Surface Links with Free Abelian Link Groups

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Abstract

It is known that if a classical link group is a free abelian group, then its rank is at most two, and a μ-component 2-link group for $\mu > 1$ is not a free abelian group. In this paper we give examples of surface links whose link groups are free abelian groups of rank three or four. Moreover we show that the examples of rank three are infinitely many and one of them has the triple point number four. [1]

0 Introduction

Closed 1-manifolds embedded locally flatly in the Euclidean 3-space $\mathbb{R}^3$ are called *classical links*, and closed 2-manifolds embedded locally flatly in the Euclidean 4-space $\mathbb{R}^4$ are called *surface links*. A surface link whose each component is of genus zero (resp. one) is called a *2-link* (resp. *T^2-link*). Two classical links (resp. surface links) are *equivalent* if one is carried to the other by an ambient isotopy of $\mathbb{R}^3$ (resp. $\mathbb{R}^4$).

It is known that if a classical link group is a free abelian group, then its rank is at most two (cf. [11] Theorem 6.3.1). It is also known that a μ-component 2-link group for $\mu > 1$ is not a free abelian group (cf. [7] Corollary 2 of Chapter 3).

In this paper in Section 2 we give examples of surface links whose link groups are free abelian groups of rank three (Theorem 2.1) or four (Theorem 2.2). These examples are link groups of *torus-covering T^2-links*, which are T^2-links in $\mathbb{R}^4$ which can be described in braid forms over the standard torus (see Definition 1.4).

In Section 3 we study the torus-covering-links S_n of Theorem 2.1, i.e. the torus-covering T^2-links whose link groups are free abelian groups of rank three, where n are integers. Computing quandle cocycle invariants, we show that S_n is not equivalent to S_m if $n \neq m$ (Theorem 3.1). Using the quandle cocycle invariant together with a BW orientation for the singularity set of a surface diagram, we can moreover determine the triple point number of S_0 of Theorem 2.1. In fact, the triple point number of S_0 is four, and its associated torus-covering-chart $\Gamma_{T,0}$ realizes the surface diagram with triple points whose number is the triple point number (Theorem 3.2).

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1 Definitions and Preliminaries

As preliminaries, we give the definitions of braided surfaces, charts and *torus-covering-links* (Definition 1.4) (cf. [12]). We can compute the link groups of torus-covering T^2-links (Lemma 1.8). Throughout this paper, let $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$ be the standard generators of the braid group of degree m.

Definition 1.1. A compact and oriented 2-manifold S_B embedded properly and locally flatly in $D_1^2 \times D_2^2$ is called a *braided surface* of degree m if S_B satisfies the following conditions:

(i) $\mathrm{pr}_2|_{S_B} : S_B \to D_2^2$ is a branched covering map of degree m,

(ii) ∂S_B is a closed m-braid in $D_1^2 \times \partial D_2^2$, where D_1^2, D_2^2 are 2-disks, and $\mathrm{pr}_2 : D_1^2 \times D_2^2 \to D_2^2$ is the projection to the second factor.

A braided surface S_B is called *simple* if every singular index is two. Two braided surfaces of the same degree are *equivalent* if there is a fiber-preserving ambient isotopy of $D_1^2 \times D_2^2$ rel $D_1^2 \times \partial D_2^2$ which carries one to the other.

There is a *chart* which represents a simple surface braid.

Definition 1.2. Let m be a positive integer, and Γ be a graph on a 2-disk D_2^2. Then Γ is called a *surface link chart* of degree m if it satisfies the following conditions:

(i) $\Gamma \cap \partial D_2^2 = \emptyset$.

(ii) Every edge is oriented and labeled, and the label is in $\{1, \ldots, m-1\}$.

(iii) Every vertex has degree 1, 4, or 6.

(iv) At each vertex of degree 6, there are six edges adhering to which, three consecutive arcs oriented inward and the other three outward, and those six edges are labeled i and $i+1$ alternately for some i.

(v) At each vertex of degree 4, the diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$ (Fig. 1.1).

A vertex of degree 1 (resp. 6) is called a *black vertex* (resp. *white vertex*). A black vertex (resp. white vertex) in a chart corresponds to a branch point (resp. triple point) in the surface diagram of the associated simple surface braid by the projection pr_2.

A chart with a boundary represents a simple braided surface.

There is a notion of *C-move equivalence* between two charts of the same degree. The following theorem is well-known.

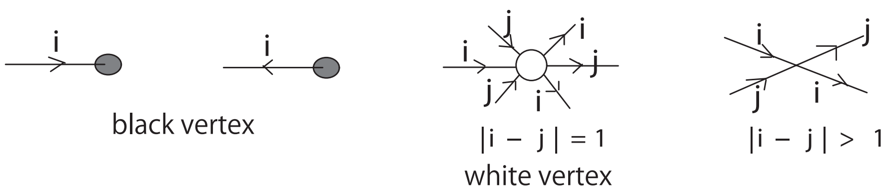

Figure 1.1: Vertices in a chart

Theorem 1.3 ([9, 10])**.** *Two charts of the same degree are C-move equivalent if and only if their associated simple braided surfaces are equivalent.*

Now we will give the definition of *torus-covering-links* (cf. [12]).

Definition 1.4. Let D^2 be a 2-disk, and S^1 a circle. First, embed $D^2 \times S^1 \times S^1$ into $\mathbb{R}^4$ naturally, and identify $D^2 \times S^1 \times S^1$ with $D^2 \times I_3 \times I_4/\sim$, where $(x, 0, v) \sim (x, 1, v)$ and $(x, u, 0) \sim (x, u, 1)$ for $x \in D^2$, $u \in I_3 = [0, 1]$ and $v \in I_4 = [0, 1]$.

Let us consider a surface link S embedded in $D^2 \times S^1 \times S^1$ such that $S \cap (D^2 \times I_3 \times I_4)$ is a simple braided surface. We call S a *torus-covering-link* (Fig. 1.2). In particular, if each component of a torus-covering-link is of genus one, then we call it a *torus-covering T^2-link*.

A torus-covering-link is associated with a chart on the standard torus, i.e. a chart Γ_T in $I_3 \times I_4$ such that $\Gamma_T \cap (I_3 \times \{0\}) = \Gamma_T \cap (I_3 \times \{1\})$ and $\Gamma_T \cap (\{0\} \times I_4) = \Gamma_T \cap (\{1\} \times I_4)$. Denote the classical braids represented by $\Gamma_T \cap (I_3 \times \{0\})$ and $\Gamma_T \cap (\{0\} \times I_4)$ by a and b respectively. We will call Γ_T a *torus-covering-chart with boundary braids a and b*. In particular, a torus-covering T^2-link is associated with a torus-covering-chart without black vertices, and the torus-covering T^2-link is determined from the boundary braids a and b, which are commutative. In this case we will call Γ_T a torus-covering-chart *without black vertices and with boundary braids a and b*.

