

# A GENERALIZED KOSZUL THEORY AND ITS APPLICATION

LIPING LI

ABSTRACT. Let  $A$  be a graded algebra. In this paper we develop a generalized Koszul theory by assuming that  $A_0$  is self-injective instead of semisimple and generalize many classical results. The application of this generalized theory to directed categories and finite EI categories is described.

## 1. INTRODUCTION

The work described in this paper originated in the exploration of homological properties of finite EI categories. We want to apply Koszul theory, which has been proved to be very useful in the representation theory of algebras, to study the Ext groups of representations of finite EI categories. Examples of such applications can be found in [19] and [21], where Koszul theory has been applied to incidence algebras of posets. In general this theory applies to graded algebras, and we do not assume that the degree 0 part of the algebra is semisimple, unlike the classical Koszul theory described in [4] [9] [10] [15]. This generalization is necessary so that we can apply the theory to finite EI categories.

There do already exist several generalized Koszul theories where the degree 0 part  $A_0$  of a graded algebra  $A$  is not required to be semisimple, see [11] [13] [14] and [21]. Each Koszul algebra  $A$  defined by Woodcock in [21] is supposed to satisfy that  $A$  is both a left projective  $A_0$ -module and a right projective  $A_0$ -module. This requirement is too strong for us. Indeed, even the category algebra  $k\mathcal{E}$  of a standardly stratified finite EI category  $\mathcal{E}$  (studied in [20]) does not satisfy this requirement:  $k\mathcal{E}$  is a left projective  $k\mathcal{E}_0$ -module but in general not a right projective  $k\mathcal{E}_0$ -module. In Madsen's paper [14],  $A_0$  is supposed to have finite global dimension. But for a finite EI category  $\mathcal{E}$ , this happens in our context if and only if  $k\mathcal{E}_0$  is semisimple. The theory developed by Green, Reiten and Solberg in [11] works in a very general framework, but some efforts are required to fill the gap between their theory and our applications.

Thus we want to develop a generalized Koszul theory which can inherit many useful results of the classical theory, and can be applied to structures with nice properties such as finite EI categories. Let  $A$  be a graded  $k$ -algebra with  $A_0$  being self-injective instead of being semisimple. Then we define generalized *linear modules* (or *Koszul modules*), *Koszul algebras* in a similar way to the classical case. That is, a graded  $A$ -module  $M$  is *Koszul* if  $M$  has a linear projective resolution, and  $A$  is a *Koszul algebra* if  $A_0$  viewed as a graded  $A$ -module is Koszul. We also define *quasi-Koszul modules* and *quasi-Koszul algebras*:  $M$  is *quasi-Koszul* if the  $\text{Ext}_A^*(A_0, A_0)$ -module  $\text{Ext}_A^*(M, A_0)$  is generated in degree 0, and  $A$  is a *quasi-Koszul algebra* if  $A_0$  is a quasi-Koszul  $A$ -module. It turns out that this generalization works nicely for our goal. Many classical results described in [4] [9] [10] and [15] generalize to our context. In particular, we obtain the Koszul duality both on the category of linear modules and on the derived category.

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We then focus on the applications of this generalized Koszul theory. First we define *directed categories*. A directed category  $\mathcal{C}$  is a  $k$ -linear category equipped with a partial order  $\leq$  on  $\text{Ob}\mathcal{C}$  such that for each pair of objects  $x, y \in \text{Ob}\mathcal{C}$ , the space of morphisms  $\mathcal{C}(x, y)$  is non-zero only if  $x \leq y$ . Directed categories include the  $k$ -linearizations of finite EI categories as special examples. This partial order determines a canonical pre-order  $\preceq$  on the isomorphism classes of simple representations. Following the technique in [20], we develop a stratification theory for directed categories, describe the structures of standard modules and characterize every directed category  $\mathcal{C}$  standardly stratified with respect to the canonical pre-order:

**Theorem 1.1.** *Let  $\mathcal{C}$  be a directed category with respect to a partial order  $\leq$ . Then  $\mathcal{C}$  is standardly stratified for the canonical pre-order  $\preceq$  if and only if the morphism space  $\mathcal{C}(x, y)$  is a projective  $\mathcal{C}(y, y)$ -module for every pair of objects  $x, y \in \text{Ob}\mathcal{C}$ .*

By the correspondence between graded  $k$ -linear categories and graded algebras described in [16], we can view a directed category as a directed algebra and vice-versa. Therefore, all of our results on graded algebras can be applied to graded directed categories. In particular, the following theorem relates Koszul theory to stratification theory:

**Theorem 1.2.** *Let  $\mathcal{C}$  be a graded directed category with  $\mathcal{C}_0 = \bigoplus_{x \in \text{Ob}\mathcal{C}} \mathcal{C}(x, x)$  being a self-injective algebra. Then:*

- (1)  $\mathcal{C}$  is standardly stratified with respect to the canonical pre-order  $\preceq$  if and only if  $\mathcal{C}$  is a projective  $\mathcal{C}_0$ -module.
- (2)  $\mathcal{C}$  is a Koszul category if and only if  $\mathcal{C}$  is a quasi-Koszul category standardly stratified for  $\preceq$ .
- (3) If  $\mathcal{C}$  is standardly stratified for  $\preceq$ , then a graded  $\mathcal{C}$ -module  $M$  generated in degree 0 is linear if and only if it is a quasi-Koszul  $\mathcal{C}$ -module and a projective  $\mathcal{C}_0$ -module.

Applying the homological dual functor  $E = \text{Ext}_{\mathcal{C}}^*(-, \mathcal{C}_0)$  to a graded directed category  $\mathcal{C}$ , we construct the *Yoneda category*  $E(\mathcal{C}_0) = \text{Ext}_{\mathcal{C}}^*(\mathcal{C}_0, \mathcal{C}_0)$ . We prove that if  $\mathcal{C}$  is a Koszul directed category with  $\mathcal{C}_0$  being self-injective, then  $E(\mathcal{C}_0)$  is also a Koszul directed category.

We acquire a very nice correspondence between the classical Koszul theory and our generalized Koszul theory for directed categories.

**Theorem 1.3.** *Let  $\mathcal{C}$  be a graded directed category with  $\mathcal{C}_0$  being a self-injective algebra. Define  $\mathcal{D}$  to be the subcategory of  $\mathcal{C}$  obtained by replacing each endomorphism ring by  $k \cdot 1$ , the span of the identity endomorphism. Then:*

- (1)  $\mathcal{C}$  is a Koszul category in our sense if and only if  $\mathcal{C}$  is standardly stratified for the canonical pre-order  $\preceq$  and  $\mathcal{D}$  is a Koszul category in the classical sense.
- (2) If  $\mathcal{C}$  is a Koszul category, then a graded  $\mathcal{C}$ -module  $M$  is linear if and only if  $M$  is a projective  $\mathcal{C}_0$ -module, and  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is a linear  $\mathcal{D}$ -module.

Finite EI categories have nice combinatorial properties. These properties can be used to define length gradings on the sets of morphisms. We discuss the possibility to put such a grading on an arbitrary finite EI category, and prove the following result for *finite free EI categories* (defined in [12]):

**Theorem 1.4.** *Let  $\mathcal{E}$  be a finite free EI category. Then the following are equivalent:*

- (1)  $\text{pdk}\mathcal{E}_0 < \infty$ ;
- (2)  $\text{pdk}\mathcal{E}_0 \leq 1$ ;
- (3)  $\mathcal{E}$  is standardly stratified in a sense defined in [20];

(4)  $k\mathcal{E}$  is a Koszul algebra.

Moreover,  $k\mathcal{E}$  is a quasi-Koszul algebra if and only if  $\text{Ext}_{k\mathcal{E}}^i(k\mathcal{E}_0, k\mathcal{E}_0) = 0$  for each  $i \geq 2$ .

We then give a sufficient condition for the category algebra of a finite EI category  $\mathcal{E}$  to be quasi-Koszul. An object  $x \in \text{Ob } \mathcal{E}$  is called *left regular* if for every morphism  $\alpha : z \rightarrow x$ , the stabilizer of  $\alpha$  in  $\text{Aut}_{\mathcal{E}}(y)$  has order invertible in  $k$ . Similarly we define *right regular* objects. Then:

**Theorem 1.5.** *Let  $\mathcal{E}$  be a finite free EI category. If every object in  $\mathcal{E}$  is either left regular or right regular, then  $k\mathcal{E}$  is a quasi-Koszul algebra.*

Motivated by the fact that the Yoneda category  $E(\mathcal{C}_0)$  of a directed Koszul category  $\mathcal{C}$  is still a directed Koszul category, and hence is standardly stratified, we ask whether the Koszul dual algebra  $\Gamma = \text{Ext}_A^*(A_0, A_0)$  of a graded algebra  $A$  standardly stratified for a partial order  $\leq$  is still standardly stratified for  $\leq$  (or  $\leq^{op}$ ). This question has only been studied for the case that  $A_0$  is semisimple, see [1] [2] [8] [17] and [18]. By assuming that  $A_0$  is a self-injective algebra, and supposing that all standard modules are concentrated in degree 0 and linear, we get a sufficient condition for  $\Gamma$  to be standardly stratified with respect to  $\leq^{op}$ .

The layout of this paper is as follows. The generalized Koszul theory is developed in the first three sections. In Section 2 we define Koszul modules and quasi-Koszul modules, which generalize linear modules and Koszul modules in the classical theory, and describe their basic properties. Since Koszul modules and quasi-Koszul modules do not coincide in our context, we also give a relation between these two concepts. Koszul algebras are studied in Section 3. The Koszul dualities are proved in Section 4. Most results in these three sections are generalized from works in [4] [9] [10] [15]. Some results can be deduced from the paper [11] of Green, Reiten and Solberg, who worked in a more general context, but we present full arguments for the sake of completeness.

The last three sections are on the application of the general theory developed before. Directed categories are defined in Section 5. Their stratification properties and Koszul properties are discussed in details in this section as well. The main content of Section 6 to apply the Koszul theory and stratification theory to finite EI categories, which have nice combinatorial structures. In Section 7 we modify the technique of [1] to study standardly stratified algebras with linear standard modules.

Here are the notation and conventions we use in this paper. All algebras are  $k$ -algebras with  $k$  being algebraically closed. All modules are finitely generated left modules. Let  $A$  be a graded algebra. A graded  $A$ -module  $M$  is said to be *locally finite* if  $\dim_k M_i < \infty$  for each degree  $i \in \mathbb{Z}$ . By  $A\text{-mod}$  and  $A\text{-gmod}$  we denote the category of all finitely generated  $A$ -modules and the category of all locally finite graded  $A$ -modules. Let  $M$  and  $N$  be two  $A$ -modules. By  $\text{Hom}_A(M, N)$  we denote the space of module homomorphisms from  $M$  to  $N$ . If furthermore  $M$  and  $N$  are graded, we use  $\text{hom}_A(M, N)$  to denote all *graded module homomorphisms* from  $M$  to  $N$ , i.e., the homomorphisms  $\varphi \in \text{Hom}_A(M, N)$  such that  $\varphi(M_i) \subseteq N_i$  for all  $i \in \mathbb{Z}$ . When  $\varphi$  is said to be a homomorphism between two graded modules, it is supposed to be a graded homomorphism. If  $M$  is a graded  $A$ -module, its  $s$ -th *shift*  $M[s]$  is defined in the following way:  $M[s]_i = M_{i-s}$  for all  $i \in \mathbb{Z}$ . If  $M$  is generated in degree  $s$ , then  $\bigoplus_{i \geq s+1} M_i$  is a graded submodule of  $M$ , and  $M_s \cong M / \bigoplus_{i \geq s+1} M_i$  as vector spaces. We then view  $M_s$  as an  $A$ -module by identifying it with this quotient module. We also regard the zero module 0 as a projective module since with this convention the expressions of many results can be simplified.

## 2. GENERALIZED KOSZUL MODULES

Throughout this section  $A$  is a non-negatively graded and locally finite associative  $k$ -algebra with identity 1 generated in degrees 0 and 1, i.e.,  $A = \bigoplus_{i=0}^{\infty} A_i$  such that  $A_i \cdot A_j = A_{i+j}$  for all  $i, j \geq 0$ ; each  $A_i$  is finite-dimensional. We also suppose that  $A_0$  is a *self-injective* algebra, i.e., every projective  $A_0$ -module is injective as well. Define  $J = \bigoplus_{i=1}^{\infty} A_i$ , which is a two-sided ideal of  $A$ . An  $A$ -module  $M$  is called *graded* if  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  such that  $A_i \cdot M_j \subseteq M_{i+j}$ . We say  $M$  is *generated in degree  $s$*  if  $M = A \cdot M_s$ . It is clear that  $M$  is generated in degree  $s$  if and only if  $JM \cong \bigoplus_{i \geq s+1} M_i$ , which is equivalent to  $J^l M \cong \bigoplus_{i \geq s+l} M_i$  for all  $l \geq 1$ .

Most results in this section are generalized from [9] [10] and [15]. We suggest that the reader refer to these papers.

We collect some preliminary results in the following lemma.

**Lemma 2.1.** *Let  $A$  be as above and  $M$  be a locally finite graded  $A$ -module. Then:*

- (1)  $J = \bigoplus_{i \geq 1} A_i$  is contained in the graded radical of  $A$ ;
- (2)  $M$  has a graded projective cover;
- (3) the graded syzygy  $\Omega M$  is also locally finite.

*Proof.* By definition, the graded radical  $\text{grad } A$  is the intersection of all maximal proper graded submodules of  $A$ . Let  $L \subsetneq A$  be a maximal proper graded submodule. Then  $L_0$  is a proper subspace of  $A_0$ . We claim that  $A_i = L_i$  for all  $i \geq 1$ . Otherwise, we can define  $\tilde{L} \subseteq A$  in the following way:  $\tilde{L}_0 = L_0$  and  $\tilde{L}_i = A_i$  for  $i \geq 1$ . Then  $L \subsetneq \tilde{L} \subsetneq A$ , so  $L$  is not a maximal proper graded submodule of  $A$ . This contradiction tells us that  $L_i = A_i$  for all  $i \geq 1$ . Therefore,  $J \subseteq \text{grad } A$ , and the first statement is proved.

We use the following fact to prove the second statement: every primitive idempotent of  $A_0$  (as an algebra) can be lifted to a primitive idempotent of  $A$ . Consequently, a projective  $A_0$ -module concentrated in some degree  $d$  can be lifted to a graded projective  $A$ -module generated in degree  $d$ .

Define  $\bar{M} = M/JM$ , which is also a locally finite graded  $A$ -module. Write  $\bar{M} = \bigoplus_{i \geq 0} \bar{M}_i$ . Then each  $\bar{M}_i$  is a finite-dimensional graded  $A$ -module since  $J\bar{M} = 0$  and  $A_0\bar{M}_i = \bar{M}_i$  for all  $i \geq 0$ . Therefore, there is a decomposition of  $\bar{M}$  in which each indecomposable summand (which must be of finite dimension) is concentrated in a certain degree. Moreover, for each  $i \in \mathbb{Z}$ , there are only finitely many summands which are concentrated in degree  $i$ .

Take  $L$  to be such an indecomposable summand and without loss of generality suppose that it is concentrated in degree 0. As an  $A_0$ -module,  $L$  is finitely generated and has a finitely generated projective cover  $P_0$ . By the lifting property,  $P_0$  can be lifted to a finitely generated (hence locally finite) graded projective module  $P$  generated in degree 0.

We claim that  $P$  is a graded projective cover of  $L$ . Indeed, there is a graded surjective homomorphism  $\varphi : P \rightarrow L$  by mapping  $P_0$  onto  $L = L_0$  and  $JP$  to 0. Furthermore, this surjection is minimal. That is, if  $Q$  is a proper summand of  $P$ , then the induced map  $\varphi' : Q \rightarrow L$  is not surjective. Otherwise, the proper submodule  $Q_0 \subsetneq P_0$  can be mapped onto  $L_0$ . But this contradicts the fact that  $P_0$  is a projective cover of  $L$  as an  $A_0$ -module. In conclusion, we proved that  $P$  is a graded projective cover of  $L$ .

Take the direct sum of these projective covers  $P$  when  $L$  ranges over all indecomposable summands of  $\bar{M}$ . In this way we obtain a graded projective cover  $\tilde{P}$  of  $\bar{M}$ . We claim that  $\tilde{P}$  is also a graded projective cover of  $M$ . On one hand, the surjection  $p : \tilde{P} \rightarrow \bar{M}$  can be factored through the quotient homomorphism  $q : M \rightarrow \bar{M}$  and induces a graded homomorphism  $p' : \tilde{P} \rightarrow M$  such that  $p = qp'$ ,

as showed in the following diagram:

$$\begin{array}{ccc} & & M \\ & \nearrow p' & \downarrow q \\ P & \xrightarrow{p} & \bar{M} \end{array}$$

Define  $M'$  to be the image of  $p'$ . Viewed as vector spaces,  $\bar{M} = M/JM \subseteq M'$  and  $M = \bar{M} + JM$ . Therefore,  $M = JM + M'$  as graded  $A$ -modules. By the graded Nakayama's lemma and the first statement, we deduce that  $M' = M$ , and hence  $p'$  is a surjection. On the other hand, if there is a proper summand  $\tilde{Q} \subsetneq \tilde{P}$  such that the induced map

$$\tilde{Q} \xrightarrow{\iota} \tilde{P} \xrightarrow{p'} M$$

is a graded surjection, where  $\iota$  is the inclusion, then the map  $\iota p' q = \iota p$  is a graded surjection from  $\tilde{Q}$  to  $\bar{M} = M/JM$ . This contradicts the fact that  $\tilde{P}$  is a graded projective cover of  $\bar{M}$ . In conclusion,  $\tilde{P}$  is a graded projective cover of  $M$ , and the second statement is proved.

Now we turn to the third statement. By the above proof, the graded projective cover  $\tilde{P}$  of  $M$  can be written as a direct sum  $\bigoplus_{i \geq 0} P^i$  of graded projective modules, where  $P^i$  is generated in degree  $i$ . For each fixed degree  $i \geq 0$ , there are only finitely many indecomposable summands  $L$  of  $\bar{M}$  which are concentrated in degree  $i$ , and the graded projective cover of each  $L$  is finitely generated. Consequently,  $P^i$  as the direct sum of graded projective covers of those  $L$  concentrated in degree  $i$  is finitely generated, and hence locally finite.

For a fixed  $n \geq 0$ , we have  $\tilde{P}_n = \bigoplus_{i \geq 0} P_n^i = \bigoplus_{0 \leq i \leq n} P_n^i$ . Since each  $P^i$  is locally finite,  $\dim_k P_n^i < \infty$ . Therefore,  $\dim_k \tilde{P}_n < \infty$ , and  $\tilde{P}$  is locally finite as well. As a submodule (up to isomorphism) of  $\tilde{P}$ , the graded syzygy  $\Omega M$  is also locally finite.  $\square$

The results in the previous lemma will be used frequently in the rest of this paper without being mentioned.

**Lemma 2.2.** *Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be an exact sequence of graded  $A$ -modules. Then:*

- (1) *If  $M$  is generated in degree  $s$ , so is  $N$ .*
- (2) *If  $L$  and  $N$  are generated in degree  $s$ , so is  $M$ .*
- (3) *If  $M$  is generated in degree  $s$ , then  $L$  is generated in degree  $s$  if and only if  $JM \cap L = JL$ .*

*Proof.* (1): This is obvious.

(2): Let  $P$  and  $Q$  be graded projective covers of  $L$  and  $N$  respectively. Then  $P$  and  $Q$ , and hence  $P \oplus Q$  are generated in degree  $s$ . In particular, each graded projective cover of  $M$ , which is isomorphic to a direct summand of  $P \oplus Q$ , is generated in degree  $s$ . Thus  $M$  is also generated in degree  $s$ .

(3): We always have  $JL \subseteq JM \cap L$ . Let  $x \in L \cap JM$  be a homogeneous element of degree  $i$ . Since  $M$  is generated in degree  $s$ , we have  $i \geq s + 1$ . If  $L$  is generated in degree  $s$ , then  $x \in J^{i-s}L \subseteq JL$ . Thus  $L \cap JM \subseteq JL$ , so  $JL = L \cap JM$ .

Conversely, the identity  $JL = L \cap JM$  gives us the following commutative diagram where all rows and columns are exact:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & JL & \longrightarrow & JM & \longrightarrow & JN \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L/JL & \longrightarrow & M/JM & \longrightarrow & N/JN \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Consider the bottom sequence. Notice that  $(M/JM) \cong M_s$  is concentrated in degree  $s$ . Thus  $L/JL$  is also concentrated in degree  $s$ , i.e.,  $L/JL \cong L_s$ . Let  $I = A \cdot L_s$ . Then  $L \subseteq I + JL \subseteq L$ , so  $I + JL = L$ . Notice that  $J$  is contained in the graded Jacobson radical of  $A$ . Therefore, by the graded Nakayama lemma,  $I = A \cdot L_s = L$ , so  $L$  is generated in degree  $s$ .  $\square$

**Corollary 2.3.** *Suppose that each graded  $A$ -module in the short exact sequence  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  is generated in degree 0. Then  $J^i M \cap L = J^i L$  for all  $i \geq 0$ .*

*Proof.* Since all modules  $L$ ,  $M$  and  $N$  are generated in degree 0, all  $J^s L$ ,  $J^s M$  and  $J^s N$  are generated in degree  $s$  for  $s \geq 0$ . The exactness of the above sequence implies  $JL = L \cap JM$ , which in turns gives the exactness of  $0 \rightarrow JL \rightarrow JM \rightarrow JN \rightarrow 0$ . By the above lemma,  $J^2 M \cap JL = J^2 L$  and this implies the exactness of  $0 \rightarrow J^2 L \rightarrow J^2 M \rightarrow J^2 N \rightarrow 0$ . The conclusion follows from induction.  $\square$

Now we introduce generalized *linear modules* (or called *Koszul modules*).

**Definition 2.4.** *A graded  $A$ -module  $M$  generated in degree 0 is called a linear module (or a Koszul module) if it has a (minimal) projective resolution*

$$\cdots \longrightarrow P^n \longrightarrow P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

*such that  $P^i$  is generated in degree  $i$  for all  $i \geq 0$ .*

A direct consequence of this definition and the previous lemma is:

**Corollary 2.5.** *Let  $M$  be a linear module. Then  $\Omega^i(M)/J\Omega^i(M) \cong \Omega^i(M)_i$  is a projective  $A_0$ -module for each  $i \geq 0$ , or equivalently,  $\Omega^i(M) \subseteq JP^{i-1}$  where  $P^{i-1}$  is a graded projective cover of  $\Omega^{i-1}(M)$ , where  $\Omega$  is the Heller operator.*

*Proof.* Since  $M$  is linear,  $\Omega^i(M)$  is generated in degree  $i$ , and  $\Omega^i(M)/J\Omega^i(M) \cong \Omega^i(M)_i$ . Moreover, all  $\Omega^i(M)[-i]$  are linear  $A$ -modules as well. By induction, it is sufficient to prove  $\Omega M \subseteq JP^0$ . But this is obvious since  $\Omega M$  is generated in degree 1. From the following commutative diagram we deduce that  $\Omega M \subseteq JP^0$  if and only if the bottom sequence is exact, or equivalently  $M/JM \cong P^0/JP^0 \cong P_0^0$

is a projective  $A_0$ -module.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Omega M & \longrightarrow & JP^0 & \longrightarrow & JM \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Omega M & \longrightarrow & P^0 & \longrightarrow & M \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & P^0/JP^0 & \longrightarrow & M/JM \longrightarrow 0
\end{array}$$

□

There are several characterizations of linear modules.

**Proposition 2.6.** *Let  $M$  be a graded  $A$ -module generated in degree 0. Then the following are equivalent:*

- (1)  $M$  is linear.
- (2) The syzygy  $\Omega^i(M)$  is generated in degree  $i$  for every  $i \geq 0$ .
- (3) For all  $i > 0$ ,  $\Omega^i(M) \subseteq JP^{i-1}$  and  $\Omega^i(M) \cap J^2P^{i-1} = J\Omega^i(M)$ , where  $P^{i-1}$  is a graded projective cover of  $\Omega^{i-1}(M)$ .
- (4)  $\Omega^i(M) \subseteq JP^{i-1}$  and  $\Omega^i(M) \cap J^{s+1}P^{i-1} = J^s\Omega^i(M)$  for all  $i > 0, s \geq 0$ .

*Proof.* The equivalence of (1) and (2) is clear. It is also obvious that (3) is the special case of (4) for  $s = 1$ . Now we show (1) implies (4). Indeed, if  $M$  is a linear module, then both  $JP^0$  and  $\Omega M$  are generated in degree 1 and  $\Omega M \subseteq JP^0$ . Therefore we have the following exact sequence

$$0 \longrightarrow \Omega M \longrightarrow JP^0 \longrightarrow JM \longrightarrow 0$$

in which all modules are generated in degree 1. By Corollary 2.5  $J^{s+1}P^0 \cap \Omega M = J^s\Omega M$  for all  $s > 0$ . Notice that all syzygies of  $M$  are also linear with suitable grade shifts. Replacing  $M$  by  $\Omega^i(M)[-i]$  and using induction we get (4).

Finally we show (3) implies (2) to finish the proof. Since  $\Omega M \subseteq JP^0$  we still have the above exact sequence. Notice that both  $JM$  and  $JP^0$  are generated in degree 1 and  $J^2P^0 \cap \Omega M = J\Omega M$ , by Lemma 2.2,  $\Omega M$  is generated in degree 1 as well. Now the induction procedure gives us the required conclusion. □

The condition that  $\Omega^i(M) \subseteq JP^{i-1}$  (or equivalently,  $\Omega^i(M)/J\Omega^i(M) \cong \Omega^i(M)_i$  is a projective  $A_0$ -module) in (3) of the previous proposition is necessary, as shown by the following example:

**Example 2.7.** *Let  $G$  be a finite cyclic group of prime order  $p$  and  $k$  be an algebraically closed field of characteristic  $p$ . Let the group algebra  $kG$  be concentrated on degree 0, so  $J = 0$ . Consider the trivial  $kG$ -module  $k$ . Obviously,  $k$  is not a linear module. But since  $J = 0$ , the condition  $J\Omega^i(k) = J^2P^{i-1} \cap \Omega^i(k)$  holds trivially.*

**Remark 2.8.** *We do not use the property that  $A_0$  is a self-injective algebra up to now. Therefore, all results described before still hold for a non-negatively graded, locally finite graded algebra  $A$  with  $A_0$  being an arbitrary finite-dimensional algebra.*

**Proposition 2.9.** *Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be a short exact sequence of graded  $A$ -modules such that  $L$  is a linear  $A$ -module. Then  $M$  is linear if and only if  $N$  is linear.*

*Proof.* We verify the conclusion by using statement (2) in the last proposition. That is, given that  $\Omega^i(L)$  is generated in degree  $i$  for each  $i \geq 0$ , we want to show that  $\Omega^i(M)$  is generated in degree  $i$  if and only if so is  $\Omega^i(N)$ .

Consider the given exact sequence. By Lemma 2.2,  $M$  is generated in degree 0 if and only if  $N$  is generated in degree 0. Therefore we have the following diagram in which all rows and columns are exact:

$$(2.1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega L & \longrightarrow & M' & \longrightarrow & \Omega N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P & \longrightarrow & P \oplus Q & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where  $P$  and  $Q$  are graded projective covers of  $L$  and  $N$  respectively. In general  $P \oplus Q$  is not a graded projective cover of  $M$ , and hence  $M' \not\cong \Omega M$ . However, under the hypothesis of this proposition we claim that  $P \oplus Q$  is indeed a graded projective cover of  $M$  and  $M' \cong \Omega M$ . To see this, we point out that the given exact sequence induces a short exact sequence of  $A_0$ -modules:

$$0 \longrightarrow L_0 \longrightarrow M_0 \longrightarrow N_0 \longrightarrow 0.$$

Observe that  $L_0 = \Omega^0(L)_0$  is projective (by Corollary 2.5) and hence injective (since  $A_0$  is self-injective) as an  $A_0$ -module. Therefore, the above sequence splits, and  $M_0 \cong L_0 \oplus N_0$ . On one hand, a graded projective cover  $\tilde{P}$  of  $M$  should be isomorphic to a direct summand of  $P \oplus Q$ . On the other hand, since  $\tilde{P}$  induces a projective cover  $\tilde{P}_0$  of  $M_0 \cong L_0 \oplus N_0$ ,  $\tilde{P}$  should contain a summand isomorphic to  $Q \oplus P$ . This forces  $\tilde{P} \cong P \oplus Q$ , and our claim is proved.

