Bound states of the Klein-Gordon equation in D-dimensions with some physical scalar and vector exponential-type potentials including orbital centrifugal term

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Abstract

The approximate analytic bound state solutions of the Klein-Gordon equation with equal scalar and vector exponential-type potentials including the centrifugal potential term are obtained for any arbitrary orbital angular momentum number l and dimensional space D. The relativistic/non-relativistic energy spectrum equation and the corresponding unnormalized radial wave functions, in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$, where $\alpha > -1$, $\beta > -1$ and $z \in [-1, +1]$ or the generalized hypergeometric functions $_2F_1(a, b; c; z)$, are found. The Nikiforov-Uvarov (NU) method is used in the solution. The solutions of the Eckart, Rosen-Morse, Hulthén and Woods-Saxon potential models can be easily obtained from these solutions. Our results are identical with those ones appearing in the literature. Finally, under the PT-symmetry, we can easily obtain the bound state solutions of the trigonometric Rosen-Morse potential.

Keywords: Approximation schemes, Eckart-type potentials, Rosen-Morse-type potentials, trigonometric rosen-morse potential, Klein-Gordon equation, NU method

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I. INTRODUCTION

The exact solutions of the wave equations (non-relativistic or relativistic) are very important since they contain all the necessary information regarding the quantum system under consideration. However, analytical solutions are possible only in a few simple cases such as the hydrogen atom and the harmonic oscillator [1,2]. Most quantum systems could be solved only by using approximation schemes like rotating Morse potential via Pekeris approximation [3] and the generalized Morse potential by means of an improved approximation scheme [4]. Recently, the study of exponential-type potentials have attracted much attention from many authors [5-26]. These potentials include the Woods-Saxon [5,6], Hulthén [7-16], Manning-Rosen [17-22], the Eckart [23-25] and the Rosen-Morse [26] potentials.

The spherically symmetric Eckart-type potential model [27] is a molecular potential model which has been widely applied in physics [28] and chemical physics [29,30] and is generally expressed as

$$V(r;q) = V_1 \cos ech_q^2 \alpha r - V_2 \coth_q \alpha r,\ \ V_1, V_2 > 0,\ \ -1 \leq q < 0 \text{ or } q > 0, \tag{1}$$

where the coupling parameters V_1 and V_2 describe the depth of the potential well, while the screening parameter α is related to the range of the potential. It is a special case of the five-parameter exponential-type potential model [31,32]. The range of parameter q was taken as $q > 0$ by Ref. [33] and has been extended to $-1 \leq q < 0$ or $q > 0$ or even complex by Ref. [34]. The deformed hyperbolic functions given in (1) have been introduced for the first time by Arai [35] for real q values. When q is complex, the functions in (1) are called the generalized deformed hyperbolic functions. The Eckart-type potentials (1) can also be written in the exponential form as

$$V(r;q) = 4V_1 \frac{e^{-2\alpha r}}{(1-qe^{-2\alpha r})^2} - V_2 \frac{1+qe^{-2\alpha r}}{1-qe^{-2\alpha r}}. \tag{2}$$

The study of the bound and scattering states for the Eckart-type potential has raised a great deal of interest in the non-relativistic as well as in relativistic quantum mechanics. The s-wave ($l = 0$) bound-state solution of the Schrödinger equation for the Eckart potential has been widely investigated by using various methods, such as the supersymmetric (SUSY) shape invariance technology [36], point cannonical transformation (PCT) method [37] and SUSY Wentzel-Kramers-Brillouin (WKB) approximation approach [38]. The bound state solutions of the s-wave Klein-Gordon (KG) equation with equally mixed Rosen-Morse-type (Eckart and Rosen–Morse well) potentials have been studied [39]. The bound state solutions of the s-wave Dirac equation with equal vector and scalar Eckart-type potentials in terms of the basic concepts of the shape-invariance approach in the SUSYQM have also been studied [24]. The spin symmetry and pseudospin symmetry in the relativistic Eckart potential have been investigated by solving the Dirac equation for mixed potentials [25]. Unfortunately, the wave equations for the Eckart-type potential can only be solved analytically for zero angular momentum states because of the centrifugal potential term. Some authors [23-25] studied the analytical approximations to the bound state solutions of the Schrödinger equation with Eckart potential by using the usual existing approximation scheme proposed by Greene and Aldrich [40] for the centrifugal potential term. This approximation has also been used to study analytically the arbitrary l-wave scattering state solutions of the Schrödinger equation for the Eckart potential [41,42]. The same approximation scheme for the spin-orbit coupling term has been used to study the spin symmetry and pseudospin symmetry analytical solutions of the Dirac equation with the Eckart potential using the AIM [43]. Overmore, the pseudospin symmetry analytical solutions of the Dirac equation for the Eckart potential have been found by using the SUSY WKB formalism [44]. Very recently, for the first time, the approximation scheme for the centrifugal potential term has also been used in [45] to obtain the approximate analytical solution of the KG equation for equal scalar and vector Eckart potentials for arbitrary l-states by means of the functional analysis method.

This approximation for the centrifugal potential term [7,14,40] has also been used to solve the Schrödinger equation [7,14], KG [8,15] and Dirac equation [15] for the Hulthén potential. Recently, the KG and Dirac equations have been solved in the presence of the Hulthén potential, where the energy spectrum and the scattering wave functions were obtained for spin-0 and spin-$\frac{1}{2}$ particles, using a more general approximation scheme for the centrifugal potential [15]. They found that the good approximation, however, occurs when the screening parameter α and the dimensionless parameter γ are taken as $\alpha = 0.1$ and γ =1, respectively, which is simply the case of the usual approximation [7,14]. Also, other authors have recently proposed an alternative approximation scheme for the centrifugal potential to solve the Schrödinger equation for the Hulthén potential [46]. Taking $\omega = 1$, their approximation can be reduced to the usual approximation [7,14]. Very recently, we have also proposed a new approximation scheme for the centrifugal term [9].

The Nikiforov-Uvarov (NU) method [47] and other methods have also been used to solve the D-dimensional Schrödinger equation [48] and relativistic D-dimensional KG equation [49], Dirac equation [4,10,26,50] and spinless Salpeter equation [51].

The aim of this work is to employ the usual approximation scheme [40,45] in order to solve the D-dimensional radial KG equation for any orbital angular momentum number l for the scalar and vector Eckart-type potentials using a general mathematical model of the NU method. This offers a simple, accurate and efficient scheme for the exponential-type potential models in quantum mechanics. We consider the following relationship between the scalar and vector potentials: $V(r) = V_0 + \beta S(r)$, where V_0 and β are arbitrary constants [52]. Under the restriction of equally mixed potentials $S(r) = V(r)$, the KG equation turns into a Schrödinger-like equation and thus the bound state solutions are very easily obtained through the well-known methods developed in the non-relativistic quantum mechanics. It is interesting to note that, this restriction include the case where $V(r) = 0$ when both constants vanish, the situation where the potentials are equal ($V_0 = 0; \beta = 1$) and also the case where the potentials are proportional [53] when $V_0 = 0$ and $\beta = \pm 1$, which provide the equally-mixed scalar and vector potential case $V(r) = \pm S(r)$. Very recently, we have obtained an approximate analytic solution of the KG equation in the presence of equal scalar and vector generalized deformed hyperbolic potential functions by means of parameteric generalization of the NU method. Furthermore, for the equally-mixed scalar and vector potential case $V(r) = \pm S(r)$, we have obtained the approximate bound state rotational-vibrational (ro-vibrational) energy levels and the corresponding normalized wave functions expressed in terms of the Jacobi polynomial $P_n^{(\mu,\nu)}(x)$, where $\mu > -1$, $\nu > -1$ and $x \in [-1, +1]$ for a spin-zero particle in a closed form [54].

