The classification theorem for groups of homeomorphisms of the line. Nonamenability of Thompson's group F. *

L. A. Beklaryan
Moscow Institute of Physics and Technology,
Central Economics and Mathematics Institute
Moscow, Russia
beklar@cemi.rssi.ru, beklaryan@mailfrom.ru

This paper allows one to obtain a criterion for the existence of a projectively invariant measure formulated in terms of combinatorial properties of a group (amenability of some canonical quotient group). Such necessary and sufficient condition is a basis of the classification scheme for groups of homeomorphisms of the line. In particular, a nonamenability of Thompson's group F follows from the obtained criterion.

Key words: groups of homeomorphisms of the line (circle), projectively invariant measure, Stone-Čech compactification, amenability.

Introduction

Groups of homeomorphisms of the line (circle) occur in studying various problems in geometry, groups of quasiconformal mapping, functional differential equations, wave theory, calculus of variations etc.[1],[2]-[3], [4], [5]-[9], [10]. They can be classified on the basis of various characteristics. As such characteristics it is possible to consider an amenability property, a growth function of a group (for finitely generated groups), a structure of the orbits (for groups of homeomorphisms of locally compact space) etc. For groups of homeomorphisms of the line the most detailed classification is based on the characteristics of a series of metric invariants [12]-[15]. Therefore, it is important to have criteria for the existence of such invariants in various terms of an initial group such as topological characteristics, combinatorial characteristics, characteristics of canonical subgroups. In [12] the criterion for the existence of an invariant measure was obtained in terms of topological characteristics of an initial group of homeomorphisms of the line. In [14] the criterion for the existence of a projectively invariant measure was obtained in

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terms of properties of canonical subgroups of an initial group of homeomorphisms of the line. In [15] the criterion for the existence of an ω-projectively invariant measure was obtained in terms of topological properties of an initial group of homeomorphisms of the line. In [11] the criterion for the existence of an invariant measure for groups of homeomorphisms of the circle was obtained and formulated on the basis of combinatorial characteristics of a group (lack of free subgroups with two generators for some canonical quotient group). In the same paper there is a reformulation of the noted criterion by dint of the other combinatorial properties of a group. Though, classification possibilities of such criterion appear limited.

In the present paper the criterion for the existence of a projectively invariant measure is obtained (Theorem A) on the basis of combinatorial characteristics of an initial group of homeomorphisms of the line (amenability of some canonical quotient group). There is a reformulation of the noted criterion in this paper (Theorem B). This criterion is an elaboration of previously obtained criterion for a wider class of groups. Expansion of the class of groups requires some weakening of conditions of the criterion. In this case the condition of the existence of a free subgroup with two generators is replaced by the condition of nonamenability. Such criterion is the most important classification instrument and it is a basis of the classification scheme for groups of homeomorphisms of the line. It is proved that there is not a projectively invariant measure for Thompson's group F, that entails, owing to offered criterion, a nonamenability of this group (Theorem D). An example is given in the form of Brin's group $\mathcal{B}$, which is amenable group. This example shows that the conditions, stated in Theorem B, are precise.

For implementation of the mentioned tasks there is obtained a new criterion for the existence of a projectively invariant measure, which is formulated as a property of the commutator. There is a reformulation of this criterion in terms of the graphs of homeomorphisms belonging to an initial group (§2). It is shown that in case of absence of a projectively invariant measure an initial group contains a special subgroup $\Lambda =< p, q >$ with two generators, where one of the generators p is a freely acting homeomorphism (§5). By using a property of the orbits, it is constructed a special induced representation for the right-invariant mean on the canonical quotient group of homeomorphisms of the line. For this representation we define and study singular right-invariant means. For groups, which contain a freely acting homeomorphism, right-invariant mean is singular (§3). For their studying we use the Stone-Čech compactification of a canonical quotient group. Singular functionals are permanent functionals, therefore, in the integral representation of the right-invariant mean a support of an invariant probability measure belongs to the remainder of the Stone-Čech compactification (§4). The absence of a projectively invariant measure and, consequently, the existence of a special subgroup with two generators allows us to construct a certain map of "transposition" on the line. Such transposition map induces a continuous mapping of involution in the Stone-Čech compactification of a canonical quotient group, with the remainder goes into the remainder. We study the structure of the remainder of the Stone-Čech compactification of the canonical quotient group Λ/H_Λ with respect to the action of the shift $[p^{-1}]$ on this extension, and, in particular, an existence of an invariant set for this action (§5). From the existence of such involution map it follows that the singular right-invariant mean is simultaneously nonsingular. This contradiction proves the nonamenability of the canonical quotient group in case of the absence of a projectively invariant measure (§6).

§1. Preliminaries

Through $Homeo_+(\mathbb{X})$, $\mathbb{X} = \mathbb{R}$, $\mathbb{S}^1$ we denote a group of all orientation-preserving homeomorphisms of $\mathbb{X}$. Let's define an important canonical subset G^S of the group G as a union of stabilizers

$$G^S = \bigcup_{t \in \mathbb{X}} St_{\,G(t)}.$$

The set G^S is not obliged to be a group. The following embeddings are obvious

$$G^S \subseteq < G^S > \subseteq G. \tag{1}$$

It turns out that such chain of embeddings is characterized by extreme property, which is described in the following lemma.

Lemma 1 ([12]). *Let $G \subseteq Homeo_+(\mathbb{R})$. Then either $G^S =< G^S >$ or $< G^S >= G$.* ■

The alternative, formulated in this lemma, is not strict as there are groups, for which equality $G^S = G$ is true. The given lemma underlies the important theorem about a quotient group.

Theorem 1 ([12]). *Let $G \subseteq Homeo_+(\mathbb{X})$. Then the quotient group $G/{<}G^S{>}$ is commutative and isomorphic to some subgroup of the additive group of $\mathbb{X}$.* ■

Remark 1. *Let $G \subseteq Homeo_+(\mathbb{X})$. If $(G/{<}G^S{>}) \neq < e >$, then in the group G there is a freely acting element.* ■

The theorem of the structure of a quotient group is determining in the study of groups of homeomorphisms of the line (circle). At the same time, such characteristic, as the quotient group $G/{<}G^S{>}$, cannot be a universal classification instrument for groups of homeomorphisms of the line, since this quotient group is nontrivial only for groups with an invariant measure (Theorem 9). In particular, for the solvable group $G =< t+1, 2\,t >$ with two generators the quotient group $G/{<}G^S{>}$ is trivial and does not carry any new information about the initial group G.

1.1. Minimal sets.

For the group $G \subseteq Homeo_+(\mathbb{X})$ its minimal set is an important topological characteristic.

Definition 1. *A minimal set of the group $G \subseteq Homeo(\mathbb{X})$ it is a closed G-invariant subset of the $\mathbb{X}$, which does not contain proper closed G-invariant subsets. If there is not a nonempty minimal set then, by definition, we assume that the minimal set is empty.* ■

An importance of the minimal sets is determined by the fact that if they are nontrivial then the orbits of the points have certain canonical properties, and the minimal sets themselves determine the supports of the metric invariants.

Another important topological characteristic is a set:

$$Fix\, G^S = \{t \in \mathbb{X} : \quad \forall g \in G^S, \quad g(t) = t\}.$$

Theorem 2 ([14])**.** *Let $G \subseteq Homeo_+(\mathbb{R})$. Then one of the following mutually exclusive statements is true:*

a) any minimal set is discrete and belongs to the set $Fix\ G^S$, and the set $Fix\ G^S$ consists of the union of the minimal sets;

b) a minimal set is a perfect nowhere dense subset of the $\mathbb{R}$. In that case, it is a unique minimal set, and it is contained in a closure of the orbit $\overline{G(t)}$ of an arbitrary point $t \in \mathbb{R}$;

c) a minimal set coincides with $\mathbb{R}$;

d) a minimal set is empty. ■

A nondiscrete minimal set of the group G (such set is unique) we denote through $E(G)$.

Let's formulate a criterion of nonemptiness of a minimal set. First let's define a set

$$G^S_\infty = \{g \in G^S : \quad \sup\{t : g(t) = t\} = +\infty, \quad \inf\{t : g(t) = t\} = -\infty\}.$$

Proposition 1 ([14],[15])**.** *Let $G \subseteq Homeo_+(\mathbb{R})$ and at least one of the conditions is fulfilled:*

a) G is a finitely generated group;

b) $Fix\ G^S \neq \emptyset$,

c) $G \neq G^S_\infty$.

Then there is a nonempty minimal set. ■

Theorem 3 ([15])**.** *Let $G \subseteq Homeo_+(\mathbb{X})$. If for the subgroup $\Gamma \subseteq G$ the minimal set $E(\Gamma)$ is neither empty nor discrete then $E(\Gamma) \subseteq E(G)$.* ■

Let's give an amplification of the last theorem for the case of the normal subgroups.

Theorem 4 ([15])**.** *Let $G \subseteq Homeo_+(\mathbb{X})$. If for the normal subgroup $\Gamma \subseteq G$ the minimal set $E(\Gamma)$ is neither empty nor discrete then it coincide with minimal set of the initial group G, i.e. $E(\Gamma) = E(G)$.* ■

As shown, the minimal sets of normal subgroups of the initial group $G \subseteq Homeo_+(\mathbb{X})$ have an *extremal property*: they are either discrete (possibly empty) or coincide with a minimal set of the group G.

By Theorem 4, the normal subgroup Γ and the initial group G have the same topological complexity. This fact is especially useful when we need to reduce an algebraic complexity with saving all topological properties.

There is another very important question: Do the minimal sets have an inheritance property? The answer to this question is the following theorem.

Theorem 5. *Let $G \subseteq Homeo_+(\mathbb{X})$ with nonempty minimal set. Then for the normal subgroup $\Gamma \subseteq G$ its minimal set is also nonempty.*

Proof.
(i) The first case. Let $Fix\ \Gamma^S \neq \emptyset$. Then, by Proposition 1, the minimal set of the subgroup Γ is nonempty.
(ii) The second case. Let $Fix\ \Gamma^S = \emptyset$. Then the condition $Fix\ G^S = \emptyset$ is fair, and for such group G the minimal set is not discrete. It is obvious, that for the subgroup Γ the condition $\Gamma \neq < e >$ is fair. Owing to Proposition 1, in case $\Gamma \neq \Gamma^S_\infty$ the minimal set of the subgroup Γ is also nonempty.

Therefore, it remains to consider the case when $\Gamma = \Gamma^S_\infty$. Let's consider the closed interval I with the property $I \cap E(G) \neq \emptyset$. If we show that for an arbitrary point $\bar{t} \in \mathbb{R}$ the orbit $\Gamma(\bar{t})$ satisfies the condition

$$I \cap \Gamma(\bar{t}) \neq \emptyset, \tag{2}$$

then it implies the nonemptiness of the minimal set of the subgroups Γ.

Let's consider a point $\tau \in E(G)$. There is an element $\hat{\gamma} \in \Gamma$ which satisfies following conditions

$$\tau \in (t_1,\, t_2), \quad \hat{\gamma}(t_1) = (t_1), \quad \hat{\gamma}(t_2) = (t_2), \qquad \hat{\gamma}(t) > t, \quad t \in (t_1,\, t_2). \tag{3}$$

As I, we take the interval $[t_1,\, t_2]$. Let's consider an arbitrary point $\bar{t} \in \mathbb{R}$. If $\bar{t} \in I$ then this point satisfies the condition (2).

Let $\bar{t} \notin I$. If $\bar{t} > t_2$ then we consider a family P^+ of elements of the group Γ

$$P^+ = \{\gamma :\ \gamma = g\hat{\gamma}g^{-1}, \quad g \in G, \quad g(t_1) > t_1\} \tag{4}$$

and a corresponding family of open intervals

$$\mathbb{P}^+ = \{(g(t_1),\, g(t_2)) :\ g \in G, \quad g(t_1) > t_1\}. \tag{5}$$

Each element $\gamma \in P^+$ on the corresponding interval $(g(t_1),\, g(t_2))$ satisfies following conditions

$$\gamma(g(t_1)) = g(t_1), \quad \gamma(g(t_2)) = g(t_2), \quad \gamma(t) > t, \quad t \in (g(t_1),\, g(t_2)). \tag{6}$$

Obviously, the family of open intervals $\mathbb{P}^+$ is a covering for a closed interval $[t_2,\, \bar{t}]$. From it we can select a finite subcover $(g_j(t_1),\, g_j(t_2)), \quad j = 1, ..., K$. It is not difficult to see, that an element $\bar{\gamma} \in \Gamma$ with a property $\bar{\gamma}^{-1}(\bar{t}) \in I$ can be formed using the elements $\gamma_j = g_j \hat{\gamma} g_j^{-1}, \quad j = 1, ..., K$, that is for the point $\bar{t}$ the condition (2) is satisfied.

If $\bar{t} < t_1$ then we consider a family P^- of elements of the subgroup Γ

$$P^- = \{\gamma: \ \gamma = g\hat{\gamma}g^{-1}, \quad g \in G, \quad g(t_2) < t_2\} \tag{7}$$

and a corresponding family of open intervals

$$\mathbb{P}^- = \{(g(t_1),\, g(t_2)): \ g \in G, \quad g(t_2) < t_2\}. \tag{8}$$

Each element $\gamma \in P^-$ on the corresponding interval $(g(t_1),\, g(t_2))$ satisfies following conditions

$$\gamma(g(t_1)) = g(t_1), \quad \gamma(g(t_2)) = g(t_2), \quad \gamma(t) > t, \quad t \in (g(t_1),\, g(t_2)). \tag{9}$$

Obviously, the family of open intervals $\mathbb{P}^-$ is a covering for a closed interval $[\bar{t},\, t_1]$. From it we can select a finite subcover $(g_j(t_1),\, g_j(t_2)), \quad j = 1, ..., K$. It is not difficult to see, that an element $\bar{\gamma} \in \Gamma$ with a property $\bar{\gamma}^{-1}(\bar{t}) \in I$ can be formed using the elements $\gamma_j = g_j \hat{\gamma} g_j^{-1}, \quad j = 1, ..., K$, that is for the point $\bar{t}$ the condition (2) is satisfied. The theorem is proved. □

Let's define a very important canonical subgroup of an initial group related with a minimal set.

Definition 2. *For the group $G \subseteq Homeo_+(\mathbb{X})$ the normal subgroup H_G is defined as follows:*

1) if the minimal set is neither empty nor discrete then

$$H_G = \{h \in G: \quad E(G) \subseteq Fix \ < h >\};$$

2) if the minimal set is nonempty and discrete then $H_G = G^S$
(from the discreteness of the minimal set it follows that the set $Fix\, G^S$ is not empty, from the nonemptiness of the set $Fix\, G^S$ it follows that G^S is a normal subgroup);

3) if the minimal set is empty then $H_G = < e >$. ■

Let's note that if the minimal set coincide with the whole line then $H_G = < e >$.

1.2. Metric invariants.

Let's define an important metric invariant. Let $\mathcal{M}$ is the space of charges on $\mathbb{X}$, which are finite on compacts ($\mathcal{M}^+$ is a cone of Borel measures). In case $\mathbb{X} = \mathbb{R}$ the space $\mathcal{M}$ is considered as conjugate space to the space $\mathcal{R}(\mathbb{R})$ of continuous functions on $\mathbb{R}$ with compact support and with topology of inductive limit [16]. In case $\mathbb{X} = \mathbb{S}^1$ the space $\mathcal{M}$ is considered as conjugate space to the space $C(\mathbb{S}^1)$ of continuous functions on $\mathbb{S}^1$. For the group $G \subseteq Homeo_+(\mathbb{X})$ let's denote through G_* a group of continuous linear operators, which are acting on the space $\mathcal{M}$.

The group G_* is isomorphic to the group $G \subseteq Homeo_+(\mathbb{X})$ and isomorphism $\theta : G \longrightarrow G_*$, where $\theta(g) = g_*$, is defined as follows: for any measure μ and for any Borel set B

$$g_*\mu(B) = \mu(g^{-1}(B)).$$

Let's note that the cone of positive measures $\mathcal{M}^+$ is invariant, relatively to the group of continuous linear operators G_*. For any measure $\mu \in \mathcal{M}^+$ through $\mathcal{K}_\mu(G)$ we denote a closed convex cone, generated by the orbit $G_*(\mu) = \{g_*\mu\}_{g\in G}$ of measure μ, i.e.

$$\mathcal{K}_\mu(G) = \overline{conv\,\{\lambda\, g_*\mu\}}_{g\in G,\, \lambda\in\mathbb{R}_+}.$$

It's obvious, that the cone $\mathcal{K}_\mu(G)$ is invariant, relatively to the group of continuous linear operators G_*.

Definition 3. *Let $G \subseteq Homeo_+(\mathbb{X})$ and $\mu \in \mathcal{M}^+$. The cone $\mathcal{K}_\mu(G)$ is called minimal if for any measure $\bar{\mu} \in \mathcal{K}_\mu(G)$ the condition $\mathcal{K}_\mu(G) = \mathcal{K}_{\bar{\mu}}(G)$ is fulfilled.* ■

Let $\mathcal{K} \subseteq \mathcal{M}^+$ is the cone and $\mu \in \mathcal{K}$. A ray $\lambda\mu$, $\lambda > 0$ is called *extreme*, if there aren't measures $\mu_1, \mu_2 \in \mathcal{K}\backslash\mu$ and nonnegative numbers λ_1, λ_2 such that $\mu = \lambda_1\mu_1 + \lambda_2\mu_2$.

Definition 4. *Borel measure $\mu \in \mathcal{M}^+$, which is finite on compacts, is called ω-projectively invariant, relatively to the group $G \subseteq Homeo_+(X)$, if the convex cone $\mathcal{K}_\mu(G)$ is minimal, the ray $\lambda\mu$ is extreme, and ω is a cardinality of the set of extreme rays.* ■

In case $\omega = 1$ the invariant cone $\mathcal{K}_\mu(G)$ is one-dimensional and such measure is called *projectively invariant measure*. If the invariant cone $\mathcal{K}_\mu(G)$ is one-dimensional and also is fixed, relatively to the group of linear operators G_*, then such projectively invariant measure is *invariant measure*. It's obvious, that in case of the circle $\mathbb{X} = \mathbb{S}^1$ any projectively invariant measure is an invariant measure.

It's important to formulate various criteria for the existence of an ω-projectively invariant measure without a priori assumptions about nature of a group. Let's note that for the group with an invariant or projectively invariant measure the minimal set is not empty.

Theorem 6 ([15])**.** *Let $G \subseteq Homeo_+(\mathbb{R})$. If the minimal set of the group G is not empty then there is such cardinal number ω that for the group G there is an ω-projectively invariant measure.* ■

It is obvious, that following estimate holds

$$1 \leq \omega \leq \infty.$$

It's interesting to know, in which cases is there an ω-projectively invariant measure with finite ω? Answer to this question is given by the next theorem about *extreme property* of an ω-projectively invariant measure.

Theorem 7 ([17]). *Let $G \subseteq Homeo_+(\mathbb{R})$ and the minimal set of the group G is neither empty nor discrete. Then for the ω-projectively invariant measure μ the cardinal number ω is equal to either 1 or infinity.* ■

There is another interesting question: under what conditions from the existence of an ω-projectively invariant measure does the existence of an invariant or a projectively invariant measure follow? Below we are going to describe obstacles, related with the existence of an invariant or a projectively invariant measure in terms of topological characteristics, algebraic characteristics or combinatorial characteristics.

1.2.1. Invariant measures.

Let's formulate a criterion of the existence of an invariant measure in terms of topological characteristics.

Theorem 8 ([12]). *Let $G \subseteq Homeo_+(\mathbb{X})$. Then the following statements are equivalent:*

1) there is a Borel (probabilistic, in case of $\mathbb{X} = \mathbb{S}^1$) measure μ, which is finite on compacts and invariant, relatively to the group G;

2) the set $Fix\ G^S$ is nonempty. ■

Let's formulate a criterion of the existence of an invariant measure in terms of the quotient group $G/ < G^S >$.

