On the Maximum Number of Colors for Links

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May 07, 2012

Abstract

For each odd prime p, and for each non-split link admitting non-trivial p-colorings, we prove that the maximum number of Fox colors is p. We also prove that we can assemble a non-trivial p-coloring with any number of colors, from the minimum to the maximum number of colors. Furthermore, for any rational link, we prove that there exists a non-trivial coloring of a 2-bridge diagram of it, modulo its determinant, which uses all colors available. If this determinant is an odd prime, then any non-trivial coloring of the 2-bridge diagram, modulo the determinant, uses all available colors. Facts about torus links and their colorability are also proved.

Keywords:

MSC 2010: 57M27

1 Introduction

Given a positive integer, say p, and a link L, a p-coloring of L is an assignment of integers modulo p to the arcs of a diagram of L such that at each crossing, the integers assigned to the arcs meeting at this crossing satisfy the coloring condition. This condition states that, in the given modulus, twice the integer assigned to the over-arc equals the sum of the integers assigned to the under-arcs (see Figure 1). In a coloring, the integers assigned to the arcs are called colors. Colorings were introduced by Fox in [2] (see also [5] and [6]). The number of colorings is an invariant under the Reidemeister moves and for any positive integer p and for any link L there are always the so-called trivial colorings. In a trivial coloring each of the arcs of the diagram bears the same color. Furthermore, the fact that a link admits or not non-trivial colorings in a given modulus is also an invariant. In this sense, the interesting colorings are the non-trivial colorings

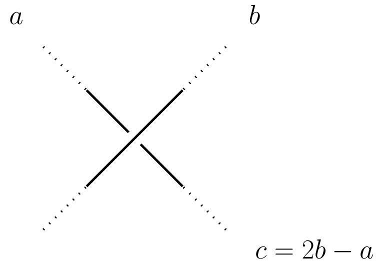

Figure 1: Detail of crossing of a diagram endowed with a non-trivial p-coloring, for odd prime p. The colors meeting at this crossing are all distinct.

i.e., those that use more than one color. How many distinct colors are needed to obtain a non-trivial coloring in a given modulus, for a given link? This question was first raised in [3]. This number, called the minimum number of colors, is clearly a knot invariant. It is also a fascinating invariant from the point of view of its computattion. In fact, by definition, for each diagram of the link under study, one should look for the minimum number of colors it takes to obtain a non-trivial coloring in that diagram. Repeat this operation for each diagram of the link under study and finally obtain the answer by taking the minimum of all these minima. But each link has infinitely many diagrams. Therefore it is unreasonable to apply the definition to calculate this invariant which makes this a challenging task.

In [7] the last two authors obtained upper bounds for the minimum number of colors it takes to assemble a non-trivial coloring for the torus knots of type $(2, n)$. These upper bounds were improved in [9]. There is now a considerable number of references in the literature devoted to the minimum number of colors. Some are concerned with the calculation of (upper bounds of) minima for specific families of knots ([7], [12]). Others are concerned with establishing the exact value of the minima for specific modulus but for any link ([7], [16], [14], [15], [11]). Others still are concerned with establishing transformations which allow the systematic reduction of the number of colors ([7], [9]).

In this article we look into the dual problem. Given a positive modulus and a link, which is the maximum number of colors one can assign to the arcs of a diagram in order to obtain a non-trivial coloring in the given modulus?

Definition 1.1 (Maximum number of colors) *Given an integer $p > 1$, assume there are non-trivial p-colorings on a given link L. Assume further that D is a diagram of L. We let $n_{p,L}(D)$ denote the maximum number of distinct colors assigned to the arcs of D it takes to produce a non-trivial p-coloring on D. We denote* $\text{maxcol}_p L$ *the maximum of these maxima over all diagrams D of L:*

$$\text{maxcol}_p L := \max\{n_{p,L}(D) \mid D \text{ is a diagram of } L\}$$

For each L, we call $\text{maxcol}_p L$ *the* **maximum number of colors of L, mod** p*. In the sequel, we will drop the "mod p" whenever it is clear which p we are referring to. Note that* $\text{maxcol}_p L$ *is tautologically a topological invariant of L.*

On the other hand what are the possible numbers of colors a link can admit under a given modulus?