The paper is structured as follows: In section 2, we derive a general model of the NU method valid for any central or non-central potential. In section 3, the approximate analytical solutions of the D-dimensional radial KG equation with arbitrary l-states for equally-mixed scalar and vector Eckart-type potentials and other typical potentials are obtained by means of the NU method. Also, the exact s-wave KG equation has also been solved for the Rosen-Morse-type potentials and other typical potentials. The relative convenience of the Eckart-type potential (Rosen-Morse-type potential) with the Hulthén potential (Woods-Saxon potential) has been studied, respectively. We make some remarks on the energy equations and the corresponding wavefunctions for the Eckart and Rosen-Morse well potentials in various dimensions and their non-relativistic limits in section 4. Section 5 contains the summary and conclusions.

II. NU METHOD

The NU method is briefly outlined here and the details can be found in [47]. This method was proposed to solve the second-order differential wave equation of the hypergeometric-type:

$$\sigma^2(z)\psi_n''(z) + \sigma(z)\widetilde{\tau}(z)\psi_n'(z) + \widetilde{\sigma}(z)\psi_n(z) = 0, \tag{3}$$

where $\sigma(z)$ and $\widetilde{\sigma}(z)$ are at most second-degree polynomials and $\widetilde{\tau}(z)$ is a first-degree polynomial. The prime denotes the differentiation with respect to z. To find a particular solution of Eq. (3), one can decompose the wave function $\psi_n(z)$ as follows:

$$\psi_n(z) = \phi_n(z)y_n(z), \tag{4}$$

leading to a hypergeometric type equation

$$\sigma(z)y_n''(z) + \tau(z)y_n'(z) + \lambda y_n(z) = 0, \tag{5}$$

where

$$\lambda = k + \pi'(z), \tag{6}$$

and $y_n(z)$ satisfies the Rodrigues relation

$$y_n(z) = \frac{A_n}{\rho(z)}\frac{d^n}{dz^n}\left[\sigma^n(z)\rho(z)\right]. \tag{7}$$

In the above equation, A_n is a constant related to the normalization and $\rho(z)$ is the weight function satisfying the condition

$$\sigma(z)\rho'(z) + \left(\sigma'(z) - \tau(z)\right)\rho(z) = 0, \tag{8}$$

with

$$\tau(z) = \widetilde{\tau}(z) + 2\pi(z), \tau'(z) < 0. \tag{9}$$

Since $\rho(z) > 0$ and $\sigma(z) > 0$, the derivative of $\tau(z)$ should be negative [47] which is the essential condition for a proper choice of solution. The other part of the wavefunction in Eq. (4) is defined as

$$\sigma(z)\phi'(z) - \pi(z)\phi(z) = 0, \tag{10}$$

where

$$\pi(z)=\frac{1}{2}\left[\sigma'(z)-\widetilde{\tau}(z)\right]\pm\sqrt{\frac{1}{4}\left[\sigma'(z)-\widetilde{\tau}(z)\right]^2-\widetilde{\sigma}(z)+k\sigma(z)}. \tag{11}$$

The determination of k is the essential point in the calculation of $\pi(z)$, for which the discriminant of the square root in the last equation is set to zero. This results in the polynomial $\pi(z)$ which is dependent on the transformation function $z(r)$. Also, the parameter λ defined in Eq. (6) takes the following form

$$\lambda=\lambda_n=-n\tau'(z)-\frac{1}{2}n\left(n-1\right)\sigma''(z),\quad n=0,1,2,\cdots. \tag{12}$$

We may construct a general recipe of the NU method valid for any central and non-central potential. We begin by comparing the following hypergeometric equation

$$\left[z\left(1-c_3z\right)\right]^2\psi_n''(z)+\left[z\left(1-c_3z\right)\left(c_1-c_2z\right)\right]\psi_n'(z)+\left(-Az^2+Bz-C\right)\psi_n(z)=0, \tag{13}$$

with its counterpart Eq. (3), we then obtain [54]

$$\widetilde{\tau}(z)=c_1-c_2z,\ \sigma(z)=z\left(1-c_3z\right),\ \widetilde{\sigma}(z)=-Az^2+Bz-C. \tag{14}$$

Substituting Eq. (14) into Eq. (11), we find

$$\pi(z)=c_4+c_5z\pm\left[\left(c_6-c_3k_{+,-}\right)z^2+\left(c_7+k_{+,-}\right)z+c_8\right]^{1/2}, \tag{15}$$

where

$$c_4=\frac{1}{2}\left(1-c_1\right),\ c_5=\frac{1}{2}\left(c_2-2c_3\right),\ c_6=c_5^2+A,\ c_7=2c_4c_5-B,\ c_8=c_4^2+C. \tag{16}$$

The discriminant under the square root sign must be set to zero and the resulting equation must be solved for k, it yields

$$k_{+,-}=-\left(c_7+2c_3c_8\right)\pm2\sqrt{c_8c_9}, \tag{17}$$

where

$$c_9=c_3\left(c_7+c_3c_8\right)+c_6. \tag{18}$$

Inserting Eq. (17) into Eq. (15) and solving the resulting equation, we make the following choice of parameters:

$$\pi(z)=c_4+c_5z-\left[\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)z-\sqrt{c_8}\right], \tag{19}$$

$$k_{-} = -\left(c_7 + 2c_3c_8\right) - 2\sqrt{c_8c_9}. \tag{20}$$

Further, from Eq. (9), we get

$$\tau(z) = 1 - \left(c_2 - 2c_5\right)z - 2\left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)z - \sqrt{c_8}\right], \tag{21}$$

whose derivative must be negative:

$$\tau'(z) = -2c_3 - 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) < 0, \tag{22}$$

in accordance with essential requirement of the method [47]. Solving Eqs. (6) and (12), we get the energy equation:

$$\left(c_2 - c_3\right)n + c_3n^2 - (2n+1)\,c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0, \tag{23}$$

for the potential under investigation. Let us now turn to the wave functions. The solution of the differential equation (8) for the weight function $\rho(z)$ is

$$\rho(z) = z^{c_{10}}\left(1 - c_3z\right)^{c_{11}}, \tag{24}$$

and consequently from Eq. (7), the first part of the wave function becomes

$$y_n(z) = P_n^{(c_{10},c_{11})}(1 - 2c_3z), \;\; \mathrm{Re}(c_{10}) > -1, \;\; \mathrm{Re}(c_{11}) > -1, \tag{25}$$

where

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} - 1, \;\; c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9}, \tag{26}$$

and $P_n^{(a,b)}(1 - c_3z)$ are Jacobi polynomials. The second part of the wave function (4) can be found from the solution of the differential equation (10) as

$$\phi(z) = z^{c_{12}}\left(1 - c_3z\right)^{c_{13}}, \tag{27}$$

where

$$c_{12} = c_4 + \sqrt{c_8}, \;\; c_{13} = -c_4 + \frac{1}{c_3}\left(\sqrt{c_9} - c_5\right). \tag{28}$$

Hence, the general wave functions (4) read as

$$u_l(z) = N_n z^{c_{12}}\left(1 - c_3z\right)^{c_{13}} P_n^{(c_{10},c_{11})}(1 - 2c_3z), \tag{29}$$

where N_n is a normalization constant.