Theorem 9 ([12]). *Let $G \subseteq Homeo_+(\mathbb{R})$. If the quotient group $G/ < G^S >$ is nontrivial then there is a measure μ, which is finite on compacts and invariant, relatively to the group G.* ■

Remark 2. *Owing to Proposition 1 and Theorem 8, for the group $G \subseteq Homeo_+(\mathbb{R})$ from the existence of an invariant measure it follows that the minimal set is nonempty.* ■

Remark 3. *Let $G \subseteq Homeo_+(\mathbb{X})$ and for the group G the minimal set is nonempty. Then for the group G there is an inclusion $H_G \subseteq G^S$. If there is a Borel measure μ, which is finite on compacts and invariant, relatively to the group G, then the equality $H_G = G^S$ holds. If the condition $H_G = G^S$ is satisfied, then for the group G there is an invariant measure.* ■

Remark 4. *If for the group $G \subseteq Homeo_+(\mathbb{X})$ the minimal set is discrete then there is a Borel measure μ, which is finite on compacts and invariant, relatively to the group G.* ■

We have already noted that such characteristic as the quotient group $G/ < G^S >$ cannot be a universal classification instrument for groups of homeomorphisms of the line since this quotient group is nontrivial only for groups with an invariant measure. On the other hand, this characteristic, like a topological characteristic from Theorem 8, doesn't carry any information about groups, which don't satisfy these criteria. This lack can be overcome by reformulation of the given criteria in form of alternatives.

In terms of combinatorial characteristics, let's formulate a criterion of the existence of an invariant Borel measure on the circle in form of a strict alternative, which is unimprovable amplification of the Krylov-Bogolyubov-Day theorem on the existence of invariant measure for an amenable group, acting on the circle.

Theorem 10 ([11])**.** *For the group of homeomorphisms of the circle $G \subseteq Homeo\left(\mathbb{S}^1\right)$ either the quotient group G/H_G contains a free subgroup with two generators or there is a probabilistic Borel measure, which is invariant, relatively to the group G. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.* ■

Let's give an equivalent reformulation of Theorem 10 where all statements are given in terms of combinatorial characteristics.

Theorem 11 ([11])**.** *For the group of homeomorphisms of the circle $G \subseteq Homeo\left(\mathbb{S}^1\right)$ either the quotient group G/H_G contains a free subgroup with two generators or the quotient group G/H_G is commutative. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.* ■

The quotient group G/H_G is a universal instrument for classification for groups of homeomorphisms of the circle. Such characteristic splits the set of all groups of homeomorphisms of the circle in two classes. First class consists of groups with simple combinatorial characteristics of the quotient group G/H_G, another class is formed by groups with complex ones. Unfortunately this classification instrument is not applicable to study the canonical normal subgroup H_G. More detailed study of normal subgroup H_G leads to the study of induced groups of homeomorphisms of the *line*, with which such instrument is not useful. Therefore, a problem of a formulation of a similar characteristic for classification of groups of homeomorphisms of the line is actual.

Nevertheless, it's also possible to formulate a criterion of the existence of invariant measure for groups of homeomorphisms of the line $G \subseteq Homeo_+\left(\mathbb{R}\right)$ in terms of combinatorial characteristics. It's obvious, that in that case, in contrast to the criterion of the existence of an invariant measure for groups of homeomorphisms of the circle, the conditions on the quotient group G/H_G must be more hard.

Theorem 12 ([11])**.** *For the group of homeomorphisms of the line $G \subseteq Homeo_+\left(\mathbb{R}\right)$ with nonempty minimal set either the quotient group G/H_G contains a free subsemigroup with two generators or there is a Borel measure, which is finite on compacts and invariant, relatively to the group G. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.* ■

Let's give an equivalent reformulation of Theorem 12 also with statements in form of combinatorial characteristics.

Theorem 13 ([11]). *For the group of homeomorphisms of the line $G \subseteq Homeo_+(\mathbb{R})$ with nonempty minimal set the quotient group G/H_G either contains a free subsemigroup with two generators or is commutative. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.* ■

For groups of homeomorphisms of the line the properties of the quotient group G/H_G from Theorem 13 can be a basis of the classification scheme. At the same time, such scheme has some lacks. The quotient group G/H_G, which contains a free subsemigroup with two generators, either can be solvable (for example, a simple group $G =< t + 1, 2t >$) or contains a free subgroup with two generators (such group has maximal combinatorial complexity). The reason of this lack is revealed in Theorem 12. By the noted theorem, the characteristic, in form of the quotient group G/H_G, splits the set of groups of homeomorphisms of the line in three classes: groups with invariant measure; groups with nonempty minimal set but without invariant measure; groups with empty minimal set. Therefore, a problem of a formulation of a more acceptable characteristic for classification of groups of homeomorphisms of the line is still actual.

1.2.2. Projectively invariant measures.

Let's formulate an important criterion of the existence of a projectively invariant measure in terms of both topological and algebraic characteristics. For any element $q \in Homeo_+(\mathbb{R})$ we use the notation:

$$T_q = \sup\{t : \ q(t) = t\}, \quad t_q = \inf\{t : \ q(t) = t\}, \text{ if } Fix < q > \neq \emptyset;$$
$$T_q = t_q = -\infty, \quad \text{if } Fix < q >= \emptyset.$$

Let's define one more canonical subset of the group $G \subseteq Homeo_+(\mathbb{R})$

$$C_G = (G \setminus G^S) \cup G^S_\infty.$$

Theorem 14 ([13]). *Let for the group $G \subseteq Homeo_+(\mathbb{R})$ there is not an invariant Borel measure, which is finite on compacts. Then there is a projectively invariant Borel measure, relatively to the group G, which is finite on compacts, if and only if following conditions are satisfied simultaneously:*

1) the set G^S_∞ is a subgroup and the quotient group G/G^S_∞ is not commutative;

2) for any $g \in G^S \backslash C_G$ following conditions are fulfilled:
t_g, T_g are finite and for any $t \in]-\infty, t_g[$, $T \in]T_g, +\infty[$ it's true that
$sign[g(t) - t] = -sign[g(T) - T]$;

3) for any g_1, $g_2 \in G^S \backslash C_G$ either $t_{g_1} = t_{g_2}$ and $T_{g_1} = T_{g_2}$ or $[t_{g_1}, T_{g_1}] \bigcap [t_{g_2}, T_{g_2}] = \emptyset$. Specified alternative is strict. ■

For groups, which don't have an invariant measure, we can formulate another criterion of the existence of a projectively invariant measure also in terms of the canonical subsets G^S_∞ and C_G of the initial group $G \subseteq Homeo_+(\mathbb{R})$.

Theorem 15 ([14])**.** *Let for the group $G \subseteq Homeo_+(\mathbb{R})$ there is not an invariant Borel measure which, is finite on compacts. Then there is a projectively invariant Borel measure, relatively to the group G, which is finite on compacts, if and only if the subsets G^S_∞ and C_G are subgroups and it's true that $C_G \neq G^S_\infty$.* ■

Remark 5 ([14],[17])**.** *If for the group $G \subseteq Homeo_+(\mathbb{R})$ there is a projectively invariant Borel measure μ, relatively to the group G, which is finite on compacts, but there is not an invariant Borel measure, which is finite on compacts, then the following statements hold: $G^S_\infty = H_G$; the quotient groups G/C_G and C_G/G^S_∞ are commutative and the quotient group C_G/G^S_∞ is not cyclic; measure μ for the group C_G is invariant.* ■

For groups without an invariant measure we can formulate another result about existence of a projectively invariant measure, based only on presence of a special normal subgroup.

Theorem 16 ([15])**.** *Let $G \subseteq Homeo_+(\mathbb{R})$ and there is not an invariant Borel measure, relatively to the group G, which is finite on compacts. There is a projectively invariant Borel measure μ, relatively to the group G, which is finite on compacts, if and only if there is a normal subgroup $\Gamma \subseteq G$ with following properties: for the group Γ there is an invariant measure; the quotient group $\Gamma/{<}\Gamma^S{>}$ is not cyclic.* ■

Remark 6. *Owing to Remarks 2 and 3, in Theorem 16 for the normal subgroup Γ there is a condition $\Gamma^S = {<}\Gamma^S{>}$. From Remark 1 it follows that the subgroup Γ contains a freely acting element.* ■

Remark 7. *By Proposition 1 and Theorem 14, for the group $G \subseteq Homeo_+(\mathbb{R})$ the nonemptiness of the minimal set follows from the existence of a projectively invariant measure.* ■

1.3. On the maximal subgroups with invariant measure.

In the study of groups of homeomorphisms of the line the existence of a maximal normal subgroup with an invariant measure is essential. Let's define such subgroups and also describe their structure.

Definition 5. *Let $G \subseteq Homeo_+(\mathbb{R})$. The normal subgroup M_G of the group G is called 0-maximal if for the M_G there is an invariant measure and M_G is not a proper subset of some normal subgroup of the group G, for which there is an invariant measure.* ■

Theorem 17 ([15],[17])**.** *Let $G \subseteq Homeo_+(\mathbb{R})$ and for the group G there is a nonempty minimal set. Then for the group G there is a unique 0-maximal subgroup M_G and also M_G:*

1) *coincide with the group G if for the group G there is an invariant measure;*

2) *coincide with the normal subgroup C_G if for the group G there is not an invariant measure, but there is a projectively invariant measure;*

3) contains the normal subgroup H_G if for the group G there is not a projectively invariant measure and $G = C_G$; moreover, it's true that $M_G^S = H_G$ and the quotient group M_G/H_G is either trivial or infinite cyclic group. Specified alternative is strict;

4) coincide with the normal subgroup H_G if for the group G there is not a projectively invariant measure and $G \neq C_G$.

Moreover, any normal subgroup with an invariant measure is contained in 0-maximal subgroup. ■

Remark 8. *If in the conditions of Theorem 17 the quotient group M_G/H_G is an infinite cyclic group then there is a freely acting element in the maximal normal subgroup M_G.* ■

Let's formulate a criterion of the existence of a projectively invariant measure for special class of groups of homeomorphisms of the line in terms of combinatorial characteristics.

Theorem 18 ([15])**.** *Let the group $G \subseteq Homeo_+(\mathbb{R})$ contains a normal subgroup Γ with invariant measure and with freely acting element. Then either the quotient group G/H_G contains a free subgroup with two generators or there is a projectively invariant measure. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.* ■

Let's give an equivalent reformulation of Theorem 18 with statements in form of combinatorial characteristics.

Theorem 19 ([15])**.** *Let the group $G \subseteq Homeo_+(\mathbb{R})$ contains a normal subgroup Γ with invariant measure and with freely acting element. Then either the quotient group G/H_G contains a free subgroup with two generators or the group G/H_G is a solvable group of solvability length not greater than 2. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.* ■

§2. Criterions for the existence of an invariant and a projectively invariant measures in terms of commutator-group $[G, G]$.

Let's formulate criterions for the existence of an invariant and a projectively invariant measures in terms of commutator-group $[G, G]$. Further will be given an equivalent reformulation of the criterion of existence of a projectively invariant measure in terms of the geometry of graphs of homeomorphisms of the initial group G and also in terms of commutator-group $[G, G]$. In the future such reformulation will be central for obtaining a criterion of existence of a projectively invariant measure in form of combinatorial characteristics.

Theorem 20. *Let $G \subseteq Homeo_+(\mathbb{R})$ and a minimal set of the group G is nonempty. Then there is an invariant Borel measure, which is finite on compacts, if and only if*

$$[G, G] \subseteq H_G. \tag{10}$$

Proof.
Necessity. Let for the group G there is an invariant measure. By Remark 3, we have the equality $H_G = G^S$. Consequently, $< G^S >= G^S$. Then, by Theorem 1, the quotient group G/H_G is commutative that implies the inclusion (10).
Sufficiency. Let the minimal set is discrete. Then, by Theorems 2 and 8, for the group G there is an invariant measure.

Let the minimal set is not discrete. From the condition (10) it follows that the quotient group G/H_G is commutative. The quotient group G/H_G can be realized as a group of homeomorphisms on the minimal set $E(G)$. The action of the left coset $[g]$ of the group G by the normal subgroup H_G is defined by the following rule: $[g](t) = g(t)$ for any $t \in E(G)$. Then, from the commutativity of the quotient group G/H_G, it follows that any two elements $g_1, g_2 \in G$ commute on the minimal set $E(G)$, i.e. for any $t \in E(G)$ commutativity rule $g_1g_2(t) = g_2g_1(t)$ holds. Let's note that for any element $g \in G^S$ there is a point $\bar{t} \in E(G)$ such that $g(\bar{t}) = \bar{t}$. In view of the marked commutativity rule, for any $q \in G$ the following chain of equalities $g(q(\bar{t})) = q(g(\bar{t})) = q(\bar{t})$ is fulfilled. It means that along with the point $\bar{t}$ the point $q(\bar{t})$ also belongs to the set $Fix\, g$. Since this holds for any element $q \in G$ so next inclusion is fair

$$E(G) \subseteq Fix\, g, \quad g \in G^S. \tag{11}$$

But for such group G it is true that $Fix\, G^S \neq \emptyset$. Hence, by Theorem 8, for such group G there is an invariant measure. □

Let's formulate criterion for the existence of a projectively invariant measures in terms of commutator-group $[G, G]$.

Theorem 21. *Let $G \subseteq Homeo_+(\mathbb{R})$ and a minimal set of the group G is nonempty. Then there is a projectively invariant Borel measure, relatively to the group G, which is finite on compacts, if and only if there is the following inclusion*

$$[G, G] \cap G^S \subseteq H_G. \tag{12}$$

Proof.
Necessity. Let there is a projectively invariant measure.
(i) The first case. For the group G there is an invariant measure. By Theorem 20, it is equivalent to the condition $[G, G] \subseteq H_G$ that implies the inclusion (12).
(ii) The second case. For the group G there is not an invariant measure. By Remark 5, the quotient group G/C_G is commutative and, thus, $[G, G] \subseteq C_G$. By the same Remark 5, it is true that $G^S_\infty = H_G$. By the definition for the group C_G, it is true that $C^S_G = G^S_\infty$ that implies the inclusion (12).

Sufficiency. Let the inclusion (12) is fulfilled.
(i) The first case. Let the following condition is satisfied

$$[G,G] \subseteq H_G. \tag{13}$$

Then, by Theorem 20, for the group G there is an invariant measure.
(ii) The second case. Let the following condition is satisfied

$$[G,G] \backslash H_G \neq \emptyset. \tag{14}$$

For the commutator-group let's use the notation $Q = [G,G]$. It is obvious, that $Q^S = Q \cap G^S$. Then from the inclusion (12) it follows that $Q^S \subseteq H_G$. Consequently, the following chain of relations $< Q^S >= Q^S = Q^S_\infty \subseteq H_G$ is fulfilled. From the property (14), it follows that $Q \neq< Q^S >$. Therefore, by Remark 1, in the group Q there is a freely acting element and, owing to Theorem 9, for the group Q there is an invariant measure. In that case, owing to Theorem 18, the quotient group G/H_G either contains a free subgroup with two generators or is a solvable group of solvability length not greater than 2.

The commutator-group Q is normal in the group G and for any $g \in G$ there is a relation $gG^Sg^{-1} = G^S$. Then, from the representation $< Q^S >= Q \cap G^S$, it follows that $< Q^S >$ is a normal subgroup of the group G and, accordingly, it is normal in H_G. There is an isomorphism

$$(G/{<}Q{>})/(H_G/{<}Q{>}) \cong G/H_G. \tag{15}$$

The quotient group G/Q is commutative by construction and, by Theorem 1, the quotient group Q/Q^S is also commutative. Then, owing to the isomorphism

$$(G/{<}Q^S{>})/(Q/{<}Q^S{>}) \cong G/Q, \tag{16}$$

the quotient group $G/{<}Q^S{>}$ is a solvable group of solvability length not greater than 2. In that case, owing to the isomorphism (17), the quotient group G/H_G can't contain a free subgroup with to generators and, by Theorem 18, for the group G there is a projectively invariant measure. □

Next question is quite interesting. Can we reformulate the condition (12) in terms of local properties of the graphs of homeomorphisms themselves from the commutator-group $[G,G]$ or from the initial group G?

Definition 6. *Let $G \subseteq Homeo_+(\mathbb{R})$ is a group with nonempty minimal set. An element $g \in G$ is called plane if there is not a triple of points t_1, t_2, t_3, $t_2 \in E(G)$ in case of nondiscrete minimal set, or a triple of points t_1, t_2, t_3, $t_2 \in Fix\, G^s$ in case of discrete minimal set, for which next conditions are satisfied*

$$t_1 < t_2 < t_3, \qquad g(t_1) = t_1, \quad g(t_2) \neq t_2, \quad g(t_3) = t_3. \tag{17}$$

■

The following statement holds.

Lemma 2. *Let $G \subseteq Homeo_+(\mathbb{R})$ is a group with nonempty minimal set. In order to satisfy the inclusion*

$$[G, G] \cap G^S \subseteq H_G\,, \tag{18}$$

it is necessary and sufficient that all elements of the commutator-group $[G, G]$ are plane.

Proof. For the commutator-group we use the notation $Q = [G, G]$. It is not difficult to see, that in the case of finite values of T_q, (t_q) for the element $q \in G$ the point T_q, (t_q) always belongs to the minimal set of the group, i.e. T_q, $(t_q) \in E(G)$ in case of nondiscrete minimal set, and T_q, $(t_q) \in Fix\, G^s$ in case of discrete minimal set.
Necessity. Let the inclusion (18) takes place. It is obvious, that $Q^S \subseteq G^S$. Then, from the inclusion (18), it follows that $Q^S \subseteq H_G$ and, hence, all elements of Q^S are plane. Each element $q \in Q \backslash Q^S$ is freely acting and, thus, is plane. Therefore, all elements of commutator-group Q are plane.
Sufficiency. Let's suppose that the commutator-group is plane and let's prove by contradiction. Let the condition (18) is violated , i.e. there is a relation

$$([G, G] \cap G^S) \backslash H_G \neq \emptyset. \tag{19}$$

Step 1. *The minimal set of the group G is nondiscrete.*

Indeed, from the condition (19), it follows that $[G, G] \backslash H_G \neq \emptyset$. Owing to Theorem 20, for the group G there is not an invariant measure and, by Remark 4, the minimal set of such group is nondiscrete.

Step 2. *For any element $q \in Q$ it is true that*

$$E(G) \cap (t_q,\, T_q) \subseteq Fix\ q. \tag{20}$$

(In the case of finite t_q or T_q inclusion can be enhanced. Instead of the interval (t_q, T_q) should be written $[t_q, T_q)$, $(t_q, T_q]$, $[t_q, T_q]$, respectively.)

The property (20) follows immediately from the condition that all elements of the group Q are plane.

Step 3. *The set Q^S is invariant, relatively to the operation of conjugation, i.e. $gQ^S g^{-1} = Q^S$, $g \in G$.*

This is a consequence of the invariance of the set G^S, relatively to the operation of conjugation, i.e. of the condition $gG^S g^{-1} = G^S$, $g \in G$ and the normality of the subgroup Q of the group G.

Step 4. *The set Q^S_∞ is a normal subgroup of the group G and it is true that $Q^S_\infty \subseteq H_G$.*

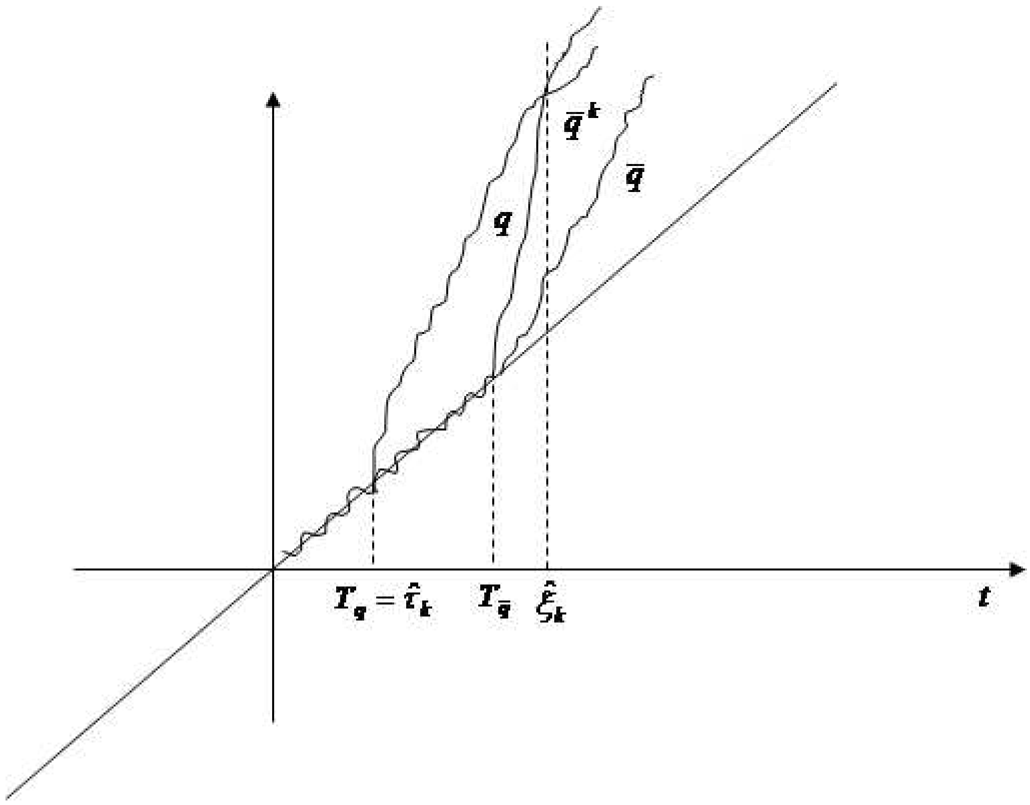
Indeed, from the condition (20), it follows that for each element $q \in Q^S_\infty$ it is true that $q \in H_G$, i.e. $Q^S_\infty \subseteq H_G$. Hence, Q^S_∞ is a subgroup. Since Q is a normal subgroup of the group G then the set Q^S_∞ is invariant, relatively to the operation of conjugation, i.e. $gQ^S_\infty g^{-1} = Q^S_\infty$, $g \in G$. Therefore, the subgroup Q^S_∞ is a normal subgroup of the group G.