Now let us consider the top row of diagram 2.1. Since  $M' \cong \Omega M$ , and  $\Omega L$  is generated in degree 1,  $\Omega M$  is generated in degree 1 if and only if  $\Omega N$  is generated in degree 1 by Lemma 2.2. Replace  $L$ ,  $M$  and  $N$  by  $(\Omega L)[-1]$ ,  $(\Omega M)[-1]$  and  $(\Omega N)[-1]$  (all of which are linear) respectively in the short exact sequence. Repeating the above procedure and using the fact that the Heller operator  $\Omega$  and the grade shift functor  $[-]$  commute, we conclude that  $\Omega^2(M)[-1]$  is generated in degree 1 if and only if  $\Omega^2(N)[-1]$  is generated in degree 1, i.e.,  $\Omega^2(M)$  is generated in degree 2 if and only if  $\Omega^2(N)$  is generated in degree 2. By induction,  $M$  is linear if and only if  $N$  is linear.  $\square$

The condition that  $L$  is linear in this proposition is necessary. Indeed, quotient modules of a linear module might not be linear.

**Lemma 2.10.** *Let  $M$  be a graded  $A$ -module generated in degree  $s$ . If  $M_s$  is a projective  $A_0$ -module, then  $\text{Ext}_A^i(M, A_0) \cong \text{Ext}_A^{i-1}(\Omega M, A_0)$  for all  $i \geq 1$ .*

*Proof.* It is true for  $i > 1$ . When  $i = 1$ , consider the following exact sequence:

$$0 \rightarrow \text{Hom}_A(M, A_0) \rightarrow \text{Hom}_A(P, A_0) \rightarrow \text{Hom}_A(\Omega M, A_0) \rightarrow \text{Ext}_A^1(M, A_0) \rightarrow 0.$$

As a graded projective cover of  $M$ ,  $P$  is also generated in degree  $s$ . Since  $M_s$  is a projective  $A_0$ -module,  $P_s \cong M_s$ . So

$$\text{Hom}_A(M, A_0) \cong \text{Hom}_{A_0}(M_s, A_0) \cong \text{Hom}_{A_0}(P_s, A_0) \cong \text{Hom}_A(P, A_0).$$

Thus  $\text{Hom}_A(\Omega M, A_0) \cong \text{Ext}_A^1(M, A_0)$ .  $\square$



**Proposition 2.11.** *Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be an exact sequence of linear modules. Then it induces the following short exact sequence:*

$$0 \longrightarrow \text{Ext}_A^*(N, A_0) \longrightarrow \text{Ext}_A^*(M, A_0) \longrightarrow \text{Ext}_A^*(L, A_0) \longrightarrow 0.$$

*Proof.* As in the proof of Proposition 2.9, the above exact sequence gives exact sequences  $0 \rightarrow \Omega^i(L) \rightarrow \Omega^i(M) \rightarrow \Omega^i(N) \rightarrow 0$ ,  $i \geq 0$ . For a fixed  $i$ , the sequence  $0 \rightarrow \Omega^i(L)_i \rightarrow \Omega^i(M)_i \rightarrow \Omega^i(N)_i \rightarrow 0$  splits since all terms are projective  $A_0$ -modules by Corollary 2.5. Applying the exact functor  $\text{Hom}_{A_0}(-, A_0)$  we get an exact sequence

$$0 \rightarrow \text{Hom}_{A_0}(\Omega^i(N)_i, A_0) \rightarrow \text{Hom}_{A_0}(\Omega^i(M)_i, A_0) \rightarrow \text{Hom}_{A_0}(\Omega^i(L)_i, A_0) \rightarrow 0$$

which is isomorphic to

$$0 \rightarrow \text{Hom}_A(\Omega^i(N), A_0) \rightarrow \text{Hom}_A(\Omega^i(M), A_0) \rightarrow \text{Hom}_A(\Omega^i(L), A_0) \rightarrow 0$$

since all modules are generated in degree  $i$ . By Lemma 1.9, it is isomorphic to

$$0 \longrightarrow \text{Ext}_A^i(N, A_0) \longrightarrow \text{Ext}_A^i(M, A_0) \longrightarrow \text{Ext}_A^i(L, A_0) \longrightarrow 0.$$

Putting them together we have:

$$0 \longrightarrow \text{Ext}_A^*(N, A_0) \longrightarrow \text{Ext}_A^*(M, A_0) \longrightarrow \text{Ext}_A^*(L, A_0) \longrightarrow 0.$$

□

If a graded  $A$ -module  $M$  is linear, so are all syzygies  $\Omega^i(M)[-i]$ ,  $i \geq 0$ . The next example shows that the linear property of  $M$  in general does not imply the linear property of  $J^i M[-i]$ :

**Example 2.12.** *Let  $\mathcal{E}$  be a finite EI category with two objects  $x$  and  $y$  such that:  $\text{Aut}_{\mathcal{E}}(x) = \langle g \rangle \cong \text{Aut}_{\mathcal{E}}(y) = \langle h \rangle$  are cyclic groups of order 2;  $\text{Hom}_{\mathcal{E}}(x, y)$  has one element  $\alpha$  on which both  $\text{Aut}_{\mathcal{E}}(x)$  and  $\text{Aut}_{\mathcal{E}}(y)$  act trivially; and  $\text{Hom}_{\mathcal{E}}(y, x) = \emptyset$ . Let  $k$  be an algebraically closed field of characteristic 2. We put the following grade on the category algebra  $A = k\mathcal{E}$ :  $A_0$  is spanned by  $\{g, 1_x, h, 1_y\}$  and  $A_1$  is spanned by  $\alpha$ . Consider the projective  $k\mathcal{E}$ -module  $P_x = k\mathcal{E}1_x$ . Obviously,  $P_x$  is linear, but  $JP_x \cong k_y$  is not linear.*

However, for some special cases, we can get a conclusion as follows.

**Proposition 2.13.** *Suppose that  $A_0$  is a linear  $A$ -module. If  $M$  is a linear  $A$ -module, then  $J^i M[-i]$  and  $M_0$  are also linear  $A$ -modules. In particular,  $M$  is projective viewed as an  $A_0$ -module (or equivalently,  $M_i$  is a projective  $A_0$ -module for every  $i \geq 0$ ).*

*Proof.* Without loss of generality we assume that  $M$  is indecomposable. Since  $M$  is linear,  $M_0$  is a projective  $A_0$ -module and is contained in  $\text{add}(A_0)$ , the category of all  $A$ -modules each of which is isomorphic to a direct summand of  $A_0^{\oplus m}$  for some  $m \geq 0$ . But  $A_0$  is linear, so is  $M_0$ .

Notice that  $M$  and  $M_0$  have the same graded projective cover (up to isomorphism) as  $A$ -modules. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & \Omega(M_0) & \longrightarrow & JM \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P^0 & \xrightarrow{id} & P^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & JM & \longrightarrow & M & \longrightarrow & M_0 \longrightarrow 0 \end{array}$$

Consider the top sequence. The modules  $M$  and  $M_0$  are linear, so are  $\Omega M[-1]$  and  $\Omega(M_0)[-1]$ . Therefore,  $JM[-1]$  is also linear by Proposition 2.9. Now replacing  $M$  by  $JM[-1]$  and using recursion, we conclude that  $J^i M[-i]$  is a linear  $A$ -module for every  $i \geq 0$ . This proves the first statement.

Since  $J^i M[-i]$  is a linear  $A$ -module,  $(J^i M[-i])_0 \cong (J^i M)_i \cong M_i$  is a projective  $A_0$ -module for each  $i \geq 0$  by Corollary 2.5. Since  $M = \bigoplus_{i \geq 0} M_i$  as an  $A_0$ -module, we deduce that  $M$  is a projective  $A_0$ -module if and only if  $M_i$  is a projective  $A_0$ -module for every  $i \geq 0$ .  $\square$

The following lemma is very useful.

**Lemma 2.14.** *Let  $M$  be a non-negatively graded  $A$ -module and suppose that  $A$  is a projective  $A_0$ -module. Then the following are equivalent:*

- (1) *all  $\Omega^i(M)_j$  are projective  $A_0$ -modules,  $i, j \geq 0$ ;*
- (2) *all  $\Omega^i(M)_i$  are projective  $A_0$ -modules,  $i \geq 0$ ;*
- (3) *all  $M_i$  are projective  $A_0$ -modules,  $i \geq 0$ .*

*Proof.* It is clear that (1) implies (2).

(3) implies (1): Suppose that  $M_i$  is a projective  $A_0$ -module for all  $i \geq 0$ . Let  $P$  be a graded projective cover of  $M$ . The surjective homomorphism  $\varphi : P \rightarrow M$  gives a surjective homomorphism  $\varphi_j : P_j \rightarrow M_j$  with kernel  $(\Omega M)_j$ . By the hypothesis,  $P_j$  and  $M_j$  are projective  $A_0$ -modules for all  $j \geq 0$ . Then  $P_j \cong M_j \oplus (\Omega M)_j$ , so all  $(\Omega M)_j$  are projective  $A_0$ -modules for  $j \geq 0$ . Replacing  $M$  by  $\Omega M$  and using recursion, we conclude that all  $\Omega^i(M)_j$  are projective  $A_0$ -modules for  $i, j \geq 0$ . In particular, all  $\Omega^i(M)_i$  are projective  $A_0$ -modules.

(2) implies (3): Conversely, suppose that  $\Omega^i(M)_i$  is a projective  $A_0$ -modules for every  $i \geq 0$ . We use contradiction to show that all  $M_i$  are projective  $A_0$ -modules. If this not the case, we can find the minimal number  $n \geq 0$  such that  $M_n$  is not a projective  $A_0$ -module. As above, consider  $\varphi_n : P_n \rightarrow M_n$  with kernel  $(\Omega M)_n$ . We claim that this kernel is not a projective  $A_0$ -module. Indeed, if it is a projective  $A_0$ -module, then it is injective as well, so  $P_n \cong (\Omega M)_n \oplus M_n$ . Consequently,  $M_n$  is isomorphic to a summand of the projective  $A_0$ -module  $P_n$  and must be a projective  $A_0$ -module, too. This is impossible. Therefore,  $(\Omega M)_n$  is not a projective  $A_0$ -module. Now replacing  $M$  by  $\Omega M$  and using induction, we deduce that  $\Omega^n(M)_n$  is not a projective  $A_0$ -module. This contradicts our assumption. Therefore, all  $M_i$  are projective  $A_0$ -modules.  $\square$

Now we define *quasi-Koszul modules* over the graded algebra  $A$ .

**Definition 2.15.** *A non-negatively graded  $A$ -module  $M$  is called quasi-Koszul if*

$$\text{Ext}_A^1(A_0, A_0) \cdot \text{Ext}_A^i(M, A_0) = \text{Ext}_A^{i+1}(M, A_0)$$

*for all  $i \geq 0$ . The algebra  $A$  is called a quasi-Koszul algebra if  $A_0$  as an  $A$ -module is quasi-Koszul.*

A graded  $A$ -module  $M$  is quasi-Koszul if and only if as a graded  $\text{Ext}_A^*(A_0, A_0)$ -module  $\text{Ext}_A^*(M, A_0)$  is generated in degree 0. The graded algebra  $A$  is a quasi-Koszul algebra if and only if the cohomology ring  $\text{Ext}_A^*(A_0, A_0)$  is generated in degree 0 and degree 1.

The quasi-Koszul property is preserved by the Heller operator. Explicitly, if  $M$  is a quasi-Koszul  $A$ -module with  $M_0$  being a projective  $A_0$ -module, then its syzygy  $\Omega M$  is also quasi-Koszul. This is because for each  $i \geq 1$ , we have:

$$\begin{aligned} \text{Ext}_A^i(\Omega M, A_0) &\cong \text{Ext}_A^{i+1}(M, A_0) \\ &= \text{Ext}_A^1(A_0, A_0) \cdot \text{Ext}_A^i(M, A_0) \\ &= \text{Ext}_A^1(A_0, A_0) \cdot \text{Ext}_A^{i-1}(\Omega M, A_0). \end{aligned}$$

The identity  $\text{Ext}_A^i(M, A_0) \cong \text{Ext}_A^{i-1}(\Omega M, A_0)$  is proved in Lemma 2.10.

If  $A_0$  is a semisimple  $k$ -algebra, quasi-Koszul modules generated in degree 0 coincide with linear modules. This is not true if  $A_0$  is only self-injective. Actually, by the following theorem, every Koszul module is quasi-Koszul, but the converse does not hold in general. For example, let  $kG$  be the group algebra of a finite group concentrated in degree 0. The reader can check that every  $kG$ -module generated in degree 0 is quasi-Koszul, but only the projective  $kG$ -modules are Koszul. If  $|G|$  is not invertible in  $k$ , then all non-projective  $kG$ -modules generated in degree 0 are quasi-Koszul but not Koszul.

The following theorem gives us a close relation between quasi-Koszul modules and Koszul modules.

**Theorem 2.16.** *A graded  $A$ -module  $M$  generated in degree 0 is Koszul if and only if it is quasi-Koszul and  $\Omega^i(M)_i$  is a projective  $A_0$ -module for every  $i \geq 0$ .*

The following lemma will be used in the proof of this theorem.

**Lemma 2.17.** *Let  $M$  be a graded  $A$ -module generated in degree 0 with  $M_0$  being a projective  $A_0$ -module. Then  $\Omega M$  is generated in degree 1 if and only if every  $A$ -module homomorphism  $\Omega M \rightarrow A_0$  extends to an  $A$ -module homomorphism  $JP \rightarrow A_0$ , where  $P$  is a graded projective cover of  $M$ .*

*Proof.* The short exact sequence  $0 \rightarrow \Omega M \rightarrow P \rightarrow M \rightarrow 0$  induces an exact sequence  $0 \rightarrow (\Omega M)_1 \rightarrow P_1 \rightarrow M_1 \rightarrow 0$ . Applying the exact functor  $\text{Hom}_{A_0}(-, A_0)$  we get another exact sequence

$$0 \rightarrow \text{Hom}_{A_0}(M_1, A_0) \rightarrow \text{Hom}_{A_0}(P_1, A_0) \rightarrow \text{Hom}_{A_0}((\Omega M)_1, A_0) \rightarrow 0.$$

Since  $M_0$  is a projective  $A_0$ -module,  $M_0 \cong P_0$ , so  $(\Omega M)_0 = 0$ . Therefore,  $\Omega M$  is generated in degree 1 if and only if  $\Omega M/J(\Omega M) \cong (\Omega M)_1$ , if and only if the above sequence is isomorphic to

$$0 \rightarrow \text{Hom}_{A_0}(M_1, A_0) \rightarrow \text{Hom}_{A_0}(P_1, A_0) \rightarrow \text{Hom}_{A_0}(\Omega M/J\Omega M, A_0) \rightarrow 0.$$

Here we use the fact that  $A_0$  is self-injective and the functor  $\text{Hom}_{A_0}(-, A_0)$  is a dual functor. But the above sequence is isomorphic to

$$0 \longrightarrow \text{Hom}_A(JM, A_0) \longrightarrow \text{Hom}_A(JP, A_0) \longrightarrow \text{Hom}_A(\Omega M, A_0) \longrightarrow 0$$

since  $JM$  and  $JP$  are generated in degree 1. Therefore,  $\Omega M$  is generated in degree 1 if and only if every (non-graded)  $A$ -module homomorphism  $\Omega M \rightarrow A_0$  extends to a (non-graded)  $A$ -module homomorphism  $JP \rightarrow A_0$ .  $\square$

Now let us prove the theorem.

*Proof. The only if part.* Let  $M$  be a Koszul  $A$ -module. Without loss of generality we can suppose that  $M$  is indecomposable. Notice that all syzygies  $\Omega^i(M)[-i]$  are also Koszul. Therefore,  $\Omega^i(M)_i \cong (\Omega^i(M)[-i])_0$  is a projective  $A_0$ -module for all  $i \geq 0$ .

Now we show that  $M$  is quasi-Koszul, i.e.,

$$\text{Ext}_A^{i+1}(M, A_0) = \text{Ext}_A^1(A_0, A_0) \cdot \text{Ext}_A^i(M, A_0)$$

for all  $i \geq 0$ . By Lemma 2.10, we have  $\text{Ext}_A^{i+1}(M, A_0) \cong \text{Ext}_A^1(\Omega^i(M), A_0)$  and  $\text{Ext}_A^i(M, A_0) \cong \text{Hom}_A(\Omega^i(M), A_0)$ . Therefore, it suffices to show  $\text{Ext}_A^1(M, A_0) = \text{Ext}_A^1(A_0, A_0) \cdot \text{Hom}_A(M, A_0)$  since the conclusion follows immediately if we replace  $M$  by  $(\Omega M)[-1]$  recursively.

To prove this identity, we first identify  $\text{Ext}_A^1(M, A_0)$  with  $\text{Hom}_A(\Omega M, A_0)$  by Lemma 2.10. Take an element  $x \in \text{Ext}_A^1(M, A_0)$  and let  $g : \Omega M \rightarrow A_0$  be the corresponding homomorphism. Since  $M$  is linear,  $M_0$  is a projective  $A_0$ -module,

and  $\Omega M$  is generated in degree 1. Thus by the previous lemma,  $g$  extends to  $JP^0$ , and hence there is a homomorphism  $\tilde{g} : JP^0 \rightarrow A_0$  such that  $g = \tilde{g}\iota$ , where  $P^0$  is a graded projective cover of  $M$  and  $\iota : \Omega M \rightarrow JP^0$  is the inclusion.

$$\begin{array}{ccc} \Omega M & \xrightarrow{\iota} & JP^0 \\ \downarrow g & \nearrow \tilde{g} & \\ A_0 & & \end{array}$$

We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & P^0 & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow \iota & & \parallel & & \downarrow p \\ 0 & \longrightarrow & JP^0 & \longrightarrow & P^0 & \longrightarrow & P_0^0 \longrightarrow 0 \end{array}$$

where the map  $p$  is defined to be the projection of  $M$  onto  $M_0 \cong P_0^0$ .

The map  $\tilde{g} : JP^0 \rightarrow A_0$  gives a push-out of the bottom sequence. Consequently, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & P^0 & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow \iota & & \parallel & & \downarrow p \\ 0 & \longrightarrow & JP^0 & \longrightarrow & P^0 & \longrightarrow & P_0^0 \longrightarrow 0 \\ & & \downarrow \tilde{g} & & \downarrow & & \parallel \\ 0 & \longrightarrow & A_0 & \longrightarrow & E & \longrightarrow & P_0^0 \longrightarrow 0. \end{array}$$

Since  $P_0 \in \text{add}(A_0)$ , we can find some  $m$  such  $P_0$  can be embedded into  $A_0^{\oplus m}$ . Thus the bottom sequence  $y \in \text{Ext}_A^1(P_0, A_0) \subseteq \bigoplus_{i=1}^m \text{Ext}_A^1(A_0, A_0)$  and we can write  $y = y_1 + \dots + y_m$  where  $y_i \in \text{Ext}_A^1(A_0, A_0)$  is represented by the sequence

$$0 \longrightarrow A_0 \longrightarrow E_i \longrightarrow A_0 \longrightarrow 0.$$

Composed with the inclusion  $\epsilon : P_0 \rightarrow A_0^{\oplus m}$ , the map  $\epsilon \circ p = (p_1, \dots, p_m)$  where each component  $p_i$  is defined in an obvious way. Consider the pull-backs:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_0 & \longrightarrow & F_i & \longrightarrow & M \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow p_i \\ 0 & \longrightarrow & A_0 & \longrightarrow & E_i & \longrightarrow & A_0 \longrightarrow 0. \end{array}$$

Let  $x_i$  be the top sequence. Then  $x = \sum_{i=1}^m x_i = \sum_{i=1}^m y_i p_i \in \text{Ext}_A^1(A_0, A_0) \cdot \text{Hom}_A(M, A_0)$  and hence  $\text{Ext}_A^1(M, A_0) \subseteq \text{Ext}_A^1(A_0, A_0) \cdot \text{Hom}_A(M, A_0)$ . The other inclusion is obvious.

**The if part.** By Proposition 2.6, it suffices to show that  $\Omega^i(M)$  is generated in degree  $i$ ,  $i > 0$ . But we observe that if  $M$  is quasi-Koszul and  $\Omega^i(M)_i$  are projective  $A_0$ -modules for all  $i \geq 0$ , then each  $\Omega^i(M)$  has these properties as well. Thus we only need to show that  $\Omega M$  is generated in degree 1 since the conclusion follows if we replace  $M$  by  $\Omega M$  recursively. By the previous lemma, it suffices to show that each (non-graded)  $A$ -module homomorphism  $g : \Omega M \rightarrow A_0$  extends to  $JP^0$ .

The map  $g$  gives a push-out  $x \in \text{Ext}_A^1(M, A_0)$  as follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & P^0 & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow g & & \downarrow & & \parallel \\ 0 & \longrightarrow & A_0 & \longrightarrow & E & \longrightarrow & M \longrightarrow 0 \end{array}$$

Since  $M$  is quasi-Koszul,  $x$  is contained in  $\text{Ext}_A^1(A_0, A_0) \cdot \text{Hom}_A(M, A_0)$ . Thus  $x = \sum_i y_i h_i$  with  $y_i \in \text{Ext}_A^1(A_0, A_0)$  and  $h_i \in \text{Hom}_A(M, A_0)$ , and each  $y_i h_i$  gives the following commutative diagram, where the bottom sequence corresponds to  $y_i$ :

$$(2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & A_0 & \longrightarrow & E_i & \longrightarrow & M \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow h_i \\ 0 & \longrightarrow & A_0 & \longrightarrow & F_i & \longrightarrow & A_0 \longrightarrow 0 \end{array}$$

By the natural isomorphism  $\text{Ext}_A^1(M, A_0) \cong \text{Hom}_A(\Omega M, A_0)$  (see Lemma 2.10), each  $y_i h_i$  corresponds an  $A$ -homomorphism  $g_i : \Omega M \rightarrow A_0$  such that the following diagram commutes:

$$(2.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & P^0 & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow g_i & & \downarrow & & \parallel \\ 0 & \longrightarrow & A_0 & \longrightarrow & E_i & \longrightarrow & M \longrightarrow 0 \end{array}$$

Diagrams 2.2 and 2.3 give us:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega M & \xrightarrow{\iota} & JP^0 & \longrightarrow & JM \longrightarrow 0 \\ & & \parallel & & \downarrow \tilde{j} & & \downarrow j \\ 0 & \longrightarrow & \Omega M & \longrightarrow & P^0 & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow g_i & & \downarrow \tilde{h}_i & & \downarrow h_i \\ 0 & \longrightarrow & A_0 & \xrightarrow{\rho} & F_i & \longrightarrow & A_0 \longrightarrow 0 \end{array}$$

Since  $JM$  is sent to 0 by  $h_i j$ , there is a homomorphism  $\varphi_i$  from  $JP_0$  to the first term  $A_0$  of the bottom sequence such that  $\rho \varphi_i = \tilde{h}_i \tilde{j}$ . Then  $g_i$  factors through  $\varphi_i$ , i.e.,  $g_i = \varphi_i \iota$ . Since  $g = \sum_i g_i$ , we know that  $g$  extends to  $JP^0$ . This finishes the proof.  $\square$

An easy corollary of the above theorem is:

**Corollary 2.18.** *Suppose that  $A$  is projective viewed as an  $A_0$ -module. Then a graded  $A$ -module  $M$  is linear if and only if it is quasi-Koszul as an  $A$ -module and projective as an  $A_0$ -module (or equivalently all  $M_i$  are projective  $A_0$ -modules).*

*Proof.* By Lemma 2.14, all  $\Omega^i(M)_i$  are projective  $A_0$ -modules for  $i \geq 0$  if and only if  $M_s$  is a projective  $A_0$ -module for every  $s \geq 0$ . The conclusion follows from the previous theorem.  $\square$

In particular, if  $A_0$  is a linear  $A$ -module, then by letting  $M = A$  in Proposition 2.13, we deduce that all  $A_i$  are projective  $A_0$ -modules for  $i \geq 0$ .

### 3. GENERALIZED KOSZUL ALGEBRAS

In this section we generalize to our context some useful results on classical Koszul algebras which appear in [4]. As before, throughout this section  $A$  is a non-negatively graded, locally finite associative  $k$ -algebra with  $A_0$  being a self-injective algebra. For two graded  $A$ -modules  $M$  and  $N$ , we use  $\text{Hom}_A(M, N)$  and

$\text{hom}_A(M, N)$  to denote the space of all module homomorphisms and the space of graded module homomorphisms respectively. The derived functors  $\text{Ext}$  and  $\text{ext}$  correspond to  $\text{Hom}$  and  $\text{hom}$  respectively.

Recall that  $A$  a *quasi-Koszul algebra* if  $A_0$  is quasi-Koszul as an  $A$ -module. In particular, if  $A_0$  is a linear  $A$ -module, then  $A$  is a quasi-Koszul algebra.

**Theorem 3.1.** *The graded algebra  $A$  is quasi-Koszul if and only if the opposite algebra  $A^{op}$  is quasi-Koszul.*

*Proof.* Since the quasi-Koszul property is invariant under the Morita equivalence, without loss of generality we can suppose that  $A$  is a basic algebra. Therefore,  $A_0$  is also a basic algebra. Let  $M$  and  $N$  be two graded  $A$ -modules. We claim  $\text{Ext}_A^i(M, N) \cong \text{Ext}_{A^{op}}^i(DN, DM)$  for all  $i \geq 0$ , where  $D$  is the duality functor  $\text{Hom}_k(-, k)$ . Indeed, Let

$$\cdots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

be a projective resolution of  $M$ . Applying the functor  $\text{Hom}_A(-, N)$  we get the following chain complex  $C^*$ :

$$0 \longrightarrow \text{Hom}_A(P^0, N) \longrightarrow \text{Hom}_A(P^1, N) \longrightarrow \cdots$$

Using the natural isomorphism  $\text{Hom}_A(P^i, N) \cong \text{Hom}_{A^{op}}(DN, DP^i)$ , we get another chain complex  $E^*$  isomorphic to the above one:

$$0 \longrightarrow \text{Hom}_{A^{op}}(DN, DP^0) \longrightarrow \text{Hom}_{A^{op}}(DN, DP^1) \longrightarrow \cdots$$

Notice that all  $DP^i$  are injective  $A^{op}$ -modules. Thus

$$\text{Ext}_A^i(M, N) \cong H^i(C^*) \cong H^i(E^*) \cong \text{Ext}_{A^{op}}^i(DN, DM)$$

which is exactly our claim.

Now let  $M = N = A_0$ . Then  $\text{Ext}_A^i(A_0, A_0) \cong \text{Ext}_{A^{op}}^i(DA_0, DA_0)$ . Since  $A_0$  is self-injective and basic, it is a Frobenius algebra. Therefore,  $DA_0$  is isomorphic to  $A_0^{op}$  as a left  $A_0^{op}$ -module (and hence as a left  $A^{op}$ -module). Consequently,  $A_0$  is a quasi-Koszul  $A$ -module if and only if  $A_0^{op}$  is a quasi-Koszul  $A^{op}$ -module.  $\square$

However, if  $A_0$  is a linear  $A$ -module,  $A_0^{op}$  need not be a linear  $A^{op}$ -module, as shown by the following example.

**Example 3.2.** Let  $\mathcal{E}$  be a finite EI category with two objects  $x$  and  $y$  such that:  $\text{Aut}_{\mathcal{E}}(x) = \langle g \rangle$  is a cyclic group of order 2;  $\text{Aut}_{\mathcal{E}}(y)$  is a trivial group;  $\text{Hom}_{\mathcal{E}}(x, y) = \{\alpha\}$  (thus  $\alpha \circ g = \alpha$ ) and  $\text{Hom}_{\mathcal{E}}(y, x) = \emptyset$ . Let  $k$  be an algebraically closed field of characteristic 2. The category algebra  $A = k\mathcal{E}$  is of dimension 4. Let  $A_0$  be the space spanned by  $1_x, g$  and  $1_y$ , and let  $A_1$  be the one-dimensional space spanned by  $\alpha$ . The reader can check that  $A_0$  is a linear  $A$ -module. The opposite algebra  $A^{op} = k\mathcal{E}^{op}$  can also be graded in a similar way, but  $A_0^{op}$  is not a linear  $A^{op}$ -module. However, we will show in Section 6 that  $A^{op}$  is a quasi-Koszul algebra.