Definition 1.2 (Spectrum of a Link for a given modulus) *Given an integer $p > 1$, assume there are non-trivial p-colorings on a given link L. We let*

$$\text{Spec}_p L := \{n \in \mathbf{Z}^+ \mid \text{there is a diagram, } D\text{, of } L \text{ with a non-trivial } p\text{-coloring which uses } n \text{ colors}\}$$

For each L, we call $\text{Spec}_p L$ *the* **spectrum of L, mod** p*. In the sequel, we will drop the "mod p" whenever it is clear which p we are referring to. Note that* $\text{Spec}_p L$ *is a topological invariant of L.*

A **split link** is a link which can be made to be contained in two disjoint balls in 3-space. A **non-split link** is a link which is not a split link.

For split links the number of colorings in a given modulus equals the product of the number of colorings of each non-split component of the link. On the other hand, for split links the question of minimum number of colors is trivial. In fact, one may consider the disjoint union of a certain number of (non-split) links (which may even not admit non-trivial colorings under a given modulus). The disjoint union of these non-split links is non-trivially colored in the indicated modulus by coloring one of the split components with color 0 and each of the others with color 1. We conclude that the minimum number of colors of any split link is 2.

The question of maximum number of colors for split links is slippery since the different split components give independent contributions. It all comes down to investigating what are the contributions of each of the split components i.e., of the non-split links.

For these reasons, in the present article we consider only non-split links.

The questions that ultimately lead to the results in this article were motivated by extensive calculations performed with the help of the programme [10]. This programme includes functions that were written for the purpose of this article.

The main results of this article are the following.

Theorem 1.1 *Let p be an odd prime. Let L be a non-split link which admits non-trivial p-colorings. Then*

$$\mathrm{maxcol}_p L = p$$

Corollary 1.1 *We keep the hypothesis of Theorem 1.1 although allowing p to be composite.*

1. *If L admits a non-trivial p-coloring with a crossing as in Figure 2, then the result follows.*
2. *If L admits a non-trivial p-coloring with a crossing as in Figure 1, and $b - a$ is invertible mod p, then the result follows.*

Rational knots are also 2-bridge knots i.e., knots admitting diagrams with exactly two over-arcs, see [17]. Such a diagram is known as a Schubert normal form. The 2-bridge diagrams of rational links simplify the process of finding the diagram which maximizes the number of colors used.

Theorem 1.2 *Let R be a rational link.*

There is always a non-trivial coloring of a 2-bridge diagram of R, modulo its determinant, which uses all available colors.

Corollary 1.2 *Let R be a rational knot with odd prime determinant p.*

Any non-trivial p-coloring of a 2-bridge diagram of R uses all available colors.

Torus knots also provide interesting examples concerning the number of colors used in a coloring. We keep the following convention. When we refer to the torus link of type (r, s), notation $T(r, s)$, we are implicitly considering the representation of this link by the closure of the r-braid $(\sigma_{r-1}\sigma_{r-2}\cdots\sigma_1)^s$ (see Figures 10 and 11). The classification of torus links according to their determinants has been done in [4].

Theorem 1.3 *Let k and l be positive integers.*

- *Consider $T(2k+1, 2l)$. It admits non-trivial $(2k+1)$-colorings using all colors available, with a uniform distribution of colors over the diagram.*
- *Consider $T(2l, 2k+1)$. It admits non-trivial $(2k+1)$-colorings using all colors available. For such a diagram, such a $(2k+1)$-coloring using all colors available does not exhibit a uniform distribution of the colors over this diagram. Furthermore, if additionally $\gcd(2l, 2k+1) = 1$, a $(2k+1)$-coloring of such a diagram using all colors available cannot exhibit uniform distribution of the colors.*
- *Consider $T(2k, 2l)$ with $\gcd(2k, 2l) = 2$. The determinant here is $2kl$. $T(2k, 2l)$ are both colorable modulo any factor of $2kl$. Modulo $2kl$ there is no uniform distribution of colors; for the other factors there is uniform distribution of colors.*

Theorem 1.4 *Let p be an odd prime. Let L be a non-split link which admits non-trivial p-colorings. Then*

$$\mathrm{Spec}_p L = \{\mathrm{mincol}_p L, 1 + \mathrm{mincol}_p L, 2 + \mathrm{mincol}_p L, \dots, \mathrm{maxcol}_p L\}$$