Step 5. *For any element* $q \in Q^S \backslash Q^S_\infty$ *the values* t_q, T_q *are finite.*

It is obvious, that for an element q one of the values t_q, T_q is finite. For definiteness let the value T_q is finite and $q(t) > t, \quad t \in (T_q, +\infty)$. Let's suppose that $t_q = -\infty$. Since the minimal set $E(G)$ is nondiscrete then there is an element $g \in G$ such that $g(T_q) \neq T_q$. For definiteness let $g(T_q) > T_q$. Otherwise, this condition will be satisfied by the inverse element. Let's consider an element $\bar{q} = gqg^{-1}$. From the steps 3 and 4, the inclusion $\bar{q} \in Q^S \backslash Q^S_\infty$ follows. For such element it is true that $T_{\bar{q}} = g(T_q),\ t_{\bar{q}} = -\infty, \quad \bar{q}(t) > t, \quad t \in (T_{\bar{q}}, +\infty)$. Since $T_q \in E(G) \cap (-\infty, T_{\bar{q}})$ then, owing to the step 2, there is a condition $\bar{q}(T_q) = T_q$. Since $g(T_q) > T_q$ then for any $k = 1, 2, ...$ there is a point $\hat{\tau}_k, \quad T_q \leq \hat{\tau}_k < T_{\bar{q}}$ for which following conditions are fulfilled

$$q(\hat{\tau}_k) = \bar{q}^k(\hat{\tau}_k), \quad q(t) > \bar{q}^k(t), \quad t \in (\hat{\tau}_k, T_{\bar{q}}), \quad k = 1, 2, ... \ . \tag{21}$$

(In fact there is a stronger property that we do not need. Since all elements of the group Q are plane, the minimal set $E(G)$ is nondiscrete and $\bar{q}^k(t) = t, \quad t \in E(G) \cap (-\infty, T_{\bar{q}}]$ then there is an equality $\hat{\tau}_k = T_q$).

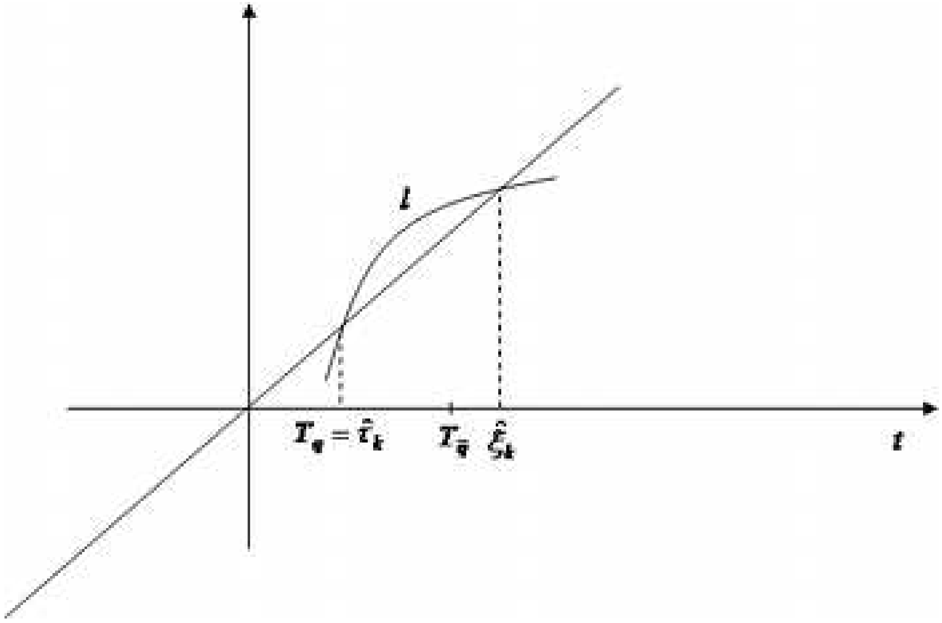


Let's consider an element $\bar{q}^k$ for sufficiently large fixed k. For such element there is a point $\hat{\xi}_k > T_{\bar{q}}$ such that

$$q(\hat{\tau}_k) = \bar{q}^k(\hat{\tau}_k), \quad q(\hat{\xi}_k) = \bar{q}^k(\hat{\xi}_k), \quad q(t) > \bar{q}^k(t), \quad t \in (\hat{\tau}_k, \hat{\xi}_k), \quad k = 1, 2, ... \ . \tag{22}$$

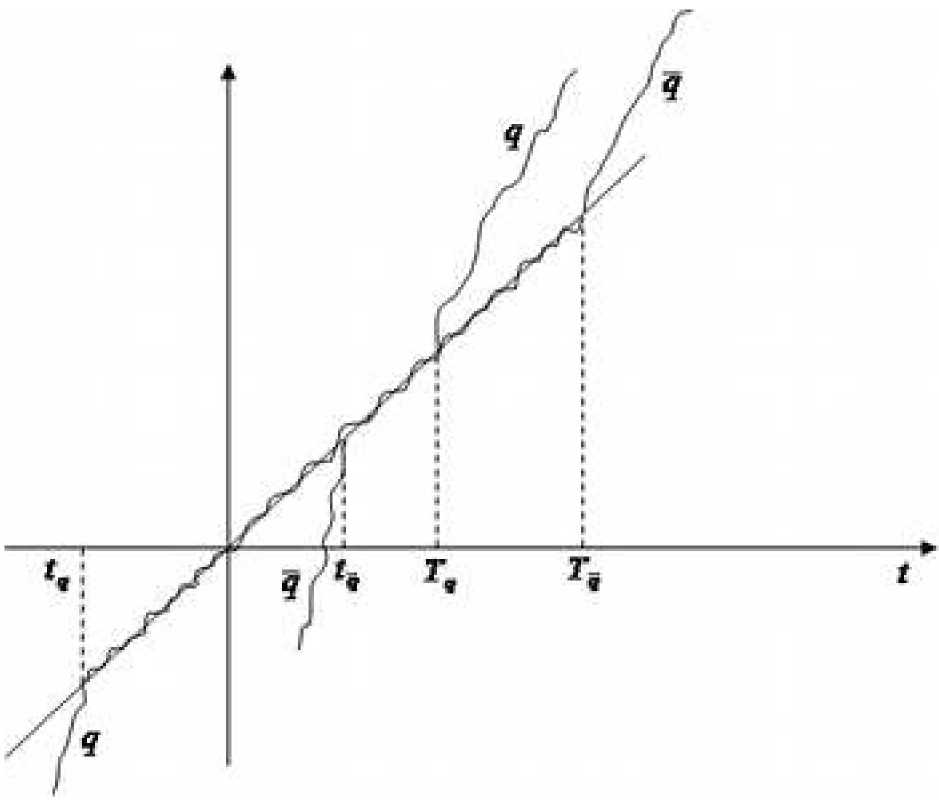
Then for the element $l = \bar{q}^{-k} q$ it is true that

$$l(\hat{\tau}_k) = \hat{\tau}_k, \quad l(\hat{\xi}_k) = \hat{\xi}_k, \quad l(t) > t, \quad t \in (\hat{\tau}_k, \hat{\xi}_k). \tag{23}$$



Since $T_{\bar{q}} \in E(G)$ and $T_{\bar{q}} \in (\hat{\tau}_k, \hat{\xi}_k)$ then $\hat{\tau}_k, \hat{\xi}_k$ are also belong to the minimal set $E(G)$. Hence, the element $l \in Q$ can't be plane. Contradiction. Therefore, the value t_q is also finite.
Step 6. *For any* $q, \bar{q} \in Q^S \backslash Q^S_\infty$ *either* $[t_q, T_q] \cap [t_{\bar{q}}, T_{\bar{q}}] = \emptyset$, *or* $t_q = t_{\bar{q}}$, $T_q = T_{\bar{q}}$.

Let's prove by contradiction. For definiteness let it is true that $t_{\bar{q}} \leq T_q < T_{\bar{q}}$. Also for definiteness let for the elements $q, \bar{q}$ it is true that: $q(t) > t,\ t > T_q$; $q(t) > t,\ t > T_{\bar{q}}$. Otherwise, this condition will be satisfied by the inverse elements.

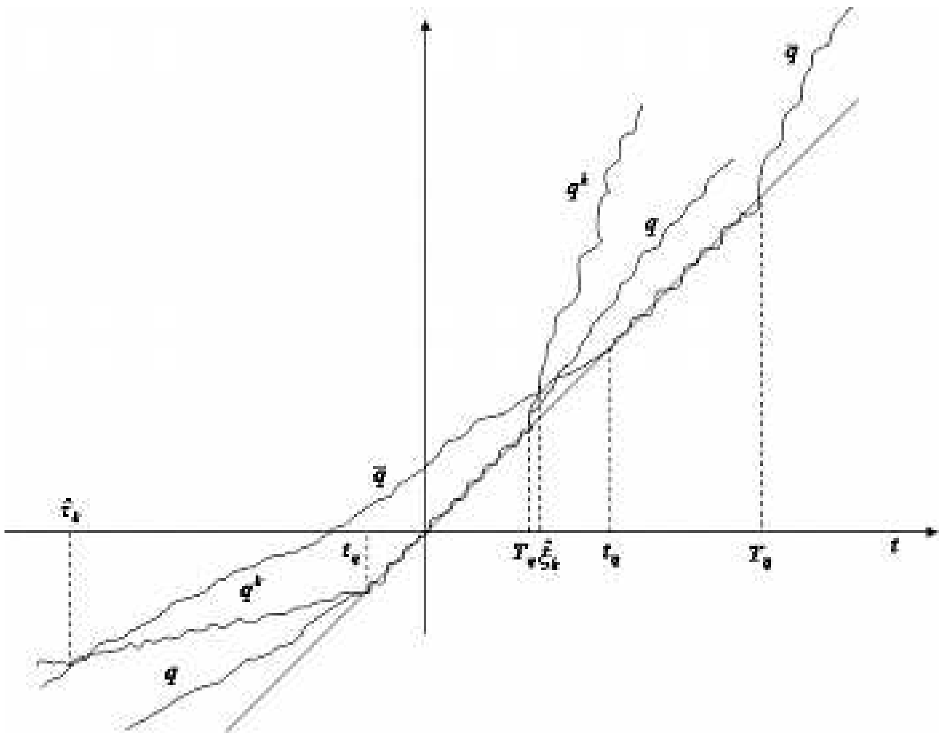


Then, literally repeating the proof of the step 5 for the elements $q, \bar{q}$, we obtain a contradiction with the fact that elements of the subgroup Q are plane. The case when $t_{\bar{q}} < T_q \leq T_{\bar{q}}$ is studied in a similar way. Consequently, such mutual arrangement of the elements $q, \bar{q}$ can not be.
<u>Step 7.</u> *For any element* $q \in Q^S \backslash Q^S_\infty$ *it is true that*:

$$sign[g(t) - t] = -sign[g(T) - T], \qquad t \in (-\infty, t_g), \quad T \in (T_g, +\infty). \tag{24}$$

Let's prove by contradiction. For definiteness let $q(T) > T,\ \ T \in (T_g, +\infty)$. Otherwise, this condition will be satisfied by the inverse element. Let the condition (24) is violated. It means that for the element q it is true that $q(t) > t,\ \ t \in (-\infty, t_g)$. Since the minimal set $E(G)$ of the group G is nondiscrete then there is an element $g \in G$ for which $g(T_g) \neq T_g$. Let $g(T_g) > T_g$. Otherwise, this condition will be satisfied by the inverse element. Let's consider an element $\bar{q} = gqg^{-1}$. It is obvious, that $\bar{q} \in Q^S \backslash Q^S_\infty$ and $T_{\bar{q}} = g(T_g)$. In view of the step 6, it is true that $T_g < t_{\bar{q}}$. Let's note that for the element $\bar{q}$ the condition $\bar{q}(t) > t,\ \ t \in (-\infty, t_{\bar{q}}) \cup (T_{\bar{q}}, +\infty)$ is also fulfilled. For sufficiently large positive integer k let's consider an element q^k. For such element there are points $\hat{\tau}_k, \hat{\xi}_k, \ \ \hat{\tau}_k < t_q,\ \ T_q < \hat{\xi}_k < t_{\bar{q}}$ such that

$$q^k(\hat{\tau}_k) = \bar{q}(\hat{\tau}_k), \quad q^k(\hat{\xi}_k) = \bar{q}(\hat{\xi}_k), \quad \bar{q}(t) > q^k(t), \quad t \in (\hat{\tau}_k, \hat{\xi}_k). \tag{25}$$



Let's form an element $l = q^{-k}\bar{q}$. For this element it is true that

$$l(\hat{\tau}_k) = \hat{\tau}_k, \quad l(\hat{\xi}_k) = \hat{\xi}_k, \quad l(t) > t, \quad t \in (\hat{\tau}_k, \hat{\xi}_k). \tag{26}$$

Since the points $t_q, T_g \in E(G) \cap (\hat{\tau}_k, \hat{\xi}_k)$ then the points $\hat{\tau}_k, \hat{\xi}_k$ are also belong to the minimal set $E(G)$. In that case, the element $l \in Q$ can't be plane. Consequently, for the elements $q \in Q^S \backslash Q^S_\infty$ the condition (24) is satisfied.

Step 8. *The quotient group Q/Q^S_∞ is not commutative.*

From the condition (19) and from $Q^S_\infty \subseteq H_G$ (see step 3), it follows that there is an element $\tilde{q}$, which satisfies the condition $\tilde{q} \in Q^S \backslash Q^S_\infty$. For such element the values $t_{\tilde{q}},\ T_{\tilde{q}}$ are finite (step 5). As it was noted in the previous steps, since the minimal set $E(G)$ is nondiscrete then there is an element $g \in G$ with the property $g(T_{\tilde{q}}) > T_{\tilde{q}}$. Let's form an element $\bar{q} = g\tilde{q}g^{-1}$. It is obvious, that $\bar{q} \in Q^S \backslash Q^S_\infty$ and there is equalities $t_{\bar{q}} = g(t_{\tilde{q}}),\ \ T_{\bar{q}} = g(T_{\tilde{q}})$ and, owing to the step 6, we obtain that $t_{\bar{q}} > T_{\tilde{q}}$. Since $Q^S_\infty \subseteq H_G$ the quotient group Q/Q^S_∞ can be realized as a group acting on the minimal set $E(G)$. For each coset $[q]$ of the group Q by the subgroup Q^S_∞ and for any $t \in E(G)$ we assume that $[q](t) = q(t)$. Obviously, that two formed cosets $[\tilde{q}],\ \ [\bar{q}]$ are different. Moreover, their actions on the minimal set $E(G)$ do not commute. Thus, the quotient group Q/Q^S_∞ is not commutative.

Step 9. *For the group Q there is not an invariant measure.*

In the step 8 we mentioned that owing to the condition (19) in the group Q there is an element $\tilde{q} \in Q^S \backslash Q^S_\infty$. In the same place by the element $\tilde{q}$ we formed the element $\bar{q} \in Q^S \backslash Q^S_\infty$. For this elements the condition $[t_{\tilde{q}}, T_{\tilde{q}}] \cap [t_{\bar{q}}, T_{\bar{q}}] = \emptyset$ is fulfilled and thus $Fix\ \ Q = \emptyset$. By Theorem 8, for such group there is not an invariant measure.

Step 10. *For the group Q there is a projectively invariant measure.*

Indeed, owing to the steps 4-8, the group Q satisfies all conditions of Theorem 14 that implies the existence of a projectively invariant measure.

Step 11. *The existence of a projectively invariant measure for the group Q contradicts the assumption* (19).

In the step 4 we mentioned that Q^S_∞ is a normal subgroup of the group G. For any freely acting element $\bar{g} \in G$ its conjugate elements $g\bar{g}g^{-1}$ are also freely acting. For the group Q there is a projectively invariant measure μ and, by Remark 5, the set C_Q is a subgroup of the group Q (normal subgroup). Since $C_Q \subseteq Q$ and it consists of either elements of Q^S_∞ or freely acting elements, and Q is a normal subgroup of the group G, then C_Q is a normal subgroup of the group G. Since for the group Q there is a projectively invariant measure μ, but there is not an invariant measure then, by Remark 5, the measure μ is invariant, relatively to the subgroup C_Q, and the quotient group C_Q/Q^S_∞ is not cyclic. Then, by Theorem 16, for the group G there is a projectively invariant measure $\tilde{\mu}$.

Since for the subgroup Q there is not an invariant measure then there is not an invariant measure also for the initial group G. Thus, by Remark 5, the projectively invariant measure $\tilde{\mu}$ is invariant on the subgroup C_G. By the same Remark 5, the quotient group G/C_G is commutative and hence the inclusion $Q = [G, G] \subseteq C_G$ is fair. Therefore, the measure $\tilde{\mu}$ is also invariant on Q. But in the step 9 we obtained that for the group Q there is not an invariant measure. Contradiction. Consequently, the condition (19) can't take place. Lemma is proved. □

Lemma 2 can be reformulated. Instead of the commutator-group $[G, G]$ the initial group G

will appear.

Lemma 3. *Let $G \subseteq Homeo_+(\mathbb{R})$ is a group with nonempty minimal set. In order to satisfy the inclusion*

$$[G,G] \cap G^S \subseteq H_G\,, \tag{27}$$

it is necessary and sufficient that all elements of the group G are plane.

Proof.
Sufficiency. It follows directly from Lemma 2.
Necessity. Let for the group G there is a projectively invariant measure.
(i) The first case. For the group G there is an invariant measure. Then, by Theorem 8, it is true that $Fix\,G^S \neq \emptyset$ and from the definition of the set $Fix\,G^S$ it follows that for any element $g \in G^S$ the inclusion

$$Fix\,G^S \subseteq Fix\,g \tag{28}$$

is fulfilled. If the minimal set is discrete then, from (28), it follows that each element of the group G is plane. Let the minimal set is nondiscrete. From the definition of the minimal set, it follows that inclusion $E(G) \subseteq Fix\,G^S$ is fair. Then, from (28), it follows that each element of the group G is plane.
(ii) The second case. For the group G there is not an invariant measure. By Remark 5, for such group the equality $G^S_\infty = H_G$ is satisfied. Thus, any element $g \in G^S_\infty$ is plane. It remains to show that any element $g \in G^S \backslash G^S_\infty$ is also plane. Since $G^S \backslash G^S_\infty = G^\backslash C_G$ then, by Theorem 14, for an element $g \in G^S \backslash G^S_\infty$ the values t_g, T_g are finite. Obviously, the points t_g, T_g belong to the minimal set, i.e. $t_g,\,T_g \in E(G)$. If we will show that there is not a point $\hat{t} \in (t_g,\,T_g) \cap E(G)$, then Lemma will be proved.

