi inx} \mid x=0)}$ (resp. $\boldsymbol{\sigma_{n_R} = 2\lambda^{(nr)}(n)\ e^{-2\pi inx} \mid x=0)}$) corresponds to the degeneracy of the (irreducible) circle $\boldsymbol{2\lambda^{(nr)}(n)\ e^{2\pi inx}}$ (resp. $\boldsymbol{2\lambda^{(nr)}(n)\ e^{-2\pi inx}}$) on $\boldsymbol{2n}$ left (resp. right) transcendental quanta.

Remark that the one-to-one correspondence between the global elliptic semimodule $2\phi_L^{(1),nr}(x)$ (resp. $2\phi_R^{(1),nr}(x)$) and the left (resp. right) zeta function $\zeta_L(s_+) = \sum\limits_n n^{-s_+}$ (resp. $\zeta_R(s_-) = \sum\limits_n n^{-s_-}$) is clear if it is noted that $n^s = n^{\sigma+i\tau}$ can be written

$$e^{(\sigma+i\tau)\ell nn} = e^{\sigma\ell nn} \cdot e^{i\tau\ell nn}$$

where $e^{\sigma\ell nn}$ can correspond to the radius of a circle with phase $e^{i\tau\ell nn}$ in such a way that $1/e^{\sigma\ell nn}$ maps into $2\lambda^{(nr)}(n)$ and $1/e^{i\tau\ell nn}$ maps into $e^{2\pi inx}$ for each term of $\zeta_L(s_+)$ and of $2\phi_L^{(1),nr}(x)$. ■

3.25 Proposition

Let $D_{4n^2,i^2}\cdot\varepsilon_{4n^2}$ be a coset representative of the Lie (bisemi)algebra of the decomposition (bisemi)group $D_{i^2}(\mathbb{Z})_{|4n^2}$ and let α_{4n^2} be the split Cartan subgroup (bi)element.

Then, **the products of the pairs of the trivial zeros of the Riemann zeta functions $\boldsymbol{\zeta_R(s_-)}$ and $\boldsymbol{\zeta_L(s_+)}$ are mapped into the products of the corresponding pairs of the nontrivial zeros according to:***:*

$$\begin{array}{llcl}\{D_{4n^2,i^2}\cdot\varepsilon_{4n^2}\}: & \{\det(\alpha_{4n^2})\}_n & \longrightarrow & \{\det(D_{4n^2,i^2}\cdot\varepsilon_{4n^2}\cdot\alpha_{4n^2})_{ss}\}_n \\ & \{(-2n)\times(-2n)\}_n & \longrightarrow & \{\lambda_+^{(nr)}(4n^2,i^2,E_{4n^2})\times\lambda_-^{(nr)}(4n^2,i^2,E_{4n^2})\}_n\ , \\ &&& \forall\ n\in\mathbb{N}\end{array}$$

where "ss denotes the semisimple form.

Proof: Let

$$\boldsymbol{\alpha_{4n^2}} = \begin{pmatrix} \boldsymbol{4n^2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{1}\end{pmatrix}$$

be the (split) Cartan subgroup element associated with the square of the global residue degree $f_{v_{2n}} = 2n$.

Let

$$\boldsymbol{D_{4n^2,i^2}} = \begin{pmatrix} \mathbf{1} & \boldsymbol{i} \\ \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \ \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \\ \boldsymbol{i} & \mathbf{1} \end{pmatrix}$$

be the coset representative of the Lie (bisemi)algebra [Pie3] **of the decomposition (bisemi)group acting on $\boldsymbol{\alpha_{4n^2}}$** :
It corresponds to the coset representative of an unipotent Lie (bisemi)algebra mapping in the topological Lie (bisemi)algebra $\mathrm{gl}_1(F^{(nr)}_{\overline{v}} \times F^{(nr)}_{v})$ consisting in vector fields on the Lie subgroup $\mathrm{GL}_1(F^{(nr)}_{\overline{v}} \times F^{(nr)}_{v})$.
Let

$$\boldsymbol{\varepsilon_{4n^2}} = \begin{pmatrix} \boldsymbol{E_{4n^2}} & \mathbf{0} \\ \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

be the infinitesimal (bi)generator of this Lie (bisemi)algebra corresponding to the square of the global residue degree $f_{v_{2n}} = 2n$.