**Proposition 3.3.** *The graded algebra  $A$  is Koszul if and only if  $\Omega^i(A_0)_i$  are projective  $A_0$ -modules for all  $i \geq 0$ , and whenever  $\text{ext}_A^i(A_0, A_0[n]) \neq 0$  we have  $n = i$ .*

*Proof.* If  $A$  is a Koszul algebra, then  $A_0$  is a linear  $A$ -module, and by Corollary 2.5 all  $\Omega^i(A_0)_i$  are projective  $A_0$ -modules. Moreover, there is a linear projective resolution

$$\cdots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow A_0 \longrightarrow 0$$

with  $P^i$  being generated in degree  $i$ . Applying  $\text{hom}_A(-, A_0[n])$  we find that all terms in this complex except  $\text{hom}_A(P^n, A_0[n])$  are 0. Consequently,  $\text{ext}_A^i(A_0, A_0[n]) \neq 0$  unless  $i = n$ .

Conversely, suppose that  $\Omega^i(A_0)_i$  is a projective  $A_0$ -module for each  $i \geq 0$  and  $\text{ext}_A^i(A_0, A_0[n]) = 0$  unless  $n = i$ , we want to show that  $\Omega^i(A_0)$  is generated in degree  $i$  by induction. Obviously,  $\Omega^0(A_0) = A_0$  is generated in degree 0. Suppose that  $\Omega^j(A_0)$  is generated in degree  $j$  for  $0 \leq j \leq i$ . Now consider  $\Omega^{i+1}(A_0)$ . By applying the graded version of Lemma 2.10 recursively, we have

$$\text{hom}_A(\Omega^{i+1}(A_0), A_0[n]) = \text{ext}_A^{i+1}(A_0, A_0[n]).$$

The right-hand side is 0 unless  $n = i + 1$ , so  $\Omega^{i+1}(A_0)$  is generated in degree  $i + 1$ . By induction we are done.  $\square$

The reader can check that the conclusion of this proposition is also true for linear modules. i.e.,  $M$  is a linear  $A$ -module if and only if  $\text{ext}_A^i(M, A_0[n]) \neq 0$  implies  $n = i$ .

We can define a tensor algebra  $T(A)$  generated by  $A_1$ , which is a  $(A_0, A_0)$ -bimodule. Explicitly,

$$T(A) = A_0 \oplus A_1 \oplus (A_1 \otimes A_1) \oplus (A_1 \otimes A_1 \otimes A_1) \oplus \dots,$$

where all tensors are over  $A_0$  and we use  $\otimes$  rather than  $\otimes_{A_0}$  to simplify the notation. This tensor algebra has a natural grading. Clearly,  $A$  is a quotient algebra of  $T(A)$ . Let  $I$  be the kernel of the quotient map  $q : T(A) \rightarrow A$ . We say that  $A$  is a *quadratic algebra* if the ideal  $I$  has a set of generators contained in  $A_1 \otimes A_1$ .

**Theorem 3.4.** *If  $A$  is a Koszul algebra, then it is a quadratic algebra.*

*Proof.* This proof is a modification of the proofs of Theorem 2.3.2 and Corollary 2.3.3 in [4]. First, consider the exact sequence

$$0 \longrightarrow W \longrightarrow A \otimes A_1 \longrightarrow A \longrightarrow A_0 \longrightarrow 0$$

where  $W$  is the kernel of the multiplication. Clearly,  $\Omega(A_0) \cong J = \bigoplus_{i \geq 1} A_i$ . Since the image of  $(A \otimes A_1)_1 = A_0 \otimes A_1$  under the multiplication is exactly  $A_1 = \Omega(A_0)_1$ ,  $A \otimes A_1$  is a projective cover of  $\Omega(A_0)$  and  $\Omega^2(A_0) = W \subseteq J \otimes A_1$ . Therefore,  $W$  is generated in degree 2, and hence  $W/JW \cong W_2$  is concentrated in degree 2. Observe that  $A$  is a quotient algebra of  $T(A)$  with kernel  $I$ . Let  $R_n$  be the kernel of the quotient map  $A_1^{\otimes n} \rightarrow A_n$ .

If  $A$  is not quadratic, we can find some  $x \in R_n$  with  $n > 2$  such that  $x$  is not contained in the two-sided ideal generated by  $\sum_{i=2}^{n-1} R_i$ . Consider the following composite of maps:

$$A_1^{\otimes n} = A_1^{\otimes n-1} \otimes A_1 \xrightarrow{p} A_{n-1} \otimes A_1 \xrightarrow{m} A_n.$$

Clearly  $p(x) \in W$  since  $m(p(x)) = 0$ . We show  $p(x) \notin JW$  by contradiction.

Indeed, if  $p(x) \in JW$ , then  $p(x) \in A_1 W$  since  $JW \cong \bigoplus_{i \geq 3} W_i = A_1 W$  (notice that  $W$  is generated in degree 2). Therefore, we can express  $p(x)$  as a linear combination of vectors of the form  $\lambda \cdot w$  with  $\lambda \in A_1$  and  $w \in W$ . But  $W \subseteq J \otimes A_1$ , so each  $w$  can be expressed as  $\sum_i w'_i \otimes \lambda'_i$  with  $w'_i \in A_{n-2}$ ,  $\lambda'_i \in A_1$  such that  $\sum_{i=1}^s w'_i \cdot \lambda'_i = 0$  by the definition of  $W$ .

Since there is a surjective product map  $\varphi : A_1^{\otimes n-2} \rightarrow A_{n-2}$ , we can choose a pre-image  $v_i^1 \otimes \dots \otimes v_i^{n-2} \in \varphi^{-1}(w'_i)$  for each  $i$  and define

$$\tilde{w} = \sum_{i=1}^s v_i^1 \otimes \dots \otimes v_i^{n-2} \otimes \lambda'_i$$

which is contained in  $R_{n-1}$  clearly. Observe that  $p(\lambda \otimes \tilde{w}) = \lambda \cdot w$ . Since  $p(x)$  is a linear combination of vectors of the form  $\lambda \cdot w$ , by the above process we can get some  $y$  which is a linear combination of vectors of the form  $\lambda \otimes \tilde{w}$  such that  $p(y) = p(x)$ . Clearly,  $p(x - y) = 0$  and  $y \in A_1 \otimes R_{n-1}$ .

Consider the following short exact sequence

$$0 \longrightarrow R_{n-1} \longrightarrow A_1^{\otimes n-1} \longrightarrow A_{n-1} \longrightarrow 0.$$

Since  $A_0$  is a linear  $A$ -module, so is  $J[-1] = \Omega(A_0)[-1]$ . Therefore, by Corollary 2.5,  $A_1 \cong J[-1]_0$  is a projective  $A_0$ -module. Therefore, the following sequence is also exact:

$$0 \longrightarrow R_{n-1} \otimes A_1 \longrightarrow A_1^{\otimes n} \xrightarrow{p} A_{n-1} \otimes A_1 \longrightarrow 0.$$

Thus  $x - y \in R_{n-1} \otimes A_1$  since  $p(x - y) = 0$ . It follows  $x \in A_1 \otimes R_{n-1} + R_{n-1} \otimes A_1$ , which contradicts our choice of  $x$ .

We proved  $x \notin JW$ . Then  $p(x) \in W/JW \cong W_2$  is of degree 2. But this is impossible since  $p$  as a graded homomorphism sends  $x \in R_n$  with  $n > 2$  to an element of degree  $n$ .  $\square$

We can define the *Koszul complex* for  $A$  in a similar way to the classical situation.

Let  $A \cong T(A)/(R)$  be quadratic with  $R \subseteq A_1 \otimes A_1$  being a set of relations. Define  $P_n^n = \bigcap_{i=0}^{n-2} A_1^{\otimes i} \otimes R \otimes A_1^{\otimes n-i-2} \subseteq A_1^{\otimes n}$ . In particular,  $P_0^0 = A_0$ ,  $P_1^1 = A_1$  and  $P_2^2 = R$ . Let  $P^n = A \otimes P_n^n$  such that  $A_0 \otimes P_n^n \cong P_n^n$  is in degree  $n$ . Define  $d^n : P^n \rightarrow P^{n-1}$  to be the restriction of  $A \otimes A_1^{\otimes n} \rightarrow A \otimes A_1^{\otimes n-1}$  by  $a \otimes v_1 \otimes \dots \otimes v_n \mapsto av_1 \otimes v_2 \otimes \dots \otimes v_n$ . The reader can check  $d^{n-1}d^n = 0$  for  $n \geq 1$ . Therefore we get the following Koszul complex  $K^*$ :

$$\dots \longrightarrow P^3 \xrightarrow{d^3} A \otimes R \xrightarrow{d^2} A \otimes A_1 \xrightarrow{d^1} A \longrightarrow 0.$$

**Theorem 3.5.** *Let  $A \cong T(A)/(R)$  be a quadratic algebra. Then  $A$  is a Koszul algebra if and only if the Koszul complex is a projective resolution of  $A_0$ .*

*Proof.* One direction is trivial. Now suppose that  $A_0$  is a linear  $A$ -module. The Koszul complex  $K^*$  of  $A$  has the following properties:

(1). Let  $Z^n$  be the kernel of  $d^n : P^n \rightarrow P^{n-1}$ . The restricted map  $d_n^n :$

$$P_n^n = A_0 \otimes \left( \bigcap_{i=0}^{n-2} A_1^{\otimes i} \otimes R \otimes A_1^{\otimes n-i-2} \right) \rightarrow P_n^{n-1} = A_1 \otimes \left( \bigcap_{i=0}^{n-3} A_1^{\otimes i} \otimes R \otimes A_1^{\otimes n-i-3} \right)$$

is injective. Therefore  $Z_i^n = 0$  for every  $i \leq n$ .

(2).  $Z_{n+1}^n$ , the kernel of the map  $d_n^{n+1} :$

$$P_{n+1}^n = A_1 \otimes \left( \bigcap_{i=0}^{n-2} A_1^{\otimes i} \otimes R \otimes A_1^{\otimes n-i-2} \right) \rightarrow P_{n+1}^{n-1} = A_2 \otimes \left( \bigcap_{i=0}^{n-3} A_1^{\otimes i} \otimes R \otimes A_1^{\otimes n-i-3} \right)$$

is

$$A_1 \otimes \left( \bigcap_{i=0}^{n-2} A_1^{\otimes i} \otimes R \otimes A_1^{\otimes n-i-2} \right) \cap (R \otimes A_1^{\otimes n-1}) = \bigcap_{i=0}^{n-1} A_1^{\otimes i} \otimes R \otimes A_1^{\otimes n-i-1}$$

which is exactly  $P_{n+1}^{n+1}$  (or  $d_{n+1}^{n+1}(P_{n+1}^{n+1})$  since  $d_{n+1}^{n+1}$  is injective by the last property).

We claim that each  $P^n = A \otimes P_n^n$  is a projective  $A$ -module. Clearly, it is enough to show that each  $P_n^n = Z_n^{n-1}$  is a projective  $A_0$ -module. We prove the following stronger conclusion. That is,  $Z_i^n$  are projective  $A_0$ -modules for  $i \in \mathbb{Z}$  and  $n \geq 0$ . We use induction on  $n$ .

Since  $A_0$  is a linear  $A$ -module, by Proposition 2.13,  $A_i$  are projective  $A_0$ -modules for all  $i \geq 0$ . The conclusion is true for  $Z^0 \cong J$  since  $J_0 = 0$  and  $J_m = A_m$  for  $m \geq 1$ . Suppose that it is true for  $l \leq n$ . That is, all  $Z_i^l$  are projective  $A_0$ -modules for  $l \leq n$  and  $i \in \mathbb{Z}$ . Consider  $l = n + 1$ . By the second property described above,  $P_{n+1}^{n+1} = Z_{n+1}^n$ , which is a projective  $A_0$ -module by the induction hypothesis.



Therefore,  $P^{n+1} = A \otimes P_{n+1}^{n+1}$  is a projective  $A$ -module, so  $P_i^{n+1}$  are all projective  $A_0$ -modules for  $i \in \mathbb{Z}$ . But the following short exact sequence of  $A_0$ -modules splits

$$0 \longrightarrow Z_i^{n+1} \longrightarrow P_i^{n+1} \longrightarrow Z_i^n \longrightarrow 0$$

since  $Z_i^n$  is a projective  $A_0$ -module by the induction hypothesis. Now as a direct summand of  $P_i^{n+1}$  which is a projective  $A_0$ -module,  $Z_i^{n+1}$  is a projective  $A_0$ -module as well. Our claim is proved by induction.

We claim that this complex is acyclic. First, the sequence

$$P^1 = A \otimes A_1 \longrightarrow P^0 = A \otimes A_0 \longrightarrow A_0 \longrightarrow 0$$

is right exact. By induction on  $n > 1$

$$\text{ext}_A^{n+1}(A_0, A_0[m]) = \text{coker}(\text{hom}_A(P^n, A_0[m]) \rightarrow \text{hom}_A(Z^n, A_0[m])).$$

By Property (1),  $Z_m^n = 0$  if  $m < n + 1$ . If  $m > n + 1$ ,  $\text{hom}_A(P^n, A_0[m]) = 0$  since  $P^n$  is generated in degree  $n$ , so  $\text{ext}_A^{n+1}(A_0, A_0[m]) = \text{hom}_A(Z^n, A_0[m])$  by the above identity. But the left-hand side of this identity is non-zero only if  $m = n + 1$  since  $A_0$  is linear. Therefore,  $\text{hom}_A(Z^n, A_0[m]) = 0$  for  $m > n + 1$ . Consequently,  $Z^n$  is generated in degree  $n + 1$ . By property (2),  $Z_{n+1}^n = d_{n+1}^{n+1}(P_{n+1}^{n+1})$ , so  $Z^n = d^{n+1}(P^{n+1})$  since both modules are generated in degree  $n + 1$ . Therefore, the Koszul complex is acyclic, and hence is a projective resolution of  $A_0$ .  $\square$

#### 4. GENERALIZED KOSZUL DUALITY

In this section we prove the Koszul duality. As before,  $A$  is a non-negatively graded, locally finite algebra with  $A_0$  being a self-injective algebra. Define  $\Gamma = \text{Ext}_A^*(A_0, A_0)$  which has a natural grading. Notice that  $\Gamma_0 \cong A_0^{op}$  is also a self-injective algebra. Let  $M$  be a graded  $A$ -module. Then  $\text{Ext}_A^*(M, A_0)$  is a graded  $\Gamma$ -module. Thus we can define a functor  $E = \text{Ext}_A^*(-, A_0)$  from  $A\text{-gmod}$  to  $\Gamma\text{-gmod}$ .

**Theorem 4.1.** *If  $A$  is a Koszul algebra, then  $E = \text{Ext}_A^*(-, A_0)$  gives a duality between the category of linear  $A$ -modules and the category of linear  $\Gamma$ -modules. That is, if  $M$  is a linear  $A$ -module, then  $E(M)$  is a linear  $\Gamma$ -module, and  $E_\Gamma EM = \text{Ext}_\Gamma^*(EM, \Gamma_0) \cong M$ .*

*Proof.* Since  $M$  and  $A_0$  both are linear, by Proposition 2.13  $M_0$  and  $JM[-1]$  are linear, where  $J = \bigoplus_{i \geq 1} A_i$ . Furthermore, we have the following short exact sequence of linear modules:

$$0 \longrightarrow \Omega M[-1] \longrightarrow \Omega(M_0)[-1] \longrightarrow JM[-1] \longrightarrow 0.$$

As in the proof of Proposition 2.9, this sequence induces exact sequences recursively (see diagram 2.1):

$$0 \longrightarrow \Omega^i(M)[-i] \longrightarrow \Omega^i(M_0)[-i] \longrightarrow \Omega^{i-1}(JM[-1])[1-i] \longrightarrow 0,$$

and gives exact sequences of  $A_0$ -modules:

$$0 \longrightarrow \Omega^i(M)_i \longrightarrow \Omega^i(M_0)_i \longrightarrow \Omega^{i-1}(JM[-1])_{i-1} \longrightarrow 0.$$

Applying the exact functor  $\text{Hom}_{A_0}(-, A_0)$  and using the following isomorphism for a graded  $A$ -module  $N$  generated in degree  $i$

$$\text{Hom}_A(N, A_0) \cong \text{Hom}_A(N_i, A_0) \cong \text{Hom}_{A_0}(N_i, A_0),$$

we get:

$$0 \rightarrow \text{Hom}_A(\Omega^{i-1}(JM[-1]), A_0) \rightarrow \text{Hom}_A(\Omega^i(M_0), A_0) \rightarrow \text{Hom}_A(\Omega^i M, A_0) \rightarrow 0.$$

By Lemma 2.10, this sequence is isomorphic to

$$0 \rightarrow \text{Ext}_A^{i-1}(JM[-1], A_0) \rightarrow \text{Ext}_A^i(M_0, A_0) \rightarrow \text{Ext}_A^i(M, A_0) \rightarrow 0.$$

Now let the index  $i$  vary and put these sequences together. We have:

$$0 \longrightarrow E(JM[-1])[1] \longrightarrow E(M_0) \longrightarrow EM \longrightarrow 0.$$

Let us focus on this sequence. We claim  $\Omega(EM) \cong E(JM[-1])[1]$ . Indeed, since  $M_0$  is a projective  $A_0$ -module and the functor  $E$  is additive,  $E(M_0)$  is a projective  $\Gamma$ -module. Since  $JM[-1]$  is Koszul,  $JM[-1]$  is quasi-Koszul and hence  $E(JM[-1])$  as a  $\Gamma$ -module is generated in degree 0. Thus  $E(JM[-1])[1]$  is generated in degree 1, and  $E(M_0)$  is minimal. This proves the claim. Consequently,  $\Omega(EM)$  is generated in degree 1 as a  $\Gamma$ -module. Moreover, replacing  $M$  by  $JM[-1]$  (which is also linear) and using the claimed identity, we have

$$\Omega^2(EM) = \Omega(E(JM[-1])[1]) = \Omega(E(JM[-1])[1]) = E(J^2M[-2])[2],$$

which is generated in degree 2. By recursion, we know that  $\Omega^i(EM) \cong E(J^iM[-i])[i]$  is generated in degree  $i$  for all  $i \geq 0$ . Thus  $EM$  is a linear  $\Gamma$ -module. In particular for  $M = A$ ,

$$EA = \text{Ext}_A^*(A, A_0) = \text{Hom}_A(A, A_0) = \Gamma_0$$

is a linear  $\Gamma$ -module.

Since  $\Omega^i(EM)$  is generated in degree  $i$  and

$$\begin{aligned} \Omega^i(EM)_i &\cong E(J^iM[-i])[i]_i \cong E(J^iM[-i])_0 \\ &= \text{Hom}_A(J^iM[-i], A_0) \cong \text{Hom}_A(M_i, A_0), \end{aligned}$$

we have

$$\begin{aligned} \text{Hom}_\Gamma(\Omega^i(EM), \Gamma_0) &\cong \text{Hom}_{\Gamma_0}(\Omega^i(EM)_i, \Gamma_0) \\ &\cong \text{Hom}_{\Gamma_0}(\text{Hom}_A(M_i, A_0), \Gamma_0) \\ &\cong \text{Hom}_{\Gamma_0}(\text{Hom}_{A_0}(M_i, A_0), \Gamma_0) \\ &\cong M_i. \end{aligned}$$

The last isomorphism holds because  $A_0$  is self-injective and  $\Gamma_0 \cong A_0^{op}$ .

We have proved that  $EM$  is a linear  $\Gamma$ -module. Therefore,  $(\Omega^i(EM))_i$  is a projective  $\Gamma_0$ -module for every  $i \geq 0$ . Applying Lemma 2.10 recursively,  $\text{Ext}_\Gamma^i(EM, \Gamma_0) \cong \text{Hom}_\Gamma(\Omega^i(EM), \Gamma_0) \cong M_i$  for every  $i \geq 0$ . Adding them together,  $E_\Gamma E(M) \cong \bigoplus_{i=0}^\infty M_i \cong M$ .

Now we have  $E_\Gamma(E(A)) = E_\Gamma(\Gamma_0) \cong A$ . Moreover,  $\Gamma$  is a graded algebra with  $\Gamma_0 \cong A_0^{op}$  being self-injective as an algebra and linear as a  $\Gamma$ -module. Using this duality, we can exchange  $A$  and  $\Gamma$  in the above reasoning and get  $EE_\Gamma(N) \cong N$  for an arbitrary linear  $\Gamma$ -module  $N$ . Thus  $E$  induces an equivalence between these two categories.  $\square$

**Remark 4.2.** We can also use  $\text{hom}_A(-, A_0)$  to define the functor  $E$  on the category of linear  $A$ -modules, namely  $E := \bigoplus_{i \geq 0} \text{ext}_A^i(-, A_0[i])$ . Indeed, for a linear  $A$ -module  $M$ , we have:

$$\begin{aligned} \text{Ext}_A^*(M, A_0) &= \bigoplus_{i \geq 0} \text{Ext}_A^i(M, A_0) \\ &= \bigoplus_{i \geq 0} \bigoplus_{j \in \mathbb{Z}} \text{ext}_A^i(M, A_0[j]) \\ &= \bigoplus_{i \geq 0} \text{ext}_A^i(M, A_0[i]) \end{aligned}$$

since  $\text{ext}_A^i(M, A_0[j]) = 0$  for  $i \neq j$ , see the paragraph after Proposition 3.3.

Now we want to study the Koszul duality on the level of derived categories. We introduce the following notation: let  $D_{gr}^b A$  be the bounded derived category of  $A$ -gmod, and define  $\mathcal{T}$  to be the category formed by objects  $T[i]$  where  $T$  ranges over indecomposable summand of  $A_0$  and  $i \in \mathbb{Z}$ .

**Lemma 4.3.** *Let  $A$  be a locally finite graded algebra with  $A_0$  being a self-injective algebra and a linear  $A$ -module. Let  $\Gamma = \text{Ext}_A^*(A_0, A_0)$ . Then we have:*

- (1) *If  $\text{pd}_A A_0 < \infty$ , then  $\dim_k \Gamma < \infty$ .*
- (2) *If the global dimension  $\text{gldim } A < \infty$ , then  $\text{gldim } A_0 < \infty$ ,  $\text{pd}_A A_0 < \infty$ ;*
- (3) *Conversely, if  $\text{gldim } A_0 < \infty$ ,  $\text{pd}_A A_0 < \infty$ , and  $\dim_k A < \infty$ , then  $\text{gldim } A < \infty$ .*

*Proof.* Notice that  $\Gamma_i = \text{Ext}_A^i(A_0, A_0) = 0$  for all  $i \geq m+1$ . If  $0 \leq i \leq m$ , then

$$\Gamma_i = \text{Ext}_A^i(A_0, A_0) \cong \text{Hom}_A(\Omega^i(A_0), A_0)$$

which is finite-dimensional since  $\Omega^i(A_0)$  is a finitely generated  $A$ -module and  $A$  is locally finite. Therefore,  $\dim_k \Gamma < \infty$ . This proves the first statement.

If  $\text{gldim } A = m < \infty$ , clearly  $\text{pd}_A A_0 \leq m$ . Take an arbitrary  $A_0$ -module  $M$ . Viewed as a  $A$ -module concentrated in degree 0,  $A$  has a projective resolution

$$0 \longrightarrow P^m \longrightarrow P^{n-1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow M \longrightarrow 0,$$

which induces

$$0 \longrightarrow P_0^m \longrightarrow P_0^{n-1} \longrightarrow \cdots \longrightarrow P_0^0 \longrightarrow M_0 \longrightarrow 0.$$

This is a projective resolution of the  $A_0$ -module  $M = M_0$ . Therefore,  $\text{pd}_{A_0} M \leq m$ . So  $\text{gldim } A_0 \leq m$ , and (2) is proved.

Conversely, assume that  $\text{pd}_A A_0 < \infty$ ,  $\text{gldim } A_0 < \infty$  and  $\dim_k A < \infty$ . To prove  $\text{gldim } A < \infty$ , it suffices to show  $\text{pd}_A S < \infty$  for each simple  $A$ -module  $S$  since every  $A$ -module has a composition series of finite length. Clearly,  $S$  is still simple viewed as a  $A_0$ -module. Therefore, there exists a number  $m \geq 0$  such that  $\Omega_{A_0}^{m+1}(S) = 0$  but  $\Omega_{A_0}^m(S)$  is a non-zero projective  $A_0$ -module. Consider the exact sequence of  $A_0$ -modules:

$$0 \longrightarrow \Omega_{A_0}^m(S) \longrightarrow P \longrightarrow \Omega_{A_0}^{m-1}(S) \longrightarrow 0.$$

Each module appearing in this sequence can be viewed as an  $A$ -module concentrated in degree 0. Notice that the first two terms are projective  $A_0$ -modules and hence have finite projective dimensions as  $A$ -modules. Therefore,  $\text{pd}_A \Omega_{A_0}^{m-1}(S) < \infty$ . By induction,  $\text{pd}_A S < \infty$ .  $\square$

**Theorem 4.4.** *Suppose that  $A$  is a Koszul algebra such that  $\text{gldim } A_0$ ,  $\text{pd}_A A_0$ ,  $\text{gldim } \Gamma_0$  and  $\text{pd}_\Gamma \Gamma_0$  are all finite. Then  $D_{gr}^b A \cong D_{gr}^b \Gamma^{op}$  as triangulated categories.*

*Proof.* By the above lemma, both  $A$  and  $\Gamma$  are finite-dimensional algebras. Therefore,  $\text{gldim } A$  and  $\text{gldim } \Gamma$  are also finite. Consequently, the category  $\mathcal{T}$  generates  $D_{gr}^b A$  as a triangulated category, and the category of all indecomposable graded projective  $\Gamma$ -modules generates  $D_{gr}^b \Gamma$ . Now the conclusion can be deduced from Theorem 4.3.4 in [14] by letting  $T = A_0$ . We remind the reader that functor  $G_{A_0}$  in [14] is  $\text{Ext}_A^*(A_0, -)$  instead of  $E = \text{Ext}_A^*(-, A_0)$  used by us throughout this paper, and the algebra  $\Gamma$  in his paper is set to be  $\text{Ext}_A^*(A_0, A_0)^{op}$ .  $\square$

## 5. APPLICATION TO DIRECTED CATEGORIES

In this section we will apply the general theory to a type of structures called *directed categories*, for which there exist very nice relations between stratification theory and Koszul theory and a nice correspondence between our generalized Koszul theory and the classical theory. All categories  $\mathcal{C}$  we consider in this section are locally finite  $k$ -linear categories with finitely many objects, that is, for  $x, y \in \text{Ob } \mathcal{C}$ , the set of morphisms  $\mathcal{C}(x, y)$  is a finite-dimensional  $k$ -vector space. To simplify the technical part, we suppose furthermore that  $\mathcal{C}$  is *skeletal*, i.e.,  $x \cong y$  implies  $x = y$  for  $x, y \in \text{Ob } \mathcal{C}$ .

**Definition 5.1.** *A locally finite  $k$ -linear category  $\mathcal{C}$  is a directed category if there is a partial order  $\leq$  on  $\text{Ob } \mathcal{C}$  such that  $\mathcal{C}(x, y) \neq 0$  only if  $x \leq y$ .*

Correspondingly, we define *directed algebras*.

**Definition 5.2.** *A finite-dimensional algebra  $A$  is called a directed algebra with respect to a partially ordered set of orthogonal idempotents  $\{e_i; \leq\}_{i=1}^n$  if  $\sum_{i=1}^n e_i = 1$  and  $\text{Hom}_A(Ae_i, Ae_j) \cong e_i Ae_j \neq 0$  implies  $e_j \leq e_i$ .*

Notice that in the above definition we do not require the idempotents  $e_i$  to be primitive. Clearly, every algebra  $A$  is always directed with respect to the trivial set  $\{1\}$ .