Let's prove by contradiction. Let there is a point $\hat{t} \in (t_g,\,T_g) \cap E(G)$. Then there is an element $\hat{g} \in G$ for which the point $\hat{g}(T_g)$ is in a sufficiently small neighborhood of the point $\hat{t}$ and, in particular, the condition $\hat{g}(T_g) \in (t_g,\,T_g)$ is fulfilled. Let's consider an element $l = \hat{g}g\hat{g}^{-1}$. For such element the points t_l, T_l are also finite and there is a condition $T_l = \hat{g}(T_g)$. But it contradicts the condition 3) of Theorem 14. Thus, such point $\hat{t} \in (t_g,\,T_g) \cap E(G)$ doesn't exist. Lemma is proved. ☐

§3. Right-invariant mean on discrete quotient group G/H_G.

Let $\mathbb{Y}$ is a locally compact Hausdorff space. Through $C_b(\mathbb{Y})$ we denote a Banach space of bounded continuous real functions on $\mathbb{Y}$ with topology of uniform convergence and through $C_{0b}(\mathbb{Y})$ we denote a space of bounded continuous real functions on $\mathbb{Y}$ with compact support. Previously (§1 section 1.2) we already consider the space $\mathcal{R}(\mathbb{R})$ of continuous real functions on $\mathbb{R}$ with compact support and with topology of inductive limit. Let's describe a general construction, which allows us to specify the nature of right-invariant mean of the quotient group $G/H_G, \quad G \subseteq Homeo_+(\mathbb{R})$.

3.1. Construction of a measure corresponding to the right-invariant mean on the quotient group G/H_G.

Let there is a group $G \subseteq Homeo_+(\mathbb{R})$ with nonempty minimal set. If the minimal set is discrete then we fix a point $\bar{t} \in Fix\ G^S$ (owing to Theorem 2, the set $Fix\ G^S$ is a union of all minimal sets, each of which is discrete). If minimal set is nondiscrete then we fix a point $\bar{t} \in E(G)$, where $E(G)$ is a minimal set. For each continuous function $f \in C_b(\mathbb{R})$ through f_G we denote a function defined on the quotient group G/H_G by the rule: for any left coset $[g]$ of the group G by subgroup H_G

$$f_G([g]) = f(g^{-1}(\bar{t})). \tag{29}$$

The correctness of this definition follows immediately from the definition of the normal subgroup H_G.

It is obvious, that the function f_G belongs to the space $B(G/H_G)$ of all bounded complex functions on the discrete quotient group G/H_G. In that case, we have a a continuous linear map

$$\mathcal{L}_{\bar{t}}: \ C_b(\mathbb{R}) \to B(G/H_G), \qquad \mathcal{L}_{\bar{t}} f = f_G. \tag{30}$$

It is obvious, that

$$ker\ \mathcal{L}_{\bar{t}} = \{f(.): \ f(.) \in C_b(\mathbb{R});\ f(t) = 0 \ \text{ for all } \ t \in E(G)\}. \tag{31}$$

For the space $\mathcal{R}(\mathbb{R})$ let's consider an inclusion map

$$\mathcal{I}: \ \mathcal{R}(\mathbb{R}) \to C_b(\mathbb{R}). \tag{32}$$

It is obvious, that

$$Im\ \mathcal{I} = C_{0b}(\mathbb{R}). \tag{33}$$

If the quotient group G/H_G (discrete) is amenable then there is a right-invariant mean m on the discrete group G/H_G (see [21]), i.e. there is a continuous linear functional on the Banach space $B(G/H_G)$ of bounded complex functions with the properties

$$m(\bar{F}(.)) = \overline{m(F(.))} \tag{34}$$

$$m(F(.)) \geq 0, \quad F(.) \geq 0, \qquad m(1) = 1, \tag{35}$$

$$m(F_\varrho(.)) = m(F(.)), \qquad F_\varrho(\rho) = F(\rho\varrho) \quad \text{for all} \quad \varrho, \rho \in G/H_G. \tag{36}$$

Then $\mathcal{L}^*_{\bar{t}} m, \quad \mathcal{I}^* \mathcal{L}^*_{\bar{t}} m$ are positive (owing to the condition (35)) continuous linear functionals on the spaces $C_b(\mathbb{R}),\ \mathcal{R}(\mathbb{R})$ respectively. Moreover, there is a condition $\mathcal{L}^*_{\bar{t}} m \neq 0$. The space, conjugate to the space $C_b(\mathbb{R})$, is a space $\mathbb{M}_b(\mathbb{R})$ of regular bounded finitely additive charges on $\mathbb{R}$. The space, conjugate to the space $\mathcal{R}(\mathbb{R})$, is a space $\mathcal{M}(\mathbb{R})$ of regular bounded countably additive charges on $\mathbb{R}$, which are finite on compacts. In that case, there is a regular bounded finitely additive Borel measure $\nu \in \mathbb{M}_b^+(\mathbb{R})$ such that for any $f(.) \in C_b(\mathbb{R})$ there is a representation

$$\mathcal{L}_{\bar{t}}^* m(f(.)) = \int_{\mathbb{R}} f(t) d\nu(t). \tag{37}$$

Since for any $f(.) \in ker\, \mathcal{L}_{\bar{t}}$ the equality

$$\int_{\mathbb{R}} f(t) d\mu(t) = 0 \tag{38}$$

is fulfilled then for the support of the measure ν there is an inclusion condition

$$supp\ \nu \subseteq E(G). \tag{39}$$

Exactly the same there is a countably additive Borel measure $\mu \in \mathcal{M}^+(\mathbb{R})$, which is finite on compacts, such that for any $f(.) \in \mathcal{R}(\mathbb{R})$ there is a representation

$$\mathcal{I}^* \mathcal{L}_{\bar{t}}^* m(f(.)) = \int_{\mathbb{R}} f(t) d\mu(t). \tag{40}$$

Since for any $f(.) \in ker\, \mathcal{L}_{\bar{t}} \mathcal{I}$ the equality

$$\int_{\mathbb{R}} f(t) d\mu(t) = 0 \tag{41}$$

is fulfilled then for the support of the measure μ there is an inclusion condition

$$supp\ \mu \subseteq E(G). \tag{42}$$

3.2. On invariance of measures ν and μ.

In previous section 2.1 we constructed a bounded finitely additive Borel measure $\nu \in \mathbb{M}_b^+(\mathbb{R})$ and a countably additive Borel measure $\mu \in \mathcal{M}^+(\mathbb{R})$, which is finite on compacts. Let's show that the measures ν и μ are invariant, relatively to the group G.

For each function $f(.) \in C_b(\mathbb{R})$ and an arbitrary element $g \in G$ let's define a shift map by the rule: ${}_g f(t) = f(g^{-1}(t))$. A shift map on the space $B(G/H_G)$ was defined in (36).

Lemma 4. *Let $G \subseteq Homeo_+(\mathbb{R})$ is a group with nonempty minimal set. It is given a point $\bar{t} \in Fix\ G^S$, in case of discrete minimal set, otherwise it is given a point $\bar{t} \in E(G)$, in case of nondiscrete minimal set. For any element $g \in G$, from the corresponding left coset $[g]$ of the group G by the subgroup H_G, and for an arbitrary function $f(.) \in C_b(\mathbb{R})$ there is a commute rule*

$$\mathcal{L}_{\bar{t}}\ {}_g f = (\mathcal{L}_{\bar{t}}\ f)_{[g]}. \tag{43}$$

For any element $g \in G$, from the corresponding left coset $[g]$ of the group G by the subgroup H_G, and for an arbitrary function $f(.) \in \mathcal{R}(\mathbb{R})$ there is also a commute rule

$$\mathcal{L}_{\bar{t}}\ \mathcal{I}\ {}_g f = (\mathcal{L}_{\bar{t}}\ \mathcal{I}\ f)_{[g]}. \tag{44}$$

Proof. For an arbitrary element $g \in G$ and for an arbitrary function $f(.) \in C_b(\mathbb{R})$ there are relations: for any left coset $[\bar{g}]$ of the group G by the subgroup H_G

$$(\mathcal{L}_{\bar{t}}\ {}_g f)([\bar{g}]) = ({}_g f)_G([\bar{g}]) = {}_g f(\bar{g}^{-1}(\bar{t})) = f(g^{-1}\bar{g}^{-1}(\bar{t})), \tag{45}$$
$$(\mathcal{L}_{\bar{t}}\ f)_{[g]}([\bar{g}]) = (f_G)_{[g]}([\bar{g}]) = f_G([\bar{g}][g]) = f(g^{-1}\bar{g}^{-1}(\bar{t})). \tag{46}$$

Exactly the same for an arbitrary element $g \in G$ and for an arbitrary function $f(.) \in \mathcal{R}(\mathbb{R})$ there are relations: for any left coset $[\bar{g}]$ of the group G by the subgroup H_G

$$(\mathcal{L}_{\bar{t}}\ \mathcal{I}\ {}_g f)([\bar{g}]) = f(g^{-1}\bar{g}^{-1}(\bar{t})), \tag{47}$$
$$(\mathcal{L}_{\bar{t}}\ \mathcal{I}\ f)_{[g]}([\bar{g}]) = f(g^{-1}\bar{g}^{-1}(\bar{t})). \tag{48}$$

The right-hand sides of the formulas (45), (46) are equal. Similarly, the right-hand sides are equal in the formulas (47), (48). Hence, the left sides are also equal

$$(\mathcal{L}_{\bar{t}}\ {}_g f)([\bar{g}]) = (\mathcal{L}_{\bar{t}}\ f)_{[g]}([\bar{g}]), \qquad f(.) \in C_b(\mathbb{R}), \tag{49}$$
$$(\mathcal{L}_{\bar{t}}\ \mathcal{I}\ {}_g f)([\bar{g}]) = (\mathcal{L}_{\bar{t}}\ \mathcal{I}\ f)_{[g]}([\bar{g}]), \qquad f(.) \in \mathcal{R}(\mathbb{R}). \tag{50}$$

From these relations the lemma follows. □

For each element $g \in G$ and a measure $\nu \in \mathbb{M}_b^+(\mathbb{R})$ let's define a measure $g_*\nu$ by the following rule

$$\int_{\mathbb{R}} f(t) d g_*\nu(t) = \int_{\mathbb{R}} {}_g f(t) d\nu(t), \qquad f(.) \in C_b(\mathbb{R}). \tag{51}$$

It is obvious, that by such rule the group G induces a group G_* of continuous linear operators in the space $\mathbb{M}_b(\mathbb{R})$. The G_* of continuous linear operators in the space $\mathcal{M}_b(\mathbb{R})$ we defined earlier (§1 section 1.2).

Lemma 5. *Let $G \subseteq Homeo_+(\mathbb{R})$ is a group with nonempty minimal set and with right-invariant mean m on the quotient group G/H_G. It is given a point $\bar{t} \in Fix\ G^S$, in case of discrete minimal set, otherwise it is given a point $\bar{t} \in E(G)$, in case of nondiscrete minimal set. Then a bounded finitely additive Borel measure $\nu \in \mathbb{M}_b^+(\mathbb{R})$ and a countably additive Borel measure $\mu \in \mathcal{M}^+(\mathbb{R})$, which is finite on compacts, induced by right-invariant mean m, are invariant, relatively to the group G.*

Proof. From the relations (43), (44) and from the conditions

$$m\,(\mathcal{L}_{\bar{t}}\ f)_{[g]} = m\,(\mathcal{L}_{\bar{t}}\ f), \quad m\,(\mathcal{L}_{\bar{t}}\ \mathcal{I}\ f)_{[g]} = m\,(\mathcal{L}_{\bar{t}}\ \mathcal{I}\ f),$$

which follow from the definition of the right-invariant mean m, we get equalities

$$\mathcal{L}_{\bar{t}}^*\ m\,({}_g f) = \mathcal{L}_{\bar{t}}^*\ m\,(f), \qquad f(.) \in C_b(\mathbb{R}), \tag{52}$$
$$\mathcal{I}^*\mathcal{L}_{\bar{t}}^*\ m\,({}_g f) = \mathcal{I}^*\mathcal{L}_{\bar{t}}^*\ m\,(f), \qquad f(.) \in \mathcal{R}(\mathbb{R}). \tag{53}$$

Thus, the functional $\mathcal{L}_{\bar{t}}^{*}\, m$ is invariant, relatively to the action of the group G on the space $C_b(\mathbb{R})$, and the functional $\mathcal{I}^{*}\mathcal{L}_{\bar{t}}^{*}\, m$ is invariant, relatively to the action of the group G on the space $\mathcal{R}(\mathbb{R})$. The functional $\mathcal{L}_{\bar{t}}^{*}\, m \in C_b^{*}(\mathbb{R})$ induces a finitely additive Borel measure $\nu \in \mathbb{M}_b^{+}(\mathbb{R})$, which implements the integral representation in (37) for the functional $\mathcal{L}_{\bar{t}}^{*}\, m$. The functional $\mathcal{I}^{*}\mathcal{L}_{\bar{t}}^{*}\, m \in \mathcal{R}^{*}(\mathbb{R})$ induces a countably additive Borel measure $\mu \in \mathcal{M}_b^{+}(\mathbb{R})$, which implements the integral representation in (40) for the functional $\mathcal{I}^{*}\mathcal{L}_{\bar{t}}^{*}\, m$. Then, from the equalities (52) и (53), next equalities are follow

$$\int_{\mathbb{R}} {}_g f(t) d\nu(t) = \int_{\mathbb{R}} f(t) d\nu(t), \qquad f(.) \in C_b(\mathbb{R}), \tag{54}$$

$$\int_{\mathbb{R}} {}_g f(t) d\mu(t) = \int_{\mathbb{R}} f(t) d\mu(t), \qquad f(.) \in \mathcal{R}(\mathbb{R}). \tag{55}$$

From the expressions (54) и (55), it follows that for any $g_* \in G_*$ it is true that

$$g_*\, \nu = \nu, \tag{56}$$

$$g_*\, \mu = \mu, \tag{57}$$

i.e. the measures ν и μ are invariant, relatively to the group G. □

3.3. The criterion of singularity of right-invariant mean on the quotient group G/H_G.

Let us remind that for any right-invariant mean m on the quotient group G/H_G there is the condition $\mathcal{L}_{\bar{t}}^{*}\, m \neq 0$, i.e. an induced measure ν is always nontrivial.

Definition 7. *Let $G \subseteq Homeo_{+}(\mathbb{R})$ is a group with nonempty minimal set and with right-invariant mean m on the quotient group G/H_G. The right-invariant mean m is called singular if for any $\bar{t}$ ($\bar{t} \in Fix\ G^S$, in case of discrete minimal set, otherwise $\bar{t} \in E(G)$, in case of nondiscrete minimal set) it is true that $\mathcal{I}^{*}\mathcal{L}_{\bar{t}}^{*}\, m = 0$. Otherwise, the right-invariant mean is called nonsingular.* ■

Definition 8. *Linear functional $l \in C_b^{*}(\mathbb{Y})$ is called permanent if*

$$l(f(.)) \geq 0, \quad f(.) \geq 0, \tag{58}$$

$$l(1) = 1, \tag{59}$$

$$l(f(.)) = 0, \quad f(.) \in C_{b0}(\mathbb{Y}). \tag{60}$$

■

Let's formulate a criterion, for which every right-invariant mean on the quotient group G/H_G is singular.

Lemma 6. *Let $G \subseteq Homeo_{+}(\mathbb{R})$ is a group with nonempty minimal set. Then every right-invariant mean on the quotient group G/H_G is singular if and only if $G/H_G \neq < e >$.*

Proof. Let every right-invariant mean on the quotient group G/H_G is singular. Let's prove by contradiction. We assume that $G/H_G =< e >$. Then, by Remark 3, $H_G = G^S$ and for the group G there is an invariant Borel measure on the line, which is finite on compacts. In that case, by Theorem 8, the condition $Fix\ G^S \neq \emptyset$ is fair. Let's choose a point $\bar{t} \in Fix\ G^S$. The measure μ, concentrated at the point $\bar{t}$ with value $\mu(\bar{t}) = 1$, is an invariant measure for the group G. For any right-invariant mean m on the quotient group G/H_G the representation (40) is fair, which implies that $\mathcal{I}^*\mathcal{L}^*_{\bar{t}}\, m \neq 0$. Contradiction. Hence, it is true that $G/H_G \neq< e >$.

The converse. Let $G/H_G \neq< e >$. Let's prove by contradiction. We assume that there is a right-invariant mean m, which is nonsingular, i.e. $\mathcal{I}^*\mathcal{L}^*_{\bar{t}}\, m \neq 0$ for some $\bar{t}$, and $\mu \in \mathcal{M}^+(\mathbb{R})$ is nontrivial countably additive Borel measure, which is finite on compacts, and for which there is a representation (40). By Lemma 5, the measure μ is invariant ,relatively to the group G. Then, by Theorem 8, there is a condition $Fix\ G^S \neq \emptyset$ and, owing to the definition, there is an equality $H_G = G^S$. In that case, from the condition $G/H_G \neq< e >$ it follows that $G \neq G^S$. Thus, in the group G there is a freely acting element $g \in G \backslash G^S$.

Without loss of generality, we assume that $g(t) = t + 1$. Since the measure μ is invariant, relatively to the freely acting element g, then $\mu([0, 1]) \neq 0$ and $\mu(\mathbb{R}) = +\infty$. In that case, for each function $f_N(.) \in \mathcal{R}(\mathbb{R}), \quad N \in \mathbb{Z}^+$

$$f_N(t) = \begin{cases} 0, & t \in (-\infty,\, -(N+1)), \\ (N+1)+t, & t \in [-(N+1),\, -N), \\ 1, & t \in [-N, N], \\ (N+1)-t, & t \in (N,\, N+1], \\ 0, & t \in (N+1,\, +\infty). \end{cases}$$

we have the estimate

$$\mathcal{I}^*\mathcal{L}^*_{\bar{t}}\ m\,(f_N(.)) = \int_{\mathbb{R}} f_N(t)d\mu(t) \geq N\mu([0, 1])\,, \tag{61}$$

while for the right-invariant mean m we have that $m(1) = 1$. Contradiction. Thus there is not such nontrivial measure μ, i.e. there is not nonsingular right-invariant mean m. □

Consequence 1. *Let $G \subseteq Homeo_+(\mathbb{R})$ is a group with nonempty minimal set. For any right-invariant mean m on the quotient group G/H_G and for any $\bar{t}$ ($\bar{t} \in Fix\ G^S$, in case of discrete minimal set, otherwise $\bar{t} \in E(G)$, in case of nondiscrete minimal set) the linear functionals $m,\ \mathcal{L}^*_{\bar{t}}\ m$ are permanent.* ■

§4. On the Stone-Čech compactification.

Further the group $G \subseteq Homeo_+(\mathbb{R})$ and the quotient group G/H_G we will consider as discrete groups. Here are some concepts and facts related to the Stone-Čech compactification of a locally compact Hausdorff space.

Let $\mathbb{Y}$ is a locally compact Hausdorff space. Earlier (see §3) we considered the spaces of functions $C_b(\mathbb{Y})$, $C_{0b}(\mathbb{Y})$. Through $C_{\mathbb{Y}} \subseteq C_b(\mathbb{Y})$ we denote a subset of bounded continuous functions with values in the interval $[0,1]$. Through $[0,1]^{C_{\mathbb{Y}}}$ we denote the complete direct product with the Tikhonov topology. The elements of the space $[0,1]^{C_{\mathbb{Y}}}$ are infinite-dimensional vectors $\{\alpha_\varphi\}_{\varphi\in C_{\mathbb{Y}}}$, where $0 \leq \alpha_\varphi \leq 1$, $\varphi \in C_{\mathbb{Y}}$. It is obvious, that such space is Hausdorff and compact. There is a canonical embedding

$$\Phi : \mathbb{Y} \to [0,1]^{C_{\mathbb{Y}}}, \tag{62}$$

$$\Phi(y) = \{\Phi_\varphi(y)\}_{\varphi\in C_{\mathbb{Y}}} = \{\varphi(y)\}_{\varphi\in C_{\mathbb{Y}}}, \quad y \in \mathbb{Y}. \tag{63}$$

The closure of the image $\Phi(\mathbb{Y})$ (embedding) in the space $[0,1]^{C_{\mathbb{Y}}}$ is denoted by $\beta\mathbb{Y}$ and it is compact. The set $\beta\mathbb{Y}\backslash\mathbb{Y}$ is called the remainder. The embedding $\Phi : \mathbb{Y} \to \beta\mathbb{Y}$ of the topological space $\mathbb{Y}$ in the topological space $\beta\mathbb{Y}$ is a homeomorphism, and the image $\Phi(\mathbb{Y})$ is open in $\beta\mathbb{Y}$.