Every root of the Lie (bisemi)algebra is determined by the (equivalent) eigenvalues $\boldsymbol{\lambda^{(nr)}_{\pm}(4n^2, i^2, E_{4n^2})}$ of

$$\boldsymbol{D_{4n^2,i^2} \cdot \varepsilon_{4n^2} \cdot \alpha_{4n^2}} = \left[\begin{pmatrix} \mathbf{1} & \boldsymbol{i} \\ \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \ \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \\ \boldsymbol{i} & \mathbf{1} \end{pmatrix}\right] \begin{pmatrix} \boldsymbol{E_{4n^2}} & \mathbf{0} \\ \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \ \begin{pmatrix} \boldsymbol{4n^2} & \mathbf{0} \\ \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

given by

$$\boldsymbol{\lambda^{(nr)}_{\pm}(4n^2, i^2, E_{4n^2})} = \frac{\boldsymbol{1 \pm i\sqrt{(16n^2 \cdot E_{4n^2}) - 1}}}{\mathbf{2}}$$

where $D_{4n^2,i^2} \cdot \varepsilon_{4n^2}$ is the coset representative of the Lie (bisemi)algebra $\mathrm{Lie}(D_{4n^2}(\mathbb{Z})_{|4n^2})$ of the decomposition group $D_{i^2}(\mathbb{Z})_{|4n^2}$.

According to proposition 3.24, the squares $(-2n)^2$ of the trivial zeros of $\zeta_R(s_-)$ and $\zeta_L(s_+)$ are the squares of the global residue degrees $f_{v_{2n}} = 2n$.
As $D_{4n^2} \cdot \varepsilon_{4n^2}$ is of Galois type, it maps squares of trivial zeros $(-2n)^2$ into products of corresponding pairs of other zeros $\lambda^{(nr)}_{+}(4n^2, i^2, E_{4n^2}) = \lambda^{(nr)}_{-}(4n^2, i^2, E_{4n^2})$ which are nontrivial zeros since they have real parts localized on the line $\sigma = \frac{1}{2}$.
So, $D_{4n^2,i^2} \cdot \epsilon_{4n^2}$ maps a lattice of transcendental biquanta on the considered Lie (bisemi) group into a lattice of energies of these biquanta on the associated Lie (bisemi)algebra. ■

3.26 Corollary

The eigenvalues $\lambda_+^{(nr)}(4n^2,i^2,E_{4n^2})$ *and* $\lambda_-^{(nr)}(4n^2,i^2,E_{4n^2})$ *of* $(D_{4n^2,i^2}\cdot\varepsilon_{4n^2}\cdot\alpha_{4n^2})$ *for all* $n\in I\!N$ *are the nontrivial zeros of the Riemann zeta function* $\zeta(s)=\sum\limits_n n^{-s}$ *.*

Proof: The set $\{-2n\}_n$, being the trivial zeros of the right and left zeta functions $\zeta_R(s_-)$ and $\zeta_L(s_+)$, constitutes also the set of trivial zeros of the classical Riemann zeta function $\zeta(s)=\sum\limits_n n^{-s}$.

So, the eigenvalues $\lambda_+^{(nr)}(4n^2,i^2,E_{4n^2})$ and $\lambda_-^{(nr)}(4n^2,i^2,E_{4n^2})$ of $(D_{4n^2,i^2}\cdot\varepsilon_{4n^2}\cdot\alpha_{4n^2})$, generated from the corresponding trivial zeros "$-2n$", are:

a) the nontrivial zeros of $\zeta(s)$ since they are localized on the line $\sigma=\frac{1}{2}$ and disposed symmetrically on this line with respect to $\tau=0$, $s=\sigma+i\tau$;

b) the relevant zeros of the energy density function $\zeta(s)$ which is an inverse space function, i.e. an "energy" function, on transcendental quanta labelled by the integers "n". ∎

3.27 Proposition

Let the nontrivial zeros of $\zeta(s)$, $\lambda_+^{(nr)}(4n^2,i^2,E_{4n^2})$ *and* $\lambda_-^{(nr)}(4n^2,i^2,E_{4n^2})$ *be written compactly and classically according to* $\frac{1}{2}+i\gamma_j$ *and* $\frac{1}{2}-i\gamma_j$, $j\hookleftarrow n$.

1) *Then,* **the consecutive spacings**

$$\boldsymbol{\delta\gamma_j=\gamma_{j+1}-\gamma_j\ ,\qquad j=1,2,\dots}$$

between the nontrivial zeros of $\zeta(s)$ are equivalently:

a) **the infinitesimal generators on one quantum of the Lie subsemialgebra $\mathrm{gl}_1(F_{v^1}^{(nr)})$ (or $\mathrm{gl}_1(F_{\overline{v}^1}^{(nr)})$) of the linear parabolic unitary semigroup $P_1(F_{v^1}^{(nr)})\subset \mathrm{GL}_1(F_{v}^{(nr)})\equiv T_1(F_{v}^{(nr)})$ (or $P_1(F_{\overline{v}^1}^{(nr)})\subset \mathrm{GL}_1(F_{\overline{v}}^{(nr)})\equiv T_1^t(F_{\overline{v}}^{(nr)})$).**

b) **the energies of one (free) quantum from subsemilattice of $(j+1)$ quanta in such a way that $\delta_{\gamma_j}>\frac{\gamma_{j+1}}{j+1}$ where $\frac{\gamma_{j+1}}{j+1}$ is the energy of one quantum bound to j quanta.**