III. BOUND-STATE SOLUTIONS

The D-dimensional time-independent arbitrary l-states radial KG equation with scalar and vector potentials $S(r)$ and $V(r)$, respectively, where $r = |\mathbf{r}|$ describing a spinless particle takes the general form [3,49]:

$$\boldsymbol{\nabla}_D^2 \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(\mathbf{x}) + \frac{1}{\hbar^2 c^2}\left\{\left[E_{nl} - V(r)\right]^2 - \left[Mc^2 + S(r)\right]^2\right\}\psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(\mathbf{x}) = 0,$$

$$\nabla_D^2 = \sum_{j=1}^{D}\frac{\partial^2}{\partial x_j^2},\ \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(\mathbf{x}) = R_l(r) Y_{l_1 \cdots l_{D-2}}^{(l)}(\theta_1, \theta_2, \cdots, \theta_{D-1}), \tag{30}$$

where E_{nl}, M and $\boldsymbol{\nabla}_D^2$ denote the KG energy, the mass and the D-dimensional Laplacian, respectively. In addition, $\mathbf{x}$ is a D-dimensional position vector. Let us decompose the radial wave function $R_l(r)$ as follows:

$$R_l(r) = r^{-(D-1)/2} u_l(r), \tag{31}$$

we, then, reduce Eq. (30) into the D-dimensional radial Schrödinger-like equation with arbitrary orbital angular momentum number l as

$$\frac{d^2 u_l(r)}{dr^2} + \frac{1}{\hbar^2 c^2}\left\{\left[E_{nl} - V(r)\right]^2 - \left[Mc^2 + S(r)\right]^2 - \frac{l'(l'+1)\hbar^2 c^2}{r^2}\right\} u_l(r) = 0, \tag{32}$$

where we have set $l'(l'+1) = \left[(\mathcal{M}-2)^2 - 1\right]/4$ and $\mathcal{M} = D + 2l$ where $l = 0, 1, 2, \cdots$. Under the equally mixed potentials $S(r) = \pm V(r)$, the KG turns into a Schrödinger-like equation and thus the bound state solutions are very easily obtained with the help of the well-known methods developed in the non-relativistic quantum mechanics. We use the existing approximation for the centrifugal potential term in the non-relativistic model [7,14] which is valid only for $q = 1$ value [49,55]:

$$\widetilde{V}(r) = \frac{l'(l'+1)}{r^2} \approx 4\alpha^2 l'(l'+1)\frac{e^{-2\alpha r}}{\left(1 - qe^{-2\alpha r}\right)^2},\ l' = (\mathcal{M} - 3)/2, \tag{33}$$

in the limit of small α and l'.

A. The Eckart-type model

At first, let us rewrite Eq. (2) in a form to include the Hulthén potential,

$$V(r;q)=4V_1\frac{e^{-2\alpha r}}{\left(1-qe^{-2\alpha r}\right)^2}-V_2\frac{1}{1-qe^{-2\alpha r}}-V_3\frac{qe^{-2\alpha r}}{1-qe^{-2\alpha r}}, \tag{34}$$

and then follow the model used in Refs. [49,55,56] by inserting the above equation and the approximate potential term (33) into (32), we obtain

$$\frac{d^2u_l(r)}{dr^2}+$$

$$\frac{1}{\hbar^2c^2}\left\{-\frac{\left[8\left(E_{nl}\pm Mc^2\right)V_1+4\alpha^2\hbar^2c^2l'(l'+1)\right]e^{-2\alpha r}}{(1-qe^{-2\alpha r})^2}+\frac{2\left(E_{nl}\pm Mc^2\right)\left(V_2+qV_3e^{-2\alpha r}\right)}{(1-qe^{-2\alpha r})}\right\}u_l(r)$$

$$=\frac{1}{\hbar^2c^2}\left[\left(Mc^2\right)^2-E_{nl}^2\right]u_l(r),\ u_l(0)=0, \tag{35}$$

which is now amenable to the NU solution. We further use the following ansätze in order to make the above differential equation more compact

$$z(r)=e^{-2\alpha r},\ \varepsilon_{nl}=\frac{\sqrt{\left(Mc^2\right)^2-E_{nl}^2}}{Q},\ \beta=\frac{8\left(E_{nl}\pm Mc^2\right)V_1}{Q^2}+l'(l'+1),$$

$$\gamma=\frac{2\left(E_{nl}\pm Mc^2\right)V_2}{Q^2},\ \lambda=\frac{2\left(E_{nl}\pm Mc^2\right)V_3}{Q^2},\ Q=2\hbar c\alpha. \tag{36}$$

Notice that $|E_{nl}|\leq Mc^2$. The KG equation can then be reduced to

$$[z(1-qz)]^2\frac{d^2u_l(z)}{dz^2}+z(1-qz)^2\frac{du_l(z)}{dz}$$

$$+\left\{-q^2(\varepsilon_{nl}^2+\lambda)z^2+(2q\varepsilon_{nl}^2+q\lambda-q\gamma-\beta)z-\left(\varepsilon_{nl}^2-\gamma\right)\right\}u_l(z)=0, \tag{37}$$

where $r\in[0,\infty)\to z\in[0,1]$. Before proceeding, the boundary conditions on the radial wave function $u_l(r)$ demand that $u_l(r\to\infty$ or $z\to 0)\to 0$ and $u_l(r=0$ or $z=1)$ is finite. Comparing Eq. (37) with Eq. (13), we obtain values for the set of parameters given in section 2:

$$c_1=1,\ c_2=c_3=q,\ c_4=0,\ c_5=-\frac{q}{2},\ c_6=q^2\left(\varepsilon_{nl}^2+\lambda+\frac{1}{4}\right),$$

$$c_7=-q\left(2\varepsilon_{nl}^2+\lambda-\gamma-\frac{\beta}{q}\right),\ c_8=\varepsilon_{nl}^2-\gamma,\ c_9=\left(\frac{q}{2}\right)^2\left(1+\frac{4\beta}{q}\right),\ c_{10}=2\sqrt{\varepsilon_{nl}^2-\gamma},$$

$$c_{11}=\sqrt{1+\frac{4\beta}{q}},\ c_{12}=\sqrt{\varepsilon_{nl}^2-\gamma},\ c_{13}=\frac{1}{2}\left(1+\sqrt{1+\frac{4\beta}{q}}\right),$$

$$A=q^2\left(\varepsilon_{nl}^2+\lambda\right),\ B=q(2\varepsilon_{nl}^2+\lambda-\gamma-\frac{\beta}{q}),\ C=\varepsilon_{nl}^2-\gamma, \tag{38}$$

and the energy equation via Eq. (23) as

$$\varepsilon_{nl}^2 = \frac{(\gamma+\lambda)^2}{4(n+\delta)^2} + \frac{(n+\delta)^2}{4} + \frac{\gamma-\lambda}{2},\ n = 0, 1, 2, \cdots, \tag{39}$$

where $\delta = \frac{1}{2}\left(1+\sqrt{1+\frac{4\beta}{q}}\right)$. Making use of Eq. (36), the above equation turns to become

$$M^2c^4 - E_{nl}^2 = (\hbar c\alpha)^2 (n+w)^2 + \frac{\left(E_{nl} \pm Mc^2\right)^2}{(2\hbar c\alpha)^2} \frac{(V_2+V_3)^2}{(n+w)^2} + \left(E_{nl} \pm Mc^2\right)(V_2 - V_3), \tag{40}$$

where $w = \frac{1}{2}\left(1+\sqrt{1+\frac{4l'(l'+1)}{q}+\frac{8(E_{nl}\pm Mc^2)V_1}{q(\hbar c\alpha)^2}}\right)$. The energy E_{nl} is defined implicitly by Eq. (40) which is a rather complicated transcendental equation having many solutions for given values of n and l. In the above equation, let us remark that it is not difficult to conclude that bound-states appear in four energy solutions; only two energy solutions are valid for the particle $E^p = E_{nl}^+$ and the second one corresponds to the anti-particle energy $E^a = E_{nl}^-$ in the Eckart-type field.