Every bounded continuous function $f(.) \in C_b(\mathbb{Y})$ on $\mathbb{Y}$ can be uniquely extended to a bounded continuous function $\hat{f}(.) \in C_b(\beta\mathbb{Y})$ on $\beta\mathbb{Y}$. Therefore, there is a natural isomorphism $C_b(\mathbb{Y}) \cong C_b(\beta\mathbb{Y})$. By this isomorphism, between all positive functionals on $C_b(\mathbb{Y})$ and all regular countably additive finite Borel measures, defined on $\beta\mathbb{Y}$, there is a bijective correspondence. Every positive functional $l \in C_b^*(\mathbb{Y})$ corresponds to the measure $\hat{\xi}$, which is such that for any $f(.) \in C_b(\mathbb{Y})$ there is an integral representation

$$l(f) = \int_{\beta\mathbb{Y}} \hat{f}\, d\hat{\xi}, \tag{64}$$

where $f\hat{(.)}$ is the extension of the continuous function $f(.)$ on the Stone-Čech compactification $\beta\mathbb{Y}$.

The measure $\hat{\xi}$ induces a measure ξ on the initial space $\mathbb{Y}$. Thus, the representation (66) can be rewritten in the following form

$$l(f) = \int_{\mathbb{Y}} f\, d\xi + \int_{\beta\mathbb{Y}\backslash\mathbb{Y}} \hat{f}\, d\hat{\xi}. \tag{65}$$

Proposition 2 ([22]). *A positive linear functional $l \in C_b^*(\mathbb{Y})$ is permanent if and only if the measure ξ, which represents it, is a Borel probability measure and has support in the Stone-Čech remainder $\beta\mathbb{Y}\backslash\mathbb{Y}$.* ■

Proposition 3 ([22]). *Let $T : \mathbb{Y} \to \mathbb{Y}$ is such topological map of a locally compact (but not compact) Hausdorff space $\mathbb{Y}$ into itself that $\mathbb{Y}\backslash T(\mathbb{Y})$ lies in some compact set. Then its continuous extension $\hat{T} : \beta\mathbb{Y} \to \beta\mathbb{Y}$ topologically maps $\beta\mathbb{Y}\backslash\mathbb{Y}$ onto itself.*

Let also T satisfies the following condition: every point $y \in \mathbb{Y}$ has such neighborhood U that the sets U, $T^{-1}U$, $T^{-2}U$, ... are pairwise disjoint. Then T-invariant permanent functionals $l \in C_b^*(\mathbb{Y})$ are exactly determined by Borel measures defined on $\beta\mathbb{Y}\backslash\mathbb{Y}$, which are invariant under the homeomorphism $\hat{T}_{\beta\mathbb{Y}\backslash\mathbb{Y}}$. The supports of these measures ξ are $\hat{T}$-invariant sets, i.e. $\hat{T}(supp\ \xi) = supp\ \xi$. ■

§5. The map of transposition on the line, induced by a special group of homeomorphisms of the line with two generators.

Let's show that a group with a freely acting element and without a projectively invariant measure contains some special subgroup. Using such special subgroup, we will construct a map of transposition on the line and an induced map of involution on Stone-Čech extension of a canonical quotient group, which will be formed by this special group.

5.1. The construction of a special group of homeomorphisms of the line with two generators.

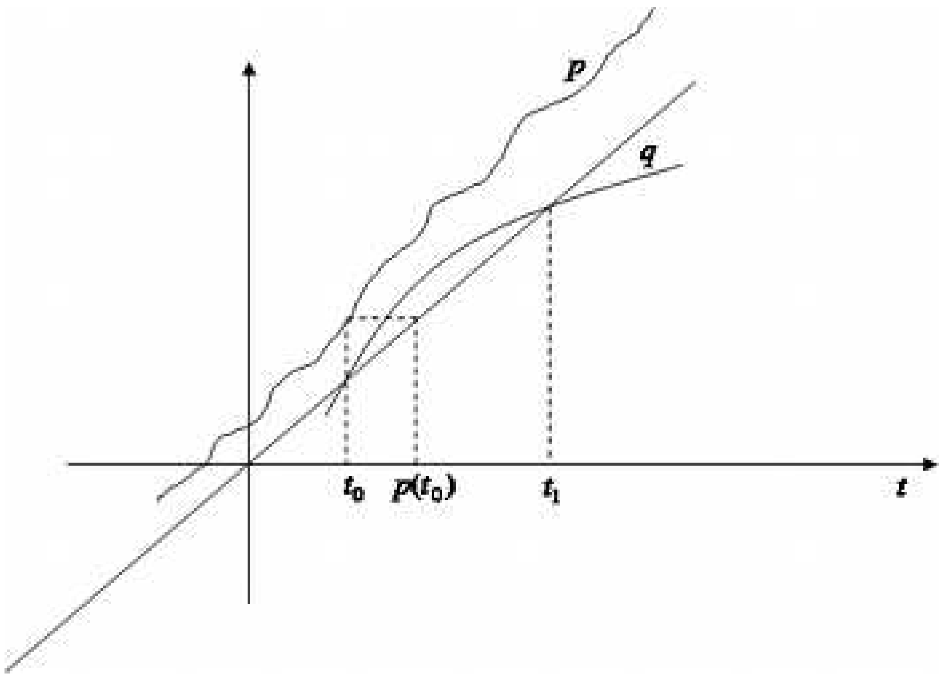
Let's formulate a lemma in which we describe the above-noted special group.

Lemma 7. *Let $G \subseteq Homeo_+(\mathbb{R})$, there is a freely acting element $\bar{g} \in G$ $(Fix\,\bar{g} = \emptyset)$ and for the group G there is not a projectively invariant measure. Then there is a subgroup $\Lambda \subseteq G$, $\quad \Lambda =< p, q >$, in which the element p is freely acting, and points $t_0, t_1 \in E(G)$, $\quad t_0 < t_1$ with following properties:*

$$q(t_0) = t_0, \quad q(t_1) = t_1, \qquad q(t) > t, \quad t \in (t_0, t_1); \tag{66}$$

$$p(t) > t, \quad t \in \mathbb{R}, \quad p(t_0) \in (t_0, t_1). \tag{67}$$

For the subgroup Λ there also is not a projectively invariant measure.

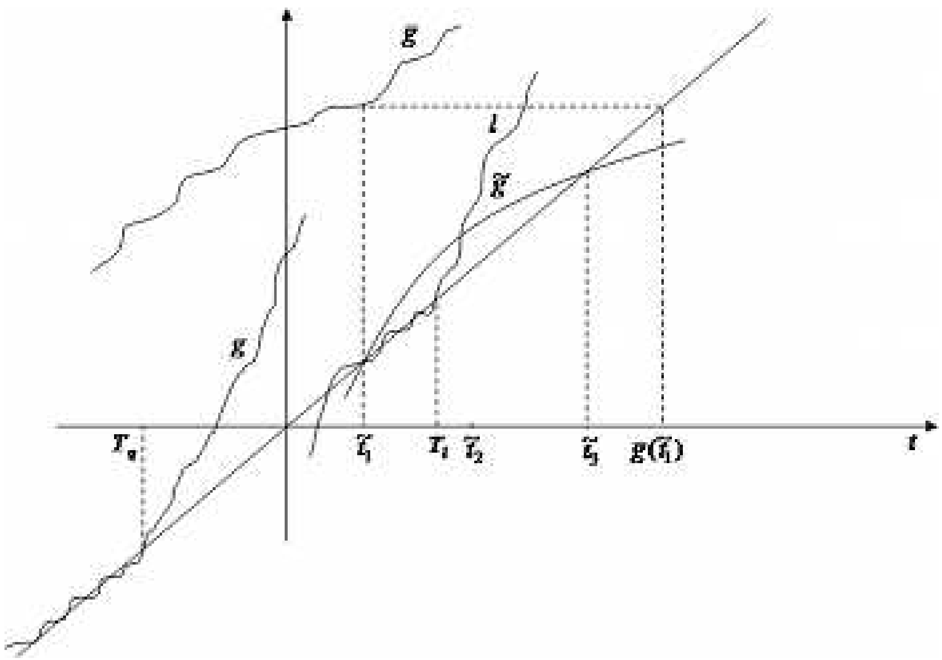


Proof. From the existence of a freely acting element and from Proposition 1, it follows that the minimal set is nonempty. Since for the group G there is not a projectively invariant measure then, by Remark 4, the minimal set is nondiscrete. Moreover, by Theorem 21 and Lemma 3, there is an element $\tilde{g}$ and a triple of points $\tilde{t}_1, \tilde{t}_2, \tilde{t}_3$ with the following properties:

$$\tilde{t}_1, \tilde{t}_2, \tilde{t}_3 \in E(G), \quad \tilde{t}_1 < \tilde{t}_2 < \tilde{t}_3, \qquad \tilde{g}(\tilde{t}_1) = \tilde{t}_1, \quad \tilde{g}(\tilde{t}_3) = \tilde{t}_3, \qquad \tilde{g}(t) > t, \quad t \in (\tilde{t}_1, \tilde{t}_3). \quad (68)$$

For definiteness, we assume that $\bar{g}(t) > t, \quad t \in \mathbb{R}$. If $\bar{g}(\tilde{t}_1) \in (\tilde{t}_1, \tilde{t}_3)$ then the group Λ is constructed. For this purpose it is enough to state $p = \bar{g}$, $q = \tilde{g}$, $t_0 = \tilde{t}_1$, $t_1 = \tilde{t}_3$.

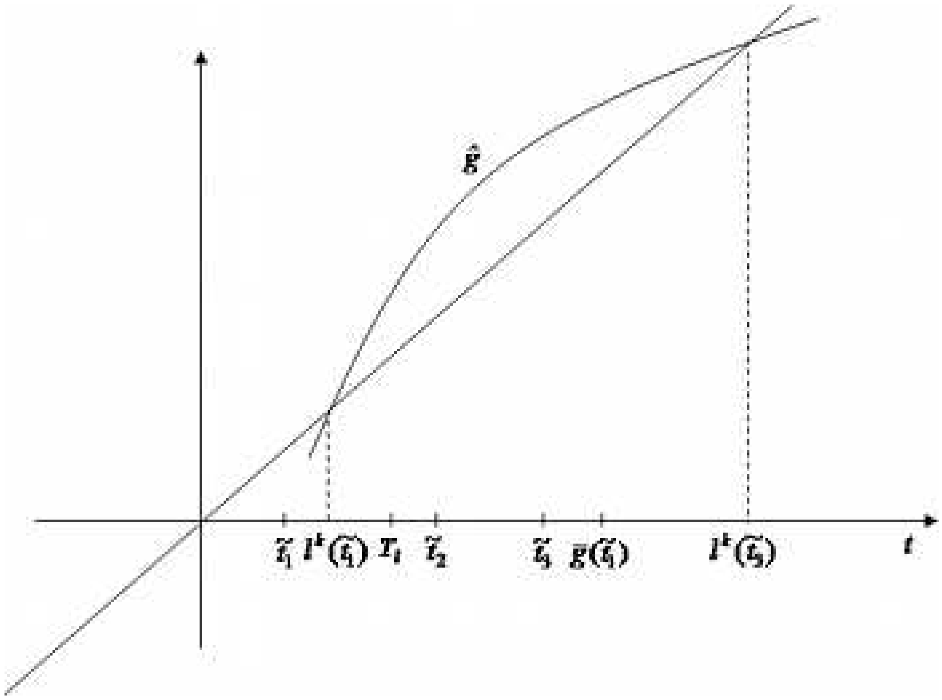
Let the condition $\bar{g}(\tilde{t}_1) > \tilde{t}_3$ is satisfied. Further we should consider two cases.
(i) The first case: $G^S \backslash G^S_\infty \neq \emptyset$. Let's choose an element $g \in G^S \backslash G^S_\infty$. For the element g one of the values t_g, T_g is finite. For definiteness, we assume that the value T_g is finite and $g(t) > t, \quad t \in (T_g, +\infty)$. It is obvious, that the point T_g belongs to the minimal set, i.e. $T_g \in E(G)$. Since $\tilde{t}_2 \in E(G) \cap (\tilde{t}_1, \tilde{t}_3)$ then, from the definition of the minimal set, it follows that there is an element $\xi \in G$, for which the inclusion $\xi(T_g) \in (\tilde{t}_1, \tilde{t}_3)$ is fair. Let's consider an element $l = \xi g \xi^{-1}$. It is obvious, that $l \in G^S \backslash G^S_\infty, \quad T_l = \xi(T_g), \qquad l(t) > t, \quad t \in (T_l, +\infty)$.



Let's form an element $\hat{g} = l^k \, \tilde{g} \, l^{-k}, \quad k \in \mathbb{Z}^+$. It is obvious, that for the element $\hat{g}$ the following conditions are fulfilled

$$\hat{g}(l^k(\tilde{t}_1)) = l^k(\tilde{t}_1), \quad \hat{g}(l^k(\tilde{t}_3)) = l^k(\tilde{t}_3), \quad l^k(\tilde{t}_1) < T_l, \quad \hat{g}(t) > t, \quad t \in (l^k(\tilde{t}_1), l^k(\tilde{t}_3)). \qquad (69)$$

By the choice of $k \in \mathbb{Z}^+$ the value $[l^k(\tilde{t}_3) - T_l]$ can be made arbitrarily large to satisfy the condition $\bar{g}(l^k(\tilde{t}_1)) \in (l^k(\tilde{t}_1), l^k(\tilde{t}_3))$.



It remains to state $p = \bar{g},\ q = \hat{g},\ t_0 = l^k(\tilde{t}_1),\ t_1 = l^k(\tilde{t}_3)$.
(ii) The second case: $G^S \backslash G^S_\infty = \emptyset, \quad (G^S = G^S_\infty)$. For such group G the equality $G = C_G$ takes place, i.e. the group consists of the union of the set G^S_∞ and the set of all freely acting elements. Moreover, for such group G the set G^S_∞ doesn't form a subgroup. Indeed, if the condition $G^S_\infty =< G^S_\infty >$ is satisfied then, in view of the existence of a freely acting element $\bar{g} \in G$, the quotient group G/G^S_∞ must be nontrivial. Then, by Theorem 9, for such group G there is an invariant measure that contradicts to conditions of this Lemma. Hence, we have that $G^S_\infty \neq< G^S_\infty >$. Then there are elements $g, g_1 \in G^S_\infty$ such that the element $\breve{g} = gg_1$ is freely acting. For definiteness, we assume that $\breve{g}(t) > t, \quad t \in \mathbb{R}$. There is an interval $[t_0, t_1]$, for which the conditions $g(t_0) = t_0, \quad g(t_1) = t_1, \qquad g(t) > t, \quad t \in (t_0, t_1)$ are satisfied. Moreover, for such interval the condition $\breve{g}(t_0) \in (t_0, t_1)$ is also fair. It remains to state $p = \breve{g}$, $q = g$. The group Λ is constructed.

Since p is a freely acting element then, by Proposition 1, the minimal set of the subgroup Λ is nonempty. Obviously, for the group Λ the set Λ^S doesn't form a group, i.e. $\Lambda^S \neq< \Lambda^S >$. Then, by Remark 3, for the group Λ there is not an invariant measure. In that case, by Theorem 8, it is true that $Fix\ \Lambda^S = \emptyset$ and, owing to Theorem 2, the minimal set is nondiscrete. From the condition $p(t_0) \in (t_0, t_1)$, it follows that $E(\Lambda) \cap (t_0, t_1) \neq \emptyset$ and hence $t_0, t_1 \in E(\Lambda)$. Therefore, the element q in the subgroup Λ is not plane. Then, by Theorem 21 and Lemma 3, for the subgroup Λ there is not a projectively invariant measure. Lemma is proved. $\square$

5.2. The construction of the map of transposition on the line and induced map of involution in the Stone-Čech extension of the discrete quotient group Λ/H_Λ.

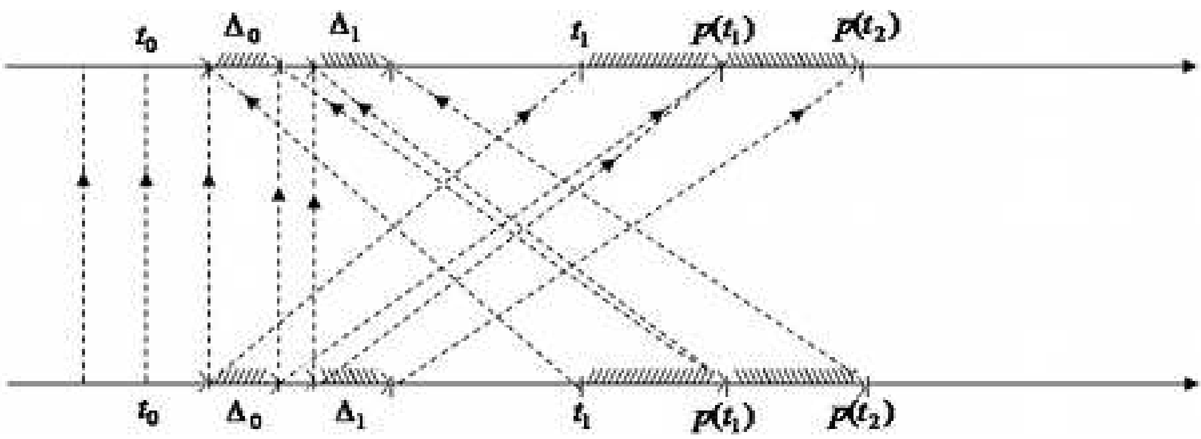
Using special group Λ, let's construct an important map of transposition on the line. Such map of transposition allows us to construct an induced map of involution in the Stone-Čech extension of the discrete quotient group Λ/H_Λ. Such constructions are necessary for studying of a organization of a support of an invariant measure on the Stone-Čech extension, which is present in the integral representation of a right-invariant mean on the discrete quotient group Λ/H_Λ. For brevity, let's use the notation $\mathbb{P} = \Lambda/H_\Lambda$

Without loss of generality, we assume that in the group Λ the freely acting element p has form $p(t) = t + 1, \quad t \in \mathbb{R}$. Using the generators p, q of the group Λ, let's form an element $b = p^{-1}q^k p^{-1}$. For a sufficiently large $k \in \mathbb{Z}$ for the homeomorphism b it is true that

$$bp(t_1) < qb(t_1). \tag{70}$$

Let's fix such large $k = \overline{k}$. Let's define a map of transposition $J_\mathbb{R} : \mathbb{R} \to \mathbb{R}$ by the following rule:

$$J_\mathbb{R}(t) = \begin{cases} q^r b p^{-r}(t), & t \in [p^r(t_1), p^{r+1}(t_1)\,), \quad r = 0, 1, ..., \\ p^r b^{-1} q^{-r}(t), & t \in [q^r b(t_1), q^r bp(t_1)\,), \quad r = 0, 1, ..., \\ t, & t \in \mathbb{R}\backslash[\cup_{r=0}^{\infty}[p^r(t_1), p^{r+1}(t_1)\,) \bigcup \cup_{r=0}^{\infty}[q^r b(t_1), q^r bp(t_1)\,)]. \end{cases} \tag{71}$$



It is obvious, that the map of transposition $J_\mathbb{R}$ performs a transposition of each unit half-interval $[p^r(t_1), p^{r+1}(t_1)\,) \quad r = 0, 1, ...$ with the half-interval $[q^r b(t_1), q^r bp(t_1))$, which belongs to the interval $[t_0, t_1)$.

For the map $J_\mathbb{R}$ it is true that $J_\mathbb{R}^2 = I$, where I is the identity map on $\mathbb{R}$, and also the commutation relation

$$J_\mathbb{R}(p(t)) = q(J_\mathbb{R}(t)), \quad t \in [t_1, +\infty) \tag{72}$$

is fair. If we fix a point $\bar{t} \in E(\Lambda)$ then for any point $t \in \Lambda(\bar{t})$ of the orbit its corresponding set

$$\{[\lambda] : \ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(\bar{t}) = t\} \tag{73}$$

by the right shift $[\tilde{\lambda}^{-1}]$ on the quotient group $\mathbb{P}$ $(\ \mathbb{P} \to \mathbb{P}[\tilde{\lambda}^{-1}]\ )$ becomes the set

$$\{[\lambda]:\ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(\bar{t}) = \tilde{\lambda}(t)\ \} \tag{74}$$

($[\lambda]$ is a left coset of the group Λ by the subgroup H_Λ).

