Exact Solutions of the Self-consistent System of Relativistic Magnetohydrodynamics Equations for an Anisotropic Plasma on the Background of Bondi-Pirani-Robinson's Metric

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Abstract. Exact solutions of the self-consistent relativistic magnetohydrodynamics equations for an anisotropic magnetized plasma on the background of Bondi-Pirani-Robinson's vacuum plane gravitational wave (PGW) metric with an arbitrary polarization are obtained, which generalize the results obtained earlier by one of the authors for the transverse polarization of a gravitational wave. Based on the reformulated energobalance equation it is shown that in the linear approximation by gravitational wave amplitude only the transverse polarization of PGW interacts with magnetized plasma.
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1. Introduction

The equations of the relativistic magnetohydrodynamics (RMHD) of a magnetoactive plasma in a gravitational field were formulated in [1]‡ using the equality requirements for the dynamic velocities of the plasma and the electromagnetic field§. These equations were obtained on the basis of the Einstein and Maxwell equations. A remarkable class of *exact solutions* of these RMHD equations was also found. It explains the motion of a magnetoactive locally isotropic plasma in the field of a plane gravitational wave (PGW). This class was called *gravimagnetic shock waves* (GMSW). It describes *essentially nonlinear processes* which do not exist in the linear approximation of magnetohydrodynamics and essentially relativistic processes in terms of predominance of the massless electromagnetic component in the magnetoactive plasma. It was shown in [2] that the GMSW in pulsar magnetospheres may be the highly effective detectors of gravitational waves from neutron stars. Particularly, giant pulses which sporadically appear in the radiation of some pulsars may be observational results of energy transfer from a gravitational wave to GMSW. Estimations made in [2]-[4] make it possible to connect giant pulses in radiation of the pulsar B0531+21 with gravitational radiation in the basic mode of oscillations from this pulsar. In fact, at present it is rather difficult to speak of identification of giant pulses as an electromagnetic display of gravimagnetic shock wave evolution in the pulsar magnetosphere and to unambiguously connect these pulses with the pulsar's gravitational radiation. Nevertheless, the idea of analyzing the effect of gravitational waves from a compact astrophysical object on its own electromagnetic radiation is highly productive for solving the problems of gravitational waves detection. In fact, the main difficulties of gravitational waves detection in the Earth environment are:

(i) An extremely small amplitude of gravitational waves on Earth ($h \lesssim 10^{-19}$) due to significant distances from relativistic astrophysical objects.

(ii) A sporadic nature of events leading to radiation of gravitational waves inside relativistic astrophysical objects with sufficient power. This does not allow one to unambiguously connect a received signal with a fact of gravitation radiation detection.

(iii) Impossibility to construct relativistic detectors with anomalous, highly effective parameters for gravitational wave detection in conditions terrestrial laboratory (superstrong magnetic fields, a highly anisotropic working body of the detector, low level of background noise etc.).

It is possible to avoid these problems if one could transfer a detector directly to a close neighborhood of a relativistic astrophysical object. In this case, one always has a ready electromagnetic signal and there is no need to convert it to other forms, which allows for conducting correlation analysis. If the working body of the detector is the magnetosphere of a relativistic astrophysical object, the optimal parameters for gravitational wave detection are achieved automatically: super-strong magnetic fields, an ultrarelativistic equation of state, highly anisotropy etc. The fundamental importance of the GMSW for gravitational theory, as a direct conversion effect of gravitational wave energy into electromagnetic energy, leads to a necessity of a more detailed and comprehensive researches. In [5], a strict foundation of the GMSW

‡ Before 2000 Yu.G. Ignatyev wrote his name as Yu.G. Ignat'ev.
§ This requirement is completely equivalent to the condition of plasma infinite conductivity, see Ref.[1].

hydrodynamic theory was formulated on the basis of the relativistic kinetic theory. As has been shown in [1]-[4], a GMSW is realized in an essentially collisionless nonequilibrium plasma in anomalously strong magnetic fields. The isotropy of a local plasma electron distribution essentially is violated under such conditions due to strong bremsstrahlung. Therefore, the anisotropy factor of a magnetoactive plasma is highly essential for the efficiency of the GMSW formation mechanism. A hydrodynamic model of GMSW in an anisotropic plasma was constructed in [6] for a specified relation between parallel and perpendicular components of the plasma pressure. The analysis in [6] was based on the general RMHD equations. Particularly, an elementary linear relation was considered in [6]. This study has revealed a strong dependence of the GMSW effect on the plasma degree of anisotropy. That fact has led to a necessity of constructing a dynamic model for the motion of an anisotropic magnetoactive plasma in the field of gravitational radiation.

Further, in [7] a detailed numerical model of GMSW has been carried out at various parameters of the anisotropic magnetoactive plasma in the computer algebra system Mathematica. This study was based on the numerical solution of the nonlinear energybalance equation by means of special methods of numerical integration. The results received in [7] have confirmed earlier made analytical estimations of the magnetoactive plasmas behavior in a field of strong gravitational wave and have defined more exactly some characteristics of GMSW.

However, at all variety of models of plasma only the case of monopolarized gravitational wave with polarization $\mathbf{e}_+$ investigated in all quoted papers. It has been thus shown that the case of monopolarized gravitational wave with polarization $\mathbf{e}_\times$ is reduced to the case of $\mathbf{e}_+$ polarization state at the coordinate and physical quantities transformations. However, the case of simultaneously existence of two polarizations of a gravitational wave was not investigated. In this paper we consider such a case. Thus it was possible to reduce some additional conditions, which in [1] have defined the structure of potential electromagnetic field in a magnetized plasma, and thereby give a more general meaning to results obtained in preceding papers. In this paper adopted a system of units where ($c = G = \hbar = 1$).

2. Self-consistent RMHD equations in a gravitational field

2.1. Frozen-in condition of magnetic field in plasma

In [1] under the assumption of equality of dynamic timelike velocity v^i of a plasma and electromagnetic field‖:

$$\overset{p}{T}_{ij}\, v^j = \varepsilon_p v_i; \quad \overset{f}{T}_{ij}\, v^j = \varepsilon_f v_i; \quad (v, v) = 1 \tag{1}$$

on the basis of conservation of the total energy-momentum tensor of a plasma and electromagnetic field,

$$T^{ij} = \overset{p}{T}{}^{ij} + \overset{f}{T}{}^{ij}, \tag{2}$$

$$T^{ij}_{\ ,j} = 0 \tag{3}$$

full self-consistent system of relativistic magnetohydrodynamic equations for magnetized plasma in arbitrary gravitational field has been obtained. It describes the

‖ The index "p" refers to the plasma, the index "f" to the field, a comma denote covariant derivatives.

motion of a relativistic plasma and an electromagnetic field in the given gravitational field.

In particular, it was shown that at positivity of the first invariant of an electromagnetic field:

$$\mathrm{Inv}_1 = F_{ij}F^{ij} = 2H^2 > 0 \tag{4}$$

and equality to zero of the second invariant:

$$\mathrm{Inv}_2 = \overset{*}{F}_{ij}\, F^{ij} = 0 \tag{5}$$

necessary and sufficient condition for the solvability of equations (1) is *the frozen-in condition of magnetic field in plasma*, i.e., equality to zero of the accompanying intensity of the electric field E_i:

$$E_i = F_{ji}v^j = 0. \tag{6}$$

In formulas (4)-(6) and further: F_{ij} - antisymmetric Maxwell tensor, $\overset{*}{F}_{ij}$ - dual Maxwell tensor.

$$\overset{*}{F}_{ij} = \frac{1}{2}\eta_{ijkl}F^{kl}, \tag{7}$$

where η_{kilm} - covariantly constant discriminant tensor (Levi-Civita's tensor) [8].