Referring to the general parametric model in section 2, we can also calculate the corresponding wave functions. The explicit form of the weight function becomes

$$\rho(z) = z^{2p}(1-qz)^{2w-1},\ p = \frac{1}{2}\left[n + w - \frac{\left(E_{nl} \pm Mc^2\right)(V_2+V_3)}{2\left(\hbar c\alpha\right)^2}\frac{1}{n+w}\right], \tag{41}$$

which gives the following Jacobi polynomials:

$$y_n(z) \to P_n^{(2p,2w-1)}(1-2qz), \tag{42}$$

as a first part of the wave functions. The second part of the wave functions can be found as

$$\phi(z) \to z^p(1-qz)^w. \tag{43}$$

Hence, the unnormalized wave functions expressed in terms of the Jacobi polynomials read

$$u_l(z) = \mathcal{N}_n z^p (1-qz)^w P_n^{(2p,2w-1)}(1-2qz), \tag{44}$$

and consequently the total radial part of the wave functions expressed in terms of the hypergeometric functions are

$$R_l(r) = \mathcal{N}_n r^{-(D-1)/2}\left(e^{-2\alpha r}\right)^p \left(1-qe^{-2\alpha r}\right)^w {}_2F_1\left(-n, n+2\left(p+w\right); 2p+1; qe^{-2\alpha r}\right), \tag{45}$$

where $\mathcal{N}_n$ is a constant related to the normalization. The relationship between the Jacobi polynomials and the hypergeometric functions is given by $P_n^{(a,b)}(1-2qx) ={}_2 F_1(-n, n+a+b+1; a+1; x)$, where ${}_2F_1(\nu,\mu;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\nu)\Gamma(\mu)}\sum\limits_{k=0}^{\infty}\frac{\Gamma(\nu+k)\Gamma(\mu+k)}{\Gamma(\gamma+k)}\frac{x^k}{k!}$.

Now, when taking $V_2 = V_3$, the energy equation (40) satisfying E_{nl} for the equally-mixed scalar and vector Eckart-type potentials becomes

$$M^2c^4 - E_{nl}^2 = (\hbar c\alpha)^2 (n+w)^2 + \frac{(E_{nl} \pm Mc^2)^2}{(\hbar c\alpha)^2} \frac{V_2^2}{(n+w)^2}, \tag{46}$$

and the wave functions:

$$u_l(z) = \mathcal{N}_n z^{\upsilon} (1-qz)^w P_n^{(2\upsilon, 2w-1)}(1-2qz), \; \upsilon = \frac{1}{2}\left[n + w - \frac{(E_{nl} \pm Mc^2) V_2}{(\hbar c\alpha)^2} \frac{1}{n+w}\right], \tag{47}$$

or the total radial wave functions in (30) are

$$R_l(r) = \mathcal{N}_n r^{-(D-1)/2} \left(e^{-2\alpha r}\right)^{\upsilon} (1 - qe^{-2\alpha r})^w \, {}_2F_1\left(-n, n + 2(\upsilon + w); 2\upsilon + 1; qe^{-2\alpha r}\right), \tag{48}$$

where $\mathcal{N}_n$ is a normalization factor. The results given in Eqs. (46) and (47) are consistent with those given in Eqs. (15) and (18) of Ref. [45].

Also, in taking $q = 1$, $2\alpha \to \alpha$, $V_1 = V_2 = 0$ and $V_3 = V_0$, Eq. (34) turns to become the Hulthén potential. Hence, we find bound state solutions for equally-mixed scalar and vector $S(r) = V(r)$ Hulthén potentials in the KG theory with any orbital angular momentum quantum number l and an arbitrary dimension D,

$$\sqrt{M^2c^4 - E_{nl}^2} = \frac{(\hbar c\alpha)(n+\nu)}{2} - \frac{(Mc^2 + E_{nl}) V_0}{\hbar c\alpha} \frac{1}{(n+\nu)}, \; \nu = \frac{D + 2l - 1}{2}, \tag{49}$$

$$u_l(z) = \mathcal{N}_n \left(e^{-\alpha r}\right)^{\varsigma} (1 - e^{-\alpha r})^{\nu} P_n^{(2\varsigma, 2\nu-1)}(1-2z), \; \varsigma = \frac{n+\nu}{2} - \frac{(Mc^2 + E_{nl}) V_0}{(\hbar c\alpha)^2} \frac{1}{n+\nu}, \tag{50}$$

and the Jacobi polynomial in the above equation can be expressed in terms of the hypergeometric function:

$$R_l(r) = \mathcal{N}_n r^{-(D-1)/2} \left(e^{-\alpha r}\right)^{\varsigma} (1 - e^{-\alpha r})^{\nu} \, {}_2F_1\left(-n, n + 2(\varsigma + \nu); 2\varsigma + 1; e^{-\alpha r}\right), \tag{51}$$

where $\mathcal{N}_n$ is a constant related to the normalization. The above results are identical to those found recently by Refs. [49,57].

In the non-relativistic limit, inserting the equally mixed Eckart-type potentials (1) into the Schrödinger equation gives

$$\frac{d^2 u_l(r)}{dr^2} + \left\{ \frac{2ME_{nl}}{\hbar^2} - \frac{[8MV_1 + 4\alpha^2\hbar^2 l'(l'+1)] e^{-2\alpha r}}{\hbar^2 (1 - qe^{-2\alpha r})^2} + \frac{2MV_2 (1 + qe^{-2\alpha r})}{\hbar^2 (1 - qe^{-2\alpha r})} \right\} u_l(r) = 0, \tag{52}$$

and further making use of the following definitions:

$$\varepsilon_{nl} = \frac{\sqrt{-2ME_{nl}}}{T}, \quad E_{nl} \leq 0, \; \beta = \frac{8MV_1}{T^2} + l'(l'+1), \; \gamma = \frac{2MV_2}{T^2}, \; T = 2\hbar\alpha, \tag{53}$$

lead us to obtain the set of parameters and energy equation given before in Eqs. (38) and (39) with $\gamma=\lambda$. Incorporating the above equation and using Eq. (39), we find the following energy eigenvalues:

$$E_{nl}=-\frac{1}{2M}\left[\hbar^2\alpha^2\left(n+w_1\right)^2+\frac{M^2V_2^2}{\hbar^2\alpha^2}\frac{1}{\left(n+w_1\right)^2}\right],\ w_1=\frac{1}{2}\left(1+\sqrt{(1+2l')^2+\frac{8MV_1}{\hbar^2\alpha^2}}\right) \tag{54}$$