Therefore, the set

$$\mathbb{P}(\ [t', t'')\ ) = \{[\lambda]:\ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(\bar{t}) \in [t', t''), \quad t', t'' \in E(\Lambda)\ \} \tag{75}$$

$$(\ \mathbb{P}(\ [t', +\infty)\ ) = \{[\lambda]:\ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(\bar{t}) \in [t', +\infty), \quad t' \in E(\Lambda)\ \}\ ) \tag{76}$$

by the right shift $[\tilde{\lambda}^{-1}]$ on the quotient group $\mathbb{P}$ becomes the set

$$\mathbb{P}(\ [\tilde{\lambda}(t'), \tilde{\lambda}(t''))\ ) = \{[\lambda]:\ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(\bar{t}) \in [\tilde{\lambda}(t'), \tilde{\lambda}(t''))\ \} \tag{77}$$

$$(\ \mathbb{P}(\ [\tilde{\lambda}(t'), +\infty)\ ) = \{[\lambda]:\ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(\bar{t}) \in [\tilde{\lambda}(t'), +\infty)\ \}\ ). \tag{78}$$

Let's suppose that $\bar{t} = t_1$. Then, owing to the discreteness of the topology on the quotient group $\mathbb{P}$, the map of transposition $J_{\mathbb{R}}$ on $\mathbb{R}$ naturally induces a continuous map of transposition $J_{\mathbb{P}} : \mathbb{P} \to \mathbb{P}$ on the quotient group $\mathbb{P}$

$$J_{\mathbb{P}}([\lambda]) = \begin{cases} [\lambda][p^r b^{-1} q^{-r}], & [\lambda] \in \{\ [\lambda]:\ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(t_1) \in [p^r(t_1), p^{r+1}(t_1)\ ) \quad r = 0, 1, ...\}, \\ [\lambda][q^r b p^{-r}], & [\lambda] \in \{\ [\lambda]:\ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(t_1) \in [q^r b(t_1), q^r b p(t_1)\ ), \quad r = 0, 1, ...\}, \\ [\lambda], & [\lambda] \in \{\ [\lambda]:\ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(t_1) \in \mathbb{R} \backslash [\cup_{r=0}^{\infty} [p^r(t_1), p^{r+1}(t_1)\ ) \bigcup \\ & \qquad \cup_{r=0}^{\infty} [q^r b(t_1), q^r b p(t_1)\ )]\ \}. \end{cases} \tag{79}$$

For the map $J_{\mathbb{P}}$ it is true that $J_{\mathbb{P}}^2 = I$, where I is the identity map on $\mathbb{P}$, and also the commutation relation

$$J_{\mathbb{P}}([\lambda][p^{-1}]) = J_{\mathbb{P}}([\lambda])[q^{-1}], \quad [\lambda] \in \{\ [\lambda]:\ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(t_1) \in [t_1, +\infty)\ \} \tag{80}$$

is fair. In turn, the continuous map of transposition $J_{\mathbb{P}}$ induces a continuous map of involution

$$\mathcal{J}\ :\ C_b(\mathbb{P}) \to C_b(\mathbb{P}) \tag{81}$$

by the following rule

$$(\mathcal{J}\varphi)(.) = \varphi(J_{\mathbb{P}}(.)), \quad \varphi(.) \in C_b(\mathbb{P}). \tag{82}$$

For the map $\mathcal{J}$ it is true that $\mathcal{J}^2 = I$, where I is the identity map on $C_b(\mathbb{P})$, and also the commutation relation

$$(\mathcal{J}\varphi)_{[p^{-1}]}([\lambda]) = (\mathcal{J}\varphi_{[q^{-1}]})([\lambda]), \quad [\lambda] \in \{\ [\lambda]:\ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(t_1) \in [t_1, +\infty)\ \} \tag{83}$$

is fair. A restriction of the involution $\mathcal{J}$ on the subspace $C_{\mathbb{P}} \subseteq C_b(\mathbb{P})$ we also denote as $\mathcal{J}$. Obviously, each function $\varphi(.) \in C_{\mathbb{P}}$ generates a function $\mathcal{J}\varphi$. Therefore, in the space $[0, 1]^{C_{\mathbb{P}}}$ with each coordinate φ there is a coordinate $\mathcal{J}\varphi$.

The elements of the space $[0,1]^{C_{\mathbb{P}}}$ we have previously denoted as an infinite-dimensional vector $\{\alpha_\varphi\}_{\varphi\in C_{\mathbb{P}}}, \quad 0\le\alpha_\varphi\le 1$. Using the continuous map of involution $\mathcal{J}$ of the space $C_{\mathbb{P}}$, in the space $[0,1]^{C_{\mathbb{P}}}$ we can also define a continuous map of involution

$$J_{C_{\mathbb{P}}}:\ [0,1]^{C_{\mathbb{P}}}\to[0,1]^{C_{\mathbb{P}}} \tag{84}$$

by the following rule

$$J_{C_{\mathbb{P}}}(\{\alpha_\varphi\}_{\varphi\in C_{\mathbb{P}}})=\{\bar\alpha_\varphi\}_{\varphi\in C_{\mathbb{P}}}=\{\bar\alpha_\varphi=\alpha_{\mathcal{J}\varphi}\}_{\varphi\in C_{\mathbb{P}}}.$$

For the map $J_{C_{\mathbb{P}}}$ it is true that $J^2_{C_{\mathbb{P}}}=I$, where I is the identity map on $[0,1]^{C_{\mathbb{P}}}$.

Let's consider the canonical embedding $\Phi:\ \mathbb{P}\to[0,1]^{C_{\mathbb{P}}}$ from (66), in which we suppose that $\mathbb{Y}=\mathbb{P}$.

Lemma 8. *Let $\Lambda=<p,q>$ is a special group from Lemma 7 and $\mathbb{P}=\Lambda/H_\Lambda$. For the canonical embedding Φ the following commutativity rule is satisfied*

$$\Phi(J_{\mathbb{P}}([\lambda]))=J_{C_{\mathbb{P}}}(\Phi([\lambda])). \tag{85}$$

Proof. By the definition, $\Phi(J_{\mathbb{P}}([\lambda]))=\{\varphi(J_{\mathbb{P}}([\lambda]))\}_{\varphi\in C_{\mathbb{P}}}$. Owing to the definition of the map of involution $\mathcal{J}$, the equality $\{\varphi(J_{\mathbb{P}}([\lambda]))\}_{\varphi\in C_{\mathbb{P}}}=\{(\mathcal{J}\varphi)([\lambda])\}_{\varphi\in C_{\mathbb{P}}}$ is fair and, from the definition of the map of involution $J_{C_{\mathbb{P}}}$, it follows that $\{(\mathcal{J}\varphi)([\lambda])\}_{\varphi\in C_{\mathbb{P}}}=J_{C_{\mathbb{P}}}(\Phi([\lambda]))$. Then from the obtained chain of equalities the statement of the lemma follows. □

For any $t\in[E(\Lambda)\cup+\infty]$ let's define a subset in the remainder $\beta\mathbb{P}\backslash\mathbb{P}$

$$\Delta_t=\bigcap_{t'<t}[\overline{\Phi(\mathbb{P}[t',t))}\cap(\beta\mathbb{P}\backslash\mathbb{P})]. \tag{86}$$

Lemma 9. *Let $\Lambda=<p,q>$ is a special group from Lemma 7 and $\mathbb{P}=\Lambda/H_\Lambda$. Then the following inclusion is fair*

$$J_{C_{\mathbb{P}}}(\Delta_{+\infty})\subseteq\Delta_{t_1}. \tag{87}$$

Proof. From the definition of the maps of transposition $J_{\mathbb{R}},\ J_{\mathbb{P}}$, it follows that for any $k=0,1,...$

$$J_{\mathbb{P}}(\,\mathbb{P}[p^k(t_1),+\infty))\subseteq\mathbb{P}(\,[q^kJ_{\mathbb{R}}(t_1),t_1)) \tag{88}$$

and, respectively,

$$\Phi(\,J_{\mathbb{P}}(\,\mathbb{P}[p^k(t_1),+\infty)))\subseteq\Phi(\,\mathbb{P}(\,[q^kJ_{\mathbb{R}}(t_1),t_1))). \tag{89}$$

Then, from the commutativity rule (85), it follows that

$$J_{C_{\mathbb{P}}}(\,\Phi(\,\mathbb{P}[p^k(t_1),+\infty)))\subseteq\Phi(\,\mathbb{P}(\,[q^kJ_{\mathbb{R}}(t_1),t_1))). \tag{90}$$

Since

$$p^k(t_1) \to +\infty, \quad q^k J_{\mathbb{R}}(t_1) \to t_1 \quad \text{for } k \to +\infty \tag{91}$$

and since the equalities

$$\Delta_{+\infty} = \bigcap_{k=0}^{+\infty} [\overline{\Phi(\mathbb{P}[p^k(t_1), +\infty))} \cap (\beta\mathbb{P}\backslash\mathbb{P})], \qquad \Delta_{t_1} = \bigcap_{k=0}^{+\infty} [\overline{\Phi(\mathbb{P}[q^k J_{\mathbb{R}}(t_1), t_1))} \cap (\beta\mathbb{P}\backslash\mathbb{P})] \tag{92}$$

are fulfilled then the statement of the lemma immediately follows from the inclusion (90). $\square$

Using the concept of the maps of transposition and involution, let's construct a series of similar maps. For each $k = 0, 1, ...$ let's define a map of transposition $J_{\mathbb{R},k} : \mathbb{R} \to \mathbb{R}$ by the following rule:

$$J_{\mathbb{R},k}(t) = \begin{cases} p^k q^r b p^{-r}(t), & t \in [p^{(k+r)}(t_1), p^{(k+r+1)}(t_1)\,), \quad r = 0, 1, ..., \\ p^r b^{-1} q^{-r}(t), & t \in [q^{(k+r)}b(t_1), q^{(k+r)}bp(t_1)\,), \quad r = 0, 1, ..., \\ t, & t \in \mathbb{R}\backslash[\cup_{r=0}^{\infty}[p^{(k+r)}(t_1), p^{(k+r+1)}(t_1)\,) \bigcup \cup_{r=0}^{\infty}[q^{(k+r)}b(t_1), q^{(k+r)}bp(t_1)\,)]. \end{cases} \tag{93}$$

It is obvious, that the map of transposition $J_{\mathbb{R},k}$ performs a transposition of each unit half-interval $[p^{(k+r)}(t_1), p^{(k+r+1)}(t_1)\,)$ $r = 0, 1, ...$ with the half-interval $[q^{(k+r)}b(t_1), q^{(k+r)}bp(t_1)\,)$, which belongs to the interval $[p^k(t_0), p^k(t_1)\,)$ и $J_{\mathbb{R},0} = J_{\mathbb{R}}$.

For the map $J_{\mathbb{R},k}$ it is true that $J^2_{\mathbb{R},k} = I$, where I is the identity map on $\mathbb{R}$, and also the commutation relation

$$J_{\mathbb{R},k}(p(t)) = q(J_{\mathbb{R},k}(t)), \quad t \in [p^k(t_1), +\infty) \tag{94}$$

is fair.

Let's suppose that $\bar{t} = t_1$. Then the map of transposition $J_{\mathbb{R},k}, \quad k = 0, 1, ...$ on $\mathbb{R}$ naturally induces a continuous map of transposition $J_{\mathbb{P},k} : \mathbb{P} \to \mathbb{P}$ on the quotient group $\mathbb{P}$

$$J_{\mathbb{P},k}([\lambda]) = \begin{cases} [\lambda][p^r b^{-1} q^{-r} p^{-k}], & [\lambda] \in \{\,[\lambda] : \ [\lambda] \in \mathbb{P}, \ \lambda^{-1}(t_1) \in [p^{(k+r)}(t_1), p^{(k+r+1)}(t_1)\,), \ r = 0, 1, ...\}, \\ [\lambda][q^r b p^{-r}], & [\lambda] \in \{\,[\lambda] : \ [\lambda] \in \mathbb{P}, \ \lambda^{-1}(t_1) \in [q^{(k+r)}b(t_1), q^{(k+r)}bp(t_1)\,), \ r = 0, 1, ...\}, \\ [\lambda], & [\lambda] \in \{\,[\lambda] : \ [\lambda] \in \mathbb{P}, \ \lambda^{-1}(t_1) \in \mathbb{R}\backslash[\cup_{r=0}^{\infty}[p^{(k+r)}(t_1), p^{(k+r+1)}(t_1)\,) \bigcup \\ & \qquad \cup_{r=0}^{\infty}[q^{(k+r)}b(t_1), q^{(k+r)}bp(t_1)\,)]\,\}. \end{cases} \tag{95}$$

It is obvious, that $J_{\mathbb{P},0} = J_{\mathbb{P}}$. For the map $J_{\mathbb{P},k}$ it is true that $J^2_{\mathbb{P},k} = I$, where I is the identity map on $\mathbb{P}$, and also the commutation relation

$$J_{\mathbb{P},k}([\lambda][p^{-1}]) = J_{\mathbb{P},k}([\lambda])[q^{-1}], \quad [\lambda] \in \{\,[\lambda] : \ [\lambda] \in \mathbb{P}, \quad \lambda^{-1}(t_1) \in [p^k(t_1), +\infty)\,\} \tag{96}$$

is fair.

In turn, the continuous map of transposition $J_{\mathbb{P},k}, \quad k=0,1,...$ induces a continuous map of involution

$$\mathcal{J}_k:\ C_b(\mathbb{P})\to C_b(\mathbb{P}) \tag{97}$$

by the following rule

$$(\mathcal{J}_k\varphi)(.)=\varphi(J_{\mathbb{P},k}(.)), \quad \varphi(.)\in C_b(\mathbb{P}). \tag{98}$$

For the map $\mathcal{J}_k$ it is true that $\mathcal{J}_k^2=I$, where I is the identity map on the space $C_b(\mathbb{P})$, and also the commutation relation

$$(\mathcal{J}_k\varphi)_{[p^{-1}]}([\lambda])=(\mathcal{J}_k\varphi_{[q^{-1}]})([\lambda]), \quad [\lambda]\in\{\,[\lambda]:\ [\lambda]\in\mathbb{P}, \quad \lambda^{-1}(t_1)\in[p^k(t_1),+\infty)\,\} \tag{99}$$

is fair. A restriction of the involution $\mathcal{J}_k$ on the subspace $C_{\mathbb{P}}\subseteq C_b(\mathbb{P})$ we also denote as $\mathcal{J}_k$. Obviously, each function $\varphi(.)\in C_{\mathbb{P}}$ generates a function $\mathcal{J}_k\varphi$. Therefore, in the space $[0,1]^{C_{\mathbb{P}}}$ with each coordinate φ there is a coordinate $\mathcal{J}_k\varphi$.

The elements of the space $[0,1]^{C_{\mathbb{P}}}$ we have previously denoted as an infinite-dimensional vector $\{\alpha_\varphi\}_{\varphi\in C_{\mathbb{P}}}, \quad 0\le\alpha_\varphi\le 1$. Using the map of involution $\mathcal{J}_k$ of the space $C_{\mathbb{P}}$, in the space $[0,1]^{C_{\mathbb{P}}}$ we can also define a continuous map of involution

$$J_{C_{\mathbb{P},k}}:\ [0,1]^{C_{\mathbb{P}}}\to[0,1]^{C_{\mathbb{P}}} \tag{100}$$

by the following rule

$$J_{C_{\mathbb{P},k}}(\{\alpha_\varphi\}_{\varphi\in C_{\mathbb{P}}})=\{\bar{\alpha}_\varphi\}_{\varphi\in C_{\mathbb{P}}}=\{\bar{\alpha}_\varphi=\alpha_{\mathcal{J}_k\varphi}\}_{\varphi\in C_{\mathbb{P}}}.$$

For the map $J_{C_{\mathbb{P},k}}$ it is true that $J^2_{C_{\mathbb{P},k}}=I$, where I is the identity map on the space $[0,1]^{C_{\mathbb{P},k}}$.

Let's consider the canonical embedding $\Phi:\ \mathbb{P}\to[0,1]^{C_{\mathbb{P}}}$ from (66), in which we suppose that $\mathbb{Y}=\mathbb{P}$. Let's formulate an analog of Lemma 8.

Lemma 10. *Let $\Lambda=<p,q>$ is a special group from Lemma 7 and $\mathbb{P}=\Lambda/H_\Lambda$. For the canonical embedding Φ the following commutativity rule is satisfied*

$$\Phi(J_{\mathbb{P},k}([\lambda]))=J_{C_{\mathbb{P},k}}(\Phi([\lambda])), \quad k=0,1,\dots\,. \tag{101}$$

Proof. By the definition, $\Phi(J_{\mathbb{P},k}([\lambda]))=\{\varphi(J_{\mathbb{P},k}([\lambda]))\,\}_{\varphi\in C_{\mathbb{P}}}$. Owing to the definition of the map of involution $\mathcal{J}_k$, the equality $\{\varphi(J_{\mathbb{P},k}([\lambda]))\,\}_{\varphi\in C_{\mathbb{P}}}=\{(\mathcal{J}_k\varphi)([\lambda])\,\}_{\varphi\in C_{\mathbb{P}}}$ is fair and ,from the definition of the map of involution $J_{C_{\mathbb{P},k}}$, it follows that $\{(\mathcal{J}_k\varphi)([\lambda])\,\}_{\varphi\in C_{\mathbb{P}}}=J_{C_{\mathbb{P},k}}(\Phi([\lambda]))$. Then from the obtained chain of the equalities the statement of the lemma follows.. ☐

Let's formulate an analog of Lemma 9.

Lemma 11. *Let $\Lambda=<p,q>$ is a special group from Lemma 7 and $\mathbb{P}=\Lambda/H_\Lambda$. Then the following inclusions are fair*

$$J_{C_{\mathbb{P},k}}(\Delta_{+\infty})\subseteq\Delta_{p^k(t_1)} \quad k=0,1,\dots\,. \tag{102}$$

Proof. From the definition of the maps of transposition $J_{\mathbb{R},k}$, $J_{\mathbb{P},k}$, it follows that for any $r = 0, 1, ...$

$$J_{\mathbb{P},k}(\,\mathbb{P}[p^{(k+r)}(t_1), +\infty)) \subseteq \mathbb{P}(\,[q^{(k+r)}J_{\mathbb{R},k}(t_1), t_1)) \tag{103}$$

and, respectively,

$$\Phi(\,J_{\mathbb{P},k}(\,\mathbb{P}[p^{(k+r)}(t_1), +\infty))) \subseteq \Phi(\,\mathbb{P}(\,[q^{(k+r)}J_{\mathbb{R},k}(t_1), t_1))). \tag{104}$$

Then, from the commutativity rule (101), it follows that

$$J_{C_{\mathbb{P}},k}(\,\Phi(\,\mathbb{P}[p^{(k+r)}(t_1), +\infty))) \subseteq \Phi(\,\mathbb{P}(\,[q^{(k+r)}J_{\mathbb{R},k}(t_1), t_1))). \tag{105}$$

Since

$$p^{(k+r)}(t_1) \to +\infty, \quad q^{(k+r)}J_{\mathbb{R},k}(t_1) \to t_1 \quad \text{for } r \to +\infty \tag{106}$$

and since the equalities

$$\Delta_{+\infty} = \bigcap_{r=0}^{+\infty} [\overline{\Phi(\mathbb{P}[p^{(k+r)}(t_1), +\infty))} \cap (\beta\mathbb{P}\backslash\mathbb{P})], \qquad \Delta_{p^k(t_1)} = \bigcap_{r=0}^{+\infty} [\overline{\Phi(\mathbb{P}[q^{(k+r)}J_{\mathbb{R},k}(t_1), t_1))} \cap (\beta\mathbb{P}\backslash\mathbb{P})], \tag{107}$$

are fulfilled then the statement of the lemma immediately follows from the inclusion (105). $\square$

In the space $[0,1]^{C_{\mathbb{P}}}$ let's define a continuous map $\mathcal{P}$ by the following rule

$$\mathcal{P}(\{\alpha_\varphi\}_{\varphi\in C_{\mathbb{P}}}) = \{\bar{\alpha}_\varphi\}_{\varphi\in C_{\mathbb{P}}} = \{\bar{\alpha}_\varphi = \alpha_{\varphi_{[p^{-1}]}}\}_{\varphi\in C_{\mathbb{P}}}.$$

Lemma 12. *Let $\Lambda =< p, q >$ is a special group from Lemma 7 and $\mathbb{P} = \Lambda/H_\Lambda$. Then the following commutativity rule is satisfied*

$$\Phi([\lambda][p^{-1}]) = \mathcal{P}\Phi([\lambda]). \tag{108}$$

A restriction of the map $\mathcal{P}$ on the Stone-Čech extension $\beta\mathbb{P}$ is a continuous extension of the right shift $[p^{-1}]$ on the group $\mathbb{P}$.