We can compute link groups of torus-covering T^2-links (Lemma 1.8). Before stating Lemma 1.8, we will give the definition of *Artin's automorphism* (Definition 1.7, cf. [10]). Let D^2 be a 2-disk, β an m-braid in a cylinder $D^2 \times [0, 1]$, Q_m the starting point set of β.

Definition 1.5. An *isotopy of D^2 associated with β* is an ambient isotopy $\{\phi_t\}_{t\in[0,1]}$ of D^2 such that

(i) $\phi_t|_{\partial D^2} = \mathrm{id}$,

(ii) $\phi_t(Q_m) = \mathrm{pr}_1(\beta \cap \mathrm{pr}_2^{-1}(t))$ for $t \in [0, 1]$, where $\mathrm{pr}_1 : D^2 \times [0, 1] \to D^2$ (resp. $\mathrm{pr}_2 : D^2 \times [0, 1] \to [0, 1]$) is the projection to the first (resp. second) factor.

Definition 1.6. A *homeomorphism of D^2 associated with β* is the terminal map $\psi = \phi_1 : D^2 \to D^2$ of an isotopy $\{\phi_t\}_{t\in[0,1]}$ of D^2 associated with β.

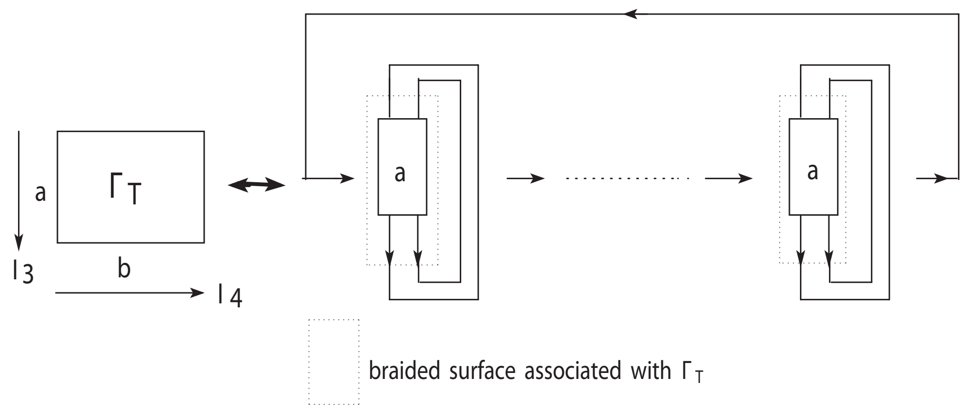

Figure 1.2: A torus-covering-link

Let q_0 be a point of ∂D^2. Identify the fundamental group $\pi_1(D^2 - Q_m, q_0)$ with the free group F_m generated by the standard generator system of $\pi_1(D^2 - Q_m, q_0)$.

Definition 1.7. *Artin's automorphism* of F_m associated with β is the automorphism of F_m induced by a homeomorphism of D^2 associated with β. We denote it by $\mathrm{Artin}(\beta)$.

We can obtain Artin's automorphism (of the free group F_m associated with an m-braid) algebraically by the following rules. Let $F_m = \langle x_1, x_2, \ldots, x_m \rangle$.

1. $\mathrm{Artin}(\beta_1\beta_2) = \mathrm{Artin}(\beta_2) \circ \mathrm{Artin}(\beta_1)$ for m-braids β_1 and β_2, and

2. $$\mathrm{Artin}(\sigma_i)(x_j) = \begin{cases} x_j & \text{if } j \neq i,\ i+1, \\ x_i x_{i+1} x_i^{-1} & \text{if } j = i, \\ x_i & \text{if } j = i+1, \end{cases}$$

   and

   $$\mathrm{Artin}(\sigma_i^{-1})(x_j) = \begin{cases} x_j & \text{if } j \neq i,\ i+1, \\ x_{i+1} & \text{if } j = i, \\ x_{i+1}^{-1} x_i x_{i+1} & \text{if } j = i+1, \end{cases}$$

   where $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots, m$.

We can compute link groups of torus-covering T^2-links.

Lemma 1.8 ([12] Lemma 3.4)**.** *Let Γ_T be a torus-covering-chart of degree m without black vertices and with boundary braids a and b. Let S be the torus-covering T^2-link associated with Γ_T. Then the link group of S is obtained as follows:*

$$\pi_1(\mathbb{R}^4 - S) = \langle\, x_1, \ldots, x_m \mid x_j = \mathrm{Artin}(a)(x_j) = \mathrm{Artin}(b)(x_j),\ j = 1, 2, \ldots, m \,\rangle,$$

where Artin$(a) : F_m \to F_m$ *(resp.* Artin(b)*) is Artin's automorphism of the free group* $F_m = \langle x_1, \dots, x_m \rangle$ *associated with the* m*-braid* a *(resp.* b*).*

2 Surface links whose link groups are free abelian

There are torus-covering T^2-links whose link groups are free abelian groups of rank three (Theorem 2.1) or four (Theorem 2.2).

Theorem 2.1. *Let* $\Gamma_{T,\,n}$ *be the torus-covering-chart of degree* 3 *without black vertices and with boundary braids* $\sigma_1^2\sigma_2^{2n}$ *and* Δ^2*, where* $\Delta = \sigma_1\sigma_2\sigma_1$ *(Garside's* Δ*) and* n *is an integer. Then the torus-covering* T^2*-link* S_n *associated with* $\Gamma_{T,n}$ *has the link group* $\pi_1(\mathbb{R}^4 - S_n) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$*.*

Theorem 2.2. *Let* Γ_T *be the torus-covering-chart of degree* 4 *without black vertices and with boundary braids* $\sigma_1^2\sigma_2^2\sigma_3^2$ *and* Δ^2*, where* $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ *(Garside's* Δ*). Then the torus-covering* T^2*-link* S *associated with* Γ_T *has the link group* $\pi_1(\mathbb{R}^4 - S) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$*.*

Proof of Theorem 2.1. Let us compute the link group $G_n = \pi_1(\mathbb{R}^4 - S_n)$ by Lemma 1.8. Let x_1, x_2 and x_3 be the generators. Then the relations concerning the boundary braid $\sigma_1^2\sigma_2^{2n}$ are

$$x_1x_2 = x_2x_1, \tag{2.1}$$
$$(x_2x_3)^{|n|} = (x_3x_2)^{|n|}. \tag{2.2}$$

The other relations concerning the other boundary braid Δ^2 are

$$\begin{aligned} x_1 &= (x_1x_2x_3)x_1(x_1x_2x_3)^{-1}, \\ x_2 &= (x_1x_2x_3)x_2(x_1x_2x_3)^{-1}, \\ x_3 &= (x_1x_2x_3)x_3(x_1x_2x_3)^{-1}, \end{aligned}$$

which are

$$x_1x_2x_3 = x_2x_3x_1, \tag{2.3}$$
$$x_2(x_1x_2x_3) = (x_1x_2x_3)x_2, \tag{2.4}$$
$$x_3x_1x_2 = x_1x_2x_3. \tag{2.5}$$