In addition, following the procedures indicated in Eqs. (41)-(45), we obtain expressions for the radial wave functions:

$$R_l(r)=\mathcal{N}_n'r^{-(D-1)/2}\left(e^{-2\alpha r}\right)^{p_1}\left(1-e^{-2\alpha r}\right)^{w_1}P_n^{(2p_1,2w_1-1)}(1-2e^{-2\alpha r}),$$
$$p_1=\frac{1}{2\hbar\alpha}\sqrt{-2M\left(E_{nl}+V_2\right)}=\frac{1}{2}\left[n+w_1-\frac{MV_2}{\hbar^2\alpha^2}\frac{1}{n+w_1}\right]. \tag{55}$$

B. The Rosen-Morse-type model

Under the replacement of q by $-q$, the Eckart-type potential model given in Eq. (1) will become the Rosen-Morse-type potential model given in Eq. (2) of Ref. [39]:

$$V(r,q)=V_1\sec h_q^2\alpha r-V_2\tanh_q\alpha r,\ V_1,V_2>0, \tag{56}$$

or alternatively [26,58]

$$V(r,q)=4V_1\frac{e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}-V_2\frac{1-qe^{-2\alpha r}}{1+qe^{-2\alpha r}}. \tag{57}$$

We may rewrite the above equation in a form to include the Woods-Saxon potential,

$$V(r,q)=4V_1\frac{e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}-V_2\frac{1}{1+qe^{-2\alpha r}}+V_3\frac{qe^{-2\alpha r}}{1+qe^{-2\alpha r}}. \tag{58}$$

Using the following definitions

$$\varepsilon_{n,0}=\frac{\sqrt{\left(Mc^2\right)^2-E_{n,0}^2}}{Q},\ \widetilde{\beta}=\beta(l\to 0)=\frac{8V_1\left(E_{n,0}\pm Mc^2\right)}{Q^2},$$
$$\widetilde{\gamma}=\gamma(l\to 0)=\frac{2\left(E_{n,0}\pm Mc^2\right)V_2}{Q^2},\ \widetilde{\lambda}=\lambda(l\to 0)=\frac{2\left(E_{n,0}\pm Mc^2\right)V_3}{Q^2}, \tag{59}$$

we write the s-wave KG equation with $S(r)=\pm V(r)$ for the potential (58) as

$$\left[z(1+qz)\right]^2\frac{d^2u_n(z)}{dz^2}+z(1+qz)^2\frac{du_n(z)}{dz}$$

$$+\left\{-q^2\left(\varepsilon_{n,0}^2+\widetilde{\lambda}\right)z^2+q\left(\widetilde{\gamma}-\widetilde{\lambda}-2\varepsilon_{n,0}^2-\frac{\widetilde{\beta}}{q}\right)z-\left(\varepsilon_{n,0}^2-\widetilde{\gamma}\right)\right\}u_n(z)=0. \quad (60)$$

Following same procedures used in the previous subsection, we obtain values for the parameters given in section 2:

$$c_1=1,\ c_2=c_3=-q,\ c_4=0,\ c_5=\frac{q}{2},\ c_6=q^2\left(\varepsilon_{n,0}^2+\widetilde{\lambda}+\frac{1}{4}\right),$$

$$c_7=q\left(2\varepsilon_{n,0}^2+\widetilde{\lambda}+\frac{\widetilde{\beta}}{q}-\widetilde{\gamma}\right),\ c_8=\varepsilon_{n,0}^2-\widetilde{\gamma},\ c_9=\left(\frac{q}{2}\right)^2\left(1-\frac{4\widetilde{\beta}}{q}\right),\ c_{10}=2\sqrt{\varepsilon_{n,0}^2-\widetilde{\gamma}},$$

$$c_{11}=-\sqrt{1-\frac{4\widetilde{\beta}}{q}},\ c_{12}=\sqrt{\varepsilon_{n,0}^2-\widetilde{\gamma}},\ c_{13}=\widetilde{\delta}=\frac{1}{2}\left(1-\sqrt{1-\frac{4\widetilde{\beta}}{q}}\right),$$

$$A=q^2\left(\varepsilon_{n,0}^2+\widetilde{\lambda}\right),\ B=-q\left(2\varepsilon_{n,0}^2+\frac{\widetilde{\beta}}{q}+\widetilde{\lambda}-\widetilde{\gamma}\right),\ C=\varepsilon_{n,0}^2-\widetilde{\gamma}, \quad (61)$$

and the energy equation

$$\varepsilon_{n,0}^2=\frac{\left(\widetilde{\gamma}+\widetilde{\lambda}\right)^2}{4(n+\widetilde{\delta})^2}+\frac{\left(n+\widetilde{\delta}\right)^2}{4}+\frac{\widetilde{\gamma}-\widetilde{\lambda}}{2}. \quad (62)$$

Inserting Eq. (59) in the above equation, we obtain energy equation satisfying $E_{n,0}$,

$$M^2c^4-E_{n,0}^2=(\hbar c\alpha)^2\left(n+\widetilde{w}\right)^2+\frac{\left(E_{n,0}\pm Mc^2\right)^2}{\left(2\hbar c\alpha\right)^2}\frac{\left(V_2+V_3\right)^2}{\left(n+\widetilde{w}\right)^2}+\left(E_{n,0}\pm Mc^2\right)\left(V_2-V_3\right),$$

$$\widetilde{w}=\frac{1}{2}\left(1-\sqrt{1-\frac{8\left(E_{n,0}\pm Mc^2\right)V_1}{q\left(\hbar c\alpha\right)^2}}\right). \quad (63)$$

The corresponding unnormalized wave functions can be calculated as before, the explicit form of the weight function becomes

$$\rho(z)=z^{2\widetilde{p}}(1-qz)^{2\widetilde{w}-1},\ \widetilde{p}=\frac{1}{2}\left[n+\widetilde{w}-\frac{\left(E_{n,0}\pm Mc^2\right)\left(V_2+V_3\right)}{2\left(\hbar c\alpha\right)^2}\frac{1}{n+\widetilde{w}}\right], \quad (64)$$

which gives the Jacobi polynomials:

$$y_n(z)\rightarrow P_n^{(2\widetilde{p},2\widetilde{w}-1)}(1+2qz), \quad (65)$$

as the first part of the wave function. The second part of the wave function can be found as

$$\phi(z)\rightarrow z^{\widetilde{p}}(1+qz)^{\widetilde{w}}. \quad (66)$$

The unnormalized wave function reads

$$u_n(z) = \widetilde{\mathcal{N}}_n z^{\widetilde{p}}(1+qz)^{\widetilde{w}} P_n^{(2\widetilde{p},2\widetilde{w}-1)}(1+2qz), \tag{67}$$

and thus the total radial part of the radial wave functions in (30) can be expressed in terms of the hypergeometric functions as

$$R_n(r) = \widetilde{\mathcal{N}}_n \left(e^{-2\alpha r}\right)^{\widetilde{p}} \left(1+qe^{-2\alpha r}\right)^{\widetilde{w}} {}_2F_1\left(-n, n+2\left(\widetilde{p}+\widetilde{w}\right); 2\widetilde{p}+1; -qe^{-2\alpha r}\right), \tag{68}$$

where $\widetilde{\mathcal{N}}_n$ is a normalization factor.