At the frozen-in condition (6) fulfilment, the condition of dynamic velocities equality (1) is always satisfied, regardless of the conditions (4)-(5). On the basis of rigorous kinetic model of plasma one of the authors has shown that the frozen-in condition is a consequence of *the drift approximation*, i.e., the smallness of Larmor length for electrons $\lambda_e = c/\omega_c$ in comparison with the characteristic inhomogeneity scale, r:

$$\Lambda = \frac{c}{r\omega_e} \ll 1; \quad \omega_c = \frac{eH}{m_e}, \tag{8}$$

where ω_c is Larmor frequency for electrons.

2.2. Self-consistent equations of magnetohydrodynamics

The complete system of self-consistent RMHD equations for a plasma in a gravitational field, obtained in [1], consists of Maxwell equations of the first group:

$$\overset{*}{F}{}^{ik}{}_{,k} = 0; \tag{9}$$

Maxwell equations of the second group:

$$F^{ik}{}_{,k} = -4\pi J^i_{\mathrm{dr}} \tag{10}$$

with spacelike *drift current*:

$$J^i_{\mathrm{dr}} = -\frac{2F^{ik}\,\overset{p}{T}{}^{l}_{k,l}}{F_{jm}F^{jm}}; \quad (J_{\mathrm{dr}}, J_{\mathrm{dr}}) < 0 \tag{11}$$

and a conservation law of the total energy-momentum of the system:

$$\overset{p}{T}{}^{ik}{}_{,k} + \overset{f}{T}{}^{ik}{}_{,k} = 0. \tag{12}$$

The continuity equation for the drift current must be satisfied in consequence of Eq. (10):

$$J^i_{\mathrm{dr}\ ,\mathrm{i}} = 0. \tag{13}$$

It should be noted some useful strict consequences of magnetohydrodynamics equations:

$$\overset{*}{F}_{ik}\, J^k_{\rm dr} = 0; \tag{14}$$

$$v^i\, \overset{p}{T}{}^{ik}_{i,k} = 0; \tag{15}$$

$$H^i\, \overset{p}{T}{}^{k}_{i,k} = 0. \tag{16}$$

2.3. Maxwell tensor representation by accompanying intensities

The components of Maxwell tensor is conveniently represented by a pair of spacelike *vectors of accompanying intensities* of electric, E_i (6), and magnetic, H_i, fields [9]:

$$E_i = F_{ji}v^j; \quad H_i = \overset{*}{F}_{ji}\, v^j, \tag{17}$$

so that:

$$(E,E) = -E^2; \quad (H,H) = -H^2; \quad (v,E) = 0; \quad (v,H) = 0. \tag{18}$$

Then Maxwell tensor and dual to it can be expressed through a pair of spacelike vectors of this accompanying intensities [9]:

$$F_{ij} = v_iE_j - v_jE_k - \eta_{ijkl}v^kH^l; \tag{19}$$

$$\overset{*}{F}_{ij} = v_iH_j - v_jH_k + \eta_{ijkl}v^kE^l, \tag{20}$$

where:

$$\frac{1}{2}F_{ij}F^{ij} = \frac{1}{2}\, \overset{*}{F}_{ij}\overset{*}{F}{}^{ij} = (E,E) - (H,H) = H^2 - E^2; \tag{21}$$

$$\frac{1}{2}F_{ij}\, \overset{*}{F}{}^{ij} = (E,H). \tag{22}$$

The energy-momentum tensor (EMT) of the electromagnetic field

$$\overset{f}{T}{}^i_k = \frac{1}{4\pi}\left(F^i{}_lF^l{}_k + \frac{1}{4}\delta^i_kF^{lm}F_{lm}\right) \tag{23}$$

can also be represented by the triplet of vectors v, E, H (see [1]). In the case of coincidence plasma's and electromagnetic field's dynamic velocities (1) the EMT is expressed through a pair of vectors, v, H [1]:

$$\overset{f}{T}{}^i_k = -\frac{1}{8\pi}\left[(\delta^i_k - 2v^iv_k)H^2 + 2H^iH_k)\right], \tag{24}$$

so:

$$\overset{f}{T} \equiv \overset{f}{T}{}^i_i = 0. \tag{25}$$

It is easy to verify that the vector v and the spacelike unit vector of the magnetic field h –

$$h^i = \frac{H^i}{H}; \quad (h,h) = -1; \quad (v,h) = 0 \tag{26}$$

– are actually the eigenvectors of the tensor $\overset{f}{T}{}^{ik}$:

$$\overset{f}{T}{}^i_kv^k = \varepsilon_Hv^i; \tag{27}$$

$$\overset{f}{T}{}^i_kh^k = \varepsilon_Hh^i, \tag{28}$$

where the invariant

$$\varepsilon_H = \frac{H^2}{8\pi} \tag{29}$$

is the energy density of the magnetic field.

2.4. Energy-momentum tensor of magnetoactive plasma

The energy-momentum tensor of a relativistic anisotropic magnetoactive plasma in gravitational and magnetic fields is (see, for example, [6]):

$$\overset{p}{T}{}^{ij} = (\varepsilon + p_\perp)v^i v^j - p_\perp g^{ij} + (p_\parallel - p_\perp)h^i h^j, \tag{30}$$

where $p_\perp, p_\parallel$ - plasmas pressure in the directions orthogonal and parallel to the magnetic field, respectively. Trace of the energy-momentum tensor (30) is:

$$\overset{p}{T} \equiv \overset{p}{T}{}^i_i = \varepsilon - p_\perp - 2p_\parallel \geq 0 \tag{31}$$

and because of the virial theorem (see [10]) it is non-negative:

$$p_\perp + 2p_\parallel \leq \varepsilon. \tag{32}$$

It is easy to verify that the vectors v and h are also eigenvectors of the energy-momentum tensor of plasma (see (27), (28))

$$\overset{p}{T}{}^{ik} v_k = \varepsilon v^i; \tag{33}$$

$$\overset{p}{T}{}^{ik} h_k = -p_\parallel h^i. \tag{34}$$

3. Solving RMHD equations in the PGW metric

3.1. The metric of a plane gravitational wave

The vacuum PGW metric is (see, for example, [11]):

$$ds^2 = 2dudv - L^2 d\Sigma^2, \tag{35}$$

where:

$$d\Sigma^2 = \cosh 2\gamma (e^{2\beta}(dx^2)^2 + e^{-2\beta}(dx^3)^2) - 2\sinh 2\gamma dx^2 dx^3 \tag{36}$$