By (2.1), (2.3) is deformed to $x_2x_1x_3 = x_2x_3x_1$. Thus we have

$$x_1x_3 = x_3x_1. \tag{2.6}$$

Similarly, by (2.4) and (2.1), we have

$$x_2x_3 = x_3x_2. \tag{2.7}$$

We can see that all the relations are generated by the three relations (2.1), (2.6) and (2.7). Thus we have

$$\begin{aligned} G_n &= \langle\, x_1,\, x_2,\, x_3 \,|\, x_1x_2 = x_2x_1,\, x_2x_3 = x_3x_2,\, x_3x_1 = x_1x_3 \,\rangle \\ &= \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}. \end{aligned}$$

□

Proof of Theorem 2.2. Similarly to the proof of Theorem 2.1, for generators x_1, x_2, x_3 and x_4, we have the following relations:

$$x_ix_{i+1} = x_{i+1}x_i, \tag{2.8}$$

where $i = 1, 2, 3$, and

$$x_i = (x_1x_2x_3x_4)x_i(x_1x_2x_3x_4)^{-1}, \tag{2.9}$$

where $i = 1, 2, 3, 4$. Using $x_1x_2 = x_2x_1$ and $x_3x_4 = x_4x_3$ of (2.8), the latter four relations (2.9) are deformed as follows:

$$x_1x_3x_4 = x_3x_4x_1, \tag{2.10}$$
$$x_2x_3x_4 = x_3x_4x_2, \tag{2.11}$$
$$x_3x_1x_2 = x_1x_2x_3, \tag{2.12}$$
$$x_4x_1x_2 = x_1x_2x_4. \tag{2.13}$$

By $x_2x_3 = x_3x_2$ of (2.8), (2.11) is deformed to $x_3x_2x_4 = x_3x_4x_2$. Thus we have

$$x_2x_4 = x_4x_2. \tag{2.14}$$

Similarly, by $x_2x_3 = x_3x_2$ of (2.8) and (2.12), we have

$$x_3x_1 = x_1x_3, \tag{2.15}$$

and by (2.14) and (2.13), we have

$$x_4x_1 = x_1x_4. \tag{2.16}$$

We can see that all the relations are generated by the relations (2.8), (2.14), (2.15) and (2.16). Thus the link group G is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. □

3 The surface links of Theorem 2.1

As surface links which can be made from classical links, there are *spun T^2-links, turned spun T^2-links*, and *symmetry-spun T^2-links.* Consider $\mathbb{R}^4$ as obtained by rotating $\mathbb{R}^3_+$ around the boundary $\mathbb{R}^2$. Then a *spun T^2-link* is obtained by rotating a classical link (cf. [2]), a *turned spun T^2-link* by turning a classical link once while rotating it (cf. [2]), and a *symmetry-spun T^2-link* by turning a classical link with periodicity rationally while rotating (cf. [15]). By definition, torus-covering-links include symmetry-spun T^2-links. Indeed, a symmetry-spun T^2-link is represented by a torus-covering-chart with no (black nor white) vertices.

It is well-known that if a classical link group is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$, then it is a Hoph link. On the other hand, by definition, the torus-covering T^2-link associated with a torus-covering-chart of degree 2 is a symmetry-spun T^2-link. Here it is known that a symmetry-spun T^2-link is either the spun T^2-link or the turned spun T^2-link of a classical link, say L, and the link group of the spun (or turned spun) T^2-link of L is isomorphic to the classical link group of L (cf. [15]). Hence we can see that if a torus-covering T^2-link has the link group $\mathbb{Z}\oplus\mathbb{Z}$ and moreover it is associated with a torus-covering-chart of degree 2, then it is either the spun or the turned spun T^2-link of a Hoph link. Thus for the torus-covering T^2-links with the link group $\mathbb{Z}\oplus\mathbb{Z}$ which are associated with torus-covering-charts of degree 2, there are just a finite number of equivalence classes.

Then what about the torus-covering T^2-links of Theorem 2.1? Are the number of the equivalence classes of them finite? The answer is no.

Theorem 3.1. *For the torus-covering T^2-links of Theorem 2.1, S_n and S_m are not equivalent for $n \neq m$, where n and m are integers.*

Before the proof, we give the definition of the *quandle cocycle invariants.* (cf. [3, 4, 5]). Let F be an oriented surface link.

Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be a generic projection. In the surface diagram $D = \pi(F)$, there are two intersecting sheets along each double point curve, one of which is higher than the other with respect to π. They are called the *over sheet* and the *under sheet* along the double point curve, respectively. In order to indicate crossing information of the surface diagram, we break the under sheet into two pieces missing the over sheet. This can be extended around a triple point. Around a triple point, the sheets are called the *top sheet*, the *middle sheet*, and the *bottom sheet* from the higher one. Then the surface diagram is presented by a disjoint union of compact surfaces which are called *broken sheets*. We denote by $B(D)$ the set of broken sheets of D.

A set X with a binary operation $* : X \times X \to X$ is called a *quandle* if it satisfies the following conditions:

(i) for any $a \in X$, $a * a = a$,

(ii) for any a, $b \in X$, there exists a unique $c \in X$ such that $a = c * b$, and

(iii) for any a, b, $c \in X$, $(a * b) * c = (a * c) * (b * c)$ (cf. [8]).

A *finite quandle* is a quandle consisting of a finite number of elements. A *trivial quandle* is a set X with the binary operation $a*b = a$ for any a, $b \in X$.

A *coloring* for a surface diagram D by a quandle X is a map $C : B(D) \to X$ such that $C(H_1) * C(H_2) = C(H_1')$ along each double point curve of D, where H_2 is the over sheet and H_1 (resp. H_1') is the under sheet such that the normal vector of H_2 points from (resp. toward) it. The image by C is called the *color*.

At a triple point of D, there exist broken sheets $J_1, J_2, J_3 \in B(D)$ uniquely such that J_1 is the bottom sheet, J_2 is the middle sheet, J_3 is the top sheet and the normal vector of J_2 (resp. J_3) points from J_1 (resp. J_2). The *color of the triple point* is the triplet $(C(J_1), C(J_2), C(J_3)) \in X \times X \times X$. The *sign* of the triple point is *positive* or $+1$ (resp. *negative* or -1) if the triplet of the normal vectors of J_1, J_2, J_3 is right-handed (resp. left-handed).