In taking $V_2 = V_3$ in Eq. (63), we find the equation for the potential in (56) satisfying $E_{n,0}$ in the s-wave KG theory,

$$M^2c^4 - E_{n,0}^2 = (\hbar c\alpha)^2 \left(n+\widetilde{w}\right)^2 + \frac{\left(E_{n,0} \pm Mc^2\right)^2}{(\hbar c\alpha)^2} \frac{V_2^2}{\left(n+\widetilde{w}\right)^2}, \tag{69}$$

and the wave functions are

$$\begin{aligned} u_n(r) &= \widetilde{\mathcal{N}}_n \left(e^{-2\alpha r}\right)^{\widetilde{p}_1} \left(1+qe^{-2\alpha r}\right)^{\widetilde{w}} P_n^{(2\widetilde{p}_1,2\widetilde{w}-1)}(1+2qe^{-2\alpha r}), \\ &= \widetilde{\mathcal{N}}_n \left(e^{-2\alpha r}\right)^{\widetilde{p}_1} \left(1+qe^{-2\alpha r}\right)^{\widetilde{w}} {}_2F_1\left(-n, n+2\left(\widetilde{p}_1+\widetilde{w}\right); 2\widetilde{p}_1+1; -qe^{-2\alpha r}\right), \\ \widetilde{p}_1 &= \frac{1}{2}\left[n+\widetilde{w} - \frac{\left(E_{n,0} \pm Mc^2\right)V_2}{(\hbar c\alpha)^2}\frac{1}{n+\widetilde{w}}\right], \end{aligned} \tag{70}$$

where $\widetilde{\mathcal{N}}_n$ is a normalization constant. After making appropriate change of the potential parameter $V_1 \to -V_1$ in Eq. (56), our results in Eqs. (69) and (70) become identical with Eqs. (13) and (14) of Ref. [39].

Also, taking $q = 1$, $2\alpha \to \alpha$, $V_1 = V_2 = 0$ and $V_3 = -V_0$, Eq. (58) turns to become the Woods-Saxon potential. Hence, we can find bound state solutions in the s-wave KG theory with equally-mixed scalar and vector $S(r) = V(r)$ for Woods-Saxon potentials as

$$\sqrt{M^2c^4 - E_{n0}^2} = \hbar c\alpha\widetilde{p}_2,\ \widetilde{p}_2 = \frac{n}{2} + \frac{\left(Mc^2+E_{n0}\right)V_0}{(\hbar c\alpha)^2}\frac{1}{n}, \tag{71}$$

and wave functions:

$$u_n(r) = \mathcal{N}_n \left(e^{-\alpha r}\right)^{\widetilde{p}_2} P_n^{(2\widetilde{p}_2,-1)}(1+2e^{-\alpha r}), \tag{72}$$

or alternatively, it can be expressed in terms of the hypergeometric function as

$$R_n(r) = \mathcal{N}_n r^{-1} \left(e^{-\alpha r}\right)^{\widetilde{p}_2} {}_2F_1\left(-n, n+2\widetilde{p}_2; 2\widetilde{p}_2+1; e^{-\alpha r}\right), \tag{73}$$

where $\mathcal{N}_n$ is a constant related to the normalization. Under appropriate parameter replacements, we obtain the non-relativistic limit of the energy eigenvalues and eigenfunctions of the above two equations are

$$E_{n,0} = -\frac{1}{2M}\left[\frac{n\hbar\alpha}{2} + \frac{2MV_0}{\hbar\alpha}\frac{1}{n}\right],\ n \neq 0, \tag{74}$$

and

$$u_n(r) = \mathcal{N}_n \left(e^{-\alpha r}\right)^{\widetilde{p}_3} {}_2F_1\left(-n, n+2\widetilde{p}_2; 2\widetilde{p}_2+1; e^{-\alpha r}\right),\ \widetilde{p}_3 = \frac{n}{2} + \frac{2MV_0}{(\hbar c\alpha)^2}\frac{1}{n}, \tag{75}$$

respectively, which is the solution of the Schrödinger equation for the potential $\Sigma(r) = V(r) + S(r) = 2V(r)$.The above results are identical to those found before by Ref. [6].

IV. DISCUSSIONS

In this section, at first, we choose appropriate parameters in the Eckart-type potential model to construct the Eckart potential, Rosen-Morse well and their PT-symmetric versions, and then discuss their energy equations in the framework of KG theory with equally mixed potentials.

A. Eckart potential

Taking $q = 1$, the potential (1) turns to the standard Eckart potential [27]

$$V(r) = V_1 \cos ech^2 \alpha r - V_2 \coth \alpha r,\ V_1, V_2 > 0. \tag{76}$$

In natural units ($\hbar = c = 1$), we can obtain the energy equation (46) for the Eckart potential in the three-dimensional spinless KG theory as

$$\begin{gathered} M^2 - E_{nl}^2 = \alpha^2 (n+w')^2 + \frac{(E_{nl} \pm M)^2}{\alpha^2}\frac{V_2^2}{(n+w')^2}, \\ w' = w(q \to 1) = \frac{1}{2}\left(1 + \sqrt{(2l'+1)^2 + \frac{8(E_{nl} \pm M)V_1}{\alpha^2}}\right), \end{gathered} \tag{77}$$

which is identical with those given in Eq. (22) of Ref. [39] under the equally-mixed potential restriction given by $S(r) = \pm V(r)$. The unnormalized wave function corresponding to the energy levels is

$$R_l(r) = \mathcal{N}'_{nl} r^{-(D-1)/2} \left(e^{-2\alpha r}\right)^{\upsilon} \left(1 - e^{-2\alpha r}\right)^{w'} P_n^{(2\upsilon, 2w'-1)}(1 - 2e^{-2\alpha r}), \tag{78}$$

where $\mathcal{N}'_{nl}$ is a normalization factor.

(i) For s-wave case, the centrifugal term $\frac{(D+2l-1)(D+2l-3)}{4r^2}=0$ and consequently the approximation term $(D+2l-1)(D+2l-3)\alpha^2\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}=0$, too. Thus, the energy eigenvalues take the following simple form

$$M^2-E_{n,0}^2=\alpha^2\left(n+w_1\right)^2+\frac{V_2^2\left(E_{n,0}\pm M\right)^2}{\alpha^2\left(n+w_1\right)^2},\ w_1=\frac{1}{2}\left(1+\sqrt{1+\frac{8\left(E_{n,0}\pm M\right)V_1}{\alpha^2}}\right). \quad (79)$$

(ii) In the non-relativistic approximation of the KG energy equation (potential energies small compared to Mc^2 and $E\simeq Mc^2$) Eq. (32) reduces into the form [59]

$$-\frac{\hbar^2}{2M}\frac{d^2u_l(r)}{dr^2}+\left\{V(r)+S(r)-\frac{l'(l'+1)\hbar^2}{r^2}\right\}u_l(r)=\left(E-Mc^2\right)u_l(r). \quad (80)$$