2) **The k-th consecutive spacings**

$$\boldsymbol{\delta_j^{(k)}=\gamma_{j+k}-\gamma_j\ ,}$$

between the nontrivial zeros of $\zeta(s)$ are equivalently:

a) **the infinitesimal generators on k quanta of the Lie subsemialgebra $\mathrm{gl}_1(F^{(nr)}_{v^k})$ (or $\mathrm{gl}_1(F^{(nr)}_{\overline{v}^k})$) of the k-th semigroup $\mathrm{GL}_1(F^{(nr)}_{v^k}) \subset \mathrm{GL}_1(F^{(nr)}_{v})$ (or $\mathrm{GL}_1(F^{(nr)}_{\overline{v}^k}) \subset \mathrm{GL}_1(F^{(nr)}_{\overline{v}})$).**

b) **the energies of k (free) quanta from subsemilattice of $(j+k)$ quanta.**

Proof: According to proposition 3.25, every nontrivial zero $\left(\frac{1}{2} \pm \gamma_{j+1}\right)$ of $\zeta_R(s_-)$, $\zeta_L(s_+)$ or $\zeta(s)$ is the infinitesimal generator of the Lie semialgebra $\mathrm{gl}_1(F^{(nr)}_{v_{j+1}})$ (or $\mathrm{gl}_1(F^{(nr)}_{\overline{v}_{j+1}})$) on $(j+1)$ compact transcendental quanta or, physically, the energy of $(j+1)$ compact transcendental quanta.
So, the consecutive spacing

$$\delta\gamma_j = \gamma_{j+k} - \gamma_j \ ,$$

between γ_{j+1} and γ_j , is the infinitesimal generator on one quantum on the Lie subsemialgebra $\mathrm{gl}_1(F^{(nr)}_{v^1})$ of the parabolic unitary semigroup $P_1(F^{(nr)}_{v^1})$ or the energy of one free quantum in a subsemilattice of $(j+1)$ transcendental compact quanta.
The k-th consecutive spacings

$$\delta^{(k)}_j = \gamma_{j+k} - \gamma_j$$

can be handled similarly. ∎

3.28 The fifth question "What is the curious connection between the spacings of the nontrivial zeros of $\zeta(s)$ and the corresponding spacings between the eigenvalues of random matrices?"

This question finds a response in the following propositions [B-K], [G-M].

3.29 Proposition

The consecutive spacings

$$\delta\gamma_j = \gamma_{j+1} - \gamma_j \ , \qquad j \in I\!N \ ,$$

between the nontrivial zeros of the Riemann zeta function $\zeta(s)$ as well as the consecutive spacings

$$\delta E^{(nr)}_{R,L}(j) = E^{(nr)}_{R,L}(j+1) - E^{(nr)}_{R,L}(j)$$

between the square roots of the pseudounramified eigenbivalues of a random matrix (of) $\mathrm{GL}_r(I\!R\times I\!R)$ *(or of* $\mathrm{GL}_r(\mathbb{C}\times\mathbb{C})$ *) or between the pseudounramified eigenvalues of a random matrix of* $\mathrm{GL}_r(I\!R)$ *, are equivalently:*

a) the infinitesimal generators on one quantum of the Lie subsemialgebra $\mathrm{gl}_1(F^{(nr)}_{v^1})$ *(or* $\mathrm{gl}_1(F^{(nr)}_{\overline{v}^1})$ *) of the linear parabolic unitary semigroup* $P_1(F^{(nr)}_{v^1})$ *(or* $P_1(F^{(nr)}_{\overline{v}^1})$ *);*

b) the energies of one transcendental pseudounramified $(N=1)$ *quantum in subsemilattices of* $(j+1)$ *transcendental pseudounramified quanta.*