There is a bijective correspondence between directed categories and directed algebras. Let  $A$  be a directed algebra with respect to a poset of orthogonal idempotents  $(\{e_i\}_{i=1}^n, \leq)$ . Then we can construct a directed category  $\mathcal{A}$  in the following way:  $\text{Ob } \mathcal{A} = \{e_i\}_{i=1}^n$  with the same partial order, and  $\mathcal{A}(e_i, e_j) = e_j Ae_i \cong \text{Hom}_A(Ae_j, Ae_i)$ . The reader can check that  $\mathcal{A}$  is indeed a directed category. We call  $\mathcal{A}$  the *associated category* of  $A$ .

Conversely, given a directed category  $\mathcal{A}$  with the poset  $(\text{Ob } \mathcal{A}, \leq)$ , we obtain an algebra  $A$  which is directed with respect to the poset of orthogonal idempotents  $(\{1_x\}_{x \in \text{Ob } \mathcal{A}}, \leq)$ , namely,  $1_x \leq 1_y$  if and only if  $x \leq y$ . As a  $k$ -vector space,  $A = \bigoplus_{x, y \in \text{Ob } \mathcal{A}} \mathcal{A}(x, y)$ . For two morphisms  $\alpha : x \rightarrow y$  and  $\beta : z \rightarrow w$ , the product  $\beta \cdot \alpha = 0$  if  $y \neq z$ , otherwise it is the composite morphism  $\beta\alpha$ . Since every vector in  $A$  is a linear combination of morphisms in  $\mathcal{A}$ , the multiplication of morphisms can be extended linearly to a well defined product in  $A$ . The reader can check that the algebra  $A$  we get in this way is indeed a directed algebra, which is called the *associated algebra* of  $\mathcal{A}$ .

It is well known that  $A\text{-mod}$  is equivalent to the category of finite-dimensional  $k$ -linear representations of  $\mathcal{A}$ . If one of  $A$  and  $\mathcal{A}$  is graded, then the other one can be graded as well. Moreover,  $A\text{-gmod}$  is equivalent to the category of finite-dimensional graded  $k$ -linear representations of  $\mathcal{A}$ . For more details, see [16]. Because of these facts, we may view a directed category  $\mathcal{A}$  as a directed algebra with respect to the set of idempotents  $\{1_x \mid x \in \text{Ob } \mathcal{A}\}$  and abuse notation and terminologies. For example, we may say idempotents in  $\mathcal{A}$ , ideals of  $\mathcal{A}$  and so on. We hope this would not cause confusions to the reader and point out that all results in previous sections can be applied to directed categories.

Directed categories generalize  $k$ -linearizations of finite EI categories. Explicitly, let  $\mathcal{E}$  be a skeletal finite EI category. Consider the category algebra  $k\mathcal{E}$  with a set of idempotents  $\{1_x\}_{x \in \text{Ob } \mathcal{E}}$  on which there is a partial order  $\leq$  such that  $1_x \leq 1_y$  if and only if  $\mathcal{E}(x, y) \neq \emptyset$ . Then the category algebra  $k\mathcal{E}$  is directed with respect to this poset of idempotents, so we can construct a directed category  $\tilde{\mathcal{E}}$  by the above correspondence. Actually,  $\tilde{\mathcal{E}}$  is precisely the  $k$ -linearization of  $\mathcal{E}$ .

Let  $\mathcal{C}$  be a directed category. A  $\mathcal{C}$ -module (or a *representation* of  $\mathcal{C}$ ) is defined to be a  $k$ -linear functor from  $\mathcal{C}$  to the category of finite-dimensional  $k$ -vector spaces.

The morphism space of  $\mathcal{C}$  can be decomposed as the direct sum of  $\mathcal{C}1_x$  with  $x$  ranging over all objects, where by  $\mathcal{C}1_x$  we denote the vector space formed by all morphisms with source  $x$ . Therefore, each  $\mathcal{C}1_x$  is a projective  $\mathcal{C}$ -module, and every indecomposable projective  $\mathcal{C}$ -module is isomorphic to a direct summand of a certain  $\mathcal{C}1_x$ . The isomorphism classes of simple  $\mathcal{C}(x, x)$ -modules with  $x$  varying within  $\text{Ob } \mathcal{C}$  give rise to isomorphism classes of simple  $\mathcal{C}$ -modules. Explicitly, let  $V$  be a simple  $\mathcal{C}(x, x)$ -module for some object  $x$ , we can construct a simple  $\mathcal{C}$ -module  $S$ :  $S(x) = V$  and  $S(y) = 0$  for  $y \neq x$ . These results are well known for finite EI categories, see [20].

Our next task is to translate some results on finite EI categories in Section 2 of [20] to directed categories. First, we want to show that every directed category is stratified with respect to the given partial order.

**Proposition 5.3.** *Let  $\mathcal{D}$  and  $\mathcal{E}$  be full subcategories of a directed category  $\mathcal{C}$  such that  $\text{Ob } \mathcal{D} \cup \text{Ob } \mathcal{E} = \text{Ob } \mathcal{C}$ ,  $\text{Ob } \mathcal{D} \cap \text{Ob } \mathcal{E} = \emptyset$ , and  $\mathcal{C}(x, y) = 0$  for  $x \in \text{Ob } \mathcal{D}$  and  $y \in \text{Ob } \mathcal{E}$ . Let  $e = \sum_{x \in \text{Ob } \mathcal{D}} 1_x$  and  $I = \mathcal{C}e\mathcal{C}$ . Then  $I$  is a stratified ideal of  $\mathcal{C}$ .*

*Proof.* The proof is similar to that of Proposition 2.2 in [20]. Clearly  $I$  is idempotent. Notice that  $\mathcal{C}e$  is the space spanned by all morphisms with sources contained in  $\text{Ob } \mathcal{D}$  and  $e\mathcal{C}e$  is the space spanned by all morphisms with both sources and targets contained in  $\text{Ob } \mathcal{D}$ . Since  $\mathcal{C}(x, y) = 0$  for  $x \in \text{Ob } \mathcal{D}$  and  $y \in \text{Ob } \mathcal{E}$ , these two spaces coincide, i.e.,  $\mathcal{C}e = e\mathcal{C}e$ . In particular,  $\mathcal{C}e$  is projective  $e\mathcal{C}e$ -module, here  $e\mathcal{C}e$  is an algebra for which the associated directed category is precisely  $\mathcal{D}$ . Therefore,  $\text{Tor}_n^{e\mathcal{C}e}(\mathcal{C}e, e\mathcal{C}) = 0$  for  $n \geq 1$ . Furthermore,

$$\mathcal{C}e \otimes_{e\mathcal{C}e} e\mathcal{C} = e\mathcal{C}e \otimes_{e\mathcal{C}e} e\mathcal{C} \cong e\mathcal{C}.$$

We claim  $e\mathcal{C} = \mathcal{C}e\mathcal{C}$ . Clearly,  $e\mathcal{C} \subseteq \mathcal{C}e\mathcal{C}$ . On the other hand, since we just proved  $\mathcal{C}e = e\mathcal{C}e$ , we have  $\mathcal{C}e\mathcal{C} = e\mathcal{C}e\mathcal{C} \subseteq e\mathcal{C}$ . Therefore,  $e\mathcal{C} = \mathcal{C}e\mathcal{C}$  as we claimed. In conclusion,  $I$  is indeed a stratified ideal of  $\mathcal{C}$ .  $\square$

**Corollary 5.4.** *Every directed category  $\mathcal{C}$  is stratified with respect to the give partial order on  $\text{Ob } \mathcal{C}$ .*

*Proof.* The partial order  $\leq$  on  $\text{Ob } \mathcal{C}$  gives a filtration on  $\text{Ob } \mathcal{C}$  in the following way: let  $S_1$  be a set containing a maximal object in  $\text{Ob } \mathcal{C}$ ,  $S_2$  is formed by adding a maximal object in  $\text{Ob } \mathcal{C} \setminus S_1$  into  $S_1$ ,  $S_3$  is formed by adding a maximal object in  $\text{Ob } \mathcal{C} \setminus S_2$  into  $S_2$ , and so on. Consider the full subcategories  $\mathcal{D}_i$  formed by  $S_i$  and let  $e_i = \sum_{x \in S_i} 1_x$ . Then the ideals  $\mathcal{C}e_i\mathcal{C}$  give a stratification of  $\mathcal{C}$  by the previous proposition.  $\square$

Now we want to describe standard modules and give a characterization of standardly stratified directed categories with respect to a particular pre-order. Before doing that, we need to define this pre-order on a complete set of primitive idempotents of  $\mathcal{C}$  (or precisely, primitive idempotents of the associated algebra). For every object  $x$ ,  $\mathcal{C}(x, x) = 1_x\mathcal{C}1_x$  is a finite-dimensional  $k$ -algebra, so we can choose a complete set of orthogonal primitive idempotents  $E_x = \{e_\lambda\}_{\lambda \in \Lambda_x}$  with  $\sum_{\lambda \in \Lambda_x} e_\lambda = 1_x$ . In this way we get a complete set of orthogonal primitive idempotents  $\bigsqcup_{x \in \text{Ob } \mathcal{C}} E_x$ . The partial order  $\leq$  on  $\text{Ob } \mathcal{C}$  can be applied to define a pre-ordered set  $(\Lambda, \preceq)$  to index all these primitive idempotents, namely for  $e_\lambda \in E_x$  and  $e_\mu \in E_y$ ,  $e_\lambda \preceq e_\mu$  if and only if  $x \leq y$ . We can check that  $\preceq$  is indeed a pre-order. We denote  $e_\lambda \prec e_\mu$  if  $e_\lambda \preceq e_\mu$  but  $e_\mu \not\preceq e_\lambda$  for  $\lambda, \mu \in \Lambda$ . Notice that indecomposable summands of  $\mathcal{C}$  (viewed as an algebra) can be indexed by these primitive idempotents in a obvious way, namely  $P_\lambda = \mathcal{C}e_\lambda$ . Therefore, the pre-ordered set  $(\Lambda, \preceq)$  can also be used to index all indecomposable summands of  $\mathcal{C}$ .

We define standard  $\mathcal{C}$ -modules in the following way:

$$\Delta_\lambda = P_\lambda / \sum_{\mu \succ \lambda, \mu \in \Lambda} \text{tr}_{P_\mu}(P_\lambda),$$

where  $\text{tr}_{P_\mu}(P_\lambda)$  is the trace of  $P_\mu$  in  $P_\lambda$ . The following proposition gives a description of standard  $\mathcal{C}$ -modules with respect to the above pre-order.

**Proposition 5.5.** *The standard  $\mathcal{C}$ -module  $\Delta_\lambda$  is only supported on  $x$  with value  $\Delta_\lambda(x) \cong 1_x \mathcal{C} e_\lambda$ , where  $x \in \text{Ob } \mathcal{C}$  and  $e_\lambda \in E_x$ .*

*Proof.* Let us first analyze the structure of  $P_\lambda = \mathcal{C} e_\lambda$ . Since  $e_\lambda \in E_x$ ,  $P_\lambda$  is a direct summand of  $\mathcal{C} 1_x$ . The value of  $\mathcal{C} 1_x$  on an arbitrary object  $y$  is  $1_y \mathcal{C} 1_x$ , the space of all morphisms from  $x$  to  $y$ . Therefore, the value of  $P_\lambda$  on  $y$  is  $1_y \mathcal{C} e_\lambda$ . By our definition of the partial order on  $\text{Ob } \mathcal{C}$ , if  $x \not\leq y$ , then there is no nontrivial morphisms from  $x$  to  $y$ . Therefore,  $1_y \mathcal{C} 1_x$  and hence  $1_y \mathcal{C} e_\lambda$  are 0. We deduce immediately that  $\Delta_\lambda$  is only supported on objects  $y$  satisfying  $x \leq y$ .

Let  $y$  be an object such that  $y > x$ . Then every  $e_\mu \in E_y$  satisfies  $e_\mu \succ e_\lambda$ . Since  $\sum_{e_\mu \in E_y} e_\mu = 1_y$ , by taking the sum we find that  $\text{tr}_{\mathcal{C} 1_y}(P_\lambda)$  is contained in  $\sum_{\mu \succ \lambda, \mu \in \Lambda} \text{tr}_{P_\mu}(P_\lambda)$ . The value on  $y$  of  $\text{tr}_{\mathcal{C} 1_y}(\mathcal{C} 1_x)$  is  $1_y \mathcal{C} 1_x$ . Since  $P_\lambda = \mathcal{C} e_\lambda$  is a direct summand of  $\mathcal{C} 1_x$ , the value on  $y$  of  $\text{tr}_{\mathcal{C} 1_y}(P_\lambda)$  is  $1_y \mathcal{C} e_\lambda$ . Consequently, the value of  $\sum_{\mu \succ \lambda, \mu \in \Lambda} \text{tr}_{P_\mu}(P_\lambda)$  on  $y$  contains  $1_y \mathcal{C} e_\lambda$ , which equals the value of  $P_\lambda$  on  $y$ . Therefore, the value of  $\sum_{\mu \succ \lambda, \mu \in \Lambda} \text{tr}_{P_\mu}(P_\lambda)$  on  $y$  is precisely  $1_y \mathcal{C} e_\lambda$ , so the value of  $\Delta_\lambda$  on  $y$  is 0.

We have proved that  $\Delta_\lambda$  is only supported on  $x$ . Clearly, its value on  $x$  is  $1_x \mathcal{C} e_\lambda$ .  $\square$

This proposition tells us that standard modules are exactly indecomposable direct summands of  $\bigoplus_{x \in \text{Ob } \mathcal{C}} \mathcal{C}(x, x)$  (viewed as a  $\mathcal{C}$ -module by identifying it with the quotient module  $\bigoplus_{x, y \in \text{Ob } \mathcal{C}} \mathcal{C}(x, y) / \bigoplus_{x \neq y} \mathcal{C}(x, y)$ ).

**Definition 5.6.** *A directed category  $\mathcal{C}$  is said to be standardly stratified if every (indecomposable) projective module  $P_\lambda$  has a  $\Delta$ -filtration by standard modules.*

To simplify the expression, we stick to the following convention from now on:

**Convention:** When we say a directed category is standardly stratified, we always refer to the preorder  $\preceq$  induced by the given partial order  $\leq$  on the set of objects.

This definition is very simple compared to the definition of standardly stratified algebras (for example, the definition in [6]). However, from the previous proposition we find that every standard module  $\Delta_\lambda$  of  $\mathcal{C}$  satisfies the following condition: if  $S_\mu$  and  $S_\nu$  are two different composition factors of  $\Delta_\lambda$ , then both  $S_\mu \preceq S_\nu$  and  $S_\nu \preceq S_\mu$  (but in general  $S_\mu \not\cong S_\nu$ ). Moreover, if  $S_\mu$  is a composition factor of the kernel  $K_\lambda$  of the surjection  $P_\lambda \rightarrow \Delta_\lambda$ , then  $S_\mu \succ S_\lambda$  since  $K_\lambda$  is only supported on objects  $y > x$ , here  $x$  is the object where  $S_\lambda$  is supported. Therefore, if  $\mathcal{C}$  satisfies the requirement in the above definition, then the associated algebra is standardly stratified as well.

The next theorem characterizes standardly stratified directed categories.

**Theorem 5.7.** *Let  $\mathcal{C}$  be a directed category. Then  $\mathcal{C}$  is standardly stratified if and only if the morphism space  $\mathcal{C}(x, y)$  is a projective  $\mathcal{C}(y, y)$ -module for every pair of objects  $x, y \in \text{Ob } \mathcal{C}$ . Moreover, if  $\mathcal{C}$  is standardly stratified, then  $\bigoplus_{x \in \text{Ob } \mathcal{C}} \mathcal{C}(x, x)$  as a  $\mathcal{C}$ -module has finite projective dimension.*

*Proof.* Suppose that  $\mathcal{C}$  is standardly stratified and take two arbitrary objects  $x$  and  $y$  in  $\mathcal{C}$ . Since 0 is regarded as a projective module, we can assume  $\mathcal{C}(x, y) \neq 0$  and want

to show that it is a projective  $\mathcal{C}(y, y)$ -module. Consider the projective  $\mathcal{C}$ -module  $\mathcal{C}1_x$ , which has a filtration with factors standard modules. Since each standard module is only supported on one object, the value of  $\mathcal{C}1_x$  on  $y$  is exactly the sum of these standard modules with non-zero values on  $y$ . This sum is direct since standard modules supported on  $y$  are non-comparable with respect to the pre-order and therefore there is no extension between them (or because by the previous proposition each of these standard modules is projective viewed as a  $\mathcal{C}(y, y)$ -module). Therefore, the value of  $\mathcal{C}1_x$  on  $y$  is a projective  $\mathcal{C}(y, y)$ -module. But the value of  $\mathcal{C}1_x$  on  $y$  is precisely  $\mathcal{C}(x, y)$ , so the only if part is proved.

Conversely, let  $P_\lambda = \mathcal{C}e_\lambda$  be an indecomposable projective  $\mathcal{C}$ -module. Its value on an arbitrary object  $y$  is  $1_y \mathcal{C}e_\lambda \cong 1_y \mathcal{C}1_x$  which is either 0 or isomorphic to a direct summand of  $\mathcal{C}(x, y)$ . If  $\mathcal{C}(x, y)$  is a projective  $\mathcal{C}(y, y)$ -module, then the value of  $P_\lambda$  on  $y$  is a projective  $\mathcal{C}(y, y)$ -module as well. This value can be expressed as a direct sum of standard modules supported on  $y$  since standard modules are exactly indecomposable direct summands of  $\bigoplus_{x \in \text{Ob } \mathcal{C}} \mathcal{C}(x, x)$ . Therefore we can get a filtration of  $P_\lambda$  by standard modules.

It is well known that the projective dimension of a standard module is finite if the algebra is standardly stratified. Since  $\bigoplus_{x \in \text{Ob } \mathcal{C}} \mathcal{C}(x, x)$  as a  $\mathcal{C}$ -module is a direct sum of standard modules, the last statement follows from this fact immediately.  $\square$

If the directed category  $\mathcal{C}$  is standardly stratified, then all standard modules have finite projective dimensions. But the converse is not true in general. However, we will prove later that for a finite EI category, all standard modules have finite projective dimension if and only if this category is standardly stratified with respect to the canonical pre-order.

From now on we suppose that  $\mathcal{C}$  is a *graded* category, that is, there is a grading on the morphisms in  $\mathcal{C}$  such that  $\mathcal{C}_i \cdot \mathcal{C}_j \subseteq \mathcal{C}_{i+j}$ , where we denote the subspace spanned by all morphisms with grade  $i$  by  $\mathcal{C}_i$ . Furthermore,  $\mathcal{C}$  is supposed to satisfy the following condition:  $\mathcal{C}_i \cdot \mathcal{C}_j = \mathcal{C}_{i+j}$ . Every vector in  $\mathcal{C}_i$  is a linear combination of morphisms with degree  $i$ . Clearly,  $\mathcal{C}_i = \bigoplus_{x, y \in \text{Ob } \mathcal{C}} \mathcal{C}(x, y)_i$ . We always suppose  $\mathcal{C}_i = 0$  for  $i < 0$  and  $\mathcal{C}_0 = \bigoplus_{x \in \text{Ob } \mathcal{C}} \mathcal{C}(x, x)$ . This is equivalent to saying that  $\mathcal{C}_0$  is the direct sum of all standard  $\mathcal{C}$ -modules by Proposition 5.5.

Given a graded directed category  $\mathcal{C}$ , we can apply the functor  $E = \text{Ext}_{\mathcal{C}}^*(-, \mathcal{C}_0)$  to construct the *Yoneda category*  $E(\mathcal{C}_0)$ :  $\text{Ob } E(\mathcal{C}_0) = \text{Ob } \mathcal{C}$  and  $E(\mathcal{C}_0)(x, y)_n = \text{Ext}_{\mathcal{C}}^n(\mathcal{C}(x, x), \mathcal{C}(y, y))$ . This is precisely the categorical version of Yoneda algebras. By the correspondence between graded algebras and graded categories, we can define *Koszul categories*, *quasi-Koszul categories*, *quadratic categories*, *Koszul modules*, *quasi-Koszul modules* for graded categories as well. We do not repeat these definitions here but emphasize that all results described in the previous sections can be applied to graded categories.

A corollary of Theorem 5.7 and Corollary 2.18 relates stratification theory to Koszul theory in the context of directed categories.

**Theorem 5.8.** *Let  $\mathcal{C}$  be a graded directed category with  $\mathcal{C}_0 = \bigoplus_{x \in \text{Ob } \mathcal{C}} \mathcal{C}(x, x)$  being a self-injective algebra. Then:*

- (1)  *$\mathcal{C}$  is standardly stratified if and only if  $\mathcal{C}$  is a projective  $\mathcal{C}_0$ -module.*
- (2)  *$\mathcal{C}$  is a Koszul category if and only if  $\mathcal{C}$  is standardly stratified and quasi-Koszul.*
- (3) *If  $\mathcal{C}$  is standardly stratified, then a graded  $\mathcal{C}$ -module  $M$  generated in degree 0 is Koszul if and only if it is a quasi-Koszul  $\mathcal{C}$ -module and a projective  $\mathcal{C}_0$ -module.*

*Proof.* Take an arbitrary pair of objects  $x, y \in \text{Ob } \mathcal{C}$ . If  $\mathcal{C}$  is standardly stratified, then  $\mathcal{C}(x, y)$  is either 0 (a zero projective module) or a projective  $\mathcal{C}(y, y)$ -module by

the previous theorem. Notice that each  $\mathcal{C}(x, y)_i$  is a  $\mathcal{C}(y, y)$ -module since  $\mathcal{C}(y, y) \subseteq \mathcal{C}_0$ , and we have the decomposition  $\mathcal{C}(x, y) = \bigoplus_{i \geq 0} \mathcal{C}(x, y)_i$ . Therefore,  $\mathcal{C}(x, y)_i$  is a projective  $\mathcal{C}(y, y)$ -module, and hence a projective  $\mathcal{C}_0$ -module since only the block  $\mathcal{C}(y, y)$  of  $\mathcal{C}_0$  acts on  $\mathcal{C}(x, y)_i$  nontrivially. In conclusion,  $\mathcal{C}_i = \bigoplus_{x, y \in \text{Ob } \mathcal{C}} \mathcal{C}(x, y)_i$  is a projective  $\mathcal{C}_0$ -module. Conversely, if  $\mathcal{C}_i$  are projective  $\mathcal{C}_0$ -modules for all  $i \geq 0$ , then each  $\mathcal{C}(x, y)_i$ , and hence  $\mathcal{C}(x, y)$  are projective  $\mathcal{C}(y, y)$ -modules, so  $\mathcal{C}$  is standardly stratified again by the previous theorem. The first statement is proved.

If  $\mathcal{C}$  is Koszul, then it is quasi-Koszul. Moreover, each  $\mathcal{C}_i$  is a projective  $\mathcal{C}_0$  module, see the last paragraph of Section 2. Therefore,  $\mathcal{C}$  is a projective  $\mathcal{C}_0$ -module, and hence is standardly stratified. Conversely, if  $\mathcal{C}$  is standardly stratified and quasi-Koszul, then  $\mathcal{C}$  is a projective  $\mathcal{C}_0$ -module, so  $\mathcal{C}_0$  is linear by Corollary 2.18, and hence  $\mathcal{C}$  is Koszul. This proves the statement.

The last part is an immediate result of Corollary 2.18.  $\square$

To each graded category  $\mathcal{C}$  we can associate an *associated quiver*  $Q$  in the following way: the vertices of  $Q$  are exactly the objects in  $\mathcal{C}$ ; if  $\mathcal{C}(x, y)_1 \neq 0$ , then we put an arrow from  $x$  to  $y$  with  $x, y$  ranging over all objects in  $\mathcal{C}$ . Clearly, the associated quiver of  $\mathcal{C}$  is completely determined by  $\mathcal{C}_0$  and  $\mathcal{C}_1$ . There is no loop in  $Q$  since  $\mathcal{C}(x, x)_1 = 0$  for each  $x \in \text{Ob } \mathcal{C}$ .

**Proposition 5.9.** *Let  $\mathcal{C}$  be a graded category with  $\mathcal{C}_0 = \bigoplus_{x \in \text{Ob } \mathcal{C}} \mathcal{C}(x, x)$  and  $Q$  be its associated quiver. Then  $\mathcal{C}$  is a directed category if and only if  $Q$  is an acyclic quiver.*

*Proof.* Assume that  $\mathcal{C}$  is directed. By the definition, there is a partial order  $\leq$  on  $\text{Ob } \mathcal{C}$  such that  $\mathcal{C}(x, y) \neq 0$  only if  $x \leq y$  for  $x, y \in \text{Ob } \mathcal{C}$ . In particular,  $\mathcal{C}(x, y)_1 \neq 0$  only if  $x < y$ . Therefore, an arrow  $x \rightarrow y$  exists in  $Q$  only if  $x < y$ . If there is an oriented cycle

$$x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n \rightarrow x_1$$

in  $Q$ , then  $x_1 < x_2 < \dots < x_n < x_1$ , which is impossible. Hence  $Q$  must be acyclic.

Conversely, if  $Q$  is acyclic, we then define  $x \leq y$  if and only if there is a directed path (including trivial path with the same source and target) from  $x$  to  $y$  in  $Q$  for  $x, y \in \text{Ob } \mathcal{C}$ . This gives rise to a well defined partial order on  $\text{Ob } \mathcal{C}$ . We claim that  $\mathcal{C}$  is a directed category with respect to this partial order, i.e.,  $\mathcal{C}(x, y) \neq 0$  implies  $x \leq y$ . Since it holds trivially for  $x = y$ , we assume that  $x \neq y$ . Take a morphism  $0 \neq \alpha \in \mathcal{C}(x, y)$  with a degree  $n$  (this is possible since  $\mathcal{C}(x, y)$  is a non-zero graded space). Since  $\mathcal{C}_n = \mathcal{C}_1 \cdot \dots \cdot \mathcal{C}_1$ , we can express  $\alpha$  as a linear combination of composite morphisms

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} x_n = y$$

with each  $\alpha_i \in \mathcal{C}_1$  and all  $x_i$  being distinct (since endomorphisms in  $\mathcal{C}$  are contained in  $\mathcal{C}_0$ ). Therefore, there is a nontrivial directed path

$$x = x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n = y$$

in  $Q$ , and we have  $x < x_1 < x_2 < \dots < y$ , which proves our claim.  $\square$

Let  $\mathcal{C}$  be a graded directed category. We define the *free cover*  $\hat{\mathcal{C}}$  of  $\mathcal{C}$  by using the associated quiver  $Q$ . Explicitly,  $\hat{\mathcal{C}}$  has the same objects and endomorphisms as  $\mathcal{C}$ . For each pair of objects  $x \neq y$  we construct  $\hat{\mathcal{C}}(x, y)$  as follows. let  $\Gamma_{x, y}$  be the set of all paths from  $x$  to  $y$  in  $Q$ . In the case that  $\Gamma_{x, y} = \emptyset$  we let  $\hat{\mathcal{C}}(x, y) = 0$ . Otherwise, take an arbitrary path  $\gamma \in \Gamma_{x, y}$  pictured as below

$$x \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{n-1} \rightarrow y,$$

and define  $(x, y)_\gamma$  to be

$$\mathcal{C}(x_{n-1}, y)_1 \otimes_{\mathcal{C}(x_{n-1}, x_{n-1})} \mathcal{C}(x_{n-2}, x_{n-1})_1 \otimes_{\mathcal{C}(x_{n-2}, x_{n-2})} \dots \otimes_{\mathcal{C}(x_1, x_1)} \mathcal{C}(x, x_1)_1.$$



Finally, we define

$$\hat{\mathcal{C}}(x, y) = \bigoplus_{\gamma \in \Gamma_{x, y}} (x, y)_{\gamma}.$$

It is clear that  $\hat{\mathcal{C}}$  is also a graded category with  $\hat{\mathcal{C}}_0 = \mathcal{C}_0$  and  $\hat{\mathcal{C}}_1 = \mathcal{C}_1$ . Therefore,  $\hat{\mathcal{C}}$  has the same associated quiver as that of  $\mathcal{C}$  and is also a directed category by the above lemma. Actually, if two grade categories  $\mathcal{C}$  and  $\mathcal{D}$  have the same degree 0 and degree 1 components, then one is a directed category if and only if so is the other.