When $V(r)=S(r)$, the energy eigenvalues obtained from Eq. (80) reduces to those energy eigenvalues obtained from the solution of the Schrödinger equation for the sum potential $\Sigma(r)=2V(r)$. In other words, the non-relativistic limit is the Schrödinger-like equation for the potential $8V_1\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}-2V_2\frac{1+e^{-2\alpha r}}{1-e^{-2\alpha r}}$. This can be achieved by making the parameter replacements $M+E_R\rightarrow 2M$ and $E_R-M\rightarrow E_{NR}$, so the non-relativistic limit of our results in Eq. (46) reduces to

$$E_{NR}=-\frac{1}{2M}\left[\alpha^2\left(n+w_2\right)^2+\frac{2M^2V_2^2}{\alpha^2\left(n+w_2\right)^2}\right], \quad (81)$$

and the corresponding wave functions in (48) become

$$R_l(r)=\mathcal{N}'_{nl}r^{-(D-1)/2}\left(e^{-2\alpha r}\right)^{\upsilon_2}\left(1-e^{-2\alpha r}\right)^{w_2}P_n^{(2\upsilon_2,2w_2-1)}(1-2e^{-2\alpha r}),$$

$$\upsilon_2=\frac{1}{2}\left[n+w_2-\frac{2MV_2}{\alpha^2}\frac{1}{n+w_2}\right],\ w_2=\frac{1}{2}\left(1+\sqrt{(1+2l')^2+\frac{16MV_1}{\alpha^2}}\right). \quad (82)$$

The above two equations are identical with the NU solution of the Schrödinger equation for a potential $V(r)$ (cf. Eqs. (54) and (55)).

B. PT-symmetric Trigonometric Rosen-Morse (tRM) potential

When we make the transformations of parameters as $\alpha\rightarrow i\alpha$, $V_2\rightarrow -iV_2$, and $V_1\rightarrow -V_1$, and using the relation between the trigonometric and the hyperbolic functions $\sin(i\alpha x)=i\sinh(\alpha x)$, the potential (1) turns to become the PT-symmetric tRM potential [60]:

$$V(x)=V_1\csc^2\alpha x-V_2\cot\alpha x,\ \ \mathrm{Re}(V_1)>0,\ \alpha=\frac{\pi}{2d},\ x=(0,d], \quad (83)$$

where $V_1 = a(a+1)$ and $V_2 = 2b$. This potential is displayed in Figure 1 which is nearly linear in $\pi/3 < \alpha x < 2\pi/3$, Coulombic in $\pi/90 < \alpha x < \pi/30$ and infinite walls at 0 and π. So it might be a prime candidate for an effective QCD potential. For a potential $V(x)$, when one makes the transformation of $x \to -x$ and $i \to -i$, if the relation $V(-x) = V^*(x)$ exists, the potential $V(x)$ is said to be PT-symmetric, where P denotes parity operator (space reflection) and T denotes time reversal [6,61]. Our point here is that $V(x)$ interpolates between the Coulomb-and the infinite wall potential [62] going through an intermediary region of linear-x-and harmonic-oscillator x^2 dependences. To see this it is quite instructive to expand the potential in a Taylor series which for appropriately small x takes the form of a Coulomb-like potential with a centrifugal-barrier like term, provided by the $\csc^2 \alpha x$ part [63],

$$V(x) \approx -\frac{V_2}{\alpha x} + \frac{V_1}{(\alpha x)^2}, \ \alpha x \ll 1. \tag{84}$$

For $\alpha x < \pi/2$ we can then take the potential (84) plus a linear like perturbation

$$\Delta V(x) = V_1/3 + V_2 x/3, \tag{85}$$

as an approximation of tRM potential. The potential (83) obviously evolves to an infinite wall as αx approaches the limits of the definition interval $0 < \alpha x < \pi$, due to the behavior of the $\cot \alpha x$ and $\csc \alpha x$ for $V_1 > 0$. The potential is essential for the QCD quark-gluon dynamics where the one gluon exchange gives rise to an effective Coulomb-like potential, while the self gluon interactions produce a linear potential as established by lattice QCD calculations of hadron properties (Cornell potential) [64]. Finally, the infinite wall piece of the tRM potential provides the regime suited for the asymptotical freedom of the quarks. Now, making the corresponding parameter replacements in Eq. (46), we end up with real energy equation for the above PT-symmetric version of the Eckart-type potentialş in the KG equation with equally mixed potentials,

$$\left(Mc^2\right)^2 - E_{nl}^2 = \frac{\left(E_{nl} \pm Mc^2\right)^2}{\left(\hbar c\alpha\right)^2} \frac{V_2^2}{\left(n+w\right)^2} - \left(\hbar c\alpha\right)^2 \left(n+w\right)^2, \tag{86}$$

and the radial wave functions build up as

$$R_l(x) = \mathcal{N}_n x^{-(D-1)/2} \left(e^{+i2\alpha x}\right)^{\upsilon} \left(1 - e^{+i2\alpha x}\right)^{w} {}_2F_1\left(-n, n + 2\left(\upsilon + w\right); 2\upsilon + 1; e^{+i2\alpha x}\right),$$

$$\upsilon = \frac{1}{2}\left[n + w + i\frac{\left(E_{nl} \pm Mc^2\right) V_2}{\left(\hbar c\alpha\right)^2}\frac{1}{n+w}\right], \ w = \frac{1}{2}\left(1 + \sqrt{(1+2l')^2 + \frac{8\left(E_{nl} \pm Mc^2\right) V_1}{\left(\hbar c\alpha\right)^2}}\right) \tag{87}$$

C. Standard Rosen-Morse well

Taking $q = 1$, $V_1 \to -V_1$ ($\widetilde{\beta} \to -\widetilde{\beta}$) and $V_2 \to -V_2$ ($\widetilde{\gamma} \to -\widetilde{\gamma}$), the potential (56) turns to the standard Rosen-Morse well [26,58]

$$V(r) = -V_1 \sec h^2 \alpha r + V_2 \tanh \alpha r, \; V_1, V_2 > 0. \tag{88}$$

This potential is useful in discussing polyatomic molecular vibrational energies. An example of its application to the vibrational states of NH_3 was given by Rosen and Morse in [26,58]. Making the corresponding parameter replacements in Eq. (69), we obtain the energy equation for the Rosen-Morse well in the s-wave KG theory with equally mixed potentials,

$$M^2 - E_{n0}^2 = \alpha^2 \left(n + \widetilde{\delta}_1\right)^2 + \frac{\left(E_{n0} \pm M\right)^2}{\alpha^2} \frac{V_2^2}{\left(n + \widetilde{\delta}_1\right)^2}, \; \widetilde{\delta}_1 = \frac{1}{2}\left(1 - \sqrt{1 + \frac{8\left(E_{n0} \pm M\right) V_1}{\alpha^2}}\right). \tag{89}$$

The unnormalized wave function corresponding to the energy levels is

$$\begin{aligned} u_n(r) &= \widetilde{\mathcal{N}}'_n \left(e^{-2\alpha r}\right)^{\widetilde{\eta}_1} \left(1 + e^{-2\alpha r}\right)^{\widetilde{\delta}_1} P_n^{\left(2\widetilde{\eta}_1, 2\widetilde{\delta}_1 - 1\right)}(1 + 2e^{-2\alpha r}), \\ \widetilde{\eta}_1 &= \frac{1}{2}\left[n + \widetilde{\delta}_1 + \frac{\left(E_{n,0} \pm M\right) V_2}{\alpha^2} \frac{1}{n + \widetilde{\delta}_1}\right], \end{aligned} \tag{90}$$

where $\widetilde{\mathcal{N}}'_n$ is a normalization constant. The results given in Eqs. (89) and (90) are consistent with those given in Eqs. (19) and (20) of Ref. [39], respectively. The s-wave energy states of the KG equation for the Rosen-Morse potential are calculated for a set of selected values parameters in Table 1.