- is a metric of "plane" (x^2, x^3); $\beta(u)$, $\gamma(u)$ - amplitudes of the polarization $\mathbf{e}_+$ and $\mathbf{e}_\times$, respectively; $u = \frac{1}{\sqrt{2}}(t - x^1)$ is the retarded time, $v = \frac{1}{\sqrt{2}}(t + x^1)$ is the advanced time. The amplitudes of PGW are arbitrary functions of the retarded time u, and $L(u)$ is *a background factor* of PGW, which defined by single nontrivial vacuum Einstein's equation¶:

$$L'' + L(\cosh^2 2\gamma \beta'^2 + \gamma'^2) = 0. \tag{37}$$

At inversion of the coordinates in the plane (x^2, x^3) and transformation of the PGW amplitude:

$$x^2 = x'^3; \quad x^3 = x'^2; \quad \beta' = -\beta; \quad \gamma' = \gamma \tag{38}$$

two-dimensional metric transforms into itself. Under rotations in the plane (x^2, x^3) by the $\pi/4$ angle:

$$x^2 = \frac{1}{\sqrt{2}}(x'^2 + x'^3); \; x^3 = \frac{1}{\sqrt{2}}(x'^3 + x'^2) \tag{39}$$

two-dimensional metric is transformed to:

$$\begin{aligned} d\Sigma'^2 = &(\cosh 2\gamma \cosh 2\beta + \sinh 2\gamma)(dx'^2)^2 + \\ &(\cosh 2\gamma \cosh 2\beta - \sinh 2\gamma)(dx'^3)^2 + 2\cosh 2\gamma \sinh 2\beta dx'^2 dx'^3. \end{aligned} \tag{40}$$

¶ The prime denotes differentiation with respect to the retarded time u.

If $\beta = 0$, i.e., in the case of PGW with a single polarization $\mathbf{e}_\times$, we get from (40):

$$d\Sigma'^2 = e^{2\gamma}(dx'^2)^2 + e^{-2\gamma}(dx'^3)^2$$

– a PGW metric with a single polarization of $\mathbf{e}_+$.

For a weak gravitational wave:

$$|\beta(u)| \ll 1;\ |\gamma(u)| \ll 1;\ L^2(u) = 1 + O^2(|\beta, \gamma|) \qquad (41)$$

rotation (39) is equivalent to the transformation of inversion:

$$\beta' = \gamma; \quad \gamma' = -\beta. \qquad (42)$$

3.2. Initial conditions

Let in the absence of PGW ($u \leq 0$):

$$\beta(u \leq 0) = 0; \quad \beta'(u \leq 0) = 0; \quad L(u \leq 0) = 1, \qquad (43)$$

plasma is homogeneous and at rest:

$$v^v(u \leq 0) = v^u(u \leq 0) = 1/\sqrt{2}; \quad v^2 = v^3 = 0;$$
$$\varepsilon(u \leq 0) = \overset{0}{\varepsilon}; \quad p_\parallel(u \leq 0) = \overset{0}{p}_\parallel; \qquad p_\perp(u \leq 0) = \overset{0}{p}_\perp, \qquad (44)$$

and homogeneous magnetic field is directed in the (x^1, x^2) plane:

$$H_1(u \leq 0) = \overset{0}{H} \cos\Omega\,; \quad H_2(u \leq 0) = \overset{0}{H} \sin\Omega\,;$$
$$H_3(u \leq 0) = 0\,; \quad E_i(u \leq 0) = 0, \qquad (45)$$

where Ω is the angle between the axis $0x^1$ (the PGW propagation direction) and the magnetic field $\mathbf{H}$.

As we noted above, the effect of a PGW with polarization $\mathbf{e}_+$ on a homogeneous plasma at the initial conditions (44)-(45) was considered in the quoted papers [1]-[6]. Taking into account transformational properties of the metric noted in section 3.1, it means that the effect of a monopolarized PGW on a homogeneous plasma has been considered earlier when the projection of vector $\overset{0}{\mathbf{H}}$ on the plane of PGW's front is parallel to the polarization axis. The case, when this projection coincides with the direction x^2 or x^3, can be reduced to the case of polarization $\mathbf{e}_+$ or $\mathbf{e}_\times$ using the substitution (38). And vice versa: the case of different polarizations $\mathbf{e}_+$ or $\mathbf{e}_\times$ can be reduced to the case with different projections on the direction x^2 or x^3 under rotation in the PGW's front plane by the angle $\pi/4$ together with rotation of the vector of magnetic field intensity. For understanding of a mechanism of strong PGW interaction with an anisotropic magnetoactive plasma it is essentially important to consider the combined case, when a PGW possesses both polarization states simultaneously, and the projection of the vector of magnetic field intensity on the PGW's front plane is parallel to the axis of one of them. The initial conditions (44)-(45) correspond to this case.

3.3. Symmetry of the problem

As is well known, the metric (35) permits the group of motions G_5, associated with three linearly independent in a point Killing vectors:

$$\underset{(1)}{\xi^i} = \delta^i_v\,; \qquad \underset{(2)}{\xi^i} = \delta^i_2\,; \qquad \underset{(3)}{\xi^i} = \delta^i_3\,. \qquad (46)$$

In consequence of the Killing vectors existence in the metric (35), all of the geometric objects, including the Christoffel symbols, the Riemann tensor, the Ricci tensor and, consequently, the energy-momentum tensor of a magnetoactive plasma, are automatically conserved at motions along the Killing's directions:

$$\underset{\xi_\alpha}{\mathrm{L}}\, g_{ij} = 0; \Rightarrow \underset{\xi_\alpha}{\mathrm{L}}\, R_{ij} = 0; \Rightarrow \underset{\xi_\alpha}{\mathrm{L}}\, T_{ij} = 0, \tag{47}$$

where $\underset{\xi}{\mathrm{L}}\, T_{ij}$ is a Lie derivative in the direction of ξ:

$$\underset{\xi}{\mathrm{L}}\, T_{ij} = T_{ij,k}\xi^k + T_{kj}\xi^{\ j}_{,i} + T_{ik}\xi^{\ k}_{,j}. \tag{48}$$

Further we require that tensors of energy-momentum of the plasma $\overset{p}{T}_{ij}$ and the electromagnetic field $\overset{f}{T}_{ij}$ inherit the symmetry separately:

$$\underset{\xi_\alpha}{\mathrm{L}}\, \overset{p}{T}_{ij} = 0; \tag{49}$$

$$\underset{\xi_\alpha}{\mathrm{L}}\, \overset{f}{T}_{ij} = 0; \quad (\alpha = \overline{1,3}). \tag{50}$$

Consequences of (50) are:

$$\underset{\xi_\alpha}{\mathrm{L}}\, F_{ij} = 0,\ \underset{\xi_\alpha}{\mathrm{L}}\, \overset{*}{F}_{ij} = 0 \Longrightarrow \underset{\xi_\alpha}{\mathrm{L}}\, H = 0,\ \underset{\xi_\alpha}{\mathrm{L}}\, E_i = 0,\ \underset{\xi_\alpha}{\mathrm{L}}\, H_i = 0. \tag{51}$$

Consequences of (49) and (51) are:

$$\underset{\xi_\alpha}{\mathrm{L}}\, \varepsilon = 0,\ \underset{\xi_\alpha}{\mathrm{L}}\, v^i = 0,\ \underset{\xi_\alpha}{\mathrm{L}}\, p_\perp = 0,\ \underset{\xi_\alpha}{\mathrm{L}}\, p_\parallel = 0. \tag{52}$$