(R,L *means* R *(right) or* L *(left)).*

Proof: The equivalence between the consecutive spacings $\delta\gamma_j$ of $\zeta(s)$ and the consecutive spacings $\delta E^{(nr)}_{R,L}(j)$ between the square roots of pseudounramified eigenbivalues of a random matrix of $\mathrm{GL}_r(I\!\!R\times I\!\!R)$ or between the pseudounramified eigenvalues of a random matrix of $\mathrm{GL}_r(I\!\!R)$, results from propositions 3.18 and 3.27.
The equivalence is exact if the eigenbivalues of a matrix of $\mathrm{GL}_r(I\!\!R\times I\!\!R)$ are pseudounramified, i.e. if they are eigenbivalues of the eigenbivalue equation (see proposition 3.18):

$$(D_R\otimes D_L)(\phi_r(G_{(Q)}(F^{(nr)}_{\overline{v}^1}\times F^{(nr)}_{v^1})))=E^{(nr)}_{R\times L}(j)(\phi_r(G_{(Q)}(F^{(nr)}_{\overline{v}^1}\times F^{(nr)}_{v^1})))$$

where $F^{(nr)}_{v^1}$ (and $F^{(nr)}_{\overline{v}^1}$) are unitary transcendental pseudounramified extensions (case $N=1$) referring to section 1.5.
Finally, remark that

$$|\delta E^{(nr)}_{R,L}|=\left|\sqrt{\delta E^{(nr)}_{R\times L}(j)}\right|$$

where $\delta E^{(nr)}_{R,L}(j)$ is a consecutive spacing between pseudounramified eigenvalues and $\sqrt{\delta E^{(nr)}_{R\times L}(j)}$ is a consecutive spacing between square roots of pseudounramified eigenbivalues. ■

3.30 Corollary

The set $\{\delta E^{(nr)}_{R,L}(j)\}_j$ *of consecutive spacings between the square roots of the eigenbivalues of a random matrix of* $\mathrm{GL}_r(I\!\!R\times I\!\!R)$ *(or of* $\mathrm{GL}_r(\mathbb{C}\times\mathbb{C})$ *) or between the eigenvalues of a random matrix of* $\mathrm{GL}_r(I\!\!R)$*, as well as the set* $\{\delta\gamma_j\}_j$ *of consecutive spacings between the nontrivial zeros of* $\zeta(s)$ *constitutes a representation of the differential inertia Galois (semi)group associated with the action of the differential operator* D_L *or* D_R*.*

Proof: This is immediate since the sets $\{\delta E^{(nr)}_{R,L}(j)\}_j$ and $\{\delta\gamma_j\}_j$ are infinitesimal generators of the Lie subsemialgebra of the linear parabolic unitary semigroup $P_1(F^{(nr)}_{v^1})$ (or $P_1(F^{(nr)}_{\overline{v}^1})$) and as the unitary parabolic bilinear semigroup $P_r(I\!\!R\times I\!\!R)\subset\mathrm{GL}_r(I\!\!R\times I\!\!R)$ corresponds to the "bilinear" representation of the product, right by left, $\mathrm{Int}_k(\phi_R(I\!\!R))\times$

$\mathrm{Int}_k(\phi_L(I\!R))$ of differential inertia Galois semigroups according to proposition 2.4 and corollaries 2.5 and 3.5, we have that:

$$P_r(I\!R \times I\!R) = \mathrm{Rep}[(\mathrm{Int}_k(\phi_R(I\!R))_{|F_{\overline{v}_r}^{-1}}) \times (\mathrm{Int}_k(\phi_L(I\!R))_{|F_{\overline{v}_r}^{-1}})] \ . \qquad \blacksquare$$

3.31 Proposition

Let $\{\delta\gamma_j\}_j$ *be the set of consecutive spacings between the nontrivial zeros of* $\zeta(s)$ *and let* $\{\delta E_{R,L}(j)\}_j$ *be the set of consecutive spacings between the square roots of the eigenbivalues of a random matrix of* $\mathrm{GL}_r(I\!R \times I\!R)$ *(or of* $\mathrm{GL}_r(\mathbb{C} \times \mathbb{C})$ *) or between the eigenbivalues of a random matrix of* $\mathrm{GL}_r(I\!R)$ *.*

Then, **there is a surjective map:**

$$\boldsymbol{IM_{E\to\gamma}} : \qquad \{\boldsymbol{\delta E_{R,L}(j)}\}_j \quad \longrightarrow \quad \{\boldsymbol{\delta\gamma_j}\}_j$$

of which kernel $\mathrm{Ker}[\boldsymbol{IM_{E\to\gamma}}]$ is equivalently the set

$$\{\boldsymbol{\delta E_{R,L}(j) - \delta E^{(nr)}_{R,L}(j)}\}_j \ , \qquad \forall\, \boldsymbol{j} \ , \quad \boldsymbol{1 \le j \le r} \ ,$$

a) **of differences of consecutive spacings between the square roots of the pseudoramified and pseudounramified eigenbivalues of a random matrix of $\mathbf{GL}_r(I\!R \times I\!R)$)(or of $\mathbf{GL}_r(\mathbb{C} \times \mathbb{C})$), or between the pseudoramified and pseudounramified eigenvalues of a random matrix of $\mathbf{GL}_r(I\!R)$;**

b) **of the energies of one compact transcendental pseudoramified ($\boldsymbol{N > 2}$) quantum in subsemilattices of $\boldsymbol{(j+1)}$ transcendental pseudoramified quanta.**

Proof: First, remark that the surjective map

$$IM_{E\to\gamma} : \qquad \{\delta E_{R,L}(j)\}_j \quad \longrightarrow \quad \{\delta\gamma_j\}_j$$

leads precisely to the map (or to the equality) between the spacing distribution between eigenvalues of a random matrix and the pair correlation of the nontrivial zeros of $\zeta(s)$ [Mon].