**Theorem 5.10.** *Let  $\mathcal{C}$  be a directed Koszul category with  $\mathcal{C}_0$  being a self-injective algebra, then the Yoneda category  $\mathcal{E} = E(\mathcal{C}_0)$  is also directed and Koszul.*

*Proof.* Applying the Koszul duality (Theorem 4.1) we know that  $\mathcal{E}$  is a Koszul category. What we need to show is that  $\mathcal{E}$  is a directed category as well. Since  $\mathcal{C}$  is standardly stratified,  $\text{pd}_{\mathcal{C}} \mathcal{C}_0 < \infty$ . Therefore, all morphisms in  $\mathcal{E}$  spans a finite-dimensional space by Lemma 4.3. In particular, for  $x, y \in \text{Ob } \mathcal{E}$ ,  $\dim_k \mathcal{E}(x, y) < \infty$ . Therefore,  $\mathcal{E}$  is a locally finite  $k$ -linear category.

Let  $\leq$  be the partial order on  $\text{Ob } \mathcal{C}$  with respect to which  $\mathcal{C}$  is directed. This partial order gives a partial order on  $\text{Ob } \mathcal{E}$  as well because  $\text{Ob } \mathcal{E} = \text{Ob } \mathcal{C}$ . We claim that  $\mathcal{E}$  is directed with respect to this partially ordered set, i.e., if  $x \not\leq y$  are two distinct objects in  $\mathcal{E}$ , then  $\mathcal{E}(x, y) = 0$ .

Since  $\mathcal{E}$  is the Yoneda category of  $\mathcal{C}$ ,  $\mathcal{E}(x, y) = {}_{1_y} \mathcal{E} 1_x \cong \text{Ext}_{\mathcal{C}}^*(\mathcal{C}_0 1_x, \mathcal{C}_0 1_y)$ . But  $\mathcal{C}$  is a Koszul category, so it is standardly stratified by the second statement of Theorem 5.8. Therefore,  $\mathcal{C}_i$  are projective  $\mathcal{C}_0$ -modules for all  $i \geq 0$  by the first part of this theorem. By Lemma 2.14, all  $\Omega^i(\mathcal{C}_0 1_x)_i$  are projective  $\mathcal{C}_0$ -modules. By Lemma 2.10, we have

$$\mathcal{E}(x, y)_i \cong \text{Ext}_{\mathcal{C}}^i(\mathcal{C}_0 1_x, \mathcal{C}_0 1_y) \cong \text{Hom}_{\mathcal{C}}(\Omega^i(\mathcal{C}_0 1_x), \mathcal{C}_0 1_y).$$

Observe that  $\mathcal{C}_0 1_y$  is only supported on the object  $y$  and  $y \not\geq x$ . If we can prove the statement that each  $\Omega^i(\mathcal{C}_0 1_x)$  is only supported on objects  $z$  with  $z \geq x$ , then our claim is proved.

Clearly,  $\Omega^0(\mathcal{C}_0 1_x) = \mathcal{C}_0 1_x = \mathcal{C}(x, x)$  is only supported on  $x$ , so the statement is true for  $i = 0$ . Now suppose that  $\Omega^n(\mathcal{C}_0 1_x)$  is only supported on objects  $z \geq x$  and consider  $\Omega^{n+1}(\mathcal{C}_0 1_x)$ . Let  $S$  be the set of objects  $z$  such that the value  $1_z \Omega^n(\mathcal{C}_0 1_x)$  of  $\Omega^n(\mathcal{C}_0 1_x)$  on  $z$  is non-zero. Then we can find a short exact sequence:

$$0 \longrightarrow N \longrightarrow \bigoplus_{z \in S} (\mathcal{C} 1_z)^{m_z} \xrightarrow{p} \Omega^n(\mathcal{C}_0 1_x) \longrightarrow 0$$

such that the map  $p$  gives a surjection  $p_z : (1_z \mathcal{C} 1_z)^{m_z} \rightarrow 1_z \Omega^n(\mathcal{C}_0 1_x)$  for  $z \in S$ . Thus  $p$  is a surjection and  $\Omega^{n+1}(\mathcal{C}_0 1_x)$  is a direct summand of  $N$ . Notice that all  $\mathcal{C} 1_z$  are supported only on objects  $w \geq z$ , and  $z \geq x$  by the induction hypothesis. Therefore, the submodule  $\Omega^{n+1}(\mathcal{C}_0 1_x) \subseteq N \subseteq \bigoplus_{z \in S} (\mathcal{C} 1_z)^{m_z}$  is only supported on objects  $w \geq x$ . Our statement is proved by induction. This finishes the proof.  $\square$

Let  $\mathcal{A}$  be the category of directed Koszul categories  $\mathcal{C}$  with  $\mathcal{C}_0 = \bigoplus_{x \in \text{Ob } \mathcal{C}} \mathcal{C}(x, x)$  being self-injective algebras. This theorem tells us that the homological dual functor  $E$  is also a dual functor from  $\mathcal{A}$  to itself. For a fixed category  $\mathcal{C} \in \text{Ob } \mathcal{A}$ , since standard modules of  $\mathcal{C}$  are exactly indecomposable summands of  $\mathcal{C}_0$ ,  $E$  interchanges standard (indecomposable projective, resp.)  $\mathcal{C}$ -modules and indecomposable projective (standard, resp.)  $E(\mathcal{C}_0)$ -modules bijectively.

The condition that  $\mathcal{C}_0$  is exactly the space spanned by endomorphisms in  $\mathcal{C}$  is crucial since it implies that standard modules are precisely indecomposable summands of  $\mathcal{C}_0$ . The following example tells us that without this assumption, the Yoneda category  $E(\mathcal{C}_0)$  might not be directed even if  $\mathcal{C}$  is a Koszul directed category.

**Example 5.11.** Let  $\mathcal{C}$  be the following category. Put an order  $x < y$  on the objects and the following grading on morphisms:  $\mathcal{C}_0 = \langle 1_x, 1_y, \beta \rangle$ ,  $\mathcal{C}_1 = \langle \alpha \rangle$ .

$$x \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} y.$$

This category is directed obviously. It is standardly stratified (actually hereditary) with  $\Delta_x \cong k_x$  and  $\Delta_y \cong k_y$ . By the exact sequence

$$0 \longrightarrow k_y[1] \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}_0 \longrightarrow 0,$$

$\mathcal{C}_0$  is a linear module. But  $\Delta_x \oplus \Delta_y \not\cong \mathcal{C}_0$ . Furthermore,  $\Delta_x \oplus \Delta_y$  is not linear since from the short exact sequence

$$0 \longrightarrow k_y[1] \oplus k_y \longrightarrow \mathcal{C} \longrightarrow \Delta_x \oplus \Delta_y \longrightarrow 0$$

we find that  $\Omega(\Delta_x \oplus \Delta_y)$  is not generated in degree 1.

By computation we get the Yoneda category  $\mathcal{D} = E(\mathcal{C}_0)$  pictured as below, with relation  $\alpha \cdot \beta = 0$ .

$$x \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} y.$$

This is not a directed category with respect to the order  $x < y$ . However, we check:  $P_y = \Delta'_y = \mathcal{D}_0 1_y$  and  $\Delta'_x = k_x \cong P_x/P_y$ . Therefore,  $\Delta'_x \oplus \Delta'_y \cong \mathcal{D}_0$ , and  $\mathcal{D}$  is standardly stratified. The exact sequence

$$0 \longrightarrow P_y[1] \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}_0 \longrightarrow 0$$

tells us that  $\mathcal{D}_0$  is a linear  $\mathcal{D}$ -module. Therefore,  $\mathcal{D}$  is standardly stratified and Koszul, but not directed.

The  $k$ -linearization of a finite EI category is a directed category by definition. However, this theorem does not hold in the context of finite EI categories. That is, if  $\mathcal{E}$  is a Koszul finite EI category,  $E(k\mathcal{E}_0)$  might not to be the category algebra of a finite EI category, as illustrated in the next example:

**Example 5.12.** Let  $\mathcal{E}$  be the following finite EI category with three objects whose endomorphism groups are all trivial. Put a grading on  $\mathcal{E}$  with  $\mathcal{E}_0 = \{1_x, 1_y, 1_z\}$ ,  $\mathcal{E}_1 = \{\alpha, \beta\}$  and  $\mathcal{E}_2 = \{\beta\alpha\}$ . Then  $(k\mathcal{E}_0) \cong k\mathcal{E}/(\beta\alpha)$  is not the category algebra of a finite EI category.

$$x \xrightarrow{\alpha} y \xrightarrow{\beta} z$$

There is a close relation between the classical Koszul theory and our generalized Koszul theory in the context of graded directed categories. Let  $\mathcal{C}$  be a graded directed category. We then define a subcategory  $\mathcal{D}$  of  $\mathcal{C}$  by replacing all endomorphism rings in  $\mathcal{C}$  by  $k \cdot 1$ , the span of the identity endomorphism. Explicitly,  $\text{Ob } \mathcal{D} = \text{Ob } \mathcal{C}$ ; for  $x, y \in \text{Ob } \mathcal{D}$ ,  $\mathcal{D}(x, y) = k\langle 1_x \rangle$  if  $x = y$  and  $\mathcal{D}(x, y) = \mathcal{C}(x, y)$  otherwise. Clearly,  $\mathcal{D}$  is also a graded directed category with  $\mathcal{D}_i = \mathcal{C}_i$  for every  $i \geq 1$ . Observe that the degree 0 component  $\mathcal{D}_0$  is semisimple.

**Theorem 5.13.** Let  $\mathcal{C}$  be a graded directed Koszul category and define the subcategory  $\mathcal{D}$  as above. If  $M$  is a linear  $\mathcal{C}$ -module, then  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is a linear  $\mathcal{D}$ -module. In particular,  $\mathcal{D}$  is a Koszul category in the classical sense.

*Proof.* We prove the conclusion by induction on the size of  $\text{Ob } \mathcal{C}$ . If the size of  $\text{Ob } \mathcal{C}$  is 1, the conclusion holds trivially. Now suppose that the conclusion is true for categories with at most  $n$  objects and let  $\mathcal{C}$  be a graded directed category with  $n + 1$  objects. Take  $x$  to be a minimal object in  $\mathcal{C}$  and define  $\mathcal{C}_x$  ( $\mathcal{D}_x$ , resp.) to be the full subcategory of  $\mathcal{C}$  ( $\mathcal{D}$ , resp.) formed by removing  $x$  from it. Clearly  $\mathcal{C}_x$  and  $\mathcal{D}_x$  have  $n$  objects.

The following fact, which is well known in the context of finite EI categories (see [22]), is essential in the proof.

**Fact:** Every graded  $\mathcal{C}_x$ -module  $N$  can be viewed as a  $\mathcal{C}$ -module with  $N(x) = 0$  by induction. Conversely, every graded  $\mathcal{C}$ -module  $M$  with  $M(x) = 0$  can also be viewed as a  $\mathcal{C}_x$ -module by restriction. Furthermore, if  $M(x) = 0$ , then  $\Omega^i(M)(x) = 0$  for all  $i \geq 0$ . The above induction and restriction preserves projective modules: a projective  $\mathcal{C}_x$ -module is still projective when viewed as a  $\mathcal{C}$ -module; conversely, a projective  $\mathcal{C}$ -module  $P$  with  $P(x) = 0$  is still projective viewed as a  $\mathcal{C}_x$ -module. Therefore, a graded  $\mathcal{C}$ -module  $M$  with  $M(x) = 0$  is linear if and only if it is linear as a  $\mathcal{C}_x$ -module. All these results hold for the pair  $(\mathcal{D}, \mathcal{D}_x)$  similarly.

By this fact, we only need to handle linear  $\mathcal{C}$ -modules  $M$  with  $M(x) \neq 0$ . Indeed, if  $M(x) = 0$ , then  $M$  is also linear regarded as a  $\mathcal{C}_x$ -module. By the induction hypothesis,  $M \downarrow_{\mathcal{D}_x}^{\mathcal{C}_x}$  is a linear  $\mathcal{D}_x$ -module. By the above fact,  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is a linear  $\mathcal{D}$ -module. Thus the conclusion is true for linear  $\mathcal{C}$ -modules  $M$  with  $M(x) = 0$ .

Firstly we consider the special case  $M = \mathcal{C}_0 1_x = \mathcal{C}(x, x)$  which is concentrated on  $x$  when viewed as a  $\mathcal{C}$ -module. It is clear that

$$\Omega(\mathcal{C}_0 1_x) \downarrow_{\mathcal{D}}^{\mathcal{C}} = \Omega(\mathcal{D}_0 1_x)$$

as vector spaces since for each pair  $u \neq v \in \text{Ob } \mathcal{C}$ ,  $\mathcal{C}(u, v) = \mathcal{D}(u, v)$ , and

$$\mathcal{C}_0 1_x \downarrow_{\mathcal{D}}^{\mathcal{C}} \cong (\mathcal{D}_0 1_x)^m \cong k_x^m,$$

where  $m = \dim_k \mathcal{C}(x, x)$ . Since  $\mathcal{C}_0 1_x$  is a linear  $\mathcal{C}$ -module,  $\Omega(\mathcal{C}_0 1_x)[-1]$  is a linear  $\mathcal{C}$ -module supported on  $\text{Ob } \mathcal{C}_x$ . By the induction hypothesis,  $\Omega(\mathcal{D}_0 1_x)[-1] = \Omega(\mathcal{C}_0 1_x) \downarrow_{\mathcal{D}}^{\mathcal{C}} [-1]$  is a linear  $\mathcal{D}_x$ -module, and hence a linear  $\mathcal{D}$ -module. Therefore,  $\mathcal{D}_0 1_x$ , and hence  $\mathcal{C}_0 1_x \downarrow_{\mathcal{D}}^{\mathcal{C}} \cong (\mathcal{D}_0 1_x)^m$  are linear  $\mathcal{D}$ -modules. In the case that  $y \neq x$ ,  $\mathcal{D}_0 1_y$  is a direct summand of  $\mathcal{C}_0 1_y \downarrow_{\mathcal{D}}^{\mathcal{C}}$ . It is linear viewed as a  $\mathcal{D}_x$ -module by the induction hypothesis, and hence is a linear  $\mathcal{D}$ -module. Consequently,  $\mathcal{D}_0$  is a linear  $\mathcal{D}$ -module, so  $\mathcal{D}$  is a Koszul category in the classical sense.

Now let  $M$  be an arbitrary linear  $\mathcal{C}$ -module with  $M(x) \neq 0$ . Consider the exact sequence

$$(5.1) \quad 0 \longrightarrow \Omega M \downarrow_{\mathcal{D}}^{\mathcal{C}} \longrightarrow P \downarrow_{\mathcal{D}}^{\mathcal{C}} \xrightarrow{p} M \downarrow_{\mathcal{D}}^{\mathcal{C}} \longrightarrow 0$$

induced by

$$0 \longrightarrow \Omega M \longrightarrow P \xrightarrow{p} M \longrightarrow 0.$$

The structures of  $\mathcal{C}$  and  $\mathcal{D}$  give the following exact sequence:

$$0 \longrightarrow {}_{\mathcal{D}} \mathcal{D} \longrightarrow {}_{\mathcal{D}} \mathcal{C} \longrightarrow \bigoplus_{x \in \text{Ob } \mathcal{C}} k_x^{m_x} \longrightarrow 0$$

where  $k_x \cong \mathcal{D}_0 1_x$  and  $m_x = \dim_k \mathcal{C}(x, x) - 1$ . Since  $P \downarrow_{\mathcal{D}}^{\mathcal{C}} \in \text{add}({}_{\mathcal{D}} \mathcal{C})$ , the above sequence gives us a corresponding sequence for  $P \downarrow_{\mathcal{D}}^{\mathcal{C}}$ :

$$(5.2) \quad 0 \longrightarrow P' \xrightarrow{\iota} P \downarrow_{\mathcal{D}}^{\mathcal{C}} \longrightarrow T \longrightarrow 0,$$

here  $P'$  is a projective  $\mathcal{D}$ -module and  $T \in \text{add}(\mathcal{D}_0)$ . Both of them are generated in degree 0.

Putting sequences 5.1 and 5.2 together we get

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Omega(M') & \longrightarrow & P' & \xrightarrow{p \circ \iota} & M' \longrightarrow 0 \\
& & \downarrow \varphi & & \downarrow \iota & & \downarrow \\
0 & \longrightarrow & \Omega(M) \downarrow_{\mathcal{D}}^{\mathcal{C}} & \longrightarrow & P \downarrow_{\mathcal{D}}^{\mathcal{C}} & \xrightarrow{p} & M \downarrow_{\mathcal{D}}^{\mathcal{C}} \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & T & \longrightarrow & M \downarrow_{\mathcal{D}}^{\mathcal{C}} / M'
\end{array}$$

where  $M' = (p \circ \iota)(P')$ . Notice that  $p$  and  $\iota$  both are injective restricted to degree 0 components. Therefore  $p \circ \iota$  is also injective restricted to the degree 0 component of  $P'$  (actually it is an isomorphism restricted to the degree 0 component). Consequently,  $P'$  is a projective cover of  $M'$  and the kernel of  $p \circ \iota$  is indeed  $\Omega(M')$ .

We claim that  $\varphi$  is an isomorphism and hence  $\Omega(M') \cong (\Omega M) \downarrow_{\mathcal{D}}^{\mathcal{C}}$ . It suffices to show  $T \cong M \downarrow_{\mathcal{D}}^{\mathcal{C}} / M'$  by the snake lemma. First, since  $T$  is concentrated in degree 0 in sequence 5.2,  $(P \downarrow_{\mathcal{D}}^{\mathcal{C}})_i = P'_i$  and

$$M'_i = (p \circ \iota)(P'_i) = p((P \downarrow_{\mathcal{D}}^{\mathcal{C}})_i) = (M \downarrow_{\mathcal{D}}^{\mathcal{C}})_i, i > 0.$$

Therefore,  $M \downarrow_{\mathcal{D}}^{\mathcal{C}} / M'$  is concentrated in degree 0. Furthermore, since  $M$  is linear,  $(P \downarrow_{\mathcal{D}}^{\mathcal{C}})_0 = P_0 = M_0 = (M \downarrow_{\mathcal{D}}^{\mathcal{C}})_0$ , and  $P'_0 = M'_0$  because  $p \circ \iota$  restricted to  $P'_0$  is an isomorphism as well. We deduce that

$$T \cong (P \downarrow_{\mathcal{D}}^{\mathcal{C}})_0 / P'_0 \cong (M \downarrow_{\mathcal{D}}^{\mathcal{C}})_0 / M'_0 = M \downarrow_{\mathcal{D}}^{\mathcal{C}} / M',$$

exactly as we claimed.

Now consider the rightmost column of the above diagram. Clearly, the bottom term  $M \downarrow_{\mathcal{D}}^{\mathcal{C}} / M' \cong T \in \text{add}(\mathcal{D}_0)$  is linear since we just proved that  $\mathcal{D}_0$  is linear. The  $\mathcal{C}$ -module  $(\Omega M)[-1]$  is linear since  $M$  is supposed to be linear. Moreover, because  $x$  is minimal and  $M$  is generated in degree 0,  $M(x) \subseteq M_0$  and hence  $(\Omega M)[-1](x) = (\Omega M)(x) = 0$ . Therefore,  $(\Omega M)[-1]$  is a linear  $\mathcal{C}$ -module supported on  $\text{Ob } \mathcal{C}_x$ , so it is also a linear  $\mathcal{C}_x$ -module. By the induction hypothesis,  $(\Omega M')[-1] \cong (\Omega M)[-1] \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is linear viewed as a  $\mathcal{D}_x$ -module, and hence linear as a  $\mathcal{D}$ -module. Thus the top term  $M'$  is a linear  $\mathcal{D}$ -module since as a homomorphic image of  $P'$  (which is generated in degree 0) it is generated in degree 0 as well. By Proposition 2.9,  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is also a linear  $\mathcal{D}$ -module since  $\mathcal{D}_0$  is semisimple by our construction. The conclusion follows from induction.  $\square$

The converse of the above theorem is also true.

**Theorem 5.14.** *Let  $\mathcal{C}$  be a graded directed category and construct the subcategory  $\mathcal{D}$  as before. Suppose that  $\mathcal{D}$  is Koszul in the classical sense. Let  $M$  be a graded  $\mathcal{C}$ -module generated in degree 0 such that  $\Omega^i(M)_i$  are projective  $\mathcal{C}_0$ -modules for all  $i \geq 0$ . Then  $M$  is a linear  $\mathcal{C}$ -module whenever  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is a linear  $\mathcal{D}$ -module.*

*Proof.* We use the similar technique to prove the conclusion. Notice that we always assume that  $\mathcal{C}_0 = \bigoplus_{x \in \text{Ob } \mathcal{C}} \mathcal{C}(x, x)$ . If  $\mathcal{C}$  has only one object, then linear modules are exactly projective modules generated in degree 0 and the conclusion holds. Suppose that it is true for categories with at most  $n$  objects. Let  $\mathcal{C}$  be a category of  $n+1$  objects and take a minimal object  $x$ . Define  $\mathcal{C}_x$  and  $\mathcal{D}_x$  as before. As in the proof of last theorem, a graded  $\mathcal{C}$ -module  $M$  with  $M(x) = 0$  is linear if and only if it is linear viewed as a  $\mathcal{C}_x$ -module by restriction. and the same result holds for the pair  $(\mathcal{D}, \mathcal{D}_x)$ . In particular,  $\mathcal{D}_x$  is a Koszul category. Therefore, we only need to show that an arbitrary graded  $\mathcal{C}$ -module  $M$  which is generated in degree 0 and satisfies the following conditions is linear:  $\Omega^i(M)_i$  is a projective  $\mathcal{C}_0$ -module for each  $i \geq 0$ ;  $M(x) \neq 0$ ; and  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is a linear  $\mathcal{D}$ -module.

Let  $M$  be such a  $\mathcal{C}$ -module and consider the commutative diagram:

$$(5.3) \quad \begin{array}{ccccccc} & & K & \xlongequal{\quad} & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega(M \downarrow_{\mathcal{D}}^{\mathcal{C}}) & \longrightarrow & P & \longrightarrow & M \downarrow_{\mathcal{D}}^{\mathcal{C}} \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & (\Omega M) \downarrow_{\mathcal{D}}^{\mathcal{C}} & \longrightarrow & \tilde{P} \downarrow_{\mathcal{D}}^{\mathcal{C}} & \longrightarrow & M \downarrow_{\mathcal{D}}^{\mathcal{C}} \longrightarrow 0 \end{array}$$

where  $P$  and  $\tilde{P}$  are projective covers of  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  and  $M$  respectively. Since  $M_0$  is a projective  $\mathcal{C}_0$ -module,  $P_0 = M_0 = (M \downarrow_{\mathcal{D}}^{\mathcal{C}})_0 = (\tilde{P} \downarrow_{\mathcal{D}}^{\mathcal{C}})_0$  as vector spaces, and the induced map  $\varphi$  restricted to  $P_0$  is an isomorphism. Therefore  $\varphi$  is surjective since both  $P$  and  $\tilde{P} \downarrow_{\mathcal{D}}^{\mathcal{C}}$  are generated in degree 0. Let  $K$  be the kernel of  $\varphi$ .

We have the following exact sequence similar to sequence 5.2:

$$0 \longrightarrow P' \longrightarrow \tilde{P} \downarrow_{\mathcal{D}}^{\mathcal{C}} \xrightarrow{\tilde{p}} T \longrightarrow 0,$$

where  $P'$  is a projective  $\mathcal{D}$ -module such that  $P'_i = (\tilde{P} \downarrow_{\mathcal{D}}^{\mathcal{C}})_i$  for every  $i \geq 1$ , and  $T \in \text{add}(\mathcal{D}_0)$ .

Let  $P''$  be a projective cover of  $T$  (as a  $\mathcal{D}$ -module). Then we obtain:

$$(5.4) \quad \begin{array}{ccccccc} & & K & \xlongequal{\quad} & K & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P' & \longrightarrow & P & \xrightarrow{p} & P'' \longrightarrow 0 \\ & & \parallel \alpha & & \downarrow \varphi & & \downarrow p'' \\ 0 & \longrightarrow & P' & \longrightarrow & \tilde{P} \downarrow_{\mathcal{D}}^{\mathcal{C}} & \xrightarrow{\tilde{p}} & T \longrightarrow 0 \end{array}$$

We give some explanations here. Since  $P$  is a projective  $\mathcal{D}$ -module and the map  $p''$  is surjective, the map  $\tilde{p} \circ \varphi$  factors through  $p''$  and gives a map  $p : P \rightarrow P''$ . Restricted to degree 0 components,  $p''$  and  $\varphi$  (see diagram 5.3) are isomorphisms and  $\tilde{p}$  is surjective. Thus  $p$  restricted to the degree 0 components is also surjective. But  $P''$  is generated in degree 0, so  $p$  is surjective. Since  $P_0 = (\tilde{P} \downarrow_{\mathcal{D}}^{\mathcal{C}})_0$  and  $P''_0 = T_0 = T$ ,  $\alpha$  restricted to the degree 0 components is an isomorphism, and hence an isomorphism of projective  $k\mathcal{D}$ -modules (notice that the middle row splits since  $P''$  is a projective  $\mathcal{D}$ -module, so the kernel should be a projective  $\mathcal{D}$ -module generated in degree 0). By the snake Lemma, the kernel of  $p''$  is also  $K$  up to isomorphism.

Let  $J = \bigoplus_{i \geq 1} \mathcal{D}_i$ . Since  $\mathcal{D}_0$  is supposed to be a linear  $\mathcal{D}$ -module,  $J[-1] \cong \Omega(\mathcal{D}_0)[-1]$  is a linear  $\mathcal{D}$ -module, too. Consider the leftmost column in diagram 5.3. The top term  $K[-1]$  is a linear  $\mathcal{D}$ -module since  $K \cong P''/T \cong P''/P'_0 \in \text{add}(J)$ . The middle term  $\Omega(M \downarrow_{\mathcal{D}}^{\mathcal{C}})[-1]$  is linear as well since  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is supposed to be linear. By Proposition 2.9, the bottom term  $(\Omega M)[-1] \downarrow_{\mathcal{D}}^{\mathcal{C}}$  must be linear.

Since  $M$  is generated in degree 0 and  $x$  is a minimal object,  $M(x) \subseteq M_0$ , so  $M(x) = (M \downarrow_{\mathcal{D}}^{\mathcal{C}})(x) \subseteq (M \downarrow_{\mathcal{D}}^{\mathcal{C}})_0$  as vector spaces. Similarly,  $P(x) \subseteq P_0$  and  $P(x) \cong (M \downarrow_{\mathcal{D}}^{\mathcal{C}})(x)$ , so  $\Omega(M \downarrow_{\mathcal{D}}^{\mathcal{C}})(x) = 0$ . Consequently,  $(\Omega M)[-1]$  is supported on  $\text{Ob } \mathcal{C}_x$  by observing the leftmost column of diagram 5.3. Moreover, we can show as in the proof of Theorem 5.10 that all of its syzygies are supported on  $\text{Ob } \mathcal{C}_x$ , and  $\Omega^i((\Omega M)[-1])_i = \Omega^{i+1}(M)_{i+1}$  are projective  $(\mathcal{C}_x)_0$ -modules. Therefore, applying

the induction hypothesis to  $(\Omega M)[-1]$  supported on  $\text{Ob } \mathcal{C}_x$  and  $(\Omega M)[-1] \downarrow_{\mathcal{D}}^{\mathcal{C}}$  supported on  $\text{Ob } \mathcal{D}_x$ , we conclude that  $(\Omega M)[-1]$  is a linear  $\mathcal{C}_x$ -module, and hence a linear  $\mathcal{C}$ -module. Clearly,  $M$  is a linear  $\mathcal{C}$ -module since it is generated in degree 0. The conclusion follows from induction.  $\square$

**Remark 5.15.** *We remind the reader that in the previous two theorems we do not require  $\mathcal{C}_0 = \bigoplus_{x \in \text{Ob } \mathcal{C}} \mathcal{C}(x, x)$  to be a self-injective algebra. By our construction,  $\mathcal{D}_0 \cong \bigoplus_{x \in \text{Ob } \mathcal{D}} k_x$  is a semisimple algebra.*

Assuming that  $\mathcal{C}_0$  is self-injective, we get the following nice correspondence.