If D has a corresponding chart, this corresponds to the following (cf. [6] Proposition 4.43 (3)). The color of the white vertex representing $\sigma_i\sigma_j\sigma_i \to \sigma_j\sigma_i\sigma_j$ ($|i-j|=1$) is (a, b, c), where a, b and c are the colors of the broken sheets of D connected with the starting points of the i'-th, $(i'+1)$-th, and $(i'+2)$-th strings of $\sigma_i\sigma_j\sigma_i$, where $i' = \min\{i, j\}$. The white vertex is positive (resp. negative) if $j > i$ (resp. $i > j$), i.e. if there is exactly one edge with the larger (resp. smaller) label oriented toward the white vertex.

Take a map $\theta : X \times X \times X \to A$, where X is a finite quandle and A is an abelian group in which the sum is written multiplicatively. Take a coloring C for the surface diagram D by the quandle X. Let $\tau_1, \tau_2, \ldots, \tau_s$ be all the triple points of D with the sign $\epsilon_i \in \{+1, -1\}$ and the color $(a_i, b_i, c_i) \in X \times X \times X$ for each τ_i. Put $W_\theta(\tau_i\,; C) = \theta(a_i, b_i, c_i)^{\epsilon_i} \in A$ for each τ_i, and $W_\theta(C) = \prod_{i=1}^{s} W_\theta(\tau_i\,; C) \in A$ for the coloring C. We call $W_\theta(\tau_i\,; C)$ the *Boltzman weight* of τ_i, and $W_\theta(C)$ the *Boltzman weight*. Since X is a finite quandle and the set of broken sheets of D is finite, so is the set of colorings for D by X. Let $C_1, C_2, \ldots, C_n$ be all the colorings, and define $\Phi_\theta(D)$ by $\Phi_\theta(D) = \sum_{j=1}^{n} W_\theta(C_j) \in \mathbb{Z}[A]$. If the map θ satisfies the following conditions ($\theta1$) and ($\theta2$), then $\Phi_\theta(D)$ does not depend on the choice of the surface diagram D of the surface link F. Then we call $\Phi_\theta(D)$ the *quandle cocycle invariant* of F associated with the 3-cocycle θ, and use the notation $\Phi_\theta(F)$. The conditions are as follows, where $x, y, z, w \in X$.

($\theta1$) $\theta(x, y, z) = 1_X$ for $x = y$ or $y = z$,

($\theta2$) $\theta(x, z, w)\cdot\theta(x, y, w)^{-1}\cdot\theta(x, y, z) = \theta(x*y, z, w)\cdot\theta(x*z, y*z, w)^{-1}\cdot \theta(x*w, y*w, z*w)$.

Proof. Consider a trivial quandle of three elements, T_3, such that the associated set is $\{0, 1, 2\}$, and take a map $\theta : T_3 \times T_3 \times T_3 \to \mathbb{Z} = \langle\, t \,\rangle$ such that

$$\theta(x, y, z) = t^{(x-y)(y-z)(z-x)z}. \tag{3.1}$$

This map θ satisfies the conditions ($\theta1$) and ($\theta2$). Let us compute the quandle cocycle invariant of S_n associated with θ by coloring the associated torus-covering-chart $\Gamma_{T,n}$.

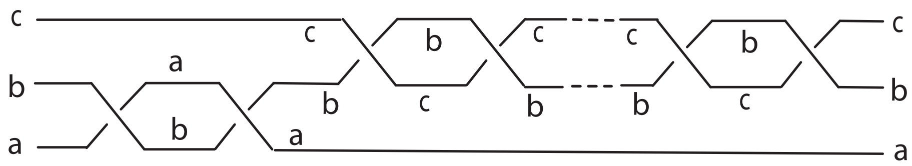

Figure 3.1: The coloring for $\sigma_1^2\sigma_2^{2n}$, if $n \geq 0$

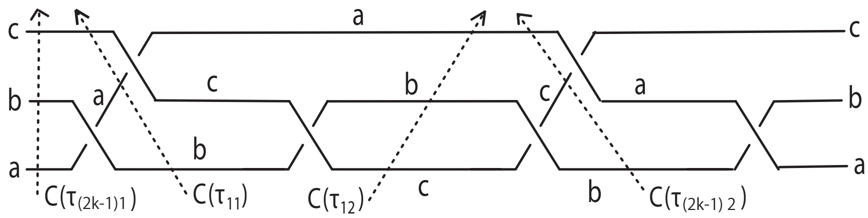

Figure 3.2: The coloring for $(\sigma_1\sigma_2\sigma_1)^2$ and the colors

First let us consider the case when n is a non-negative integer. Color the surface diagram associated with $\Gamma_{T,n}$ by T_3. Then the coloring for the diagrams of the vertical boundary braid $\sigma_1^2\sigma_2^{2n}$ and the horizontal boundary braid $(\sigma_1\sigma_2\sigma_1)^2$ is as in Fig.s 3.1 and 3.2, where a, b, $c \in \{0, 1, 2\}$, which are all the colorings. Denote the coloring by C. We can draw the part of the torus-covering-chart without black vertices and with boundary braids σ_1 and $(\sigma_1\sigma_2\sigma_1)^2$ such that it has two white vertices as in Fig. 3.3. Denote them by τ_{i1} and τ_{i2} from left to right as in Fig. 3.3, where $i = 1, 2$. The colors $C(\tau_{11})$ and $C(\tau_{12})$ of the first two white vertices τ_{11} and τ_{12} are obtained from reading the colors along the dotted paths in Fig. 3.2. Since there is exactly one edge with the larger (resp. smaller) label, i.e. the label 2 (resp. the label 1) oriented toward τ_{11} (resp. τ_{12}), we see that the sign of τ_{11} (resp. τ_{12}) is positive (resp. negative). Thus the signs and colors are $+(b, a, c)$ for τ_{11} and $-(c, b, a)$ for τ_{12}. Similarly, the color of τ_{21} (resp. τ_{22}) is obtained from that of τ_{11} (resp. τ_{12}) by exchanging a and b, and the sign of τ_{21} (resp. τ_{22}) is the same with that of τ_{11} (resp. τ_{12}). Thus we can see that the signs and colors are $+(a, b, c)$ for τ_{21} and $-(c, a, b)$ for τ_{22}.