If the kernel $\mathrm{Ker}[IM_{E\to\gamma}]$ of the map $IM_{E\to\gamma}$ is null, then

$$\delta E_{R,L}(j) = \delta E^{(nr)}_{R,L}(j) \ , \qquad j \in I\!N \ ,$$

i.e. the consecutive spacings between the eigen(bi)values of a random matrix are pseudounramified, implies the thesis of proposition 3.29, i.e. the equality between the consecutive spacings $\delta\gamma_j$ of $\zeta(s)$ and the consecutive spacings $\delta E^{(nr)}_{R,L}(j)$ between eigenvalues. $\blacksquare$

3.32 Proposition

Let

$$\begin{aligned}\delta EV^{(nr),\text{BCOE}}_{R,L}(j) &= E^{(nr),\text{BCOE}}_{R,L}(j+1) - E^{(nr),\text{BCOE}}_{R,L}(j)\\ &= \delta E^{(nr),\text{BCOE}}_{R,L}(j)\ , \qquad\qquad\qquad j \in I\!\!N\ ,\end{aligned}$$

be the consecutive spacings between the square roots of the pseudounramified eigenbivalues of a random unitary matrix of $O_r(I\!\!R\times I\!\!R)$ *or of* $U_r(\mathbb{C}\times\mathbb{C})$ *) or between the pseudounramified eigenvalues of a random unitary matrix of* $O_r(I\!\!R)$ *.*
Then, there is a surjective map:

$$IM_{\gamma\to E^{(nr)}_{\text{BCOE}}} : \qquad \{\delta\gamma_j\}_j \quad\longrightarrow\quad \{\delta EV^{(nr),\text{BCOE}}_{R,L}(j)\}_j$$

between the set $\{\delta\gamma_j\}_j$ *of consecutive spacings between the nontrivial zeros of* $\zeta(s)$ *and the set* $\{\delta EV^{(nr),\text{BCOE}}_{R,L}(j)\}_j$ *.*

Proof: According to lemma 3.20, the consecutive spacings $\delta E_{R\times L}(j)$ between the eigenbivalues of $\text{GL}_r(I\!\!R\times I\!\!R)$ decomposes into fixed and variable consecutive spacings

$$\delta E_{R\times L}(j) = \delta EF_{R\times L}(j) + \delta EV_{R\times L}(j)\ ,$$

which is also the case for the consecutive spacings $\delta E^{(nr)}_{R\times L}(j)$ between pseudounramified eigenbivalues.
As

$$\delta\gamma_j = \delta E^{(nr)}_{R,L}(j)$$

according to proposition 3.29, it results that the consecutive spacings between nontrivial zeros of $\zeta(s)$ also decomposes according to fixed and variable consecutive spacings

$$\delta\gamma_j = \delta\gamma F_j + \delta\gamma V_j\ , \qquad \forall\ j\ , \quad 1\le j\le r\ ,$$

in such a way that

$$\delta\gamma V_j = \delta EV^{(nr),\text{BCOE}}_{R,L}(j) \equiv \delta E^{(nr),\text{BCOE}}_{R,L}(j)\ .$$

And, the variable consecutive spacings $\{\delta\gamma V_j\}_j$ constitute a representation of the differential variable inertia Galois subgroup according to corollary 3.30, **the differential variable inertia Galois subgroup being a subgroup of the differential inertia Galois subgroup**. ■

3.33 Proposition (main)

Let

$$\delta\gamma_j^{(k)} = \gamma_{j+k} - \gamma_j$$

denote the k-th consecutive spacings between the nontrivial zeros of $\zeta(s)$.
Let:

- $\delta E_{R,L}^{(k)}(j) = E_{R,L}(j+k) - E_{R,L}(j)$;
- $\delta E_{R,L}^{(nr),(k)}(j) = E_{R,L}^{(nr)}(j+k) - E_{R,L}^{(nr)}(j)$;

  and
- $\delta EV_{R,L}^{(k),(nr),\mathrm{BCOE}}(j) = E_{R,L}^{(nr),\mathrm{BCOE}}(j+k) - E_{R,L}^{(nr),\mathrm{BCOE}}(j)$,