**Theorem 5.16.** *Let  $\mathcal{C}$  be a graded directed category with  $\mathcal{C}_0$  being a self-injective algebra and construct  $\mathcal{D}$  as before. Then:*

- (1)  *$\mathcal{C}$  is a Koszul category in our sense if and only if  $\mathcal{C}$  is standardly stratified and  $\mathcal{D}$  is a Koszul category in the classical sense.*
- (2) *If  $\mathcal{C}$  is a Koszul category, then a graded  $\mathcal{C}$ -module  $M$  generated in degree 0 is linear if and only if  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is a linear  $\mathcal{D}$ -module and  $M$  is a projective  $\mathcal{C}_0$ -module.*

*Proof.* If  $\mathcal{C}$  is Koszul in our sense, then it is standardly stratified by (2) of Theorem 5.8, and  $\mathcal{D}$  is Koszul in the classical sense by Theorem 5.13. Conversely, if  $\mathcal{D}$  is Koszul in the classical sense, then  $\mathcal{C}_0 \downarrow_{\mathcal{D}}^{\mathcal{C}} \in \text{add}(\mathcal{D}_0)$  is a linear  $\mathcal{D}$ -module. If  $\mathcal{C}$  is furthermore standardly stratified, then it is a projective  $\mathcal{C}_0$ -module by Theorem 5.8. Therefore, all  $\Omega^i(\mathcal{C}_0)_i$  are projective  $\mathcal{C}_0$ -modules according to Lemma 2.14. Thus  $\mathcal{C}_0$  is a linear  $\mathcal{C}$ -module by Theorem 5.14, and hence  $\mathcal{C}$  is a Koszul category. This proves the first statement.

Now suppose that  $\mathcal{C}$  is Koszul. Then  $\mathcal{C}$  is a projective  $\mathcal{C}_0$ -module. If  $M$  is a linear  $\mathcal{C}$ -module,  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is a linear  $\mathcal{D}$ -module by Theorem 5.13. Furthermore,  $M$  is a projective  $\mathcal{C}_0$ -module by Corollary 2.18. Conversely, if  $M$  is a projective  $\mathcal{C}_0$ -module and  $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$  is a linear  $\mathcal{D}$ -module, then by Lemma 2.14  $\Omega^i(M)_i$  are projective  $\mathcal{C}_0$ -modules for all  $i \geq 0$ . By Theorem 5.14  $M$  is a linear  $\mathcal{C}$ -module.  $\square$

## 6. FINITE EI CATEGORIES

When applying the generalized Koszul theory to a directed category  $\mathcal{C}$  in the previous section, we take for granted that there is already a grading on the morphisms in  $\mathcal{C}$  such that the degree 0 component is formed precisely by endomorphisms in  $\mathcal{C}$ . But in practice it is very hard to find such a grading for  $\mathcal{C}$ . Actually, we do not even know the existence of such gradings in general. In this section we will focus on finite EI categories, whose  $k$ -linearizations form a type of directed categories with combinatorial properties. These properties can be used to define a length grading on the set of morphisms and completely determine whether an arbitrary finite EI category can be graded by this length grading.

In this section we only consider skeletal and *connected* finite EI categories  $\mathcal{E}$ , i.e., for every pair  $x, y \in \text{Ob } \mathcal{E}$ , there is a chain of objects  $x_0 = x, x_1, x_2, \dots, x_n = y$  such that either  $\mathcal{E}(x_i, x_{i+1}) \neq \emptyset$  or  $\mathcal{E}(x_{i+1}, x_i) \neq \emptyset$  for every  $0 \leq i \leq n-1$ .

First we introduce some results from [12]. A morphism  $\alpha$  in  $\mathcal{E}$  is called *unfactorizable* if  $\alpha$  is not an automorphism, and whenever there is a decomposition  $\alpha = \alpha_1 \circ \alpha_2$ , either  $\alpha_1$  or  $\alpha_2$  is an automorphism. The composite morphism of an unfactorizable morphism and an automorphism is still unfactorizable. Therefore, all unfactorizable morphisms from an object  $x$  to another object  $y$  form an  $(\text{Aut}_{\mathcal{E}}(y), \text{Aut}_{\mathcal{E}}(x))$  bi-set. Every non-isomorphism can be expressed as a composite of unfactorizable morphisms. This decomposition is not unique in general. We say a finite EI category  $\mathcal{E}$  satisfies the *Unique Factorization Property* (UFP) if whenever

a non-isomorphism  $\alpha$  has two decompositions into unfactorizable morphisms:

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_m} x_m = y$$

$$x = x_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_n} y_n = y$$

then  $m = n$ ,  $x_i = y_i$ , and there are  $h_i \in \text{Aut}_{\mathcal{E}}(x_i)$  such that the following diagram commutes,  $1 \leq i \leq n-1$ :

$$\begin{array}{ccccccc} x_0 & \xrightarrow{\alpha_1} & x_1 & \xrightarrow{\alpha_2} & \dots & \xrightarrow{\alpha_{n-1}} & x_n \\ \parallel & & \downarrow h_1 & & \downarrow h_{n-1} & & \parallel \\ x_0 & \xrightarrow{\beta_1} & x_1 & \xrightarrow{\beta_2} & \dots & \xrightarrow{\beta_n} & x_n \end{array}$$

Finite EI categories with this property are called *finite free EI categories* by us. For every finite EI category  $\mathcal{E}$  there is a unique (up to isomorphism) finite free EI category  $\hat{\mathcal{E}}$  (called the *free EI cover*) and a covering functor  $\hat{F} : \hat{\mathcal{E}} \rightarrow \mathcal{E}$  such that  $\hat{F}$  is the identity map restricted to objects, isomorphisms and unfactorizable morphisms. The functor  $\hat{F}$  induces a surjective algebra homomorphism  $\varphi : k\hat{\mathcal{E}} \rightarrow k\mathcal{E}$ . Therefore,  $k\mathcal{E} \cong k\hat{\mathcal{E}}/I$ , where  $I$  is the kernel of  $\varphi$ . We have the following description of  $I$ :

**Lemma 6.1.** *The  $k\hat{\mathcal{E}}$ -ideal  $I$  as a vector space is spanned by elements of the form  $\hat{\alpha} - \hat{\beta}$ , where  $\hat{\alpha}$  and  $\hat{\beta}$  are morphisms in  $\hat{\mathcal{E}}$  with  $\hat{F}(\hat{\alpha}) = \hat{F}(\hat{\beta})$ .*

*Proof.* Let  $U$  be the vector space spanned by elements  $\hat{\alpha} - \hat{\beta}$  such that  $\hat{F}(\hat{\alpha}) = \hat{F}(\hat{\beta})$ . Clearly,  $U \subseteq I$  and we want to show the other inclusion. Let  $x \in U$ . By the definition of category algebras,  $x$  can be expressed uniquely as  $\sum_{i=1}^n \lambda_i \alpha_i$  where  $\alpha_i$  are pairwise different morphisms in  $\hat{\mathcal{E}}$  and  $\lambda_i \in k$ . Then  $\varphi(\sum_{i=1}^n \lambda_i \alpha_i) = \sum_{i=1}^n \lambda_i \hat{F}(\alpha_i) = 0$ . Those  $\hat{F}(\alpha_i)$  are probably not pairwise different in  $\mathcal{E}$ . By changing the indices if necessary, we can write the set  $\{\alpha_i\}_{i=1}^n$  as a disjoint union of  $l$  subsets:  $\{\alpha_1, \dots, \alpha_{s_1}\}$ ,  $\{\alpha_{s_1+1}, \dots, \alpha_{s_2}\}$  and so on, until  $\{\alpha_{s_{l-1}+1}, \dots, \alpha_{s_l}\}$  such that two morphisms have the same image under  $\hat{F}$  if and only if they are in the same set.

Now we have:

$$\varphi(x) = (\lambda_1 + \dots + \lambda_{s_1})\hat{F}(\alpha_{s_1}) + \dots + (\lambda_{s_{l-1}+1} + \dots + \lambda_{s_l})\hat{F}(\alpha_{s_l}) = 0.$$

Therefore,

$$\lambda_1 + \dots + \lambda_{s_1} = \dots = \lambda_{s_{l-1}+1} + \dots + \lambda_{s_l} = 0,$$

and hence

$$\begin{aligned} x &= [\lambda_2(\alpha_2 - \alpha_1) + \dots + \lambda_{s_1}(\alpha_{s_1} - \alpha_1)] + \dots \\ &\quad + [\lambda_{s_{l-1}+2}(\alpha_{s_{l-1}+2} - \alpha_{s_{l-1}+1}) + \dots + \lambda_{s_l}(\alpha_{s_l} - \alpha_{s_{l-1}+1})] \end{aligned}$$

is contained in  $U$ .  $\square$

If  $\mathcal{E}$  is a finite free EI category, we can put a *length grading* on its morphisms as follows: automorphisms and unfactorizable morphisms are given grades 0 and 1 respectively; if  $\alpha$  is a factorizable morphism, then it can be expressed (probably not unique) as a composite  $\alpha_n \alpha_{n-1} \dots \alpha_2 \alpha_1$  with all  $\alpha_i$  being unfactorizable and we assign  $\alpha$  grade  $n$ . This grading is well defined by the Unique Factorization Property of finite free EI categories. It is clear that this length grading cannot be applied to an arbitrary finite EI category. We say a finite EI category can be *graded* if this length grading is well defined on it. The following proposition gives us criterions to determine whether an arbitrary finite EI category can be graded.

**Proposition 6.2.** *Let  $\mathcal{E}$  be a finite EI category. Then the following are equivalent:*

- (1)  $\mathcal{E}$  is a graded finite EI category.
- (2) For each factorizable morphism  $\alpha$  in  $\mathcal{E}$ , whenever it has two factorizations  $\alpha_1 \circ \dots \circ \alpha_m$  and  $\beta_1 \circ \dots \circ \beta_n$  into unfactorizable morphisms, we have  $m = n$ .
- (3) Let  $\hat{\mathcal{E}}$  be the free EI cover of  $\mathcal{E}$  and  $\hat{F} : \hat{\mathcal{E}} \rightarrow \mathcal{E}$  be the covering functor. If two morphisms  $\hat{\alpha}$  and  $\hat{\beta}$  in  $\hat{\mathcal{E}}$  have the same image under  $\hat{F}$ , then they have the same length in  $\hat{\mathcal{E}}$ .

*Proof.* It is easy to see that if condition (2) holds, our grading works for  $\mathcal{E}$ , and hence (1) is true. Otherwise, if a factorizable morphism  $\alpha$  has two decompositions  $\alpha_n \circ \dots \circ \alpha_1$  and  $\beta_m \circ \dots \circ \beta_1$  with  $m \neq n$ , then  $\alpha$  should be assigned a grade  $n$  by the first decomposition, and a grade  $m$  by the second decomposition. Thus our grading cannot be applied to  $\mathcal{E}$ . This proves the equivalence of (1) and (2).

Now let  $\alpha$  be an arbitrary morphism in  $\mathcal{E}$  which has two different decompositions  $\alpha_n \circ \dots \circ \alpha_1$  and  $\beta_m \circ \dots \circ \beta_1$  into unfactorizable morphisms. Since  $\hat{\mathcal{E}}$  is the free EI cover of  $\mathcal{E}$ , these unfactorizable morphisms are also unfactorizable morphisms in  $\hat{\mathcal{E}}$ . Let  $\hat{\alpha}$  and  $\hat{\beta}$  be the composite morphisms of these  $\alpha_i$ 's and  $\beta_i$ 's in  $\hat{\mathcal{E}}$  respectively. Thus  $\hat{\alpha} - \hat{\beta}$  is contained in  $U$  since they have the same image  $\alpha$  under  $\hat{F}$ . If (3) is true, then  $m = n$  since  $\hat{\alpha}$  and  $\hat{\beta}$  have lengths  $m$  and  $n$  respectively. Therefore (3) implies (2). We can check that (2) implies (3) in a similar way.  $\square$

The following two lemmas are from [12].

**Lemma 6.3.** *Let  $\mathcal{E}$  be a finite free EI category and  $\alpha : x \rightarrow y$  be an unfactorizable morphism. Define  $H = \text{Aut}_{\mathcal{E}}(y)$  and  $H_0 = \text{Stab}_H(\alpha)$ . If  $|H_0|$  is invertible in  $k$ , then the cyclic module  $k\mathcal{E}\alpha$  is projective.*

*Proof.* This is Lemma 5.2 of [12], where we assumed that the automorphism groups of all objects are invertible in  $k$  but only used the fact that  $|H_0|$  is invertible in  $k$ . Here we give a sketch of the proof. Let  $e = \frac{1}{|H_0|} \sum_{h \in H_0} h$ . Then  $e$  is well defined since  $|H_0|$  is invertible in  $k$ , and is an idempotent in  $k\mathcal{E}$ . Now define a map  $\varphi : k\mathcal{E}e \rightarrow k\mathcal{E}\alpha$  by sending  $re$  to  $r\alpha$  for  $r \in k\mathcal{E}$ . We can check that  $\varphi$  is an  $k\mathcal{E}$ -module isomorphism. Thus  $k\mathcal{E}\alpha$  is projective. See [12] for a detailed proof.  $\square$

**Lemma 6.4.** *Let  $\mathcal{E}$  be a finite free EI category and  $\alpha : x \rightarrow y$  and  $\alpha' : x' \rightarrow y'$  be two distinct unfactorizable morphisms in  $\mathcal{E}$ . Then  $k\mathcal{E}\alpha \cap k\mathcal{E}\alpha' = 0$  or  $k\mathcal{E}\alpha = k\mathcal{E}\alpha'$ .*

*Proof.* This is Lemma 5.1 of [12]. We give a sketch of the proof. Notice that  $k\mathcal{E}\alpha$  is spanned by all morphisms of the form  $\beta\alpha$  with  $\beta : y \rightarrow z$  being a morphism starting at  $y$ . Similarly,  $k\mathcal{E}\alpha'$  is spanned by all morphisms of the form  $\beta'\alpha'$  with  $\beta' : y' \rightarrow z'$  being a morphism starting at  $y'$ . If  $x \neq x'$  or  $y \neq y'$ , then by the Unique Factorization Property of finite free EI categories we conclude that the set  $\mathcal{E}\alpha \cap \mathcal{E}\alpha' = \emptyset$ , and the conclusion follows. If  $x = x'$  and  $y = y'$ , then the set  $\mathcal{E}\alpha$  coincides with the set  $\mathcal{E}\alpha'$  if and only if there is an automorphism  $h \in \text{Aut}_{\mathcal{E}}(y)$  such that  $h\alpha = \alpha'$  again by the UFP. Otherwise, we must have  $\mathcal{E}\alpha \cap \mathcal{E}\alpha' = \emptyset$ . The conclusion follows from this observation.  $\square$

**Remark 6.5.** *The reader can check that the conclusion of Lemma 6.3 is true for any non-isomorphisms  $\alpha$  in  $\mathcal{E}$  by using the UFP. Moreover, a direct check shows that it is also true for automorphisms. Similarly, we can also prove that Lemma 6.4 still holds if we assume that  $\alpha$  and  $\alpha'$  are two morphisms with the same target and source.*

Theorem 5.8 has a corresponding version for finite EI categories.



**Proposition 6.6.** *Let  $\mathcal{E}$  be a graded finite EI category. Then  $k\mathcal{E}$  is a Koszul algebra if and only if  $k\mathcal{E}$  is a quasi-Koszul algebra and  $\mathcal{E}$  is a standardly stratified category (in a sense defined in [20]) with respect to the canonical partial order on  $\text{Ob } \mathcal{E}$ .*

*Proof.* By the decompositions

$$k\mathcal{E}_0 = \bigoplus_{x \in \text{Ob } \mathcal{E}} k\text{Aut}_{\mathcal{E}}(x), \quad k\mathcal{E}_i = \bigoplus_{x \neq y \in \text{Ob } \mathcal{E}} k\mathcal{E}(x, y)_i, \quad i > 0$$

we conclude that all  $k\mathcal{E}_i$  are projective  $k\mathcal{E}_0$ -modules if and only if  $k\mathcal{E}(x, y)_i$  are projective  $k\text{Aut}_{\mathcal{E}}(y)$ -modules for all  $i \geq 0$ ,  $x, y \in \text{Ob } \mathcal{E}$ . Notice that  $k\mathcal{E}(x, y)_i$  is spanned by morphisms from  $x$  to  $y$  with length  $i$ , and these morphisms form several orbits under the action of  $\text{Aut}_{\mathcal{E}}(y)$ . Suppose that there are  $n$  distinct orbits and take a representative  $\alpha_j$  from each orbit. Then we have a decomposition  $k\mathcal{E}(x, y)_i \cong \bigoplus_{j=1}^n k\text{Aut}_{\mathcal{E}}(y)\alpha_j$ . Thus  $k\mathcal{E}(x, y)_i$  is a projective  $k\text{Aut}_{\mathcal{E}}(y)$ -module if and only if each  $k\text{Aut}_{\mathcal{E}}(y)\alpha_j$  is a projective  $k\text{Aut}_{\mathcal{E}}(y)$ -module, and if and only if the stabilizer of  $\alpha_j$  in  $\text{Aut}_{\mathcal{E}}(y)$  has an order invertible in  $k$ . This happens if and only if  $\mathcal{E}$  is standardly stratified by Theorem 2.5 in [20]. In conclusion, all  $k\mathcal{E}_i$  are projective  $k\mathcal{E}_0$ -modules if and only if  $\mathcal{E}$  is standardly stratified in a sense defined in [20].

Notice that  $k\mathcal{E}_0$  is the direct sum of several group algebras, and hence is self-injective. If  $k\mathcal{E}$  is Koszul, then it is quasi-Koszul by Theorem 2.17. Moreover, all  $k\mathcal{E}_i$  are projective  $k\mathcal{E}_0$ -modules (see the last paragraph of Section 2). Therefore,  $\mathcal{E}$  is standardly stratified.

Conversely, if  $k\mathcal{E}$  is standardly stratified, then all  $k\mathcal{E}_i$  are projective  $k\mathcal{E}_0$ -modules. By Corollary 2.18,  $k\mathcal{E}_0$  is a linear  $k\mathcal{E}$ -module if  $k\mathcal{E}$  is quasi-Koszul.  $\square$

In the first paragraph of the above proof we have showed that a graded finite EI category  $\mathcal{E}$  is standardly stratified in the sense of [20] if and only if its  $k$ -linearization as a graded directed category is standardly stratified in our sense. Actually this is still true for an arbitrary finite EI category by comparing Theorem 5.7 in this paper and Theorem 2.5 in [20]. This is not surprising since the  $k$ -linearization of  $\mathcal{E}$  is precisely the associated category of the algebra  $k\mathcal{E}$ .

**Proposition 6.7.** *Let  $\mathcal{E}$  be a finite EI category which might not be graded. Then  $\mathcal{E}$  is standardly stratified if and only if  $M = \bigoplus_{x \in \text{Ob } \mathcal{E}} \text{Aut}_{\mathcal{E}}(x)$  viewed as a  $k\mathcal{E}$ -module has finite projective dimension.*

*Proof.* Consider the  $k$ -linearization of  $\mathcal{E}$ , which is a directed category. By Theorem 5.8 and the remark we made in the paragraph before this proposition, we conclude that  $\text{pd } M < \infty$ .

Conversely, suppose that  $\mathcal{E}$  is not standardly stratified. Then there is a non-isomorphism  $\gamma : t \rightarrow y$  such that the order of  $H_\gamma$  is not invertible in the field  $k$  by Theorem 2.5 in [20], where  $H = \text{Aut}_{\mathcal{E}}(y)$  and  $H_\gamma = \text{Stab}_H(\gamma)$ . For this object  $y$ , define  $S$  to be the set of objects  $w$  such that there is a non-isomorphism  $\beta : w \rightarrow y$  satisfying that  $|H_\beta|$  is not invertible in  $k$ . This set  $S$  is nonempty since  $t \in S$ . It is a poset equipped with the partial order inherited from the canonical partial order on  $\text{Ob } \mathcal{E}$ . Take a fixed object  $z$  which is maximal in this set and define  $I_{>z} = \{x \in \text{Ob } \mathcal{E} \mid x > z\}$ .

By our definition, for an arbitrary object  $x \in I_{>z}$  and a non-isomorphism  $\alpha : x \rightarrow y$  (if it exists), the group  $H_\alpha \leq H$  has an order invertible in  $k$ . Therefore, the  $kH$ -module  $kH\alpha$  is projective. Since the value of  $k\mathcal{E}1_x$  on  $y$  is 0 or is spanned by all non-isomorphisms from  $x$  to  $y$ , and these non-isomorphisms form a disjoint union of  $H$ -orbits, we conclude that the value of  $k\mathcal{E}1_x$  on  $y$  is a projective  $kH$ -module (notice that we always view 0 as a zero projective module). With the same reasoning, we know that the value of  $k\mathcal{E}1_z$  on  $y$  is not a projective  $kH$ -module.

Consider the  $k\mathcal{E}$ -module  $L = k\text{Aut}_{\mathcal{E}}(z)$ . We claim that  $\text{pd } L = \infty$ . If this is true, then  $\text{pd } M = \infty$  since  $L$  is a direct summand of  $M$ . We prove this claim by showing the following statement: for each  $i \geq 1$ , every projective cover of  $\Omega^i(L)$  is supported on  $I_{>z}$ ; the value  $\Omega^i(L)(y)$  of  $\Omega(L)$  on  $y$  is non-zero and is not a projective  $kH$ -module. Clearly,  $\Omega(L)$  is spanned by all non-isomorphisms starting from  $z$  and is supported on  $I_{>z}$ ;  $\Omega(L)(y)$ , spanned by all non-isomorphisms from  $z$  to  $y$ , is non-zero. Moreover,  $\Omega(L)(y)$  coincides with the value of  $k\mathcal{E}1_z$  on  $y$  and is not a projective  $kH$ -module. Therefore our statement is true for  $i = 1$ .

Suppose that this statement is true for  $n$ , and let  $P$  be a projective cover of  $\Omega^n(L)$ . The exact sequence

$$0 \longrightarrow \Omega^{n+1}(L) \longrightarrow P \longrightarrow \Omega^n(L) \longrightarrow 0$$

gives rise to an exact sequence

$$0 \longrightarrow \Omega^{n+1}(L)(y) \longrightarrow P(y) \longrightarrow \Omega^n(L)(y) \longrightarrow 0.$$

Let us focus on the above sequences. Since  $\Omega^n(L)$  is supported on  $I_{>z}$ , so are  $P$  and  $\Omega^{n+1}(L)$ . By the induction hypothesis  $\Omega^n(L)(y) \neq 0$ , Thus  $P(y) \neq 0$ . But  $P$  is supported on  $I_{>z}$ , so  $P \in \text{add}(\bigoplus_{x \in I_{>z}} k\mathcal{E}1_x)$ . Notice that the value of each  $k\mathcal{E}1_x$  on  $y$  is zero or a nontrivial projective  $kH$ -module. Therefore,  $P(y)$  is a projective  $kH$ -module. Again by the induction hypothesis,  $\Omega^n(L)(y)$  is not a projective  $kH$ -module, so  $\Omega^n(L)(y) \not\cong P(y)$ , and  $\Omega^{n+1}(L)(y)$  is non-zero. It cannot be a projective  $kH$ -module. Otherwise,  $\Omega^{n+1}(L)(y)$  is also an injective  $kH$ -module and hence the above sequence splits, so  $\Omega^n(L)(y)$  as a summand of  $P(y)$  is a projective  $kH$ -module, too. But this contradicts the induction hypothesis.

We proved the induction hypothesis for  $\Omega^{n+1}(L)$ . Thus our statement and claim are proved. Consequently,  $\text{pd } M = \infty$ .  $\square$

Now we can prove:

**Theorem 6.8.** *Let  $\mathcal{E}$  be a finite free EI category. Then the following are equivalent:*

- (1)  $\text{pd } k\mathcal{E}_0 \leq 1$ ;
- (2)  $k\mathcal{E}$  is a Koszul algebra;
- (3)  $\mathcal{E}$  is standardly stratified;
- (4)  $\text{pd } k\mathcal{E}_0 < \infty$ .

*Proof.* It is clear that  $\text{pd } k\mathcal{E}_0 = 0$  if and only if  $\mathcal{E}$  is a finite EI category with a single object since we only consider connected categories. In this situation,  $k\mathcal{E} = k\mathcal{E}_0$ ,  $J = 0$ , and all statements are trivially true. Thus without loss of generality we suppose that  $\text{pd } k\mathcal{E}_0 \neq 0$ .

Observe that  $\text{pd } k\mathcal{E}_0 = 1$  if and only if  $\Omega(k\mathcal{E}_0) = J = \bigoplus_{i \geq 1} k\mathcal{E}_i$  is projective. Since  $J$  is spanned by all non-isomorphisms in  $\mathcal{E}$  and each non-isomorphism can be written as a composition of unfactorizable morphisms, it is generated in degree 1. Thus  $k\mathcal{E}_0$  is a linear  $k\mathcal{E}$ -module, and (1) implies (2). By Proposition 6.6, (2) implies (3). The statements (3) and (4) are equivalent by Proposition 6.7.

Now we prove that (3) implies (1). If  $\mathcal{E}$  is standardly stratified, then for every morphism  $\alpha : x \rightarrow y$  in  $\mathcal{E}$ , the order of  $\text{Stab}_H(\alpha)$  is invertible in  $k$ , where  $H = \text{Aut}_{\mathcal{E}}(y)$ . By Lemma 6.4,  $J$  is a direct sum of some  $k\mathcal{E}$ -modules  $k\mathcal{E}\alpha_i$ 's with each  $\alpha_i$  being unfactorizable. By Lemma 6.3, each  $k\mathcal{E}\alpha_i$  is projective. Therefore,  $J$  is also projective, i.e.,  $\text{pd } k\mathcal{E}_0 = 1$ .  $\square$

This theorem and Theorem 5.13 give us a way to construct Koszul algebras in the classical sense. Indeed, let  $\mathcal{E}$  be a standardly stratified finite free EI category and define  $\mathcal{D}$  to be the subcategory formed by removing all non-identity automorphisms.

Then by Theorem 5.13  $k\mathcal{D}$  is a Koszul algebra in the classical sense since  $k\mathcal{E}_0$  is a linear  $k\mathcal{E}$ -module in the generalized sense by the previous theorem.

Let us get more information about the projective resolutions of  $k\mathcal{E}_0$  for arbitrary finite free EI categories. In general,  $\Omega(k\mathcal{E}_0) \cong J = \bigoplus_{i \geq 1} k\mathcal{E}_i$  is not projective, but it is still a direct sum of some  $k\mathcal{E}$ -modules  $k\mathcal{E}\alpha$ 's with each  $\alpha$  being unfactorizable by Lemma 6.4. Thus the projective resolutions of  $k\mathcal{E}_0$  is completely determined by the projective resolutions of those  $k\mathcal{E}\alpha$ 's.

**Lemma 6.9.** *Let  $\mathcal{E}$  be a finite free EI category and  $\alpha : x \rightarrow y$  be an unfactorizable morphism. Grade the  $k\mathcal{E}$ -module  $k\mathcal{E}\alpha$  by putting  $\alpha$  in degree 1, namely,  $(k\mathcal{E}\alpha)_1 = k\text{Aut}_{\mathcal{E}}(y)\alpha$ . Then  $\Omega(k\mathcal{E}\alpha)$  is 0 or is generated in degree 1, and  $\Omega(k\mathcal{E}\alpha)_1 = \Omega(k\mathcal{E}\alpha)(y)$ , the value of  $\Omega(k\mathcal{E}\alpha)$  on  $y$ .*

*Proof.* Let  $H = \text{Aut}_{\mathcal{E}}(y)$  and  $H_0 = \text{Stab}_H(\alpha)$ . If  $|H_0|$  is invertible in  $k$ , then by Lemma 6.3,  $k\mathcal{E}\alpha$  is a projective  $k\mathcal{E}$ -module, so  $\Omega^i(k\mathcal{E}\alpha) = 0$  for all  $i \geq 1$ , in particular  $\Omega(k\mathcal{E}\alpha) = 0$ . The conclusion is trivially true. Thus we only need to deal with the case that  $|H_0|$  is not invertible.