Thus, all observed physical quantities $\mathbf{P}$ *inherit the symmetry of the metric* (35):

$$\underset{\xi_\alpha}{\mathrm{L}}\, \mathbf{P} = 0; \qquad (\alpha = \overline{1,3}), \tag{53}$$

i.e., taking into account the explicit form of Killing vectors (46):

$$p = p(u); \quad \varepsilon = \varepsilon(u); \quad v^i = v^i(u); \tag{54}$$

$$F_{ik} = F_{ik}(u); \quad H_i = H_i(u); \quad h_i = h_i(u). \tag{55}$$

3.4. Maxwell tensor

In this section we obtain an expression for the vector potential of the electromagnetic field in the metric (35), taking into account the initial conditions (44)-(45). This method differs from the method used in [1]. It is based only on the first group of Maxwell equations and the initial conditions and therefore have greater generality. The vector potential conformed with the initial conditions (45) is:

$$A_v = A_u = A_2 = 0;$$
$$A_3 = \overset{0}{H}(x^1 \sin\Omega - x^2 \cos\Omega); \qquad (u \le 0). \tag{56}$$

These conditions conform with follow components of Maxwell tensor:

$$F_{23}(u \le 0) = -\overset{0}{H}\cos\Omega;\ F_{v3}(u \le 0) = \frac{1}{\sqrt{2}}\overset{0}{H}\sin\Omega;\ F_{v2}(u \le 0) = 0;$$

$$F_{u2}(u \le 0) = 0;\ F_{u3}(u \le 0) = -\frac{1}{\sqrt{2}}\overset{0}{H}\sin\Omega;\ F_{uv}(u \le 0) = 0. \tag{57}$$

As known (see [10]), the first group of Maxwell equations (9) is equivalent to the existence condition of a vector potential. It can be written as:

$$\frac{1}{\sqrt{-g}}\partial_j\sqrt{-g}\stackrel{*}{F}{}^{ij}=0. \tag{58}$$

Considering (55), we get:

$$L^2\stackrel{*}{F}{}^{u\alpha}=\mathcal{C}_{(\alpha)}\quad(=\mathrm{Const});\qquad \alpha=\{\mathrm{v},2,3\}, \tag{59}$$

setting here and further the following order of the coordinates:

$$\mathrm{Coords}:=[\mathrm{v},\mathrm{u},\mathrm{x}^2,\mathrm{x}^3], \tag{60}$$

Let us establish a connection between the components of Maxwell tensor with the components of tensor dual to it:

$$\stackrel{*}{F}{}^{uv}=-\frac{1}{L^2}F_{23};\quad \stackrel{*}{F}{}^{u2}=\frac{1}{L^2}F_{v3};\quad \stackrel{*}{F}{}^{u3}=-\frac{1}{L^2}F_{v2};$$
$$\stackrel{*}{F}{}^{v2}=\frac{1}{L^2}F_{u3};\quad \stackrel{*}{F}{}^{v3}=\frac{1}{L^2}F_{u2};\quad \stackrel{*}{F}{}^{23}=-\frac{1}{L^2}F_{uv}. \tag{61}$$

Then the initial conditions (45) give:

$$L^2\stackrel{*}{F}{}^{uv}=-F_{23}=\stackrel{0}{H}\cos\Omega; \tag{62}$$

$$L^2\stackrel{*}{F}{}^{u2}=F_{v3}=\frac{1}{\sqrt{2}}\stackrel{0}{H}\sin\Omega; \tag{63}$$

$$L^2\stackrel{*}{F}{}^{u3}=-F_{v2}=0. \tag{64}$$

Thus, the second invariant of the electromagnetic field is equal to:

$$\mathrm{Inv}_2=F_{ik}\stackrel{*}{F}{}^{ik}=\frac{2}{L^2}(F_{v3}F_{u2}-F_{23}F_{uv}), \tag{65}$$

So, taking into account (63), (64), the equality to zero of the second invariant of an electromagnetic field (5) is reduced to the relation:

$$L^2\stackrel{*}{F}{}^{v3}\equiv F_{u2}=-\sqrt{2}F_{uv}\cot\Omega. \tag{66}$$

As it is known (see for example [10]), the first group of Maxwell equations is equivalent to the existence condition of a vector potential A_i:

$$F_{ik}=\partial_iA_k-\partial_kA_i. \tag{67}$$

Let us notice that as opposed to Maxwell tensor, the components of the vector potential A_i can depend on the variables v,x^2,x^3. We write down expressions (62)-(66) relative to the vector potential A_i using definition of Maxwell tensor (67):

$$\partial_3A_2-\partial_2A_3=\stackrel{0}{H}\cos\Omega; \tag{68}$$

$$\partial_vA_3-\partial_3A_v=\frac{1}{\sqrt{2}}\stackrel{0}{H}\sin\Omega; \tag{69}$$

$$\partial_vA_2-\partial_2A_v=0; \tag{70}$$

$$\partial_uA_2-\partial_2A_u=-\sqrt{2}\cot\Omega(\partial_uA_v-\partial_vA_u)) \tag{71}$$

Introducing new functions:

$$\tilde{A}_2=A_2-\stackrel{0}{H}\cos\Omega\;x^3\equiv A_2-\delta A_2; \tag{72}$$

$$\tilde{A}_v=A_v+\frac{1}{\sqrt{2}}\stackrel{0}{H}\sin\Omega\;x^3\equiv A_v-\delta A_v, \tag{73}$$

$$\tilde{A}_3=A_3, \tag{74}$$

where:

$$\delta A_2 = \overset{0}{H} \cos\Omega\ x^3; \quad \delta A_v = -\frac{1}{\sqrt{2}} \overset{0}{H} \sin\Omega\ x^3; \quad \delta A_3 = 0, \tag{75}$$

let us reduce the relations (68) and (69) to the form similar to (70):

$$\partial_3 \tilde{A}_2 - \partial_2 A_3 = 0; \tag{76}$$

$$\partial_v A_3 - \partial_3 \tilde{A}_v = 0. \tag{77}$$

Let us notice that the renormalization of the component of the vector potential (72), (73) keeps the relation (70) invariable. But then it is possible to write down the system of equations (70), (76), (77) as:

$$\partial_\sigma \tilde{A}_\delta - \partial_\delta \tilde{A}_\sigma = 0; \quad (\sigma, \delta = v, 2, 3) \tag{78}$$

and to consider it as equations on a three-dimensional hypersurface $V^3 = \{v, x^2, x^3\}$. As it is known, the unique solution of equations (78) on V^3 is a gradient function:

$$\tilde{A}_\sigma = \partial_\sigma \Phi, \quad (\sigma = v, 2, 3), \tag{79}$$

where $\Phi = \Phi(u, v, x^2, x^3)$ is an arbitrary scalar function. The value of the potential function corresponding to the initial conditions (56) is:

$$\Phi(u \leq 0) = x^3 \overset{0}{H} \left(\frac{1}{\sqrt{2}} (v - u) \sin\Omega - x^2 \cos\Omega \right). \tag{80}$$