  $1 \leq j \leq r$, $k \leq j$,

be the k-th consecutive spacings between respectively

- *the pseudoramified eigenvalues of a random matrix of* $\mathrm{GL}_r(I\!\!R)$ *;*
- *the pseudounramified eigenvalues of a random matrix of* $\mathrm{GL}_r(I\!\!R)$ *;*
- *the pseudounramified eigenvalues of a random unitary matrix of* $O_r(I\!\!R)$ *.*

Then, we have:

1) $\delta\gamma_j^{(k)} = \delta E_{R,L}^{(nr),(k)}(j)$ **which are equivalently:**

   a) **the infinitesimal generators on k quanta of the Lie subsemialgebra $\mathrm{gl}_1(F_{v^k}^{(nr)})$ of the linear k-th semigroup $\mathrm{GL}_1(F_{v^k}^{(nr)}) \subset \mathrm{GL}_1(F_v^{(nr)})$;**

   b) **the energies of k transcendental pseudounramified $(N=1)$ quanta in subsemilattices in $(j+k)$ transcendental pseudounramified quanta;**

   c) **a representation of the differential Galois (semi)group associated with the action of the differential operator D_L or D_R on a function on k transcendental pseudounramified quanta.**

2) **a surjective map:**

$$IM_{E\to\gamma}^{(k)} : \qquad \{\delta E_{R,L}^{(k)}(j)\}_j \quad \longrightarrow \quad \{\delta\gamma_j^{(k)}\}_j$$

of which kernel $\mathrm{Ker}[IM_{E\to\gamma}^{(k)}]$ is the set $\{\delta E_{R,L}^{(k)}(j) - \delta E_{R,L}^{(nr),(k)}(j)\}_j$ *of difference of k-th consecutive spacings between the pseudoramified and pseudounramified eigenvalues of a random matrix of* $\mathrm{GL}_r(I\!\!R)$ *.*

3) **a bijective map:**

$$\boldsymbol{IM^{(k)}_{\gamma\to E^{(nr)}_{\rm BCOE}}} : \qquad \boldsymbol{\{\delta\gamma_j^{(k)}\}_j} \quad \longrightarrow \quad \boldsymbol{\{\delta EV^{(k),(nr),\rm BCOE}_{R,L}(j)\}_j}$$

where $\delta\gamma_j^{(k)}$ *denotes a* k *-th (variable) consecutive spacing verifying*

$$\delta\gamma_j^{(k)} = \delta E^{(k),(nr),\rm BCOE}_{R,L}(j)\ .$$

Proof: This proposition is a generalisation of propositions 3.29, 3.31 and 3.32 to k-th consecutive spacings. ∎

3.34 Physical interpretation of the nontrivial zeros of $\zeta(s)$

It was suggested for a long time that the nontrivial zeros of the Riemann zeta function are probably related to the eigenvalues of some wave dynamical system of which (Hamiltonian) operator is unknown [Kna].

Considering the new mathematical framework presented here taking into account the solution of the Riemann hypothesis, the connection between these two fields is rather evident. Indeed, referring to propositions 3.29 and 3.31, we see that there exists a surjective map:

$$IM_{E^{(nr)}_{R,L}\to\gamma} : \qquad E_{R,L}(j) \quad \longrightarrow \quad \gamma_j\ , \qquad \forall\ j\ , \quad 1\le j\le r\ ,$$

between square roots of eigenbivalues of a random matrix of $\mathrm{GL}_r(\mathbb{R}\times\mathbb{R})$ (or $\mathrm{GL}_r(\mathbb{C}\times\mathbb{C})$) and nontrivial zeros of $\zeta(s)$ in such a way that the kernel $\mathrm{Ker}[IM_{E^{(nr)}_{R,L}\to\gamma}]$ of $IM_{E^{(nr)}_{R,L}\to\gamma}$ is given by the set $\mathrm{Ker}[IM_{E^{(nr)}_{R,L}\to\gamma}] = \{E_{R,L}(j)-E^{(nr)}_{R,L}(j)\}_j$ of differences between the square roots of the pseudoramified and pseudounramified eigenbivalues of a matrix of $\mathrm{GL}_r(\mathbb{R}\times\mathbb{R})$.