Let us denote $\prod_{i=1}^2 \prod_{j=1}^2 W_\theta(\tau_{ij}; C)$ by $W_\theta(\Gamma_1; C)$, where $W_\theta(\tau_{ij}; C)$ is the Boltzman weight of τ_{ij} for the coloring C. Then $W_\theta(\Gamma_1; C)$ is as follows:

$$\begin{aligned} W_\theta(\Gamma_1; C) &= \theta(b, a, c) \cdot \theta(c, b, a)^{-1} \cdot \theta(a, b, c) \cdot \theta(c, a, b)^{-1} \\ &= t^{(a-b)(b-c)(c-a)(a-b)}. \end{aligned} \tag{3.2}$$

Hence we have

$$W_\theta(\Gamma_1; C) = \begin{cases} t^{-2} & \text{if } (a, b, c) = (0, 1, 2), (1, 0, 2), (1, 2, 0), (2, 1, 0) \\ t^4 & \text{if } (a, b, c) = (0, 2, 1), (2, 0, 1) \\ 1 & \text{if } \{a, b, c\} = \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}. \end{cases}$$

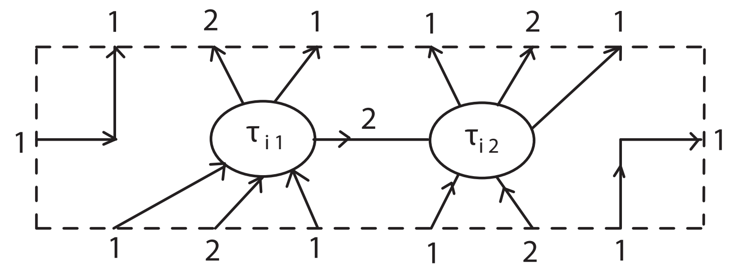

Figure 3.3: White vertices τ_{i1} and τ_{i2}, where $i = 1, 2$

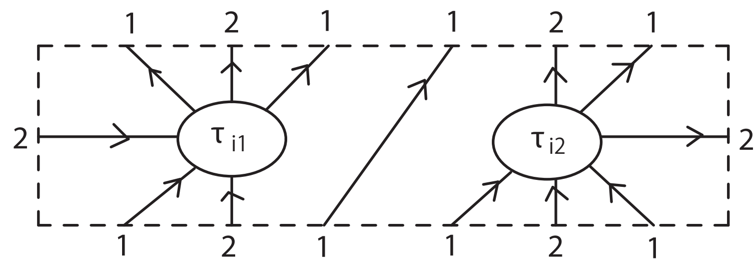

Figure 3.4: White vertices τ_{i1} and τ_{i2} ($i = 3, 4, \ldots, 2n + 2$), if $n > 0$

Similarly, We can draw the part of the torus-covering-chart without black vertices and with boundary braids σ_2 and $(\sigma_1\sigma_2\sigma_1)^2$ such that it has two white vertices as in Fig. 3.4. Denote these by τ_{i1} and τ_{i2} for $i = 3, 4, \ldots, 2n+2$ as in Fig. 3.4. Then the signs and the colors are $-(a, b, c)$ for $\tau_{(2k-1)1}$, $+(b, c, a)$ for $\tau_{(2k-1)2}$, $-(a, c, b)$ for $\tau_{(2k)1}$, and $+(c, b, a)$ for $\tau_{(2k)2}$, where $k = 2, 3, \ldots, n + 1$. Fig. 3.2 shows the colors of $\tau_{(2k-1)1}$ and $\tau_{(2k-1)2}$, and the color of $\tau_{(2k)1}$ (resp. $\tau_{(2k)2}$) is obtained from $\tau_{(2k-1)1}$ (resp. $\tau_{(2k-1)2}$) by exchanging b and c. Let us denote $\prod_{i=2k-1}^{2k}\prod_{j=1}^{2} W_\theta(\tau_{ij}; C)$ by $W_\theta(\Gamma_k; C)$, where $k = 2, 3, \ldots, n + 1$. Then we have

$$\begin{aligned} W_\theta(\Gamma_k; C) &= \theta(a, b, c)^{-1} \cdot \theta(b, c, a) \cdot \theta(a, c, b)^{-1} \cdot \theta(c, b, a) \\ &= t^{(a-b)(b-c)(c-a)(b-c)}, \end{aligned}$$

and

$$W_\theta(\Gamma_k; C) = \begin{cases} t^{-2} & \text{if } (a, b, c) = (0, 1, 2), (0, 2, 1), (2, 0, 1), (2, 1, 0) \\ t^4 & \text{if } (a, b, c) = (1, 0, 2), (1, 2, 0) \\ 1 & \text{if } \{a, b, c\} = \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}. \end{cases}$$

Since τ_{ij} for $i = 1, 2, \ldots, 2n + 2$ and $j = 1, 2$ are all the white vertices of

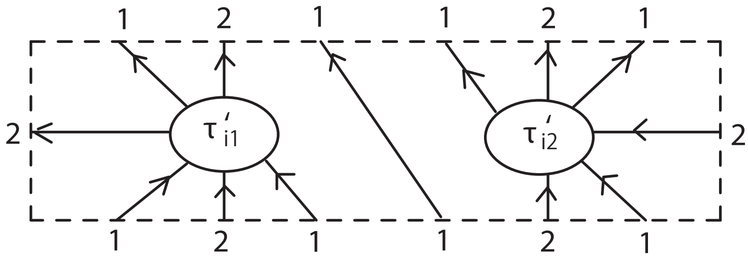

Figure 3.5: White vertices τ'_{i1} and τ'_{i2} ($i = 3, 4, \ldots, 2n+2$), if $n < 0$

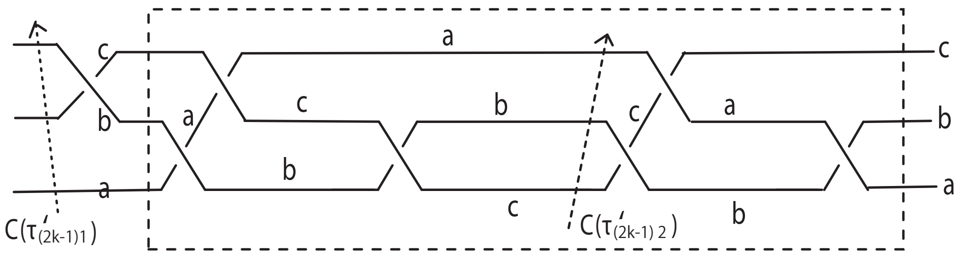

Figure 3.6: The colors of $\tau'_{(2k-1)1}$ and $\tau'_{(2k-1)2}$

$\Gamma_{T,n}$, the Boltzman weight $W_\theta(C)$ is $\prod_{k=1}^{n+1} W_\theta(\Gamma_k; C)$, which is as follows:

$$W_\theta(C) = \begin{cases} t^{-2-2n} & \text{if } (a,\, b,\, c) = (0,\, 1,\, 2), (2,\, 1,\, 0) \\ t^{4-2n} & \text{if } (a,\, b,\, c) = (0,\, 2,\, 1), (2,\, 0,\, 1) \\ t^{-2+4n} & \text{if } (a,\, b,\, c) = (1,\, 0,\, 2), (1,\, 2,\, 0) \\ 1 & \text{if } \{a,\, b,\, c\} = \{0\},\, \{1\},\, \{2\},\, \{0,\, 1\},\, \{0,\, 2\},\, \{1,\, 2\}. \end{cases}$$

Hence the quandle cocycle invariant $\Phi_\theta(S_n)$ of S_n associated with θ is

$$\Phi_\theta(S_n) = 21 + 2t^{-2-2n} + 2t^{4-2n} + 2t^{-2+4n} \tag{3.3}$$

for $n \geq 0$.