Thus

$$A_\sigma = \partial_\sigma \Phi + \delta A_\sigma. \tag{81}$$

As it is known (see, for example, [10]), it is possible to impose one gauge condition on 4 components of a vector potential. We choose this condition in the form corresponding to the initial conditions (56):

$$A_u = 0. \tag{82}$$

Then for the nonconserved components of the Maxwell tensor $F_{u\sigma}$ is valid:

$$F_{u\sigma} = \partial_{u\sigma} \Phi; \quad (\sigma = v, 2, 3). \tag{83}$$

But then condition (71) can be written down in the form:

$$\partial_u (A_2 + \sqrt{2} \cot\Omega A_v) = 0. \tag{84}$$

Integrating (84) with the initial conditions (56), we obtain:

$$A_2 + \sqrt{2} \cot\Omega A_v = 0. \tag{85}$$

Taking into account the identity:

$$\delta A_2 + \sqrt{2} \cot\Omega \delta A_v \equiv 0, \tag{86}$$

we obtain the linear differential equation from (85):

$$\partial_2 \Phi + \sqrt{2} \cot\Omega \partial_v \Phi = 0.$$

Integrating it, we obtain:

$$\Phi = \Phi(v\sqrt{2} \sin\Omega - x^2 \cos\Omega, u, x^3), \tag{87}$$

where Φ is an arbitrary function of its arguments. Using now the initial condition (80) we obtain finally:

$$\Phi = x^3 \overset{0}{H} \left(\frac{1}{\sqrt{2}} (v - \psi(u)) \sin\Omega - x^2 \cos\Omega \right), \tag{88}$$

where $\psi(u)$ is an arbitrary function of the retarded time, satisfying the initial condition:

$$\psi(u \leq 0) = u. \tag{89}$$

Thus, the final expression for components of the vector potential becomes:

$$A_2 = A_v = A_u = 0; \quad A_3 = \overset{0}{H}\left(\frac{1}{\sqrt{2}}(v - \psi(u))\sin\Omega - x^2\cos\Omega\right). \tag{90}$$

The components of the Maxwell tensor relative to the potential (90) are equal to:

$$F_{vu} = 0; \quad F_{2u} = 0; \quad F_{3u} = \frac{1}{\sqrt{2}}\overset{0}{H}\psi'\sin\Omega;$$
$$F_{2v} = 0; \quad F_{3v} = -\frac{1}{\sqrt{2}}\overset{0}{H}\sin\Omega; \quad F_{23} = -\overset{0}{H}\cos\Omega \tag{91}$$

and are defined only by one unknown function $\psi(u)$. For the components of the dual Maxwell tensor (7) we get:

$$\overset{*}{F}{}^{vu} = \frac{1}{L^2}\overset{0}{H}\cos\Omega; \quad \overset{*}{F}{}^{2u} = -\frac{1}{\sqrt{2}L^2}\overset{0}{H}\sin\Omega; \quad \overset{*}{F}{}^{3u} = 0;$$
$$\overset{*}{F}{}^{2v} = \frac{1}{\sqrt{2}L^2}\overset{0}{H}\psi'\sin\Omega; \quad \overset{*}{F}{}^{3v} = 0; \quad \overset{*}{F}{}^{23} = 0. \tag{92}$$

3.5. Accompanying intensities and the frozen-in condition

According to (17), we define the components of the vector of accompanying intensity of the electric field, E_i, as:

$$E_v = -\frac{1}{\sqrt{2}}\overset{0}{H}\sin\Omega\, v^3; \quad E_u = \frac{1}{\sqrt{2}}\overset{0}{H}\psi'\sin\Omega\, v^3;$$
$$E_2 = \overset{0}{H}\cos\Omega\, v^3; \quad E_3 = \frac{1}{\sqrt{2}}\overset{0}{H}\sin\Omega(v^v - \psi' v^u) - \overset{0}{H}\cos\Omega\, v^2. \tag{93}$$

Thus, the frozen-in condition of magnetic field in plasma (6) reduces to two equalities:

$$v^3 = 0; \quad \frac{1}{\sqrt{2}}(v_v\psi' - v_u)\sin\Omega + v^2\cos\Omega = 0. \tag{94}$$

As a result, covariant components of the Maxwell tensor, contravariant components of the dual Maxwell tensor and contravariant components of the velocity vector are defined by the expressions, obtained in [1], but now they are already defined for a more general metric of a gravitational wave and at weaker assumptions. In the quoted paper, in particular, to obtain the explicit form of the Maxwell tensor components and the velocity vector components, the analysis of the drift current components was carried out using the conservation law of this current. As it was shown above, for achievement of this purpose three assumptions are sufficient:

1. inheritance of the space symmetry by the energy-momentum tensor of the electromagnetic field and by the energy-momentum tensor of the plasma separately;
2. the equality to zero of the second invariant of the Maxwell tensor;
3. the frozen-in condition of magnetic field in plasma.

Thus, the analysis of the first group of Maxwell equations and initial conditions is sufficient.

Calculating further the covariant components of the dual Maxwell tensor, subject to (92), we get:

$$\overset{*}{F}_{uv}=\frac{\overset{0}{H}}{L^2}\cos\Omega;\ \overset{*}{F}_{u2}=\frac{\overset{0}{H}}{\sqrt{2}}e^{2\beta}\cosh 2\gamma\psi'\sin\Omega;\ \overset{*}{F}_{v2}=-\frac{\overset{0}{H}}{\sqrt{2}}e^{2\beta}\cosh 2\gamma\sin\Omega;$$

$$\overset{*}{F}_{v3}=\frac{\overset{0}{H}}{\sqrt{2}}\sinh 2\gamma\sin\Omega;\quad \overset{*}{F}_{u3}=-\frac{\overset{0}{H}}{\sqrt{2}}\sinh 2\gamma\psi'\sin\Omega;\quad \overset{*}{F}_{23}=0. \qquad (95)$$

Covariant components of the vector of magnetic field intensity relative to the Maxwell tensor (92) are equal to:

$$H_v=-\frac{\overset{0}{H}}{L^2}\left(v_v\cos\Omega+\frac{1}{\sqrt{2}}v^2\sin\Omega\right); \qquad (96)$$

$$H_u=\frac{\overset{0}{H}}{L^2}\left(v_u\cos\Omega-\frac{1}{\sqrt{2}}v^2\psi'\sin\Omega\right); \qquad (97)$$

$$H_2=-\frac{1}{\sqrt{2}}\overset{0}{H}\cosh 2\gamma e^{2\beta}\sin\Omega(v_v\psi'+v_u); \qquad (98)$$

$$H_3=\frac{1}{\sqrt{2}}\overset{0}{H}\sinh 2\gamma\sin\Omega(v_v\psi'+v_u). \qquad (99)$$

It is thus easy to show on the basis of formula (92):

$$H^3=\overset{*}{F}{}^{i3}v_i=0, \qquad (100)$$

i.e., the third contravariant coordinate of the vector of the magnetic field intensity, as well as the vector of dynamic velocity of the plasma, is equal to zero. Also it is easy to be convinced of orthogonality of the velocity vector and the magnetic field intensity (18):

$$H_iv^i\equiv 0. \qquad (101)$$

The square of the magnetic field intensity, i.e., a scalar H^2, most easier to calculate by means of the relation (21), using the explicit form of contravariant (92) and covariant (95) components of the dual Maxwell tensor:

$$H^2=\frac{\overset{0}{H}{}^2}{L^4}\left(L^2\psi'\cosh 2\gamma e^{2\beta}\sin^2\Omega+\cos^2\Omega\right). \qquad (102)$$