And thus, **if $\boldsymbol{E_{R,L}(j)=E^{(nr)}_{R,L}(j)}$, then we have that:**

$$\boldsymbol{E^{(nr)}_{R,L}(j)=\gamma_j}\ .$$

The squares $E^{(nr)}_{R\times L}(j)$ of the pseudounramified eigenbivalues $E^{(nr)}_{R,L}(j)$ of $\mathrm{GL}_r(\mathbb{R}\times\mathbb{R})$ are also eigenbivalues of the eigenbivalue equation:

$$(D_R\otimes D_L)(\phi_r(G_{(Q)}(F^{(nr)}_{\overline v^1}\times F^{(nr)}_{v^1}))) = E^{(nr)}_{R\times L}(j)(\phi_r(G_{(Q)}(F^{(nr)}_{\overline v^1}\times F^{(nr)}_{v^1})))$$

(according to proposition 3.18),

of which eigenbifunctions is the set of r-bituple

$$\langle\phi(G^{(1)}_{(Q)}(F^{(nr)}_{\overline v^1}\times F^{(nr)}_{v^1})),\dots,\phi(G^{(j)}_{(Q)}(F^{(nr)}_{\overline v^1}\times F^{(nr)}_{v^1})),\dots,\phi(G^{(r)}_{(Q)}(F^{(nr)}_{\overline v^1}\times F^{(nr)}_{v^1}))\rangle$$

referring to proposition 3.4.
And **these eigenbifunctions are the set $\boldsymbol{\Gamma(\widehat{M}^{(1)}_{v_R} \otimes \widehat{M}^{(1)}_{v_L})}$ of sections of the bisemisheaf $\boldsymbol{(\widehat{M}^{(1)}_{v_R} \otimes \widehat{M}^{(1)}_{v_L})}$ which was interpreted as the internal string field of an elementary (bisemi)particle according to section 3.6**.

These considerations lead to the following proposition.

3.35 Proposition

The squares of the nontrivial zeros γ_j of the Riemann zeta function $\zeta(s)$ are the pseudounramified eigenbivalues of the eigenbivalue (biwave) equation:

$$(D_R \otimes D_L)(\phi_r(G_{(Q)}(F^{(nr)}_{\overline{v}^1} \times F^{(nr)}_{v^1}))) = \gamma_j^2(\phi_r(G_{(Q)}(F^{(nr)}_{\overline{v}^1} \times F^{(nr)}_{v^1})))$$

of which eigenbifunctions are the sections of the bisemisheaf $(\widehat{M}^{(1)}_{v_R} \otimes \widehat{M}^{(1)}_{v_L})$ being the internal string field of an elementary (bisemi)particle.

Proof: Consequently, the squares γ_j^2 of the nontrivial zeros of $\zeta(s)$ are the eigenbivalues of a matrix of $\mathrm{GL}_r(I\!R \times I\!R)$ constituting a representation of the bilinear differential Galois semigroup associated with the action of the differential bioperator $(D_R \otimes D_L)$.
And, thus, **each nontrivial zero $\boldsymbol{\gamma_j}$ of $\boldsymbol{\zeta(s)}$ is the infinitesimal generator on $\boldsymbol{j}$ quanta of the Lie subsemialgebra $\boldsymbol{\mathrm{gl}_1(F^{(nr)}_{v_j})}$ of the Lie subsemigroup $\boldsymbol{\mathrm{GL}_1(F^{(nr)}_{v_j})}$ or the energy of $\boldsymbol{j}$ compact transcendental pseudounramified quanta $\boldsymbol{(N=1)}$**.
This can also be seen from propositions 3.25 and 3.27. ∎

References

[A-B-K] A. Aptekarev, P. Bleher, A. Kuylaerts, Large n limit of Gaussian random matrices with external source, II, *Com. Math. phys.*, **259** (2005), 367–389.

[And] Y. Andre, Quatre descriptions des groupes de Galois différentiels, *Lect. Notes Math.*, **1296** (1987), 28–41.

[A-S] M. Atiyah, I. Singer, I. The index of elliptic operators on compact manifolds, *Bull. Amer. Math. Soc.*, **69** (1963), 422–433.

[A-VM] M. Adler, P. van Moerbeke, PDEs for the joint distribution of Dyson, Airy and sine processes, *Ann. Probab.* **33** (2005), 1326–1361.

[Ble] P. Bleher, Lectures of random matrix models. The Riemann Hilbert approach, Arxiv Math-ph.0801.1858 (2008).

[B-H] E. Brezin, S. Hikani, Characteristic polynomials of random matrices, *Comm. Math. Phys.*, **214** (2000), 111–135.

[B-Z] E. Brezin, A. Zee, Universality of the correlations between eigenvalues of large random matrices, *Nucl.Phys. B40L.*, (1993), 611–627.

[Bom] E. Bombieri, The classical theory of zeta and L-functions, *Milan J. Math.*, **78** (2010), 11–59.

[B-K] M. Berry, J.P. Keating, The Riemann zeros and eigenvalue asymptotics, *SIAM Rev.*, **41** (1999), 236–266.

[Car] P. Cartier, Groupoïdes de Lie et leurs algébroïdes *Sém. Bourbaki*, **987** (2007-2008).