Consider the projective presentation

$$0 \longrightarrow N \longrightarrow k\mathcal{E}1_y[1] \xrightarrow{p} k\mathcal{E}\alpha \longrightarrow 0$$

where  $p$  maps  $1_y$  to  $\alpha$ . Since  $\Omega(k\mathcal{E}\alpha)$  is isomorphic to a direct summand of  $N$ , it is enough to show that  $N$  is generated in degree 1, and  $N_1 = N(y)$ .

Notice that  $k\mathcal{E}1_y$  is spanned by all morphisms in  $\mathcal{E}$  with source  $y$ , and  $k\mathcal{E}\alpha$  is spanned by all morphisms in  $\mathcal{E}$  of the form  $\beta\alpha$  where  $\beta$  is a morphism in  $\mathcal{E}$  with source  $y$ . We claim that  $N$  is spanned by vectors of the form  $\beta_1 - \beta_2$  with  $\beta_1\alpha = \beta_2\alpha$ , where  $\beta_1$  and  $\beta_2$  are two morphisms with source  $y$ .

Clearly, every such difference is contained in  $N$ . Conversely, let  $v \in N$ . Then  $v$  can be written as  $\sum_{i=1}^n \lambda_i \beta_i$  with  $\lambda_i \in k$  and  $\beta_i$  being pairwise different morphisms with source  $y$ . By the definition of  $p$ ,  $\sum_{i=1}^n \lambda_i \beta_i \alpha = 0$ . Those  $\beta_i \alpha$  might not be pairwise different in  $\mathcal{E}$ . Now we apply the same technique used in the proof of Lemma 6.1. By changing the indices if necessary, we can group the same morphisms together and suppose that  $\beta_1 \alpha = \dots = \beta_{s_1} \alpha$ ,  $\beta_{s_1+1} \alpha = \dots = \beta_{s_2} \alpha$  and so on, until  $\beta_{s_{l-1}+1} \alpha = \dots = \beta_{s_l} \alpha$ .

We have:

$$p(v) = (\lambda_1 + \dots + \lambda_{s_1})\beta_{s_1}\alpha + \dots + (\lambda_{s_{l-1}+1} + \dots + \lambda_{s_l})\beta_{s_l}\alpha = 0.$$

Therefore,

$$\lambda_1 + \dots + \lambda_{s_1} = \dots = \lambda_{s_{l-1}+1} + \dots + \lambda_{s_l} = 0,$$

and hence

$$\begin{aligned} v &= [\lambda_2(\beta_2 - \beta_1) + \dots + \lambda_{s_1}(\beta_{s_1} - \beta_1)] + \dots \\ &\quad + [\lambda_{s_{l-1}+2}(\beta_{s_{l-1}+2} - \beta_{s_{l-1}+1}) + \dots + \lambda_{s_l}(\beta_{s_l} - \beta_{s_{l-1}+1})]. \end{aligned}$$

So  $v$  can be written as a sum of these differences.

Now we can prove the lemma. Take an arbitrary object  $z \in \text{Ob } \mathcal{E}$  and consider the value  $N(z)$ . If it is 0, the conclusion holds trivially. Suppose that  $N(z) \neq 0$ . By the above description,  $N(z)$  is spanned by vectors  $\beta_1 - \beta_2$  such that  $\beta_1, \beta_2$  are two morphisms from  $y$  to  $z$ , and  $\beta_1\alpha = \beta_2\alpha$ . By the equivalent definition of UFP described in Remark 6.5, there is an automorphism  $h \in \text{Aut}_{\mathcal{E}}(y)$  such that  $\beta_1 = \beta_2 h$  and  $\alpha = h^{-1}\alpha$ . Therefore  $h\alpha = \alpha$ , and  $1 - h \in N(y)$ . Thus  $\beta_1 - \beta_2 = \beta_2(1 - h) \in k\mathcal{E} \cdot N(y)$ . Since  $z$  is taken to be an arbitrary object,  $N$  is generated by  $N(y)$ , which is clearly equal to  $N_1$ .  $\square$

From this lemma we can get:

**Proposition 6.10.** *Let  $\mathcal{E}$  be a finite free EI category, then  $\text{Ext}_{k\mathcal{E}}^2(k\mathcal{E}_0, k\mathcal{E}_0) = 0$ .*

*Proof.* Since  $\Omega(k\mathcal{E}_0) \cong J$ , it is enough to show  $\text{Ext}_{k\mathcal{E}}^1(J, k\mathcal{E}_0) = 0$ . The conclusion holds trivially if  $J$  is projective. Otherwise, since  $J$  is the direct sum of some  $k\mathcal{E}\alpha$ 's with  $\alpha$  being unfactorizable, by the above lemma we know that  $\Omega J$  is generated in degree 1.

Applying the functor  $\text{Hom}_{k\mathcal{E}}(-, k\mathcal{E}_0)$  to  $0 \rightarrow \Omega J \rightarrow P \rightarrow J \rightarrow 0$  we get

$$0 \rightarrow \text{Hom}_{k\mathcal{E}}(J, k\mathcal{E}_0) \rightarrow \text{Hom}_{k\mathcal{E}}(P, k\mathcal{E}_0) \rightarrow \text{Hom}_{k\mathcal{E}}(\Omega J, k\mathcal{E}_0) \rightarrow \text{Ext}_{k\mathcal{E}}^1(J, k\mathcal{E}_0) \rightarrow 0.$$

Since all modules are generated in degree 1, the sequence

$$(6.1) \quad 0 \rightarrow \text{Hom}_{k\mathcal{E}}(J, k\mathcal{E}_0) \rightarrow \text{Hom}_{k\mathcal{E}}(P, k\mathcal{E}_0) \rightarrow \text{Hom}_{k\mathcal{E}}(\Omega J, k\mathcal{E}_0)$$

is isomorphic to the sequence

$$0 \rightarrow \text{Hom}_{k\mathcal{E}_0}(J_1, k\mathcal{E}_0) \rightarrow \text{Hom}_{k\mathcal{E}_0}(P_1, k\mathcal{E}_0) \rightarrow \text{Hom}_{k\mathcal{E}_0}((\Omega J)_1, k\mathcal{E}_0)$$

obtained by applying the exact functor  $\text{Hom}_{k\mathcal{E}_0}(-, k\mathcal{E}_0)$  to the exact sequence  $0 \rightarrow (\Omega J)_1 \rightarrow P_1 \rightarrow J_1 \rightarrow 0$ . Thus the last map in sequence 6.1 is surjective, so  $\text{Ext}_{k\mathcal{E}}^1(J, k\mathcal{E}_0) = 0$ .  $\square$

The fact that  $\Omega J$  is generated in degree 1 implies  $\text{Ext}_{k\mathcal{E}}^2(k\mathcal{E}_0, k\mathcal{E}_0) = 0$ . Actually the converse statement is also true. Indeed, consider the exact sequence  $0 \rightarrow \Omega J \rightarrow P \rightarrow J \rightarrow 0$ . If  $\text{Ext}_{k\mathcal{E}}^2(k\mathcal{E}_0, k\mathcal{E}_0) = 0$ , applying the exact functor  $\text{Hom}_{k\mathcal{E}_0}(-, k\mathcal{E}_0)$  we get the exact sequence

$$0 \rightarrow \text{Hom}_{k\mathcal{E}_0}(J, k\mathcal{E}_0) \rightarrow \text{Hom}_{k\mathcal{E}_0}(P, k\mathcal{E}_0) \rightarrow \text{Hom}_{k\mathcal{E}_0}(\Omega J, k\mathcal{E}_0) \rightarrow 0,$$

which is isomorphic to

$$0 \rightarrow \text{Hom}_{k\mathcal{E}_0}(J_1, k\mathcal{E}_0) \rightarrow \text{Hom}_{k\mathcal{E}_0}(P_1, k\mathcal{E}_0) \rightarrow \text{Hom}_{k\mathcal{E}_0}(\Omega J/J(\Omega J), k\mathcal{E}_0) \rightarrow 0$$

since both  $J$  and  $P$  are generated in degree 1. Applying the functor  $\text{Hom}_{k\mathcal{E}_0}(-, k\mathcal{E}_0)$  again, we recover  $0 \rightarrow \Omega J/J(\Omega J) \rightarrow P_1 \rightarrow J_1 \rightarrow 0$ . Therefore,  $\Omega J/J(\Omega J) \cong (\Omega J)_1$ , so  $\Omega J$  is generated in degree 1.

Finite free EI categories with quasi-Koszul category algebras have very special homological properties. For example:

**Proposition 6.11.** *Let  $\mathcal{E}$  be a finite free EI category. Then the following are equivalent:*

- (1)  $\text{Ext}_{k\mathcal{E}}^i(k\mathcal{E}_0, k\mathcal{E}_0) = 0$  for all  $i \geq 2$ ;
- (2) for every unfactorizable morphism  $\alpha : x \rightarrow y$  and  $i \geq 0$ , either  $\Omega^i(k\mathcal{E}\alpha)$  are all 0, or they are all generated in degree 1 (in which case it is generated by  $\Omega^i(k\mathcal{E}\alpha)(y)$ );
- (3)  $k\mathcal{E}$  is a quasi-Koszul algebra.

*Proof.* If  $k\mathcal{E}$  is a quasi-Koszul algebra, then

$$\text{Ext}_{k\mathcal{E}}^i(k\mathcal{E}_0, k\mathcal{E}_0) = \text{Ext}_{k\mathcal{E}}^2(k\mathcal{E}_0, k\mathcal{E}_0) \cdot \text{Ext}_{k\mathcal{E}}^{i-2}(k\mathcal{E}_0, k\mathcal{E}_0)$$

for every  $i \geq 2$ . But  $\text{Ext}_{k\mathcal{E}}^2(k\mathcal{E}_0, k\mathcal{E}_0) = 0$  by Proposition 6.10, so (3) implies (1). Clearly, (1) implies (3).

Notice that  $k\mathcal{E}\alpha$  is isomorphic to a direct summand of  $J \cong \Omega(k\mathcal{E}_0)$ . Thus we only need to prove the equivalence of the following two statements:

- (1')  $\text{Ext}_{k\mathcal{E}}^i(J, k\mathcal{E}_0) = 0$  for every  $i \geq 1$ ;
- (2')  $\Omega^i(J) = 0$  or is generated in degree 1 for every  $i \geq 1$ .

Since the technique we use is similar to that in the proof of Proposition 6.10, we only give a sketched proof. In the case that  $J$  is projective, i.e.,  $\mathcal{E}$  is standardly stratified, then (1') and (2') are trivially true, hence they are equivalent. Now suppose that  $J$  is not projective. From the proof of Proposition 6.10 and the paragraph after it we conclude that  $\Omega J$  is generated in degree 1 if and only if

$\text{Ext}_{k\mathcal{E}}^1(J, k\mathcal{E}_0) = 0$ . Replacing  $J$  by  $\Omega J$  (which is also generated in degree 1 either by the induction hypothesis or by the hypothesis  $\text{Ext}_{k\mathcal{E}}^1(J, k\mathcal{E}_0) = 0$ ) and using the same technique, we get  $\Omega^2(J)$  is generated in degree 1 if and only if  $\text{Ext}_{k\mathcal{E}}^2(J, k\mathcal{E}_0) = 0$ . The equivalence of (1') and (2'), and hence the equivalence of (1) and (2), come from induction.  $\square$

The reader may guess that the category algebra of a finite free EI category is always quasi-Koszul in our sense because of the following reasons: Finite free EI categories generalize finite groups and acyclic quivers, for which the associated algebras are all quasi-Koszul; by Proposition 6.6 and Theorem 6.8, for an arbitrary finite EI category  $\mathcal{E}$ ,  $k\mathcal{E}$  is Koszul if  $\mathcal{E}$  is standardly stratified and one of the following condition holds:  $k\mathcal{E}$  is quasi-Koszul, or  $\mathcal{E}$  is a finite free EI category; and we have proved that  $\text{Ext}_{k\mathcal{E}}^2(k\mathcal{E}_0, k\mathcal{E}_0) = 0$  if  $\mathcal{E}$  is a finite free EI category. Unfortunately, this conjecture is false, as shown by the following example.

**Example 6.12.** Let  $\mathcal{E}$  be the following finite EI category where:  $\text{Aut}_{\mathcal{E}}(x) = \langle 1_x \rangle$ ,  $\text{Aut}_{\mathcal{E}}(z) = \langle 1_z \rangle$ ,  $\text{Aut}_{\mathcal{E}}(y) = \langle h \rangle$  is a group of order 2;  $\mathcal{E}(x, y) = \{\alpha\}$ ,  $\mathcal{E}(y, z) = \{\beta\}$  and  $\mathcal{E}(x, z) = \{\beta\alpha\}$ . The reader can check that  $\mathcal{E}$  is a finite free EI category and then the length grading can be applied on it. Let  $k$  be an algebraically closed field with characteristic 2.

$$x \xrightarrow{\alpha} y \xrightarrow{\beta} z.$$

The indecomposable direct summands of  $k\mathcal{E}$  and  $k\mathcal{E}_0$  are:

$$P_x = \begin{matrix} x_0 \\ y_1 \\ z_2 \end{matrix}, \quad P_y = \begin{matrix} y_0 \\ y_0 \\ z_1 \end{matrix}, \quad P_z = z_0, \quad k\mathcal{E}_0 \cong x_0 \oplus z_0 \oplus \begin{matrix} y_0 \\ y_0 \end{matrix}.$$

We use indices to mark the degrees of composition factors. The reader should bear in mind that the two simple modules  $y$  appearing in  $P_y$  have the same degree.

Take the summand  $x_0$  of  $k\mathcal{E}_0$ . By computation, we get

$$\Omega(x_0) = \begin{matrix} y_1 \\ z_2 \end{matrix}, \quad \Omega^2(x_0) = y_1, \quad \Omega^3(x_0) = y_1 \oplus z_2.$$

Applying  $\text{Hom}_{k\mathcal{E}}(-, k\mathcal{E}_0)$  to the exact sequence

$$0 \longrightarrow \Omega^3(x_0) \longrightarrow P_y[1] \longrightarrow \Omega^2(x_0) \longrightarrow 0$$

we get  $\text{Ext}_{k\mathcal{E}}^3(k\mathcal{E}_0, k\mathcal{E}_0) \neq 0$ . Consequently,  $k\mathcal{E}$  is not a quasi-Koszul algebra in our sense by the previous proposition.

We aim to characterize finite free EI categories with quasi-Koszul category algebras. For this goal, we make the following definition:

**Definition 6.13.** Let  $\mathcal{E}$  be a finite EI category. An object  $x \in \text{Ob}\mathcal{E}$  is called *left regular* if for every morphism  $\alpha$  with target  $x$ , the stabilizer of  $\alpha$  in  $\text{Aut}_{\mathcal{E}}(x)$  has an order invertible in  $k$ . Similarly,  $x$  is called *right regular* if for every morphism  $\beta$  with source  $x$ , the stabilizer of  $\beta$  in  $\text{Aut}_{\mathcal{E}}(x)$  has an order invertible in  $k$ .

**Remark 6.14.** We make some comments for this definition.

- (1) If  $x \in \text{Ob}\mathcal{E}$  is maximal, i.e., there is no non-isomorphisms with source  $x$ , then  $x$  is right regular by convention; similarly, if  $x$  is minimal, then it is trivially left regular.
- (2) The category  $\mathcal{E}$  is standardly stratified if and only if every object  $x \in \text{Ob}\mathcal{E}$  is left regular; similarly,  $\mathcal{E}^{\text{op}}$  is standardly stratified if and only if every object  $x \in \text{Ob}\mathcal{E}$  is right regular.

- (3) If  $\mathcal{E}$  is a finite free EI category and  $x \in \text{Ob } \mathcal{E}$ . Then  $x$  is left regular if and only if for every  $\alpha$  with target  $x$ , the  $k\mathcal{E}$ -module  $k\mathcal{E}\alpha$  is a left projective  $k\mathcal{E}$ -module. Similarly,  $x$  is right regular if and only if for every  $\beta$  with source  $x$ , the right  $k\mathcal{E}$ -module  $\beta(k\mathcal{E})$  is a right projective  $k\mathcal{E}$ -module.

**Lemma 6.15.** *Let  $\mathcal{E}$  be a finite free EI category and  $\beta : x \rightarrow y$  be a morphism with  $x \in \text{Ob } \mathcal{E}$  being right regular. Then there exists some idempotent  $e$  in  $k\mathcal{E}$  such that  $\beta(k\mathcal{E}) \cong e(k\mathcal{E})$  as right  $k\mathcal{E}$ -modules by sending  $e$  to  $\beta$ . In particular,  $\beta(kG)\alpha \cong e(kG)\alpha$  as vector spaces for every morphism  $\alpha$  with target  $x$ , where  $G = \text{Aut}_{\mathcal{E}}(x)$ .*

*Proof.* Let  $G_0 = \text{Stab}_G(\alpha)$  and  $e = \sum_{g \in G_0} g/|G_0|$ . This is well defined since  $x$  is right regular. Then we can prove  $\beta(k\mathcal{E}) \cong e(k\mathcal{E})$  as right  $k\mathcal{E}$ -modules in a way similar to the proof of Lemma 6.3. The isomorphism is given by sending  $er$  to  $\beta r$  for  $r \in k\mathcal{E}$ . Since the image of  $e(kG)\alpha \subseteq k\mathcal{E}$  is exactly  $\beta(k\mathcal{E})\alpha$ , we deduce that  $e(kG)\alpha \cong \beta(k\mathcal{E})\alpha$  as vector spaces.  $\square$

Using these concepts, we can get a sufficient condition for the category algebra of a finite free EI category to be quasi-Koszul.

**Theorem 6.16.** *Let  $\mathcal{E}$  be a finite free EI category such that every object  $x \in \text{Ob } \mathcal{E}$  is either left regular or right regular. Then  $k\mathcal{E}$  is quasi-Koszul.*

*Proof.* By the second statement of Proposition 6.11, it is enough to show that for each unfactorizable  $\alpha : x \rightarrow y$  and every  $i \geq 1$ ,  $\Omega^i(k\mathcal{E}\alpha)$  is 0 or generated by  $\Omega^i(k\mathcal{E}\alpha)(y)$ . Let  $H = \text{Aut}_{\mathcal{E}}(y)$  and  $H_0 = \text{Stab}_H(\alpha)$ . If  $|H_0|$  is invertible in  $k$ , then  $k\mathcal{E}\alpha$  is a projective  $k\mathcal{E}$ -module, and the conclusion follows. So we only need to deal with the case that the order of  $H_0$  is not invertible in  $k$ .

By Lemma 6.9,  $\Omega(k\mathcal{E}\alpha)$  is generated in degree 1, or equivalently, generated by its value  $\Omega(k\mathcal{E}\alpha)(y) = 1_y\Omega(k\mathcal{E}\alpha)$  on  $y$ . Now suppose that  $\Omega^i(k\mathcal{E}\alpha)$  is also generated in degree 1, or equivalently, generated by its value  $\Omega^i(k\mathcal{E}\alpha)(y) = 1_y\Omega^i(k\mathcal{E}\alpha)$  on  $y$ , where  $i \geq 1$ . We claim that  $\Omega^{i+1}(k\mathcal{E}\alpha)$  is generated by  $\Omega^{i+1}(k\mathcal{E}\alpha)(y)$ , which is clearly equal to  $\Omega^{i+1}(k\mathcal{E}\alpha)_1$ . If this is true, then conclusion follows from Proposition 6.11.

Take an arbitrary object  $z \in \text{Ob } \mathcal{E}$  such that  $\mathcal{E}(y, z) \neq \emptyset$ . (In the case  $\mathcal{E}(y, z) = \emptyset$ ,  $\Omega^s(k\mathcal{E}\alpha)(z) = 0$  for  $s \geq 0$ , and the claim is trivially true.) The morphisms in  $\mathcal{E}(y, z)$  form a disjoint union of orbits under the right action of  $H$ . By taking a representative  $\beta_i$  from each orbit we have  $\mathcal{E}(y, z) = \bigsqcup_{i=1}^n \beta_i H$ . Since  $|H_0|$  is not invertible,  $y$  is not left regular. By the assumption,  $y$  must be right regular. Therefore, by the previous lemma, for each representative morphism  $\beta_s$ ,  $1 \leq s \leq n$ , there exist some idempotent  $e_i$  such that  $\beta_s(k\mathcal{E}) \cong e_s(k\mathcal{E})$  as right projective  $k\mathcal{E}$ -modules, and  $\beta_s(k\mathcal{E})\alpha \cong e_s(k\mathcal{E})\alpha$  as vector spaces.

Consider the exact sequence

$$0 \longrightarrow \Omega^{i+1}(k\mathcal{E}\alpha) \longrightarrow P^i \longrightarrow \Omega^i(k\mathcal{E}\alpha) \longrightarrow 0,$$

where we assume inductively that  $\Omega^i(k\mathcal{E}\alpha)$  is generated in degree 1, or equivalently generated by its value on  $y$ . Thus  $P^i \in \text{add}(k\mathcal{E}1_y[1])$ . Observe that the segment of a minimal projective resolution of the  $k\mathcal{E}$ -module  $k\mathcal{E}\alpha$

$$P^{i+1} \longrightarrow P^i \longrightarrow \cdots \longrightarrow P^0 \longrightarrow k\mathcal{E}\alpha \longrightarrow 0$$

induces a minimal projective resolution of the  $kH$ -module  $kH\alpha$ :

$$P^{i+1}(y) \longrightarrow P^i(y) \longrightarrow \cdots \longrightarrow P^0(y) \longrightarrow kH\alpha \longrightarrow 0.$$

Thus  $\Omega^j(k\mathcal{E}\alpha)_1 = \Omega^j(k\mathcal{E}\alpha)(y) = \Omega_{kH}^j(kH\alpha)$  for  $1 \leq j \leq i+1$ .

Applying the exact functor  $\text{Hom}_{k\mathcal{E}}(k\mathcal{E}1_y, -)$  to the exact sequence

$$(6.2) \quad 0 \longrightarrow \Omega^{i+1}(k\mathcal{E}\alpha) \longrightarrow P^i \longrightarrow \Omega^i(k\mathcal{E}\alpha) \longrightarrow 0,$$

we get an exact sequence

$$0 \longrightarrow \Omega^{i+1}(k\mathcal{E}\alpha)(y) \longrightarrow 1_y P^i \longrightarrow \Omega^i(k\mathcal{E}\alpha)(y) \longrightarrow 0,$$

which can be identified with

$$0 \longrightarrow \Omega_{kH}^{i+1}(kH\alpha) \longrightarrow P^i(y) \longrightarrow \Omega_{kH}^i(kH\alpha) \longrightarrow 0.$$

Applying the exact functor  $\text{Hom}_{kH}(\bigoplus_{s=1}^n kHe_s, -)$  to the above sequence, we have another exact sequence

$$(6.3) \quad 0 \rightarrow \bigoplus_{s=1}^n e_s \Omega_{kH}^{i+1}(kH\alpha) \rightarrow \bigoplus_{s=1}^n e_s P^i(y) \rightarrow \bigoplus_{s=1}^n e_s \Omega_{kH}^i(kH\alpha) \rightarrow 0.$$

Since  $\Omega^i(k\mathcal{E}\alpha)$  is generated by  $\Omega^i(k\mathcal{E}\alpha)(y) = \Omega_{kH}^i(kH\alpha)$  by the induction hypothesis, the value of  $\Omega^i(k\mathcal{E}\alpha)$  on  $z$  is  $\sum_{s=1}^n \beta_s \cdot \Omega_{kH}^i(kH\alpha)$  (this is well defined as  $\Omega_{kH}^i(kH\alpha) \subseteq (kH)^{\oplus m}$  for some  $m \geq 0$ ). We check that this sum is actually direct by the UFP of  $\mathcal{E}$ . In conclusion,

$$(6.4) \quad \Omega^i(k\mathcal{E}\alpha)(z) = \bigoplus_{s=1}^n \beta_s \cdot \Omega_{kH}^i(kH\alpha) \cong \bigoplus_{s=1}^n e_s \Omega_{kH}^i(kH\alpha).$$

Similarly, the value of  $P^i$  on  $z$  is

$$(6.5) \quad P^i(z) = \bigoplus_{s=1}^n \beta_s \cdot P^i(y) \cong \bigoplus_{s=1}^n e_s P^i(y).$$

Restricted to  $z$ , sequence 6.2 gives rise to

$$(6.6) \quad 0 \longrightarrow \Omega^{i+1}(k\mathcal{E}\alpha)(z) \longrightarrow P^i(z) \longrightarrow \Omega^i(k\mathcal{E}\alpha)(z) \longrightarrow 0.$$

On one hand,  $\bigoplus_{s=1}^n \beta_s \Omega_{kH}^{i+1}(kH\alpha) \subseteq \Omega^{i+1}(k\mathcal{E}\alpha)(z)$ . On the other hand, we have:

$$\begin{aligned} \dim_k \bigoplus_{s=1}^n \beta_s \Omega_{kH}^{i+1}(kH\alpha) &= \dim_k \bigoplus_{s=1}^n e_s \Omega_{kH}^{i+1}(kH\alpha) \quad \text{by Lemma 6.15} \\ &= \dim_k \bigoplus_{s=1}^n e_s P^i(y) - \dim_k \bigoplus_{s=1}^n e_s \Omega_{kH}^i(kH\alpha) \quad \text{by sequence 6.3} \\ &= \dim_k P^i(z) - \dim_k \Omega^i(k\mathcal{E}\alpha)(z) \quad \text{by identities 6.4 and 6.5} \\ &= \dim_k \Omega^{i+1}(k\mathcal{E}\alpha)(z) \quad \text{by sequence 6.6.} \end{aligned}$$

Therefore,  $\Omega^{i+1}(k\mathcal{E}\alpha)(z) = \bigoplus_{s=1}^n \beta_s \Omega_{kH}^{i+1}(kH\alpha) = \bigoplus_{s=1}^n \beta_s \Omega_{k\mathcal{E}}^{i+1}(k\mathcal{E}\alpha)(y)$  since  $\Omega_{kH}^{i+1}(kH\alpha) = \Omega_{k\mathcal{E}}^{i+1}(k\mathcal{E}\alpha)(y)$ . That is, the value of  $\Omega^{i+1}(k\mathcal{E}\alpha)$  on  $z$  is generated by  $\Omega^{i+1}(k\mathcal{E}\alpha)(y)$ . Since  $z$  is arbitrary, our claim holds, and the conclusion follows from induction.  $\square$

## 7. STANDARDLY STRATIFIED ALGEBRAS WITH LINEAR STANDARD MODULES

Theorem 5.10 tells us that the Yoneda category  $E(\mathcal{C}_0)$  of a directed Koszul category  $\mathcal{C}$  is still a directed Koszul category, so is standardly stratified as well. Moreover, the homological dual functor  $E$  interchanges standard modules and indecomposable projective modules. Let  $A$  be a Koszul algebra which is standardly stratified with respect to a poset of orthogonal primitive idempotents  $(\{e_\lambda\}_{\lambda \in \Lambda}, \leq)$ . We may ask a similar question: is the Koszul dual algebra  $\Gamma = \text{Ext}_A^*(A_0, A_0)$  standardly stratified with respect to  $(\{e_\lambda\}_{\lambda \in \Lambda}, \leq)$  (or  $(\{e_\lambda\}_{\lambda \in \Lambda}, \leq^{op})$ ) as well? (Here

we identify the primitive idempotents of  $A$  and  $\Gamma$  in the following way: let  $e$  be a primitive idempotent of  $A$ . Then it is also a primitive idempotent of  $A_0$ . Therefore,  $A_0e$  is a projective  $A_0$ -module, and  $\text{Ext}_A^*(A_0e, A_0)$  is an indecomposable summand of  $\Gamma$ . This summand corresponds to a primitive idempotent of  $\Gamma$ , which we still denote by  $e$ .) This question has been studied in [1] [2] [8] [17] [18]. However, in all these papers  $A_0$  is supposed to be a semisimple algebra. By modifying the technique used in [1], we get a sufficient condition for the Yoneda algebra  $\Gamma$  to be standardly stratified with respect to the opposite order.