The frozen-in conditions of magnetic field in plasma (94) establishes the connection between nonzero contravariant components of the velocity vector $v^2, v^v=v_u, v^u=v_v$. Besides, there is still the normalization relation of velocity vector (1). Therefore, the only one independent coordinate of the velocity vector remains, and the electromagnetic field is defined by only one unknown function of the retarded time, $\psi(u)$. Using (96)-(102) the normalization relation of velocity vector can be written in the equivalent form:

$$\left[v_v\cos\Omega+v_2\frac{1}{\sqrt{2}}\sin\Omega\right]^2=\frac{H^2}{\overset{0}{H}{}^2}v_v^2L^4-\frac{\sin^2\Omega}{2}L^2\cosh 2\gamma e^{2\beta}. \qquad (103)$$

3.6. Drift current

Let us calculate components of a drift current, using Maxwell equations (10), considering the dependence of Maxwell tensor components only on the retarded time (55):

$$J^i_{\rm dr} = -\frac{1}{4\pi L^2}\partial_u(L^2 F^{iu}). \qquad (104)$$

Then:

$$J^v_{\rm dr} = J^u_{\rm dr} = 0\,; \qquad (105)$$

$$J^2_{\rm dr} = -\frac{\overset{0}{H}\sin\Omega}{2\sqrt{2}\pi L^2}\cosh 2\gamma\cdot\gamma'\,; \qquad (106)$$

$$J^3_{\rm dr} = -\frac{\overset{0}{H}\sin\Omega e^{2\beta}}{2\sqrt{2}\pi L^2}(\sinh 2\gamma\cdot\gamma' + \cosh 2\gamma\cdot\beta')\,. \qquad (107)$$

Calculating scalar product of the vector of the magnetic field intensity and the vector of drift current density, using (98), (99), (105), (106), (107), we get:

$$(J_{\rm dr}, H) = \frac{\overset{0}{H}{}^2}{4\pi L^2}(v_v\psi' + v_u)(\gamma' - \frac{\beta'}{2}\sinh 4\gamma). \qquad (108)$$

Thus, the presence of the second polarization of a gravitational wave leads to violation of the orthogonality of the vectors of drift current density and magnetic field intensity$^+$.

Using expression (11), it is possible to show that the equality (105) is carried out only in the case of transverse propagation of the PGW ($\Omega = \pi/2$).

3.7. Integrals of the motion

Because of existence of the motions (46), Killing equations are satisfied:

$$\underset{(\alpha)}{\xi}{}_{i,k} + \underset{(\alpha)}{\xi}{}_{k,i} = 0, \quad (\alpha = \overline{1,3}). \qquad (109)$$

Therefore conservation laws of the total EMT in a field of PGW after consistently transvection with all Killing's vectors (46) can be written down in the form:

$$\frac{1}{\sqrt{-g}}(\partial_k\sqrt{-g}\,\underset{(\alpha)}{\xi}{}^i T^k_i) = 0; \quad (\alpha = \overline{1,3}). \qquad (110)$$

Taking into account the fact that EMT components can depend only on the retarded time, we obtain following integrals [1]:

$$L^2\,\underset{(\alpha)}{\xi}{}^i T_{vi} = C_a = {\rm Const}; \quad (\alpha = \overline{1,3})\,. \qquad (111)$$

In this paper we consider only the case of *transverse propagation* of the PGW ($\Omega = \pi/2$). Then, substituting expressions for the EMT of the plasma and electromagnetic field in the integrals (111), using relations (99)-(103) and also initial conditions (43), (44), we lead integrals of the motion to the form:

$$2L^2(\varepsilon + p_\|)v_v^2 - (p_\| - p_\perp)\frac{\overset{0}{H}{}^2}{H^2}\cosh 2\gamma e^{2\beta} = (\overset{0}{\varepsilon} + \overset{0}{p})\Delta(u)\,; \qquad (112)$$

$^+$ We remind that in case of monopolarized gravitational waves these spacelike vectors are mutually orthogonal [1].

$$L^2(\varepsilon + p_{\|})v_v v_2 = 0\,; \tag{113}$$

$$L^2(\varepsilon + p_{\|})v_v v_3 = 0\,, \tag{114}$$

where:

$$\overset{0}{p} = \overset{0}{p}_{\perp}\,; \tag{115}$$

and so-called *the governing function of GMSW* is introduced:

$$\Delta(u) = 1 - \alpha^2(\cosh 2\gamma e^{2\beta} - 1)\,, \tag{116}$$

with *dimensionless parameter* α^2:

$$\alpha^2 = \frac{\overset{0}{H}{}^2}{4\pi(\overset{0}{\varepsilon} + \overset{0}{p})}\,. \tag{117}$$

Solving (112) with respect to v_v we obtain expressions for coordinates of the velocity vector as functions of the scalars : ε, $p_{\|}$, $p_{\perp}$, ψ' and the explicit functions of the retarded time:

$$v_v^2 = \frac{(\overset{0}{\varepsilon} + \overset{0}{p})}{2L^2(\varepsilon + p_{\|})}\Delta(u) + \frac{(p_{\|} - p_{\perp})}{(\varepsilon + p_{\|})}\frac{\overset{0}{H}{}^2}{H^2}\frac{\cosh 2\gamma e^{2\beta}}{2L^2}\,; \tag{118}$$

From (112) we get:

$$v_2 = 0\,. \tag{119}$$

We obtain the coordinate v_u from a normalization relation of velocity vector, using (118), (119) :

$$v_u = \frac{1}{2v_v}\,, \tag{120}$$

and from the frozen-in condition (94) we get the value of a derivative of potential ψ':

$$\psi' = \frac{1}{2v_v^2}\,, \tag{121}$$

using it, the scalar H^2 is defined from relation (102) as:

$$H^2 = \frac{\overset{0}{H}{}^2}{L^2}\frac{\cosh 2\gamma e^{2\beta}}{2v_v^2}. \tag{122}$$

Let us notice that in the case of an isotropic plasma ($p_{\perp} = p_{\|} = p$) the expression (118) becomes:

$$v_v^2 = \frac{(\overset{0}{\varepsilon} + \overset{0}{p})}{2L^2(\varepsilon + p)}\Delta(u)\,; \tag{123}$$

From RMHD system of equations it is possible to obtain a following differential equation in the PGW metric:

$$L^2\varepsilon' v_v + (\varepsilon + p_{\|})(L^2 v_v)' + \frac{1}{2}L^2(p_{\|} - p_{\perp})v_v(\ln H^2)' = 0\,. \tag{124}$$

Finally, the equation (124) is the differential equation on 3 unknown scalar functions: ε, $p_{\|}$ and $p_{\perp}$. Such underdefiniteness is a known consequence of the incompleteness of hydrodynamic description of a plasma. To solve this equation it is necessary to impose two additional connections between functions ε, $p_{\|}$, $p_{\perp}$, i.e., an equation of state:

$$p_{\|} = f(\varepsilon)\,; \quad p_{\perp} = g(\varepsilon)\,. \tag{125}$$

4. Barotropic equation of state

4.1. General formulas

Let us consider a barotropic state of an anisotropic plasma, when the connections (125) are linear:

$$p_{\|} = k_{\|}\varepsilon\,; \quad p_{\perp} = k_{\perp}\varepsilon\,, \tag{126}$$

The equation (124) is easy to integrate at the connections (126), and we get one more integral:

$$\varepsilon(\sqrt{2}L^2 v_v)^{(1+k_{\|})} H^{(k_{\|}-k_{\perp})} = \overset{0}{\varepsilon}\overset{0}{H}{}^{(k_{\|}-k_{\perp})}\,. \tag{127}$$

Thus, formally the problem is solved, as it is reduced to the solution of the algebraic equations system which, however, is still too difficult to solve and analyse. The solution is essentially defined by two dimensionless parameters: $k_{\perp}$ and $k_{\|}$. Further we consider the special cases of these parameters.