[D-H] H. Davenport, H. Heilbronn, On the zeros of certain Dirichlet series, I, *J. London Math. Soc.*, **11** (1936), 181–185.

[D-S] P. Diakonis, M. Shahshahami, On the eigenvalues of random matrices. *J. Appl. Prob.*, **31** (1994), 49–61.

[Dys] F. Dyson, Statistical theory of energy levels of complex systems I, II, III. *J. Math. Phys.*, **3** (1962), 140–156, 157–165, 166–175.

[Edw] H. Edwards, Riemann's zera function. *Academic Press*, (1974).

[Gau] M. Gaudin, Sur la loi limite de l'espacement des valeurs propres d'une matrice aléatoire. *SIAM Rev.*, **41** (1999), 236–266.

[G-M] D. Goldston, H. Montgomery, Pair correlation of zeros and primes in short intervals, *Progr. in Math.*, (1987), 183–203, Birkhauser.

[Har] R. Hartshorne, Algebraic geometry, *Grad. Texts in Math.*, **52** (1977), Springer.

[Hun] T. Hungerford, Algebra, *Grad. Texts in Math.*, **73** (1974), Springer.

[Joh] K. Johansson, The longest increasing subsequence in a random permutation and a unitary matrix model. *Math. Res. Lett.*, **5** (1998), 63–82.

[Kat] N. Katz, Exponential sums and differential equations, *Annals of Math. Studies*, **124** (1990), Princeton University Press.

[Ke-S] J.P. Keating, N.C. Snaith, Random matrix theory and $\xi\ (1/2+it)$, *Com. Math. Phys.*, **214** (2000), 57–89.

[Kna] A. Knauf, Number theory, dynamical systems and statistical mechanics, *Rev. Math. Phys.*, **11** (1999), 1027–1060.

[K-S] N. Katz, P. Sarnak, Zeros of zeta functiona and symmetry, *Bull. Amer. Math. Soc.*, **36** (1999), 1–26.

[Lan] R. Langlands, Problem in the theory of automorphic forms, *Lect. Notes Math.*, **170** (1970), 18–86.

[Meh] M. Mehta, Random matrices, *Elsevier*, (1967).

[Mez] F. Mezzadri, How to generate random matrices from the classical compact groups. Notices, *AMS*, **54, 5** (2007), 592–603.

[Mon] H. Montgomery, The pair correlation of zeros of zeta function, *Proc. Symp. Pure Math.*, **24**, (1973), 181–193.

[Odl] A. Odlyzko, The 10^{20}-th zeros of the Riemann zeta function, *Preprint ATT Bell Lab.*, (1989).

[Pie1] C. Pierre, Algebraic quantum theory. Arxiv: Math-ph/0404024 (2004).

[Pie2] C. Pierre, n-dimensional global correspondences of Langlands. Arxiv: Math.RT0510348 v.3 (2009).

[Pie3] C. Pierre, n-dimensional geometric-shifted global bilinear correspondences of Langlands on mixed motives. III. Arxiv: Math.07093383 v. 3 (2009).

[Pie4] C. Pierre, Introducing bisemistructures. Arxiv: Math.GM/0607624 (2006).

[Pie5] C. Pierre, Higher algebraic K-theories related to the global program of Langlands, Arxiv. Math.1009-2613 (2010).

[Pie6] C. Pierre, Algebraic representations of von Neumann algebras, Arxiv: Math-RT/0609232 v.2 (2006).

[Pie7] C. Pierre, From global class field concepts and modular representations to the conjectures of Shimura-Taniyama-Weil, Birch-Swinnerton-Dyer and Riemann, Arxiv: Math-RT/0608084 (2006).

[Pie8] C. Pierre, Universal mathematical and physical structures. (2009) http://www.umaps.net.

[Pol] G. Polya, Location of zeros, Collected Papers vol. II, *Ed. R.D. Boas, Cambridge, MIT Press*, (1974).

[Rie] B. Riemann, Gesammelte Werke, Zw. Aufl. (1892) , *Oeuvres Mathématiques*, Gauthiers-Viallars (1898).

[Rue] E. Ruelle, Some remarks on the location of zeros of the partition fnction for lattice systems, *Comm. Math. Phys.*, **31** (1973), 265–277.

[Tit] E. Titchmarsch, The theory of the Riemann zeta function, *Oxford University Press*, (1986).

[Wei] A. Weil, Basic number theory, *Springer*, (1974).

[Wig] E. Wigner, Random matrices in physics, *Siam Rev.*, **9** (1967), 1–23.

[Yau] Horn-Tzer Yau, Random matrices and the conjectures of Wigner and Dyson, Arxiv: Math.1007.4652 (2010).

[Zag] D. Zagier, The first 50 million prime numbers, *Math. Intell.*, **0** (1977), 7–19.

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