Proof. Indeed, for any $[\lambda] \in \mathbb{P}$ there ia a chain of equalities

$$\Phi([\lambda][p^{-1}]) = \{\varphi([\lambda][p^{-1}])\}_{\varphi\in C_{\mathbb{P}}} = \{\varphi_{[p^{-1}]}\}_{\varphi\in C_{\mathbb{P}}} = \mathcal{P}\Phi([\lambda])\,, \tag{109}$$

which implies the equality (108). Since each right shift on the group $\mathbb{P}$ is a homeomorphism of the discrete space $\mathbb{P}$ then the last statement of the lemma follows from the continuity of the map $\mathcal{P}$ and from the equality (108). $\square$

Let's remember that, by the definition, the set $\mathbb{P}(\,[p^k(t_1),+\infty)\,)$ has the following representation

$$\mathbb{P}(\,[p^k(t_1),+\infty)\,)=\{[\lambda]:\ [\lambda]\in\mathbb{P},\quad \lambda^{-1}(t_1)\in[p^k(t_1),+\infty)\ \}.$$

Lemma 13. *Let $\Lambda=<p,q>$ is a special group from Lemma 7 and $\mathbb{P}=\Lambda/H_\Lambda$. Then for each fixed $k=0,1,...$ there is the following commutativity rule:*

$$J_{\mathbb{R},k}(t)=p^k(J_{\mathbb{R}}(t)),\quad t\in[p^k(t_1),+\infty); \tag{110}$$
$$J_{\mathbb{P},k}([\lambda])=J_{\mathbb{P}}([\lambda])[p^{-k}],\ \ (\mathcal{J}_k\varphi)([\lambda])=(\mathcal{J}\varphi_{[p^{-k}]})([\lambda]),\ \ (J_{C_{\mathbb{P},k}}(\Phi([\lambda]))=J_{C_{\mathbb{P}}}\mathcal{P}^k(\Phi([\lambda])), \tag{111}$$
$$[\lambda]\in\mathbb{P}(\,[p^k(t_1),+\infty)\,). \tag{112}$$

Proof. The first two commutativity relations directly follow from the definitions of considered maps. The third commutativity relation follows from the next chain of equalities

$$(\mathcal{J}_k\varphi)([\lambda])=\varphi(J_{\mathbb{P},k}([\lambda]))=\varphi(J_{\mathbb{P}}([\lambda][p^{-k}]))=\varphi_{[p^{-k}]}(J_{\mathbb{P}}([\lambda]))=(\mathcal{J}\varphi_{[p^{-k}]})([\lambda]), \tag{113}$$

where at the third step we use the second commutativity relation. It remains to show the last commutativity relation. From the commutativity relation (101) and from the definition of the embedding Φ, it follows that

$$\Phi(J_{\mathbb{P},k}([\lambda]))=J_{C_{\mathbb{P},k}}(\Phi([\lambda]))=\{\varphi(J_{\mathbb{P},k}([\lambda]))\,\}_{\varphi\in C_{\mathbb{P}}}. \tag{114}$$

Owing to the definition of the map $\mathcal{J}_k$ and to the third commutativity relation, we obtain that

$$\{\varphi(J_{\mathbb{P},k}([\lambda]))\,\}_{\varphi\in C_{\mathbb{P}}}=\{(\mathcal{J}_k\varphi)([\lambda])\,\}_{\varphi\in C_{\mathbb{P}}}=\{(\mathcal{J}\varphi_{[p^{-k}]})([\lambda])\,\}_{\varphi\in C_{\mathbb{P}}}. \tag{115}$$

Finally, from the definitions of the maps $J_{C_{\mathbb{P}}}$, $\mathcal{P}$ and the embedding Φ, we obtain the following chain of equalities

$$\{(\mathcal{J}\varphi_{[p^{-k}]})([\lambda])\,\}_{\varphi\in C_{\mathbb{P}}}=J_{C_{\mathbb{P}}}(\,\{\varphi_{[p^{-k}]})([\lambda])\,\}_{\varphi\in C_{\mathbb{P}}}\,)=J_{C_{\mathbb{P}}}\mathcal{P}^k(\,\{\varphi([\lambda])\,\}_{\varphi\in C_{\mathbb{P}}}\,)=J_{C_{\mathbb{P}}}\mathcal{P}^k(\,\{\Phi([\lambda])\,)\,, \tag{116}$$

which implies the last commutativity relation. □

Let's formulate a result, which clarifies Lemma 11.

Proposition 4. *Let $\Lambda=<p,q>$ is a special group from Lemma 7 and $\mathbb{P}=\Lambda/H_\Lambda$. Then there is an inclusion*

$$J_{C_{\mathbb{P}}}\mathcal{P}^k(\Delta_{+\infty})\subseteq\Delta_{p^k(t_1)}\quad k=0,1,...\ . \tag{117}$$

Proof. Since the inclusion $\Delta_{+\infty}\subseteq\Phi(\mathbb{P}(p^k(t_1),+\infty)))$ takes place then, owing to the last commutativity relation from Lemma 13, there is an equality

$$J_{C_{\mathbb{P},k}}(\Delta_{+\infty})=J_{C_{\mathbb{P}}}\mathcal{P}^k(\Delta_{+\infty})\quad k=0,1,...\ . \tag{118}$$

Then the statement of the lemma follows from the corresponding inclusion from Lemma 11. □

Let's formulate some facts about a structure of the set $\Delta_{+\infty}$ from the Stone-Čech remainder $\beta\mathbb{P}\backslash\mathbb{P}$.

Proposition 5. *Let $\Lambda =< p, q >$ is a special group from Lemma 7 and $\mathbb{P} = \Lambda/H_{\Lambda}$. In the set $\Delta_{+\infty}$, which belongs to the Stone-Čech remainder $\beta\mathbb{P}\backslash\mathbb{P}$, there is not any subset, which is invariant, relatively to the map $\mathcal{P}$.*

Proof. Let's prove by contradiction. Let there is a subset $B \subseteq \Delta_{+\infty}$, for the condition $\mathcal{P}(B) = B$ is satisfied. Then from the inclusion (117) it follows that for any $k = 0, 1, ...$ the inclusion

$$J_{C_{\mathbb{P}}}(B) \subseteq \Delta_{p^k(t_1)} \tag{119}$$

is fair. Since $p^k(t_1) \to +\infty$ for $k \to +\infty$ then it follows that

$$J_{C_{\mathbb{P}}}(B) \subseteq \Delta_{+\infty}. \tag{120}$$

On the other hand, for any $t \in E(t_1)$ there is a continuous function φ on $\mathbb{P}$ of the form

$$\varphi([\lambda]) = \begin{cases} 0, & \lambda^{-1}(t_1) \in (-\infty, t], \\ 1, & \lambda^{-1}(t_1) \in (t, +\infty). \end{cases} \tag{121}$$

From the existence of such function it follows that $\Delta_t \cap \Delta_{+\infty} = \emptyset$, which contradicts to the conditions (119), (120). Consequently, there is not a subset $B \subseteq \Delta_{+\infty}$, which is invariant, relatively to the map $\mathcal{P}$. □

§6. The classification theorem for groups of homeomorphisms of the line.

For arbitrary groups of homeomorphisms of the line, let's formulate a new criterion of the existence of a projectively invariant measure in terms of amenability. Such criterion is central to the classification scheme for groups of homeomorphisms of the line. In the next section, as a consequence from this criterion, the nonamenability of Thompson's group F will be proved.

Theorem A. *Let $G \subseteq Homeo_{+}(\mathbb{R})$ is a group with nonempty minimal set and $G/H_G \neq< e >$. Then there is a projectively invariant measure if and only if the quotient group G/H_G is amenable and there is a freely acting element $\bar{g} \in G$.*

Proof.
Necessity. Let there is a projectively invariant measure.
(i) The first case. For the group G there is an invariant measure. By Remark 3, for such group it is true that $H_G = G^S$, which implies the condition $G^S =< G^S >$. Then, by Theorem 1, the quotient group G/H_G is commutative and thus it is amenable. Since $G/H_G = G/ < G^S >$, $G/H_G \neq< e >$ then it follows that there is a freely acting element $\bar{g} \in G$.

(ii) The second case. For the group G there is not an invariant measure. By Remark 5, for such group it is true that $H_G = G^S_\infty$ and the quotient groups $G/C_G, \quad C_G/G^S_\infty$ are commutative. In that case, the quotient group G/H_G is a solvable group and thus it is amenable. By the same Remark 5, it follows that the quotient group C_G/G^S_∞ is not trivial, which implies the existence of a freely acting element $\bar{g} \in C_G$.

Sufficiency. Let the quotient group G/H_G is amenable and there is a freely acting element $\bar{g} \in G$. Let's prove by contradiction. We suppose that for the group G there is not a projectively invariant measure. By Lemma 7, in the group G there is a special subgroup Λ with two generators $\Lambda =< p, \ q >$, for which there also is not a projectively invariant measure.

Step 1. *The proof of the amenability of the quotient group Λ/H_Λ.*

Since $\hat{p} \in \Lambda$ is a freely acting element, then, by Proposition 1, the minimal set of the group Λ is also nonempty. Owing to the absence of an invariant measure, from Theorem 8 it follows that $Fix\ \Lambda^S = \emptyset$. Then, by Theorem 2, the minimal set of the group Λ is nondiscrete. From the nondiscreteness of the minimal set $E(\Lambda)$ and from Theorem 3 it follows that $E(\Lambda) \subseteq E(G)$. Let's note that the subgroup $D = H_G \cap \Lambda$ is normal in Λ and contains in normal subgroup H_Λ. Then the isomorphism $(\Lambda/D)/(H_\Lambda/D) \cong \Lambda/H_\Lambda$ is fair. Obviously, the quotient groups $\Lambda/D, \ (H_\Lambda/D)$ are naturally embedded into the quotient group G/H_G. In that case, from the amenability of the quotient group G/H_G it follows the amenability of the quotient group Λ/H_Λ.

Step 2. *The representation of the right-invariant mean m on the quotient group Λ/H_Λ.*

Since the quotient group Λ/H_Λ is amenable then there is a right-invariant mean m on it. Owing to the absence of a projectively invariant measure for the group Λ, it follows that $\Lambda/H_\Lambda \neq< e >$. Then, by Lemma 6, the right-invariant mean m is singular. It means that for each given point $\bar{t} \in E(\Lambda)$ there is an equality

$$m(F_I) = 0 \tag{122}$$

for any characteristic function

$$F_I([\lambda]) = \begin{cases} 1, & \lambda^{-1}(\bar{t}) \in I, \\ 0, & \lambda^{-1}(\bar{t}) \notin I \end{cases}$$

on the quotient group Λ/H_Λ, where $[\lambda]$ is a left coset of the group Λ by the subgroup H_Λ, and $I \subset \mathbb{R}$ is an arbitrary finite interval.

The right-invariant mean is uniquely represented as a convex combination

$$m = \alpha m_- + (1-\alpha) m_+, \qquad 0 \le \alpha \le 1 \tag{123}$$

of two right-invariant means $m_-, \ m_+$, which are uniquely characterized by the following properties

$$m_-(F_+) = 0, \quad m_-(F_-) = m(F_-), \qquad m_+(F_-) = 0, \quad m_+(F_+) = m(F_+)\,, \tag{124}$$

where relations (124) must be satisfied for each of the bounded continuous functions $F_-, \ F_+ \in C_b(\Lambda/H_\Lambda)$ such that

$$F_+([\lambda]) = 0, \quad \text{if} \quad \lambda^{-1}(\bar{t}) \in (-\infty, 0); \qquad F_-([\lambda]) = 0, \quad \text{if} \quad \lambda^{-1}(\bar{t}) \in (0, \infty). \tag{125}$$

Further research will be carried out for the right-invariant mean m_+ (if $\alpha \neq 1$). Similarly, we can carry out a research for the right-invariant mean m_- (if $\alpha \neq 0$). In the previous paragraph we mentioned that, for brevity, we will use the notation $\mathbb{P} = \Lambda/H_\Lambda$, where the group $\mathbb{P}$ is considered as a discrete group. Obviously, from the singularity of the right-invariant mean m it follows the singularity of the right-invariant mean m_+ and, by Consequence 1, it is also permanent. Then, by Proposition 2 and representation (65), there is a Borel probability measure $\hat{\xi}_+$ with a support on the remainder $\beta\mathbb{P}\backslash\mathbb{P}$ such that for any bounded (continuous) function $F(.) \in C_b(\mathbb{P})$ there is a representation

$$m_+(F) = \int_{\beta\mathbb{P}\backslash\mathbb{P}} \hat{F}\ d\hat{\xi}_+, \tag{126}$$

where $\hat{F(.)}$ is an extension of the continuous function $F(.)$ on the Stone-Čech extension $\beta\mathbb{P}$.
Step 3. *The proof of the absence of the measure* $\hat{\xi}_+$.

Let the measure $\hat{\xi}_+$ exists. Since each right shift on the group $\mathbb{P}$ is a homeomorphism of the discrete space $\mathbb{P}$ then it is a topological map. Moreover, the right shift, as a topological map of the space $\mathbb{P}$, satisfies the conditions of Proposition 3. Hence the measure $\hat{\xi}_+$, corresponding to the right-invariant mean m_+ on $\mathbb{P}$, is invariant, relatively to the extension of the right shift on the Stone-Čech extension $\beta\mathbb{P}$, and a support $supp\ \xi_+$ is also invariant, relatively to the extension of the right shift on the Stone-Čech extension $\beta\mathbb{P}$. In particular, this holds for the right shift $[p^{-1}]$ on the group $\mathbb{P}$. Therefore, the support of the measure $\hat{\xi}_+$ is invariant relatively to the map $\mathcal{P}$, i.e. $\mathcal{P}(supp\ \hat{\xi}_+) = supp\ \hat{\xi}_+$. In previous step we have already mentioned that the right-invariant mean m_+ is singular. From the singularity of m_+ it follows that for the support of the measure $\hat{\xi}_+$ the inclusion $supp\ \hat{\xi}_+ \subseteq \Delta_{+\infty}$ is fulfilled. Obtained inclusion contradicts Proposition 5. Consequently, there is not such measure $\hat{\xi}_+$.
Step 4. *The proof of the absence of the measure* $\hat{\xi}_-$.

Let's define a map $\theta : \mathbb{R} \to \mathbb{R}$ in form of $\theta(t) = -t, \quad t \in \mathbb{R}$. Together with the group Λ let's consider a group $\Lambda' = \theta\Lambda\theta^{-1}$. It is also a group with two generators $\Lambda' =< p', q' >$, $p' = \theta p^{-1}\theta^{-1},\ q' = \theta q^{-1}\theta^{-1}$. The element p' is also freely acting and elements p', q' satisfy the conditions

$$q'(t_0') = t_0',\ \ q'(t_1') = t_1', \quad q'(t) > t, \quad t \in (t_0', t_1'), \quad t_0' = \theta(t_1),\ \ t_1' = \theta(t_0), \tag{127}$$

$$p'(t) > t, \quad t \in \mathbb{R}, \quad p'(t_0') \in (t_0', t_1'). \tag{128}$$

Thus, the group Λ' satisfies the same conditions as the group Λ (Lemma 7). Their minimal sets are related by the condition $\theta(E(\Lambda)) = E(\Lambda')$. Invariant mean m on the group Λ goes to the invariant mean m' on the group Λ' such that $m' = (1-\alpha)m_-' + \alpha m_+'$, where $m_-' = m_+,\ \ m_+' = m_-$. The measures $\hat{\xi}_-'\ \hat{\xi}_+'$, corresponding to the invariant mean m', are such that $\hat{\xi}_-' = \hat{\xi}_+, \quad \hat{\xi}_+' = \hat{\xi}_-$. Owing to the step 3, the measure $\hat{\xi}_+'$, which is equal to the measure $\hat{\xi}_-$, also doesn't exist.
Step 5. *Completion of the proof of the theorem.*

The absence of the measures $\hat{\xi}_-,\ \ \hat{\xi}_+$ contradicts the condition of amenability of the quotient group Λ/H_Λ, and, accordingly, the condition of amenability of the quotient group G/H_G.

Contradiction was obtained by assuming that there is not a projectively invariant measure for the group G. Therefore, for the group G there is a projectively invariant measure. Theorem is proved. $\square$

Let's give an equivalent reformulation for the criterion from Theorem A. This criterion is an elaboration of the criterion from Theorem 18 for a wider class of groups. The criterion from Theorem 18 was formulated for groups, which contain a subgroup with invariant measure and with a freely acting element, and a new one is formulated for groups, which contain only a freely acting element. Expansion of the class of groups requires some weakening of conditions of the criterion. In this case the condition of the existence of a free subgroup with two generators in canonical quotient group in Theorem 18 is replaced by the condition of nonamenability.

This criterion is an elaboration of previously obtained criterion to a wider class of groups. Expansion of the class of groups requires some weakening of conditions of the criterion. In this case the condition of the existence of a free subgroup with two generators is replaced by the condition of nonamenability.

Theorem B. *Let $G \subseteq Homeo_+(\mathbb{R})$ and there is a freely acting element $\bar{g} \in G$. Then either the quotient group G/H_G is not amenable or for the group G there is a projectively invariant measure. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.*

Proof. Let's show that Theorem B follows from Theorem A. Indeed, if in the group G there is a freely acting element then for such group the condition $G/H_G \neq < e >$ is fulfilled. Let the quotient group G/H_G is amenable. Then, by Theorem A, for the group G there is a projectively invariant measure.

The converse. Let's show that Theorem A follows from Theorem B.

Necessity. Let for the group G there is a projectively invariant measure. Let's consider two cases.

(i) The first case. For the group G there is an invariant measure. Then, by the definition, it is true that $H_G = G^S$ and, obviously, $G^S =< G^S >$. Since, by the condition of Theorem A, $G/H_G \neq < e >$ then we obtain that $G/ < G^S > \neq < e >$. Then in the group G there is a freely acting element. Moreover, by Theorem 1, the quotient group $G/ < G^S >= G/H_G >$ is commutative and, thus, it is amenable.

(ii) The second case. For the group G there is not an invariant measure. Then, owing to Theorem 15, for the group G the condition $C_G \neq G^S_{+\infty}$ is satisfied and, therefore, in the group G there is a freely acting element. And finally, by Remark 5, the quotient group G/H_G is a solvable group of solvability length not greater than 2.

Sufficiency. Let in the group G there is a freely acting element and the quotient group G/H_G is amenable. Then, by Theorem A, for the group G there is a projectively invariant measure. $\square$

Let's give a reformulation of Theorem B, using only combinatorial characteristics.

Theorem B*. *Let $G \subseteq Homeo_+(\mathbb{R})$ and there is a freely acting element $\bar{g} \in G$. Then either the quotient group G/H_G is not amenable or the quotient group G/H_G is a solvable group of solvability length not greater than 2. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.*

Proof. In the proof of Theorem B we noted that for the group with invariant measure the quotient group G/H_G is commutative. If there is a projectively invariant measure, but there is not an invariant measure, then, by Remark 5, the quotient group G/H_G is a solvable group of solvability length not greater than 2. From these facts the statement of Theorem B* follows. $\square$

The alternative, formulated in Theorem B*, allows a restatement, remaining in terms of combinatorial characteristics.