4.2. Transverse propagation of the PGW

In the case of a barotropic equation of state at the connections (126) substitution of (122) in (118) leads to result:

$$v_v^2 = \frac{1}{2}\frac{\overset{0}{\varepsilon}}{L^2\varepsilon}\Delta(u)\,. \tag{128}$$

Substituting (122), (128) in (127), we obtain the closed equation relative to the variable ε, solving which, we get definitively:

$$\varepsilon = \overset{0}{\varepsilon}\left[\Delta^{1+k_{\perp}} L^{2(1+k_{\|})}(\cosh 2\gamma e^{2\beta})^{k_{\|}-k_{\perp}}\right]^{-g_{\perp}}\,; \tag{129}$$

$$v_v = \frac{1}{\sqrt{2}}\left[\Delta L^{(k_{\|}+k_{\perp})}(\cosh 2\gamma e^{2\beta})^{\frac{k_{\|}-k_{\perp}}{2}}\right]^{g_{\perp}}\,; \tag{130}$$

$$H = \overset{0}{H}\left[\Delta L^{(1+k_{\|})}(\cosh 2\gamma e^{2\beta})^{-\frac{1-k_{\|}}{2}}\right]^{-g_{\perp}}\,, \tag{131}$$

where

$$g_{\perp} = \frac{1}{1-k_{\perp}} \in [1,2]\,. \tag{132}$$

In particular, for ultrarelativistic plasma with zero parallel pressure:

$$k_{\|} \to 0\,; \quad k_{\perp} \to \frac{1}{2} \tag{133}$$

we obtain from (129)-(132):

$$v_v = \frac{1}{\sqrt{2}} L\Delta^2 (\cosh 2\gamma e^{2\beta})^{-1/2}\,; \tag{134}$$

$$\varepsilon = \overset{0}{\varepsilon} L^{-4}\Delta^{-3}(\cosh 2\gamma e^{2\beta})\,; \quad H = \overset{0}{H} L^{-2}\Delta^{-2}(\cosh 2\gamma e^{2\beta})\,. \tag{135}$$

5. Energy balance equation

In [1] was shown that the singular state, which exists in a magnetized plasma under the condition $2\beta_0\alpha^2 > 1$ on the hypersurface:

$$\Delta(u_*) = 0\,, \tag{136}$$

is removed using the back effect of the magnetoactive plasma on the GW. That leads to the efficient absorption of GW energy by the plasma and restriction on the amplitude of the GW. An exact solution of the PGW energy transformation to the energy of the shock wave is possible only on the basis of the self-consistent system of Einstein's equations and magnetohydrodynamics equations. However, qualitative analysis of this situation can be carried out using a simple model of energy balance proposed in [2]. The energy flow of the magnetoactive plasma is directed along the direction of the PGW propagation, i.e., along the axis $0x^1$. Let $\beta_*(u)$ and $\gamma_*(u)$ are the vacuum amplitudes of the PGW. In WKB-approximation:

$$8\pi\varepsilon \ll \omega^2\,, \tag{137}$$

where ω is the characteristic PGW frequency and ε is the matter energy density, all the functions still depend only on the retarded time (see [12]). Thus, $\beta(u)$ and $\gamma(u)$ are the amplitudes of the PGW subject to absorption in plasmas; T^{ij} is the total energy-momentum tensor of the plasma and the electromagnetic field (2).

5.1. Integral law of energy conservation

Ref. [1] suggested a semiquantitative solution of this problem on the basis of a simple model of energy balance. Due to its extreme importance, we do not restrict ourselves to [1] and return to a more complete study of the problem of energy transmission from a GW to magnetoactive plasma. However, instead of solving Einstein's equations, we make use of their consequence, the conservation law of the total momentum of the system "plasma + gravitational waves". Clearly, this model is only approximate and cannot replace a rigorous solution of Einstein's equations. According to [10], an arbitrary gravitational field provides the conservation of the system's momentum:

$$p^i = \frac{1}{c}\int(-g)(T^{i4} + \overset{g}{T}{}^{i4})dV, \tag{138}$$

where $\overset{g}{T}{}^{ik}$ is the energy-momentum pseudotensor of the gravitational field and the integration covers the whole 3-dimensional space. Let us take into account that the above solution is plane-symmetric and depends on the retarded time u only. Consequently the integration over the "plane" (x^2, x^3) in (138) reduce to simply multiplying by an infinite 2-dimensional area. Dividing both sides of (138) by this area and bearing in mind that with $\Omega = \pi/2$ among the 3-dimensional flow only $P^1$is nonzero, we obtain the conservation law of the surface density of the momentum P^1_Σ:

$$P^1_\Sigma = \frac{1}{c}\int\limits_{-\infty}^{+\infty}(-g)(T^{14} + \overset{g}{T}{}^{14})dx \quad (= \mathrm{Const}). \tag{139}$$

Let the right semispace $x > 0$ be filled with magnetoactive plasma and the left one $x < 0$ with matter which does not interact with a weak GW. Let further the whole gravitational momentum be concentrated in the interval $u \in [0, u_f]$ where $t_f = \sqrt{2}u_f$ is the gravitational pulse duration. Since the integral in Eq. (139) is conserved all the time, let us consider it at $t_0 < 0$, when the GW has not yet reached the magnetoactive plasma, and $-t_f > t > 0$), when the GW has reached the plasma. Taking into account that the vacuum solution depends only on the retarded time, we get for the integral in Eq. (139):

$$\int\limits_0^{u_f} \overset{g}{T}{}_0^{14} du = \int\limits_0^{t\sqrt{2}} (T^{14} + \overset{g}{T}{}^{14}) du + \int\limits_{t/\sqrt{2}}^{u_f} \overset{g}{T}{}_0^{14} du, \qquad (140)$$

where $\overset{g}{T}{}_0^{14} = \overset{g}{T}{}^{14}(\beta_*(u), \gamma_*(u))$; $\overset{g}{T}{}^{14} = \overset{g}{T}{}^{14}(\beta(u), \gamma(u))$. Transferring one of the integrals to the left-hand side of Eq.(140), we obtain the relation:

$$\int\limits_0^{u} \overset{g}{T}{}_0^{14} du = \int\limits_0^{u} (T^{14} + \overset{g}{T}{}^{14}) du, \qquad (141)$$

where the variable $u = t/\sqrt{2} > 0$ can now take *any positive* values.

A similar law may be written for the plasma total energy; in this case instead of Eq.(141) we obtain:

$$\int\limits_0^{u} \overset{g}{T}{}_0^{44} du = \int\limits_0^{u} (T^{44} - \mathcal{E}_0 + \overset{g}{T}{}^{14}) du,$$

where $\mathcal{E}_0$ is the total energy density of the unperturbed plasma.