Similarly, if $n < 0$, then we can draw the part of the torus-covering-chart without black vertices and with boundary braids σ_2^{-1} and $(\sigma_1\sigma_2\sigma_1)^2$ such that it has two white vertices as in Fig. 3.5. Denote these by τ'_{i1} and τ'_{i2} for $i = 3, 4, \ldots, 2|n| + 2$ as in Fig. 3.5. Then the signs and colors are $+(a,\, b,\, c)$ for $\tau'_{(2k-1)1}$, $-(c,\, b,\, a)$ for $\tau'_{(2k-1)2}$, $+(a,\, c,\, b)$ for $\tau'_{(2k)1}$, and $-(b,\, c,\, a)$ for $\tau'_{(2k)2}$, where $k = 2, 3, \ldots, |n| + 1$. Fig. 3.6 shows the colors of $\tau'_{(2k-1)1}$ and $\tau'_{(2k-1)2}$, where the braid surrounded by the dotted square is the braid $(\sigma_1\sigma_2\sigma_1)^2$. The color of $\tau'_{(2k)1}$ (resp. $\tau'_{(2k)2}$) is obtained from $\tau'_{(2k-1)1}$ (resp. $\tau'_{(2k-1)2}$) by exchanging b and c. Hence we can see that the equation (3.3) holds for $n < 0$, too. Thus we see that $\Phi_\theta(S_n) \neq \Phi_\theta(S_m)$ for $n \neq m$. □

The triple point number of a surface link F is the minimum number of triple points in a surface diagram of F, for all the surface diagrams. We can moreover determine the triple point number of S_0.

Theorem 3.2. *The triple point number of S_0 of Theorem 2.1 is four, and its associated torus-covering-chart $\Gamma_{T,0}$ realizes the surface diagram with triple points whose number is the triple point number.*

Let $T_3(a, b, c)$ be a a trivial quandle of three elements such that the associated set is $\{a, b, c\}$. First let us introduce a property of a *BW orientation* for the singularity set Σ of a surface diagram D, and define $E(C;\Sigma)$ and $E(\Sigma)$ for a BW orientation for Σ and a coloring C of D by $T_3(a, b, c)$.

Let D be a surface diagram of an oriented surface link. Let Σ be the singularity set of D. A triple point (resp. branch point) of D corresponds to a vertex of degree 6 (resp. degree one) of Σ. An edge of Σ around a triple point is called a *b/m-edge*, *b/t-edge* or *m/t-edge* if it is a double point curve which is the intersection of the bottom sheet and the middle sheet, the bottom sheet and the top sheet, or the middle sheet and the top sheet respectively. The singularity set Σ admits a *BW orientation* (cf. [13]). Let us consider an edge of Σ connected with a triple point. Let us say the edge is *positive* (resp. *negative*) with respect to the triple point if it is oriented outward from (resp. toward) the triple point. Then a BW orientation orients each edge of Σ such that around each triple point, say τ, the BW orientation of the six edges around τ satisfy one of the conditions as follows: the four edges consisting of the b/m-edges and m/t-edges are positive (resp. negative), and the other two edges consisting of the b/t-edges are negative (resp. positive) with respect to τ.

Let us give a coloring C for D by $T_3(a, b, c)$, and moreover give a BW orientation to Σ. Then the color of the six edges of Σ around a triple point τ is one of the following. Let $\tau(C)$ be the color of the triple point:

(i) $(x, y), (x, y),\ (x, z), (x, z),\ (y, z), (y, z)$ if $\tau(C) = (x, y, z)$,

(ii) $(x, y), (x, y),\ (x, x), (x, x),\ (y, x), (y, x)$ if $\tau(C) = (x, y, x)$,

(iii) $(x, x), (x, x),\ (x, y), (x, y),\ (x, y), (x, y)$ if $\tau(C) = (x, x, y)$,

(iv) $(y, x), (y, x),\ (y, x), (y, x),\ (x, x), (x, x)$ if $\tau(C) = (y, x, x)$,

(v) $(x, x), (x, x),\ (x, x), (x, x),\ (x, x), (x, x)$ if $\tau(C) = (x, x, x)$,

where the first and the second colors are those of b/m-edges, the third and the fourth colors are those of b/t-edges, and the fifth and the sixth colors are those of m/t-edges, and $\{x, y, z\} = \{a, b, c\}$.

Let $\tau_1, \tau_2, \ldots, \tau_\nu$ be all the triple points of D, and let the color of τ_j be (a_j, b_j, c_j), where $a_j, b_j, c_j \in \{a, b, c\}$ and $j = 1, 2, \ldots, \nu$. Let $\epsilon_j = +1$ (resp. -1) if the BW orientation of the six edges of Σ around τ_j are as follows, where $j = 1, 2, \ldots, \nu$: the b/m-edges and m/t-edges are positive (resp. negative) and the b/t-edges are negative (resp. positive) with respect to τ. Then let us define

$$E(C;\tau_j) = \{\epsilon_j(a_j,\, b_j),\, -\epsilon_j(a_j,\, c_j),\, \epsilon_j(b_j,\, c_j)\}, \tag{3.4}$$

where $j = 1, 2, \ldots, \nu$, and $E(C;\Sigma) = \cup_{j=1}^{\nu} E(C;\tau_j)$.

Let us say the color of an edge of Σ is *degenerate* (resp. *non-degenerate*) if the color is $(x,\, x)$ (resp. $(x,\, y)$), where $x, y \in \{a,\, b,\, c\}$ with $x \neq y$. Any edge of Σ connected with a branch point has a degenerate color. Hence we can see that if an edge, say e, of Σ connected with a triple point, say τ, has a non-degenerate color, then the other end of e is connected with a triple point, say τ' (i.e. not a branch point). Moreover we see that the BW orientation of the edge e is positive with respect to τ, and negative with respect to τ'. Hence we can see that

$$E(C;\Sigma) = A \cup B, \tag{3.5}$$

where A is a set consisting of pairs in the form of $\{+(x,\, y),\, -(x,\, y)\}$, and B is a set consisting of elements in the form of $\epsilon(x,\, x)$, where $x,\, y \in \{a,\, b,\, c\}$ with $x \neq y$ and $\epsilon \in \{+1,\, -1\}$. Moreover for the above ϵ, x and y, let f be a map which maps an element of $E(C;\Sigma)$ to an integer such that $f(\epsilon(x,x)) = 0$ and $f(\epsilon(x,y)) = \epsilon$. Then let

$$E(\tau_j) = f(\epsilon_j(a_j,\, b_j)) + f(-\epsilon_j(a_j,\, c_j)) + f(\epsilon_j(b_j,\, c_j)), \tag{3.6}$$

where $j = 1, 2, \ldots, \nu$ (cf. (3.4)), and $E(\Sigma) = \sum_{j=1}^{\nu} E(\tau_j)$. Then we have

$$E(\tau_j) = \begin{cases} \epsilon_j & \text{if } \tau_j \text{ is of type (i)}, \\ 2\epsilon_j & \text{if } \tau_j \text{ is of type (ii)}, \\ 0 & \text{otherwise.} \end{cases}$$

By (3.5), we can see that

$$E(\Sigma) = 0. \tag{3.7}$$

Note that if $E(\tau_j)$ is positive (resp. negative), then $\epsilon_j = +1$ (resp. -1).