When $V_0 = S_0$, the non-relativistic limit is the solution of the Schrödinger equation for the potential $-8V_1 \frac{e^{-2\alpha r}}{\left(1+e^{-2\alpha r}\right)^2} + 2V_2 \frac{1-e^{-2\alpha r}}{1+e^{-2\alpha r}}$. In the non-relativistic limits, the energy spectrum is

$$E_{NR} = -\frac{1}{2M}\left[\alpha^2 \left(n + \widetilde{\delta}_2\right)^2 + \frac{4M^2 V_2^2}{\alpha^2 \left(n + \widetilde{\delta}_2\right)^2}\right], \; \widetilde{\delta}_2 = \frac{1}{2}\left(1 - \sqrt{1 + \frac{16 M V_1}{\alpha^2}}\right), \tag{91}$$

and the wave functions are

$$R_l(r) = \widetilde{\mathcal{N}}''_n \left(e^{-2\alpha r}\right)^{\widetilde{\eta}_2} \left(1 + e^{-2\alpha r}\right)^{\widetilde{\delta}_2} P_n^{\left(2\widetilde{\eta}_2, 2\widetilde{\delta}_2 - 1\right)}(1 + 2e^{-2\alpha r}), \; \widetilde{\eta}_2 = \frac{1}{2}\left[n + \widetilde{\delta}_2 + \frac{2MV_2}{\alpha^2} \frac{1}{n + \widetilde{\delta}_2}\right]. \tag{92}$$

V. CONCLUSIONS

We have used a parametric generalization model derived from the NU to obtain the analytic bound state solutions of the KG equation with any orbital angular momentum quantum number l for equally mixed scalar and vector Eckart-type potentials. These calculations include energy equation and the unnormalized wave functions being expressed in terms of the Jacobi polynomials or the hypergeometric functions. Furthermore, making appropriate changes in the Eckart-type potential parameters, one can generate new bound state solutions for various types of the well-known molecular potentials like the Rosen-Morse well [26], the Eckart potential, the Hulthén potential [9], the Woods-Saxon potential [5] and the Manning-Rosen potential [22] and others. It is also noted that under the PT-symmetry, the exponential potentials can be transformed into the trigonometric potentials with real bound state solutions. The KG equation with equally mixed scalar and vector Rosen-Morse-type potentials can be solved exactly for s-wave bound states. In the relativistic model, the energy equations for these potentials are complicated transcendental equations [26]. The non-relativistic limits are obtained with a proper replacements of parameters and/or by solving the original Schrödinger equation. The relativistic and non-relativistic results are identical with those ones obtained in the literature through various methods.

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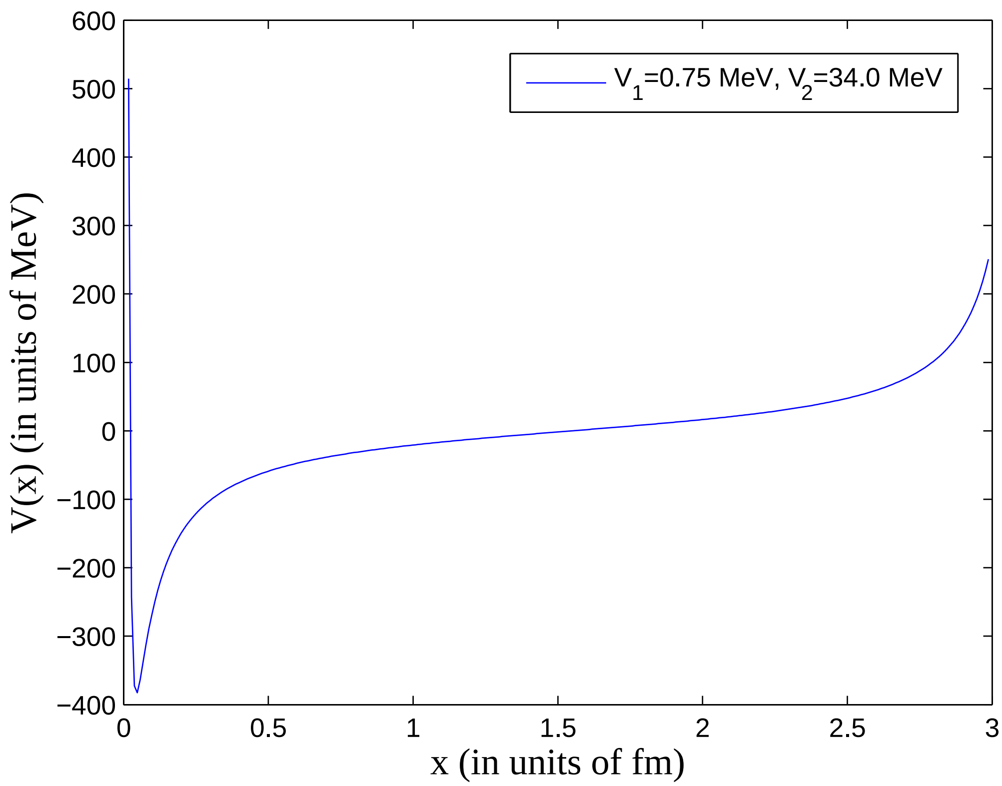
FIG. 1: Plot of the tRM potential [see Eq. (83)] for a set of parameters $a = 0.5$ and $b = 17.0$.

TABLE I: The s-wave energy spectrum of the equally mixed scalar and vector Rosen-Morse-type potentials.

n	α	q	V_1	V_2	M	E_1	E_2	E_3	E_4
1	1	1	1	−1	4	1.8137[a]	−1.9140[a]	−3.3923[a]	−3.9088[a]
2						−2.2117	−3.6791	–	–
3						−0.6606	−3.3105	–	–
4						0.8879	−2.7697	–	–
5						1.8766	−1.9765	–	–
1	1	1	2	−2	5	0.9989	−3.7763	−4.7275	−4.9351
2						−4.1746	−4.7795	–	–
3						−3.3814	−4.5376	–	–
4						−2.3989	−4.2008	–	–
5						−1.3083	−3.7529	–	–
1	0.5	1	1	−1	4	1.9558	−3.5288	−3.8460	−3.9773
2						1.9608	−2.5367	−3.5326	−3.9216
3						1.2294	−0.5126	−3.0732	−3.8358
4						−2.4823	−3.7191	–	–
5						−1.7822	−3.5695	–	–
1	1	0.5	1	−1	4	1.5783	−3.2245	−3.6502	−3.9258
2						1.9995	−1.5367	−2.9520	−3.7496
3						−1.9529	−3.4736	–	–
4						−0.7335	−3.0839	–	–
5						0.5489	−2.5528	–	–

[a]Same as in Ref. [39].



V(x) (in units of MeV)
x (in units of fm)
V1=0.75 MeV, V2=34.0 MeV
600
500
400
300
200
100
0
−100
−200
−300
−400
0
0.5
1
1.5
2
2.5
3