Theorem B.** *Let $G \subseteq Homeo_+(\mathbb{R})$ is a group with nonempty minimal set. Then either the quotient group G/H_G is a solvable group of solvability length not greater than 2 or at least one of the following conditions is satisfied: the quotient group G/H_G is not amenable; there is not a freely acting element $\bar{g} \in G$. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.*

Proof. Let's show that Theorem B** follows from Theorem B*. Let the quotient group G/H_G is not a solvable group of solvability length not greater than 2. Let's suppose that the quotient group G/H_G is amenable and there is a freely acting element $\bar{g} \in G$. Then, by Theorem B*, for such group the quotient group G/H_G is a solvable group of solvability length not greater than 2. Contradiction. Consequently, for the group G at least one of the following conditions is satisfied: the quotient group G/H_G is not amenable; there is not a freely acting element $\bar{g} \in G$.

The converse. Let's show that Theorem B* follows from Theorem B**. Let the quotient group G/H_G is not a solvable group of solvability length not greater than 2. Since, by the conditions of Theorem B*, there is a freely acting element then, by Theorem B**, the quotient G/H_G is not amenable. $\square$

We can formulate an indication of nonamenability of the quotient group G/H_G in terms of commutator $[G/H_G, G/H_G]$.

Theorem C. *Let $G \subseteq Homeo_+(\mathbb{R})$, for which there is a freely acting element $\bar{g} \in G$. If $G/H_G = [G/H_G, G/H_G]$ then the quotient group G/H_G is not amenable.*

Proof. Indeed, let's assume that the quotient group G/H_G is amenable. Then, by Theorem B*, the quotient group G/H_G is a solvable group of solvability length not greater than 2 that contradicts the condition $G/H_G = [G/H_G, G/H_G]$. Consequently, the quotient group G/H_G is not amenable. $\square$

Consequence 2. *Let $G \subseteq Homeo_+(\mathbb{R})$, for which there is a freely acting element $\bar{g} \in G$. If $G = [G, G]$ then the quotient group G/H_G is not amenable.*

Proof. Indeed, from the condition $G = [G, G]$ it follows $G/H_G = [G/H_G, G/H_G]$. Then the nonamenability of the quotient group G/H_G follows from Theorem C. □

§7. Nonamenability of Thompson's group F.

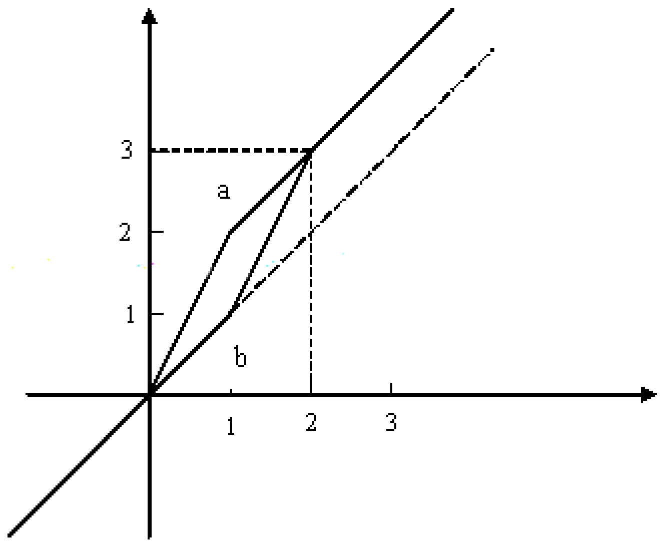
Due to the Day's problem [18], a question about existence of finitely presented and finitely generated nonamenable group, which doesn't contain a free subgroup with two generators, became actual. Thompson's group F is a potential candidate for being such group:
F is a set of piecewise-linear homeomorphisms of $[0, 1]$ which are differentiable except at finitely many points, each of these points is a dyadic rational number and, on the intervals of differentiability, the derivatives are powers of 2.

Brin and Squier [20] showed that the group F is not simple and it doesn't contain a free subgroup with two generators.

F is isomorphic to the group with two generators and two relations [19]

$$F =< \mathcal{A}, \mathcal{B} : \ [\mathcal{A}\mathcal{B}^{-1}, \mathcal{A}^{-1}\mathcal{B}\mathcal{A}], \ [\mathcal{A}\mathcal{B}^{-1}, \mathcal{A}^{-2}\mathcal{B}\mathcal{A}^2] > .$$

F can be realized as a group of homeomorphisms of the line with two generators having the following form:



The main result of this paper is represented in the next theorem.

Theorem D. *Thompson's group F is not amenable.*

Proof. Let's denote through $\bar{a}, \bar{b}$ the restrictions of homeomorphisms a, b on the ray $(0, +\infty)$. It's obvious that F is isomorphic to the group $\bar{F} =< \bar{a}, \bar{b} >$.

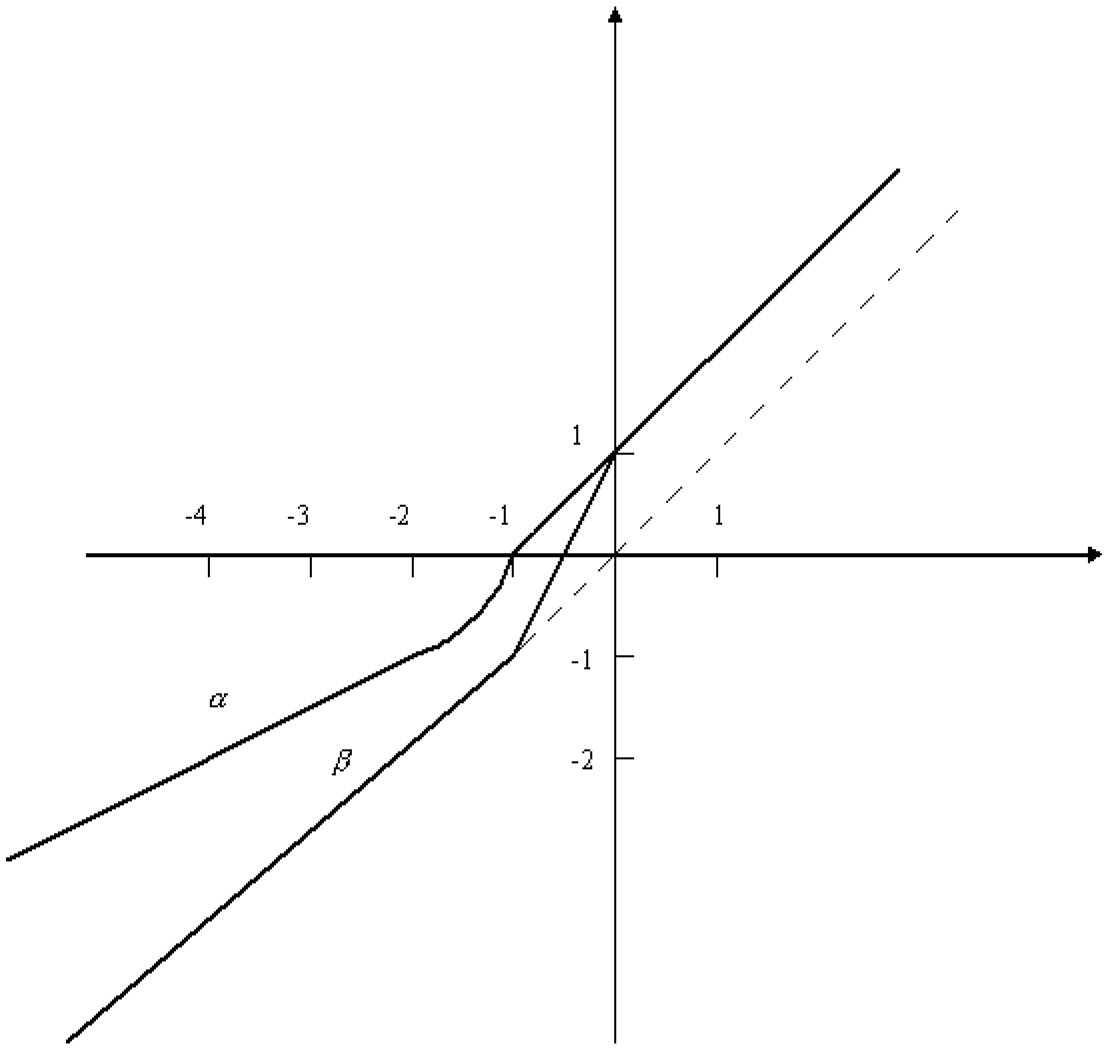
Let's define a mapping $\eta : (0, +\infty) \to \mathbb{R}$, where

$$\eta(t) = \begin{cases} -\dfrac{1}{t}, & \tau \in (0,\, 1] \\ \\ t-2, & \tau \in (1,\, +\infty], \end{cases}$$

which defines the following homeomorphisms of the line $\alpha = \eta a \eta^{-1}$, $\beta = \eta b \eta^{-1}$. The homeomorphisms α, β have next representations

$$\alpha(\tau) = \begin{cases} \dfrac{\tau}{2}, & \tau \in (-\infty,\, -2] \\ \\ -\dfrac{2}{\tau} - 2, & \tau \in (-2,\, -1] \\ \\ \tau + 1, & \tau \in (-1,\, +\infty), \end{cases} \qquad \beta(\tau) = \begin{cases} \tau, & \tau \in (-\infty,\, -1] \\ 2\tau + 1, & \tau \in (-1,\, 0] \\ \tau + 1, & \tau \in (0,\, +\infty), \end{cases}$$

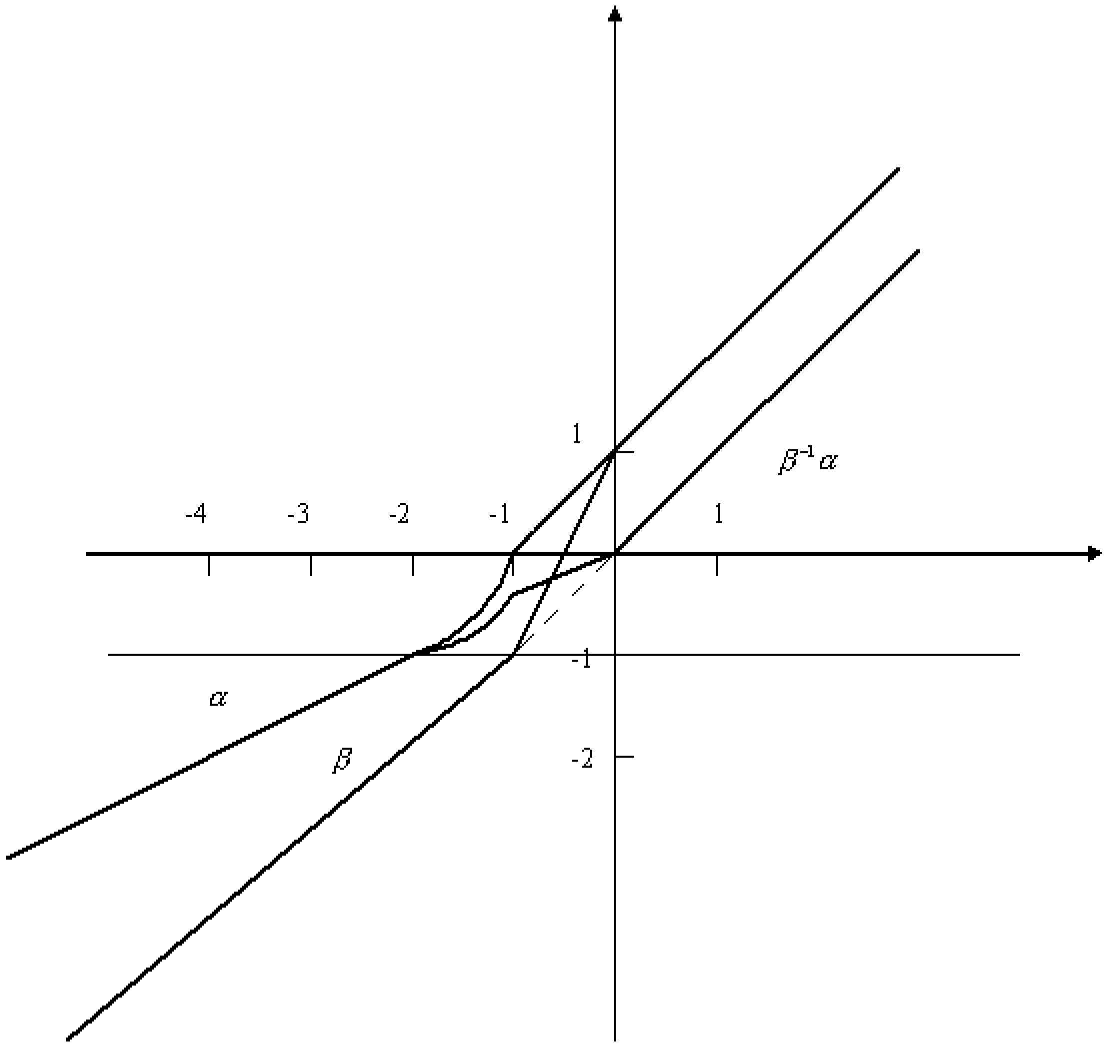
and their graphs have the form



It's obvious that the group $\bar{F}$ is isomorphic to the group $\mathcal{F} =< \alpha, \beta >$.

The element α is freely acting. Owing to Proposition 1, for the group $\mathcal{F}$ the minimal set is nonempty. Moreover, it is not difficult to see that the orbit of the point $t = 1$ is dense in the interval $[-1, 0]$. Then, by Theorem 2, the minimal set of the group $\mathcal{F}$ coincides with whole line. In that case, for the group $\mathcal{F}$ there is an equality $H_{\mathcal{F}} =< e >$ and, accordingly, it is true that $\mathcal{F}/H_{\mathcal{F}} = \mathcal{F}$.

Let's show that for the group $\mathcal{F}$ there is not both an invariant measure and a projectively invariant measure. Indeed, from the graphs of the elements β, $\beta^{-1}\alpha$ it follows that they belong



to the set of stabilizers $\mathcal{F}^S$ and $Fix < \beta >= (-\infty, -1]$, $Fix < \beta^{-1}\alpha >= [0, +\infty)$. In that case, $Fix\,\mathcal{F}^S = \emptyset$, and, by Theorem 8, for the group $\mathcal{F}$ there is not an invariant measure.

It is easy to see that $\beta \in \mathcal{F}^S \backslash C_{\mathcal{F}}$, but at the same $t_\beta = -\infty$, $T_\beta = -1$, which contradicts the condition 2) of Theorem 14. Therefore, for the group $\mathcal{F}$ there also is not a projectively invariant measure.

At this rate, owing to Theorem B, the group $\mathcal{F}$ is not amenable. Consequently, Thompson's group F, which is isomorphic to the group $\mathcal{F}$, also is not amenable. $\square$

Let's provide an example, which demonstrates the importance of the existence of a freely acting element in the alternative of Theorem B. Here are the most important properties of the group $\mathcal{F}$, which is isomorphic to Thompson's group F:

$$\mathcal{F} =< \mathcal{F}^S >, \;\; \mathcal{F} \neq \mathcal{F}^S; \quad E(\mathcal{F}) = \mathbb{R}; \quad H_{\mathcal{F}} =< e >; \quad Fix\ \mathcal{F}^S = \emptyset, \tag{129}$$

and also for the group $\mathcal{F}$ there is not a projectively invariant measure.

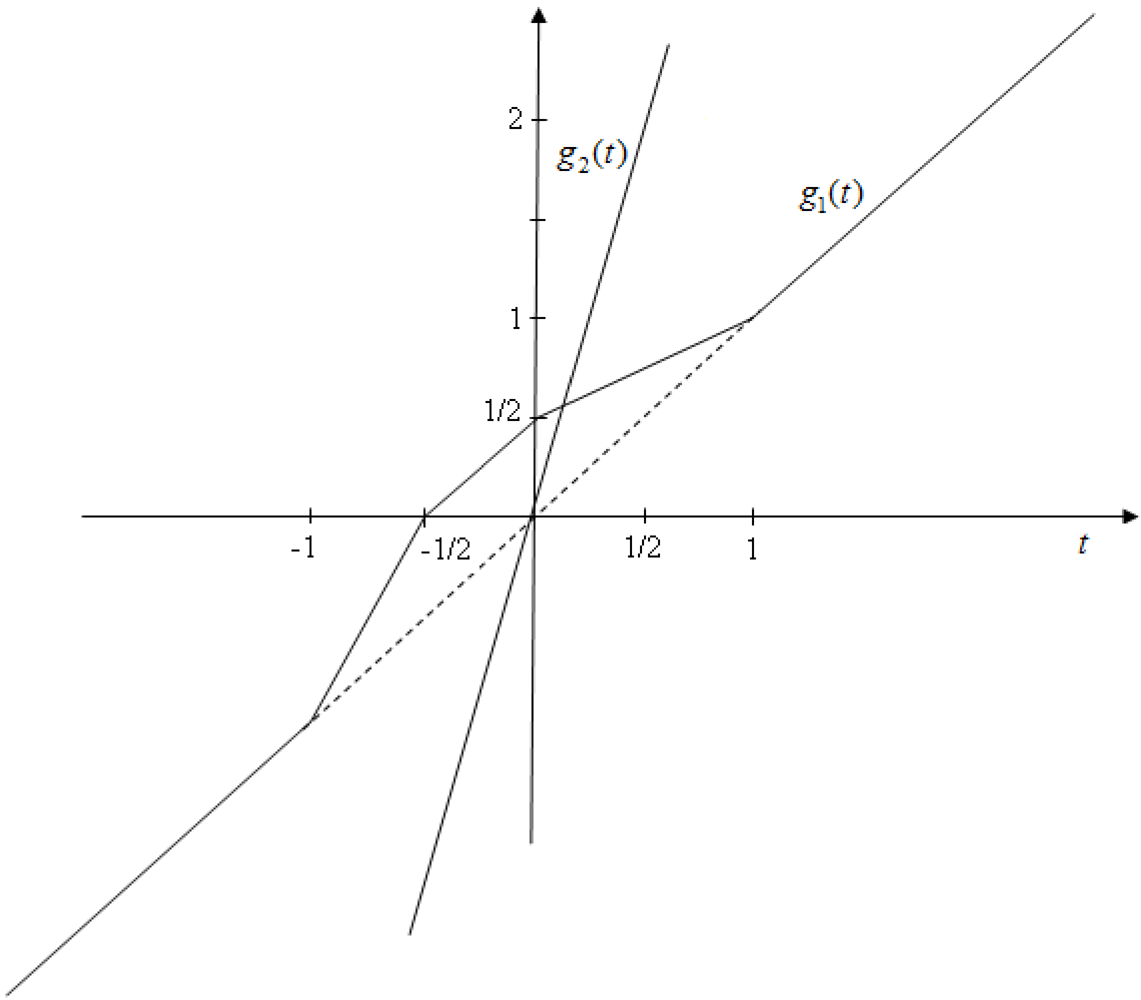
The following example was pointed out to the author by Prof. Matt Brin during the correspondence.

Example. Brin's group $\mathcal{B}$.

Let's describe the generators of Brin's group $\mathcal{B} =< g_1, g_2 >$

$$g_1(t) = \begin{cases} t, & t \in (-\infty, -1), \\ 2t+1, & t \in [-1, -\frac{1}{2}], \\ t + \frac{1}{2}, & t \in (-\frac{1}{2}, 0), \\ \frac{1}{2}t + \frac{1}{2}, & t \in [0, 1], \\ t, & t \in (1, +\infty), \end{cases} \qquad g_2(t) = 4t. \tag{130}$$

The graphs of homeomorphisms g_1, g_2 are shown in next Figure.



It's not difficult to see, that for the group $\mathcal{B}$ there are following relations

$$\mathcal{B} = \mathcal{B}^S; \quad E(\mathcal{B}) = \mathbb{R}; \quad H_{\mathcal{B}} =< e >; \quad Fix\ \mathcal{B}^S = \emptyset. \tag{131}$$

In particular, from the condition $Fix\ \mathcal{B}^S = \emptyset$ it follows that for Brin's group $\mathcal{B}$ there is not an invariant measure. Also for the group $\mathcal{B}$ the condition 2) from Theorem 14 is not satisfied. Therefore, there also is not a projectively invariant measure.

We see that, according to many of the listed basic properties, Thompson's group $\mathcal{F}$ and Brin's group $\mathcal{B}$ are similar. However, it is known that group $\mathcal{B}$ is amenable. This example of Brin's group $\mathcal{B}$ shows that the condition of the existence of a freely acting element, i.e. the condition $\mathcal{F} \neq \mathcal{F}^S$, in alternative of Theorem B is unavoidable (precise).

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