Let us say a triple point is *non-degenerate* if its color is of type (i) or (ii), and *degenerate* otherwise. By definition, if τ is a non-degenerate triple point, then $E(\tau) = \epsilon$ (resp. 2ϵ) if τ is of type (i) (resp. type (ii)), where $\epsilon \in \{+1,\, -1\}$, and if τ is a degenerate triple point (i.e. of type (iii), (iv), or (v)), then $E(\tau) = 0$.

Proof of Theorem 3.2. The quandle cocycle invariant $\Phi_\theta(S_0)$ of S_0 associated with the trivial quandle T_3 and the 3-cocycle θ of Theorem 3.1 is $\Phi_\theta(S_0) = 21 + 4t^{-2} + 2t^4$ by Theorem 3.1. Let $W_\theta(C)$ be the Boltzman weight associated with the 3-cocycle θ and a coloring C.

Let us consider another 3-cocycle $\theta' : T_3 \times T_3 \times T_3 \to \mathbb{Z} = \langle t \rangle$ such that

$$\theta'(x, y, z) = t^{(x-y)(y-z)(z-x)x}. \tag{3.8}$$

Let us denote by $W_{\theta'}(C)$ the Boltzman weight associated with the 3-cocycle θ' and a coloring C. Let C_0 be the coloring C of Theorem 3.1. Then similarly to Theorem 3.1, we can see that the Boltzman weight associated with the 3-cocycle θ' and the coloring C_0 is

$$\begin{aligned} W_{\theta'}(C_0) &= \theta'(b, a, c) \cdot \theta'(c, b, a)^{-1} \cdot \theta'(a, b, c) \cdot \theta'(c, a, b)^{-1} \\ &= t^{(a-b)(b-c)(c-a)(a-b)}, \end{aligned}$$

which is the same with $W_\theta(C_0) = W_\theta(\Gamma_1, C_0)$ (3.2) of Theorem 3.1. Hence the quandle cocycle invariant $\Phi_{\theta'}(S_0)$ of S_0 associated with θ' is the same with $\Phi_\theta(S_0)$, i.e.

$$\Phi_\theta(S_0) = \Phi_{\theta'}(S_0) = 21 + 4t^{-2} + 2t^4. \tag{3.9}$$

Let D be a surface diagram of an oriented surface link. Let $T_3(a, b, c)$ be a trivial quandle of three elements such that the associated set is $\{a, b, c\}$. Let $C(a, b, c)$ be a coloring for D by $T_3(a, b, c)$. Let us denote by $\mathcal{C}$ the set of all the colorings by T_3 described by $C(a, b, c)$, i.e. $\mathcal{C} = \{C(a, b, c) \,|\, a, b, c \in \{0, 1, 2\}\}$. By the definitions of θ (3.1) and θ' (3.8), we can see that for $C(a, b, c)$ by $T_3(a, b, c)$ and a triple point τ of D, if τ is not of type (i), then

$$W_\theta(\tau; C) = W_{\theta'}(\tau; C) = 1 \tag{3.10}$$

for any $C \in \mathcal{C}$. Now, suppose we can show that for each possible coloring $C(a, b, c)$ by $T_3(a, b, c)$ and its associated set of colorings $\mathcal{C}$, one of the following holds:

(W1) $W_\theta(C) = 1$, for each $C \in \mathcal{C}$,

(W2) For a coloring $C \in \mathcal{C}$, $W_\theta(C)$ is neither 1, t^{-2} nor t^4, or

(W3) $\Sigma_{C \in \mathcal{C}} W_{\theta'}(C) = 27$.

Then D is not a surface diagram of S_0 by the following argument. Suppose (W3) holds. Since the number of all the colorings for any surface diagram of S_0 is $\Phi_\theta(S_0)|_{t=1} = 27$, we can see that the number of all the colorings for D is 27. Since the number of the elements of $\mathcal{C}$ is $\Sigma_{C \in \mathcal{C}} W_{\theta'}(C)|_{t=1} = 27$, we have $\Phi_{\theta'}(S_0) = \Sigma_{C \in \mathcal{C}} W_{\theta'}(C)$, which contradicts (3.9) and $\Sigma_{C \in \mathcal{C}} W_{\theta'}(C) = 27$. Thus (W3) does not occur. By (3.9), the Boltzman weight $W_\theta(C)$ for a coloring C must be 1, t^{-2} or t^4. Hence $(W2)$ also does not occur. Then, if only (W1) occurs, then the quandle cocycle invariant is an integer value, which also contradicts (3.9).

Thus it follows that for a surface diagram with at most three triple points, if all the possible colorings $C(a, b, c)$ for it by $T_3(a, b, c)$ satisfies (W1), (W2) or (W3), then the surface diagram does not represent S_0, i.e. the triple point number of S_0 is at least four.

Hence from now on we will show that for a surface diagram D with at most three triple points and a set of colorings $\mathcal{C}$ for D, either (W1) or (W2) or $(W3)$ occurs. Here $\mathcal{C}$ is the set of all the colorings of T_3 described by $C(a, b, c)$, where $C(a, b, c)$ is a coloring for D by $T_3(a, b, c)$. Let us give a coloring $C(a, b, c)$ by $T_3(a, b, c)$ for D. Let Σ be the singularity set of D, and give a BW orientation to Σ. Let $\epsilon \in \{+1, -1\}$. If D does not have a triple point, then (W1) holds. Hence we can assume that D has at least one triple point.

(Case 1) If D has one triple point, then let τ be the triple point. If τ is of type (i), then $E(\Sigma) = +1$ or -1, which contradicts $E(\Sigma) = 0$ (3.7). Hence by (3.10) we have $W_\theta(\tau; C) = 1$ for any $C \in \mathcal{C}$, which is Case (W1). Indeed it is known that we can cancel the triple point (cf. [13]), i.e. there is another surface diagram which represents the same surface link represented by D such that it has no triple points.