5.2. Local analysis of the conservation law

Since the relation (141) must be valid at any values of the variable u, the corresponding local relation should be satisfied:

$$T^{41}(\beta, \gamma) + \overset{g}{T}{}^{41}(\beta, \gamma) = \overset{g}{T}{}^{41}(\beta_*, \gamma_*)\,, \qquad (142)$$

where $\overset{g}{T}{}^{41}(\beta, \gamma)$ is the energy flow of a weak GW in the direction $0x^1$ (see [10]):

$$\overset{g}{T}{}^{41} = \frac{1}{16\pi}\Big[h'^2_{23} + \frac{1}{4}(h'_{22} - h'_{33})^2\Big] = \frac{1}{4\pi}\big[(\gamma')^2 + (\beta')^2\big]. \qquad (143)$$

The prime denotes differentiation with respect to s. At substituting (143) into (142) and changing variables to v, u, we obtain:

$$2\pi\Big[T^{vv} - T^{uu}\Big] + (\gamma')^2 + (\beta')^2 = (\gamma'_*)^2 + (\beta'_*)^2. \qquad (144)$$

In the case of transversal PGW propagation and at a barotropic equation of state of an anisotropic plasma we obtain:

$$T^{vv} - T^{uu} = \left(\frac{1}{4v_v^2} - v_v^2\right)\left(\varepsilon(1 + k_\perp) + \frac{H^2}{4\pi}\right). \qquad (145)$$

Further, using the solutions of magnetohydrodynamics for a barotropic equation of state of plasma (129), (130), (131) and dimensionless parameter α^2 (117), we rewrite the energy balance equation (144) as:

$$\frac{\overset{0}{H}{}^2}{4L^2}\left(\Delta^{-\frac{4}{1-k_\perp}} L^{-\frac{4(k_\parallel + k_\perp)}{1-k_\perp}} (\cosh 2\gamma e^{2\beta})^{-\frac{2(k_\parallel - k_\perp)}{1-k_\perp}} - 1\right)\left(\frac{1}{\alpha^2} + 1\right) +$$
$$(\gamma')^2 + (\beta')^2 = (\gamma'_*)^2 + (\beta'_*)^2. \qquad (146)$$

Let us expand the expression in brackets by the smallness of the PGW amplitudes (41) but hold the term with Δ^{-1}, since the parameter α^2 in a strongly magnetized plasmas can be so large that the condition $2\alpha^2\beta > 1$ is satisfied. Then energy balance equation takes the form:

$$\frac{\overset{0}{H}{}^2}{4}\left(\Delta^{-4g_\perp}-1\right)\left(\frac{1}{\alpha^2}+1\right)+(\gamma')^2+(\beta')^2=(\gamma'_*)^2+(\beta'_*)^2. \tag{147}$$

Since in linear approximation by the smallness of the amplitudes β and γ the governing function (116) does not depend on the function $\gamma(u)$:

$$\Delta(u)=1-2\alpha^2\beta+O(\beta^2,\gamma^2)\,, \tag{148}$$

and the functions $\beta(u)$, $\gamma(u)$ are arbitrary and functionally independent, then up to β^2, γ^2, the relation (147) should be decompose into two independent parts:

$$2\overset{0}{H}{}^2 g_\perp(1+\alpha^2)\beta+(\beta')^2=(\beta'_*)^2, \tag{149}$$

$$(\gamma')^2=(\gamma'_*)^2. \tag{150}$$

Here, according to the meaning of local energy balance equation, we consider short gravitational waves (137), so we can neglect the squares of the PGW amplitudes in comparison with the squares of their derivatives with respect to the retarded time. Thus, according to (150):

$$\gamma_*(u)=\gamma(u), \tag{151}$$

i.e., in the linear approximation a weak gravitational waves with polarization $\mathbf{e}_\times$ does not interact with a magnetized plasma. This coincides with the conclusion of the paper [13].

Thus, in the linear approximation the PGW with $\mathbf{e}_\times$ polarization passes through a magnetoactive plasma without absorption, and the energy balance equation takes the form obtained in [6]. Further conclusions are similar to the case of propagation of the PGW with only one polarization $\mathbf{e}_+$.

If $\alpha^2 \gg 1$ the Eq. (147) can be written in the form (see also [2]):

$$\xi^2 V(q)+\dot{q}^2=\dot{q}_*^2\,, \tag{152}$$

where $q=\beta/\beta_0$, the dot denotes differentiation with respect to dimensionless time variable s:

$$s=\sqrt{2}\omega u, \tag{153}$$

(ω - the PGW frequency), $V(q)$ - potential function which in a weak PGW becomes:

$$V(q)=\Delta^{-4g_\perp}(q)-1, \tag{154}$$

where ξ^2 is so-called *the first parameter of GMSW* [2]:

$$\xi^2=\frac{\overset{0}{H}{}^2}{4\beta_0^2\omega^2}. \tag{155}$$

Let us introduce the new dimensionless parameter:

$$\Upsilon=2\alpha^2\beta_0 \tag{156}$$

- (*the second GMSW parameter*) and rewrite (148) in a weak PGW as:

$$\Delta(q(s))=1-2\alpha^2\beta_0 q(s)=1-\Upsilon q(s). \tag{157}$$

It leads from (157):

$$\dot{q} = -\frac{\dot{\Delta}(q)}{\Upsilon}. \tag{158}$$

To analyze the system behavior, let us suppose that the moment $s = 0$ corresponds to the front edge of a GW, while:

$$\beta_* \approx \beta_0(1 - \cos(s)) \Rightarrow q_* \approx 1 - \cos(s). \tag{159}$$

According to (157)-(159) the system starts with negative value of the governing function derivative and with function value equal to 1:

$$\begin{aligned} &\dot{\Delta}(s) \approx -\Upsilon \sin s \approx -\Upsilon s; \\ &\Delta(s) \approx 1 - \Upsilon(1 - \cos s) \approx 1 - \Upsilon\frac{s^2}{2}; \end{aligned} \qquad (s \to +0). \tag{160}$$

The energy balance equation (152) according to (154), (158), (159) becomes:

$$\dot{\Delta}^2 + \xi^2\Upsilon^2\left[\Delta^{-4g_\perp} - 1\right] = \Upsilon^2 \sin^2(s). \tag{161}$$

The minimum value of the governing function at $s = \pi/2$ is equal to:

$$\Delta_{min} = \left(\frac{1}{\xi^2} + 1\right)^{-\gamma_\perp}, \tag{162}$$

where:

$$\gamma_\perp = \frac{1}{4g_\perp} = \frac{1 - k_\perp}{4} \Rightarrow \frac{1}{8} \leq \gamma_\perp \leq \frac{1}{4}. \tag{163}$$

The maximum accessible density of a magnetic energy is

$$\left(\frac{H^2}{8\pi}\right)_{max} = \frac{\overset{0}{H}{}^2}{8\pi}\sqrt{1 + \frac{1}{\xi^2}} \tag{164}$$

and it does not depend on a plasma equation of state. Also the plasma velocity in a GMSW does not depend on an equation of state. And the maximum plasma energy density without magnetic field depends on the exponent of plasma anisotropy:

$$\varepsilon_{max} = \overset{0}{\varepsilon}\left(1 + \frac{1}{\xi^2}\right)^{\frac{1}{4}(1+k_\perp)} \tag{165}$$

It is maximum for ultrarelativistic plasma with zero parallel pressure.

6. Conclusion

Thus, the generalization of the results of [1]-[3] in the case of gravitational wave with two polarizations has been obtained and has been shown that in the linear approximation the polarization $\mathbf{e}_\times$ does not interact with a magnetized plasma. This fact is a justification for applicability of the previously obtained results for the case of arbitrarily polarized gravitational wave.

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