Throughout this section  $A$  is a graded finite-dimensional basic  $k$ -algebra with  $A_0$  being self-injective. We choose a complete set of orthogonal primitive idempotents  $\{e_\lambda\}_{\lambda \in \Lambda}$  and let  $\leq$  be a partial order on this set.

**Theorem 7.1.** *If  $A$  is standardly stratified with respect to  $\leq$  such that all standard modules are concentrated in degree 0 and linear. Then  $A_0 \cong \Delta$  and  $\Gamma = \text{Ext}_A^*(A_0, A_0)$  is standardly stratified with respect to the poset  $(\{e_\lambda\}_{\lambda \in \Lambda}, \leq^{op})$ , where  $\Delta$  is the direct sum of all standard modules.*

We show the first statement since it is relatively easier and leave the proof of the second statement to the end of this section. Let  $\Delta_\lambda$  be a standard module with graded projective cover  $P_\lambda$ . Since  $\Delta_\lambda$  is a linear  $A$ -module and concentrated in degree 0,  $\Delta_\lambda = (\Delta_\lambda)_0$  is a projective  $A_0$ -module by Corollary 2.5. The surjection  $P_\lambda \rightarrow \Delta_\lambda$  induces a surjection  $(P_\lambda)_0 \rightarrow \Delta_\lambda$ , so  $\Delta_\lambda$  is a summand of  $(P_\lambda)_0$ , and hence is isomorphic to  $(P_\lambda)_0$  since  $(P_\lambda)_0$  is indecomposable. Put all these standard modules together we find  $A_0 \cong \Delta$ .

Take a minimal element  $\mu \in \Lambda$  and let  $e = e_\mu$ . Let  $\Lambda_1 = \Lambda \setminus \{\mu\}$  and  $\epsilon = \sum_{\lambda \in \Lambda_1} e_\lambda$ . Viewed as an idempotent of  $\Gamma$ ,  $e$  is maximal with respect to  $\leq^{op}$ . The basic idea to prove the second statement is to show that  $\Gamma e \Gamma$  is a projective  $\Gamma$ -module and the quotient algebra  $\Gamma / \Gamma e \Gamma$  is standardly stratified with respect to the poset  $(\{e_\lambda\}_{\lambda \in \Lambda_1}, \leq^{op})$ . Then the conclusion follows from induction.

We collect a list of preliminary results in the following lemmas, where the algebra  $A$  is the same as in Theorem 7.1 if we do not specify it particularly.

**Lemma 7.2.** *The algebra  $\epsilon A \epsilon$  is standardly stratified with respect to the poset  $(\{e_\lambda\}_{\lambda \in \Lambda_1}, \leq)$  and has standard modules  $\epsilon A_0 e_\lambda$ ,  $\lambda \in \Lambda_1$ , which are all concentrated in degree 0. Moreover,*

$$(\epsilon A \epsilon)_0 = \bigoplus_{\lambda \in \Lambda_1} \epsilon A_0 e_\lambda = \epsilon A_0 \epsilon = \epsilon A_0$$

*is a self-injective algebra. If  $M$  is a linear  $A$ -module, then  $\epsilon M$  is a linear  $\epsilon A \epsilon$ -module. In particular, all standard modules of  $\epsilon A \epsilon$  are linear.*

*Proof.* The algebra  $\epsilon A \epsilon$  has projective modules  $\epsilon A e_\lambda$ ,  $\lambda \in \Lambda_1$ . Notice that each  $A e_\lambda$  has a  $\Delta$ -filtration and the standard module  $A_0 e \cong \Delta_\mu$  cannot appear in the filtration since  $e$  is a minimal primitive idempotent. We conclude that  $\epsilon A \epsilon$  has standard modules  $\epsilon \Delta_\lambda \cong \epsilon A_0 e_\lambda$ , and  $\epsilon A \epsilon$  has a filtration formed by  $\epsilon A_0 e_\lambda$ ,  $\lambda \in \Lambda_1$ . This proves the first statement.

Clearly,

$$(\epsilon A \epsilon)_0 = \epsilon A_0 \epsilon = \bigoplus_{\lambda \in \Lambda_1} \epsilon A_0 e_\lambda.$$

We claim  $\epsilon A_0 e = 0$ , which implies  $\epsilon A_0 = \epsilon A_0 \epsilon + \epsilon A_0 e = \epsilon A_0 \epsilon$ . Indeed,

$$\epsilon A_0 e \cong \text{Hom}_{A_0}(A_0 \epsilon, A_0 e) \cong \bigoplus_{\lambda \in \Lambda_1} \text{Hom}_A(\Delta_\lambda, \Delta_\mu) = 0$$

Since  $A$  is standardly stratified and  $\mu$  is minimal in  $\Lambda$ .



By Proposition 2.5 on page 35 of [3], the exact functor  $F = \text{Hom}_{A_0}(A_0\epsilon, -)$  gives an equivalence between a subcategory  $\mathcal{M}$  of  $A_0\text{-mod}$  and the category  $\epsilon A_0\epsilon\text{-mod}$ . Since all projective (injective as well) modules  $A_0e_\lambda$  with  $\lambda \in \Lambda_1$  are contained in  $\mathcal{M}$ , and  $F$  sends these projective (injective, resp) modules to projective (injective, resp) modules of  $\epsilon A_0\epsilon$ . Consequently,  $\epsilon A_0\epsilon = \epsilon A_0$  is self-injective.

Let  $M$  be a linear  $A$ -module and

$$\cdots \longrightarrow P^1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a linear projective resolution of  $M$ . That is, each  $P^i$  is generated in degree  $i$ . Applying the exact functor  $\text{Hom}_A(A\epsilon, -)$  we get a linear projective resolution of  $\epsilon M$  as follows

$$\cdots \longrightarrow \epsilon P^1 \longrightarrow \epsilon P^0 \longrightarrow \epsilon M \longrightarrow 0.$$

Thus  $\epsilon M$  is a linear  $\epsilon A\epsilon$ -module. Since the standard modules of  $\epsilon A\epsilon$  are indecomposable summands of  $\epsilon A_0$ , and  $A_0 = \Delta$  is a linear  $A$ -module, we conclude that every standard module of  $\epsilon A\epsilon$  is a linear  $\epsilon A\epsilon$ -module as well.  $\square$

**Lemma 7.3.** *Let  $M$  be a linear  $A$ -module up to a degree shift. Then*

- (1)  *$M$  has a  $\Delta$ -filtration.*
- (2)  *$\text{Ext}_A^i(M, A_0e) = 0$  for  $i > 0$ .*
- (3)  *$[\Omega^i(M) : \Delta_\mu] = 0$  for each  $i \geq 1$ , where  $[\Omega^i(M) : \Delta_\mu]$  is the number of  $\Delta$ -filtration factors of  $M$  isomorphic to  $\Delta_\mu$ .*
- (4)  *$M = A\epsilon M$  if and only if  $[M : \Delta_\mu] = 0$ .*

*Proof.* Without loss of generality we assume that  $M$  is linear. By Corollary 2.18  $M$  is a projective  $A_0$ -module, and hence has a  $\Delta$ -filtration since  $A_0 = \Delta$ .

To prove the second statement, it is enough to show  $\text{Ext}_A^1(M, A_0e) = 0$ . Indeed, for  $i > 1$ , by Corollary 2.5 and using Lemma 2.10 recursively we have

$$\text{Ext}_A^i(M, A_0e) \cong \text{Ext}_A^{i-1}(\Omega M, A_0e) \cong \cdots \cong \text{Ext}_A^1(\Omega^{i-1}(M), A_0e).$$

Since  $M$  is linear,  $\Omega^{i-1}(M)$  is also linear up to a degree shift. Therefore we can replace  $M$  by  $\Omega^{i-1}(M)$  and use induction.

The projective presentation  $0 \rightarrow \Omega M \rightarrow P \rightarrow M \rightarrow 0$  gives a surjective map  $\text{Hom}_A(\Omega M, A_0e) \rightarrow \text{Ext}_A^1(M, A_0e)$ . The syzygy  $\Omega M$  is linear up to a degree shift and hence has a  $\Delta$ -filtration. Since  $e$  is minimal and the algebra  $A$  is standardly stratified with respect to the poset  $(\{e_\lambda\}_{\lambda \in \Lambda}, \leq)$ , every  $\Delta$ -filtration of  $JP = \bigoplus_{i \geq 1} P_i$  has no factors isomorphic to  $A_0e$ . Therefore, the  $\Delta$ -filtration of  $\Omega M \subseteq \bigoplus_{i \geq 1} P_i$  has no factors isomorphic to  $A_0e$  either. But  $\text{Hom}_A(\Delta_\lambda, \Delta_\mu) = 0$  for  $\lambda \neq \mu$ . Thus  $\text{Hom}_A(\Omega M, \Delta_\mu) = 0$ , so  $\text{Ext}_A^1(M, A_0e) = 0$ . This proves the second statement.

To prove (3), it suffices to show  $[\Omega M : \Delta_\mu] = 0$  and the conclusion comes from induction. But this fact has been established in last paragraph.

Now we prove (4). Notice that  $A\epsilon M$  is the trace of  $A\epsilon$  in  $M$ . If  $[M : \Delta_\mu] = 0$ , then in particular  $M_0 \in \text{add}(A_0\epsilon)$ , and  $M$  is in the trace of  $A\epsilon$  since it is generated in degree 0. Conversely, if  $M = A\epsilon M$ , then  $M$  is in the trace of  $A\epsilon$ , i.e.,  $M$  is a quotient module of some  $(A\epsilon)^{\oplus m}$ . Since  $[A\epsilon : \Delta_\mu] = 0$ , we deduce that  $[M : \Delta_\mu] = 0$ .  $\square$

We define an operator  $\Pi$  on  $A\text{-gmod}$  as follows:

$$\Pi(M) = \begin{cases} A\epsilon M & \text{if } M \neq A\epsilon M, \\ \Omega M & \text{if } M = A\epsilon M. \end{cases}$$

**Lemma 7.4.** *Suppose that  $M$  and  $A\epsilon M$  are linear  $A$ -modules up to a common degree shift. Then*

(1) For all  $i \geq 1$ ,

$$\Pi^i(M) = \begin{cases} \Omega^i(M) & \text{if } M = A\epsilon M, \\ \Omega^{i-1}(A\epsilon M) & \text{if } M \neq A\epsilon M. \end{cases}$$

(2) There is some  $l \in \mathbb{Z}$  such that  $\Pi^l(M) = 0$ .

*Proof.* If  $M = A\epsilon M$ , then  $\Pi(M) = \Omega M$ . By the previous lemma,  $[\Omega M : \Delta_\mu] = 0$ , so  $\Omega M = A\epsilon \Omega M$  and  $\Pi(\Omega M) = \Omega^2(M)$ . Using induction we get  $\Pi^i(M) = \Omega^i(M)$ .

If  $M \neq A\epsilon M$ , then  $\Pi(M) = A\epsilon M$ . Clearly,  $A\epsilon(A\epsilon M) = A\epsilon M$ , so  $\Pi^2(M) = \Omega(A\epsilon M)$  by the definition. Applying statements (3) and (4) of the previous lemma recursively, we get  $\Pi^i(M) = \Omega^{i-1}(A\epsilon M)$ .

To prove (2), it suffices to show that there is a number  $l \in \mathbb{Z}$  such that  $\Pi^l(\Pi(M)) = \Omega^l(\Pi(M)) = 0$ . Since  $\Pi(M) = \Omega M$  or  $\Pi(M) = A\epsilon M$ , both of which are linear up to a degree shift,  $\Pi(M)$  has a  $\Delta$ -filtration by the above lemma. Since  $A$  is standardly stratified and finite-dimensional,  $A_0 \cong \Delta$  has finite projective dimension and every  $\Delta$ -filtration of  $\Pi(M)$  is of finite length. Thus the projective dimension of  $\Pi(M)$  is finite.  $\square$

Recall  $\Gamma = \text{Ext}_A^*(A_0, A_0)$  and the indecomposable summands of  $\Gamma$  are precisely  $\text{Ext}_A^*(A_0 e_\lambda, A_0)$ ,  $\lambda \in \Lambda$ .

**Lemma 7.5.** *Suppose that  $M$  and  $A\epsilon M$  are linear  $A$ -modules up to a common degree shift. Then the trace  $\Gamma e \text{Ext}_A^*(M, A_0)$  of  $\Gamma e$  in  $\text{Ext}_A^*(M, A_0)$  is a projective  $\Gamma$ -module. Moreover,*

$$\dim_k \text{Ext}_A^*(M, A_0) - \dim_k \Gamma e \text{Ext}_A^*(M, A_0) = \dim_k \text{Ext}_{\epsilon A \epsilon}^*(\epsilon M, \epsilon A_0).$$

*Proof.* Without loss of generality we suppose that  $M$  is generated in degree 0 and prove this lemma by induction on the least number  $l$  such that  $\Pi^l(M) = 0$ , which always exists by Lemma 7.4. If  $l = 0$ , then  $M = 0$  and the conclusion holds trivially. Now suppose that it holds for  $l \leq n-1$  and let  $M$  be a linear  $A$ -module for which this least number is  $n$ . There are two cases.

(1). If  $M \neq A\epsilon M$ , then  $\Pi(M) = A\epsilon M$ . This happens if and only if  $[M : \Delta_\mu] \neq 0$ . Since  $e = e_\mu$  is a minimal primitive idempotent, those factors isomorphic to  $\Delta_\mu = A_0 e$  can only appear as direct summands of  $M_0$  by the second statement of Lemma 7.3. Therefore we get an exact sequence  $0 \rightarrow A\epsilon M \rightarrow M \rightarrow (A_0 e)^{\oplus a} \rightarrow 0$ , where all terms are linear by our assumption. By Proposition 2.11, this sequence gives rise to:

$$0 \rightarrow \text{Ext}_A^*((A_0 e)^{\oplus a}, A_0) \rightarrow \text{Ext}_A^*(M, A_0) \rightarrow \text{Ext}_A^*(A\epsilon M, A_0) \rightarrow 0,$$

that is:

$$(7.1) \quad 0 \longrightarrow (\Gamma e)^{\oplus a} \longrightarrow \text{Ext}_A^*(M, A_0) \xrightarrow{p} \text{Ext}_A^*(A\epsilon M, A_0) \longrightarrow 0.$$

By taking the trace of  $\Gamma e$  in those terms appearing in this exact sequence, we get another exact sequence:

$$(7.2) \quad 0 \longrightarrow (\Gamma e)^{\oplus a} \longrightarrow \Gamma e \text{Ext}_A^*(M, A_0) \longrightarrow \Gamma e \text{Ext}_A^*(A\epsilon M, A_0) \longrightarrow 0.$$

Notice that  $A\epsilon M$  is linear,  $A\epsilon A\epsilon M = A\epsilon M$ , and  $\Pi^{n-1}(A\epsilon M) = \Pi^n(M) = 0$ . By the induction hypothesis,  $\Gamma e \text{Ext}_A^*(A\epsilon M, A_0)$  is a projective  $\Gamma$ -module. Then the above sequence splits and  $\Gamma e \text{Ext}_A^*(M, A_0)$  is a projective  $\Gamma$ -module as well.

Comparing sequence 7.1 to sequence 7.2, we get:

$$\begin{aligned} & \dim_k \text{Ext}_A^*(M, A_0) - \dim_k \Gamma e \text{Ext}_A^*(M, A_0) \\ &= \dim_k \text{Ext}_A^*(A\epsilon M, A_0) - \dim_k \Gamma e \text{Ext}_A^*(A\epsilon M, A_0) \\ &= \dim_k \text{Ext}_{\epsilon A \epsilon}^*(\epsilon A\epsilon M, \epsilon A_0) \end{aligned}$$

by the induction hypothesis. But  $\epsilon A \epsilon M = \epsilon M$ . Thus the conclusion is true for  $M$  by induction.

(2). If  $M = A \epsilon M$ , i.e.,  $[M : \Delta_\mu] = 0$ , then  $\Pi(M) = \Omega M$ . Let  $P$  be a graded projective cover of  $M$ . Then  $P \in \text{add}(A \epsilon)$  and  $P_0 \in \text{add}(A_0 \epsilon)$ . Since  $M$  is linear,  $M_0 \cong P_0$ , so  $\text{Ext}_A^i(M, A_0) = \text{Ext}_A^{i-1}(\Omega M, A_0)$  for  $i > 0$  by Lemma 2.10, and

$$\text{Hom}_A(M, A_0) \cong \text{Hom}_{A_0}(M_0, A_0) \cong \text{Hom}_{A_0}(P_0, A_0) \cong \text{Hom}_A(P_0, A_0).$$

Therefore, the following exact sequence:

$$0 \longrightarrow \bigoplus_{i \geq 1} \text{Ext}_A^i(M, A_0) \longrightarrow \text{Ext}_A^*(M, A_0) \longrightarrow \text{Hom}_A(M, A_0) \longrightarrow 0$$

is isomorphic to

$$(7.3) \quad 0 \longrightarrow \text{Ext}_A^*(\Omega M, A_0) \longrightarrow \text{Ext}_A^*(M, A_0) \longrightarrow \text{Hom}_A(P_0, A_0) \longrightarrow 0.$$

Applying the exact functor  $\text{Hom}_\Gamma(\Gamma e, -)$  to the above sequence, we get

$$0 \rightarrow e \text{Ext}_A^*(\Omega M, A_0) \rightarrow e \text{Ext}_A^*(M, A_0) \rightarrow e \text{Hom}_A(P_0, A_0) \rightarrow 0.$$

Notice that  $e \text{Hom}_A(P_0, A_0) = 0$  since  $A_0 e = \Delta_\mu$ ,  $P_0 \in \text{add}(A_0 \epsilon)$  and  $[A_0 \epsilon : \Delta_\mu] = 0$ . Thus  $e \text{Ext}_A^*(M, A_0) \cong e \text{Ext}_A^*(\Omega M, A_0)[-1]$  (as graded  $\Gamma$ -modules) and

$$(7.4) \quad \Gamma e \text{Ext}_A^*(M, A_0) \cong \Gamma e \text{Ext}_A^*(\Omega M, A_0)$$

Since  $[\Omega M : \Delta_\mu] = 0$ , by Lemma 7.3,  $A \epsilon \Omega M = \Omega M$ . So  $\Gamma e \text{Ext}_A^*(\Omega M, A_0)$  is a projective  $\Gamma$ -module by the induction hypothesis on  $\Pi(M) = \Omega M$ , and  $\Gamma e \text{Ext}_A^*(M, A_0)$  is projective as well.

The exact sequence  $0 \rightarrow \Omega M \rightarrow P \rightarrow M \rightarrow 0$  gives the exact sequence  $0 \rightarrow \epsilon \Omega M \rightarrow \epsilon P \rightarrow \epsilon M \rightarrow 0$ . Clearly,  $\epsilon P$  is a projective cover of  $\epsilon M$ , so  $\epsilon \Omega M = \Omega(\epsilon M)$ . Since  $\epsilon M$  is a linear  $\epsilon A \epsilon$ -module by Lemma 7.2, we have an exact sequence similar to (7.3):

$$(7.5) \quad 0 \rightarrow \text{Ext}_{\epsilon A \epsilon}^*(\Omega(\epsilon M), \epsilon A_0) \rightarrow \text{Ext}_{\epsilon A \epsilon}^*(\epsilon M, \epsilon A_0) \rightarrow \text{Hom}_{\epsilon A \epsilon}(\epsilon P_0, \epsilon A_0) \rightarrow 0.$$

Therefore,

$$\begin{aligned} & \dim_k \text{Ext}_A^*(M, A_0) - \dim_k \Gamma e \text{Ext}_A^*(M, A_0) \quad \text{by (7.3) and (7.4)} \\ &= \dim_k \text{Ext}_A^*(\Omega M, A_0) + \dim_k \text{Hom}_A(P_0, A_0) - \dim_k \Gamma e \text{Ext}_A^*(\Omega M, A_0) \\ &= \dim_k \text{Ext}_{\epsilon A \epsilon}^*(\epsilon \Omega M, \epsilon A_0) + \dim_k \text{Hom}_A(P_0, A_0) \quad \text{by induction on } \Omega M \\ &= \dim_k \text{Ext}_{\epsilon A \epsilon}^*(\epsilon M, \epsilon A_0) + \dim_k \text{Hom}_A(P_0, A_0) - \dim_k \text{Hom}_{\epsilon A \epsilon}(\epsilon P_0, \epsilon A_0), \end{aligned}$$

where the last identity comes from (7.5).

We establish the identity  $\dim_k \text{Hom}_A(P_0, A_0) = \dim_k \text{Hom}_{\epsilon A \epsilon}(\epsilon P_0, \epsilon A_0)$  and finish the proof by induction. Take an arbitrary indecomposable summand of  $P_0$ , which is isomorphic to a certain  $A_0 e_\lambda$ . Since  $A \epsilon M = M$ ,  $P_0 \cong M_0$  has no summands isomorphic to  $A_0 e$ . Therefore  $\lambda \in \Lambda_1$  and  $e_\lambda \epsilon = e_\lambda$ . Now

$$\begin{aligned} \text{Hom}_A(A_0 e_\lambda, A_0) &\cong \text{Hom}_{A_0}(A_0 e_\lambda, A_0) \cong e_\lambda A_0 = e_\lambda \epsilon A_0 \\ &\cong \text{Hom}_{\epsilon A_0 \epsilon}(\epsilon A_0 e_\lambda, \epsilon A_0) \cong \text{Hom}_{\epsilon A \epsilon}(\epsilon A_0 e_\lambda, \epsilon A_0). \end{aligned}$$

Since  $P_0$  is a direct sum of these summands, the identity holds.  $\square$

Now we can prove the second statement of Theorem 7.1.

*Proof.* We use induction on the size of the poset  $\Lambda$ . If this number is 1, then  $A$  and  $\Gamma = \Lambda^{op}$  both are local algebra. Clearly  $\Gamma$  is standardly stratified. Suppose that the conclusion is true for posets with sizes at most  $m - 1$  and let  $\Lambda$  be an poset with  $m$  elements. Take  $e$  be a minimal idempotent and define  $\epsilon, \Lambda_1$  as before.

Let  $M = A_0$  in the previous lemma. We conclude that  $\Gamma e \Gamma$  is a projective  $\Gamma$ -module. By Lemma 7.2  $\epsilon A \epsilon$  satisfies all the conditions in the theorem and  $\Lambda_1$  has only  $m-1$  elements. Therefore, by the induction hypothesis,  $\Gamma' = \text{Ext}_{\epsilon A \epsilon}^*(\epsilon A_0, \epsilon A_0)$  is standardly stratified with respect to the poset  $(\{e_\lambda\}_{\lambda \in \Lambda_1}, \leq^{op})$ . Thus it is enough to show  $\Gamma/\Gamma e \Gamma \cong \Gamma'$ .

There is an algebra homomorphism  $\varphi : \Gamma \rightarrow \Gamma'$  induced by the exact functor  $F = \text{Hom}_A(A\epsilon, -)$  in the following way:  $\varphi$  sends an  $n$ -fold exact sequence

$$0 \longrightarrow A_0 \longrightarrow M^n \longrightarrow \cdots \longrightarrow M^1 \longrightarrow A_0 \longrightarrow 0$$

representing an element  $g \in \Gamma_n$  to an  $n$ -fold exact sequence

$$0 \longrightarrow \epsilon A_0 \longrightarrow \epsilon M^n \longrightarrow \cdots \longrightarrow \epsilon M^1 \longrightarrow \epsilon A_0 \longrightarrow 0$$

representing an element  $g' \in \Gamma'_n$  for all  $n \geq 0$ . Every element  $x \in (\Gamma e)_n = \text{Ext}_A^n(A_0 e, A_0)$  can be represented by an exact sequence:

$$0 \longrightarrow A_0 \longrightarrow M^n \longrightarrow \cdots \longrightarrow M^1 \longrightarrow A_0 e \longrightarrow 0.$$

Since  $\text{Hom}_A(A\epsilon, -)$  sends  $A_0 e$  to  $\text{Hom}_A(A\epsilon, A_0 e) = 0$ ,  $\varphi$  maps every  $(\Gamma e)_n$  and hence  $\Gamma e$  to 0. Thus the ideal  $\Gamma e \Gamma$  generated by  $\Gamma e$  is also sent to 0, and  $\varphi$  gives rise to an algebra homomorphism  $\bar{\varphi} : \Gamma/\Gamma e \Gamma \rightarrow \Gamma'$ .

Clearly,  $\varphi$  maps  $\Gamma_0 = \text{End}_{A_0}(A_0)$  onto  $\text{End}_{\epsilon A \epsilon}(\epsilon A_0) = \epsilon \Gamma_0 \epsilon$ . Moreover, we have

$$\text{Ext}_A^1(A_0, A_0) \cong \text{Hom}_A\left(\bigoplus_{i \geq 1} A_i, A_0\right) \cong \text{Hom}_{A_0}(A_1, A_0) \cong \text{Hom}_{\epsilon A_0 \epsilon}(\epsilon A_1, \epsilon A_0),$$

where the last isomorphism is induced by the functor  $F$  (see the last paragraph of the proof of Lemma 7.5). But

$$\text{Hom}_{\epsilon A_0 \epsilon}(\epsilon A_1, \epsilon A_0) \cong \text{Hom}_{\epsilon A \epsilon}\left(\bigoplus_{i \geq 1} \epsilon A_i, \epsilon A_0\right) \cong \text{Ext}_{\epsilon A \epsilon}^1(\epsilon A_0, \epsilon A_0).$$

Thus, the homomorphism  $\varphi$  induced by  $F$  maps  $\text{Ext}_A^1(A_0, A_0)$  to  $\text{Ext}_{\epsilon A \epsilon}^1(\epsilon A_0, \epsilon A_0)$  surjectively. Since  $A_0$  ( $\epsilon A_0$ , resp.) is a linear  $A$ -module ( $\epsilon A \epsilon$ -module, resp.), both  $\Gamma = \text{Ext}_A^*(A_0, A_0)$  and  $\Gamma' = \text{Ext}_{\epsilon A \epsilon}^*(\epsilon A_0 \epsilon, \epsilon A_0 \epsilon)$  are generated in degree 0 and degree 1 as algebras. Therefore, the map  $\varphi : \Gamma \rightarrow \Gamma'$  is a surjective algebra homomorphism, so  $\bar{\varphi} : \Gamma/\Gamma e \Gamma \rightarrow \Gamma'$  is surjective as well.

Let  $M = A_0$  in the previous lemma. We get  $\dim_k \Gamma' = \dim_k \Gamma - \dim_k \Gamma e \Gamma$ . Therefore, as a surjective homomorphism between two  $k$ -algebras with the same dimension,  $\bar{\varphi}$  must be an isomorphism. This completes the proof.  $\square$

There is a similar conclusion in the case that  $A$  is a *quasi-hereditary* algebra, the definition for which can be found in [7].

**Corollary 7.6.** *Let  $A$  be the same as in the previous theorem. If  $A$  is quasi-hereditary with respect to the poset  $(\{e_\lambda\}_{\lambda \in \Lambda}, \leq)$ , then  $\Gamma = \text{Ext}_A^*(A_0, A_0)$  is also quasi-hereditary with respect to the poset  $(\{e_\lambda\}_{\lambda \in \Lambda}, \leq^{op})$ .*

*Proof.* We already proved that  $\Gamma$  is standardly stratified with respect to the poset  $(\{e_\lambda\}_{\lambda \in \Lambda}, \leq^{op})$ . It suffices to check that the ideal  $\Gamma e \Gamma$  is a hereditary ideal, i.e., the endomorphism algebra of  $\Gamma e$  is one-dimensional.

Since  $e$  is maximal as an idempotent of  $\Gamma$ , the standard  $\Gamma$ -module corresponding to  $e$  is exactly  $\Gamma e$ . Thus

$$\text{Hom}_\Gamma(\Gamma e, \Gamma e) \cong e \Gamma e \cong \text{Ext}_A^*(A_0 e, A_0 e) = \text{Hom}_A(A_0 e, A_0 e)$$

where the last identity follows from (2) of Lemma 7.3. Since  $A_0 e$  is a standard module of the quasi-hereditary algebra  $A$ ,  $\text{Hom}_A(A_0 e, A_0 e) \cong k$ . The conclusion follows from induction.  $\square$

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MN, 55455, USA  
 E-mail address: lixxx480@math.umn.edu