(Case 2) If D has two triple points, then considering $E(\Sigma)$, we can see that the triple points are both of type (i), or both of type (ii), or both degenerate. By (3.10), it suffices to show in the case that the triple points are both of type (i). Let τ and τ' be the triple points. If τ and τ' are both of type (i), then by (3.5), we can see that τ and τ' have the same colors. We can assume that the colors are both (a, b, c). If τ and τ' have the opposite signs, then $W_\theta(C(a, b, c)) = \theta(a, b, c)^\epsilon \cdot \theta(a, b, c)^{-\epsilon} = 1$ for any coloring $C(a, b, c) \in \mathcal{C}$. This is Case (W1). If τ and τ' have the same signs, then we have $W_\theta(C(a, b, c)) = \theta(a, b, c)^\epsilon \cdot \theta(a, b, c)^\epsilon = t^{2\epsilon(a-b)(b-c)(c-a)c}$. Then for a coloring $C(0, 1, 2) \in \mathcal{C}$ we have $W_\theta(C(0, 1, 2)) = t^{8\epsilon}$, which is neither 1, t^{-2}, nor t^4. This is Case (W2).

(Case 3) If D has three triple points, say τ_1, τ_2 and τ_3, then there are two cases. Since $E(\tau) = \epsilon$ (resp. 2ϵ) if τ is a triple point of type (i) (resp. type (ii)), and $E(\tau) = 0$ if τ is degenerate, and $E(\Sigma)$ must be zero (3.7), and since by (3.10) we can assume that there is at least one triple point of type (i), there are two cases as follows:

(Case 3.1) τ_1 and τ_2 are of type (i), and τ_3 is degenerate, or

(Case 3.2) τ_1 and τ_2 are of type (i), and τ_3 is of type (ii). Moreover we can assume that $E(\tau_1) = E(\tau_2) = +1$ and $E(\tau_3) = -2$.

(Cases 3.1) Similarly to (Case 2), this case satisfies (W1) or (W2).

(Case 3.2) Since τ_1 is of type (i), we can assume that τ_1 has the color (a, b, c). Since $E(\tau_1) = +1$, we have $E(C(a, b, c); \tau_1) = \{+(a, b), -(a, c), +(b, c)\}$. Since $E(\tau_3) = -2$, τ_3 has the color (x, y, x) of type (ii) and moreover $E(C(a, b, c); \tau_3) = \{-(x, y), +(x, x), -(y, x)\}$, where $\{x, y\} = \{a, b\}, \{a, c\}$ or $\{b, c\}$. Since $E(C(a, b, c); \tau_3) = \{-(x, y), -(y, x)\} \cup \{+(x, x)\}$, by (3.5) we can see that at least one element of $\{-(x, y), -(y, x)\}$ has the same color and the opposite BW orientation of an element of $E(C(a,\, b,\, c); \tau_1) \cup E(C(a,\, b,\, c); \tau_2)$. We can assume that the element is in $E(C(a,\, b,\, c); \tau_1)$. Then we have $\{x,\, y\} = \{a,\, b\}$ or $\{b,\, c\}$. If $\{x,\, y\} = \{a,\, b\}$, then we have

$$\begin{aligned} & E(C(a,\, b,\, c); \Sigma) \\ = \; & \{+(a,\, b), -(a,\, c), +(b,\, c), -(a,\, b), +(x,\, x), -(b,\, a)\} \cup E(C(a,\, b,\, c); \tau_2) \\ = \; & \{-(a,\, c), +(b,\, c), -(b,\, a)\} \cup E(C(a,\, b,\, c); \tau_2) \cup \{+(a,\, b), -(a,\, b)\} \cup \{+(x,\, x)\}, \end{aligned}$$

where $x = a$ or b. By (3.5) we have $E(C(a,\, b,\, c); \tau_2) = \{+(a,\, c), -(b,\, c), +(b,\, a)\}$. Hence the color of τ_2 must be $(b,\, a,\, c)$. Hence in this case $C(a,\, b,\, c) = C_1(a,\, b,\, c)$, where τ_1 has the color $(a,\, b,\, c)$, and τ_2 has the color $(b,\, a,\, c)$, and τ_3 has the color $(a,\, b,\, a)$ or $(b,\, a,\, b)$. Similarly, if $\{x,\, y\} = \{b,\, c\}$, then $C(a,\, b,\, c) = C_2(a,\, b,\, c)$, where τ_1 has the color $(a,\, b,\, c)$, and τ_2 has the color $(a,\, c,\, b)$, and τ_3 has the color $(b,\, c,\, b)$ or $(c,\, b,\, c)$.

In the case of $C(a,\, b,\, c) = C_1(a,\, b,\, c)$, if τ_1 and τ_2 have the same signs, then by (3.10) $W_\theta(C_1(a,\, b,\, c)) = \theta(a,\, b,\, c)^\epsilon \cdot \theta(b,\, a,\, c)^\epsilon = 1$. This is Case (W1). If τ_1 and τ_2 have the opposite signs, then $W_\theta(C_1(a,\, b,\, c)) = \theta(a,\, b,\, c)^\epsilon \cdot \theta(b,\, a,\, c)^{-\epsilon} = t^{2\epsilon(a-b)(b-c)(c-a)c}$, which is $t^{8\epsilon}$ when $(a,\, b,\, c) = (0,\, 1,\, 2)$. This is Case (W2).

In the case of $C(a,\, b,\, c) = C_2(a,\, b,\, c)$, if τ_1 and τ_2 have the same signs, then $W_\theta(C_2(a,\, b,\, c)) = \theta(a,\, b,\, c)^\epsilon \cdot \theta(a,\, c,\, b)^\epsilon = t^{-\epsilon(a-b)(b-c)(c-a)(b-c)}$. If $\epsilon = +1$, then $W_\theta(C_2(0,\, 1,\, 2)) = t^2$, which is Case (W2). If $\epsilon = -1$, then $\Sigma_{C \in \mathcal{C}} W_\theta(C) = 21 + 4t^{-2} + 2t^4$, which is the same with $\Phi_\theta(S_0)$. However, since $W_{\theta'}(C_2(a,\, b,\, c)) = \theta'(a,\, b,\, c)^{-1} \cdot \theta'(a,\, c,\, b)^{-1} = 1$, we have $\Sigma_{C \in \mathcal{C}} W_{\theta'}(C) = 27$. This is Case $(W3)$. If τ_1 and τ_2 have the opposite signs, then $W_\theta(C_2(a,\, b,\, c)) = \theta(a,\, b,\, c)^\epsilon \cdot \theta(a,\, c,\, b)^{-\epsilon} = t^{\epsilon(a-b)(b-c)(c-a)(b+c)}$, which is $t^{6\epsilon}$ when $(a,\, b,\, c) = (0,\, 1,\, 2)$. This is Case (W2).

Thus we have shown that the triple point number of S_0 is at least four. Since the torus-covering-chart $\Gamma_{T,0}$ has four white vertices, the triple point number of S_0 is at most four. Hence the triple point number of S_0 is four, and the associated torus-covering-chart $\Gamma_{T,\,0}$ realizes the surface diagram where the number of the triple points is the triple point number. □

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