Corollary 1.3 *We keep the hypothesis of Theorem 1.4 although allowing p to be composite.*

1. *If L admits a non-trivial p-coloring with a crossing as in Figure 2, then the result follows.*

2. *If L admits a non-trivial p-coloring with a crossing as in Figure 1, and $b - a$ is invertible mod p, then the result follows.*

Corollary 1.4 *We keep the hypothesis of Theorem 1.4 or of Corollary 1.3. In the proof of the Theorem, for each intermediate number of colors, say n with $\mathrm{mincol}_p L \leq n \leq \mathrm{maxcol}_p L$, we constructed a diagram of L, call it D_n, along with a non-trivial p-coloring using exactly n colors. This diagram cannot admit a non-trivial p-coloring with less than n colors.*

In Section 2 we prove Theorem 1.1 and Corollary 1.1; in Subsection 2.1 we look into what may happen if the hypothesis of Corollary 1.1 are not satisfied. In Section 3 we prove Theorem 1.2 and Corollary 1.2. In Section 4 we prove Theorem 1.3. In Section 5 we prove Theorem 1.4 and Corollaries 1.3 and 1.4. In Section 6 we collect a few questions for future work.

2 Proof of Theorem 1.1 and Corollary 1.1.

Proof (of Theorem 1.1).

We keep the notation and terminology of the statement of the Theorem.

If L admits a non-trivial p-coloring, there is a diagram of L along with assignments of integers mod p to its arcs such that the coloring condition is satisfied at each crossing of the diagram and using more than one color in the process. In Figure 1 we depict a crossing of this diagram whose colors are all distinct.

The p-colorings of a knot constitute a vector space since they are solutions of a system of homogeneous linear equations over the integers mod p. Moreover, there are always the so-called trivial solutions, where each arcs bears the same color. We now describe a method for passing to another non-trivial p-coloring with $0 \leftarrow a$, $1 \leftarrow b$, and $2 \leftarrow c$. This method is found in [15] and [11], we include it here for completeness.

1. If $a \neq 0$, subtract a from each color in the diagram. If $a = 0$ go to 2.;

2. If $b \neq 1$, multiply each color in the coloring obtained in 1. by the inverse of b mod p;

After applying this to the diagram under study the colors assigned to the crossing in Figure 1 become those depicted in Figure 2.

Suppose the diagram under study does not yet include color 3, mod p. Then from this diagram we obtain a new diagram endowed with a non-trivial p-coloring and including color 3. Around the crossing we have been addressing (see Figure 2), the new diagram now looks like Figure 3.

We now prove that we can obtain inductively a diagram including all colors $0, 1, 2, \cdots, p-1$. In fact, if the current diagram involves a crossing including colors $k, k+1, k+2$ as depicted on the left-hand side of Figure 4, then a type II Reidemeister move, along with the local change of colors so that the initial and final colorings are compliant, adds color $k+3$ to the existing colors, as the right-hand side of Figure 4 illustrates. This completes the proof. ■

Proof (of Corollary 1.1).

Omitted.

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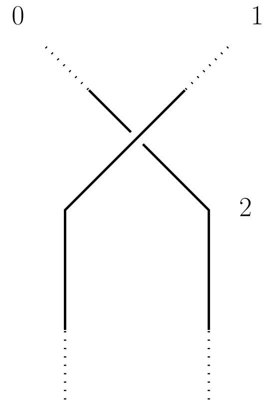

Figure 2: A new non-trivial p-coloring of the diagram. Now the crossing at issue bears the indicated colors, $0, 1, 2$.

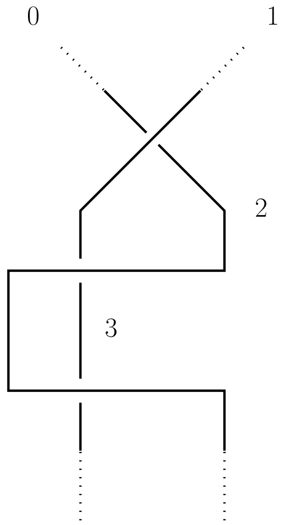

Figure 3: A new colored diagram only differing from the one in Figure 2 by a type II Reidemeister move and the insertion of color 3. The present coloring includes definitely color 3.

2.1 What may go wrong when $b - a$ is not invertible mod p.

Inspection of Figure 5 shows that, in the case of $T(2, 12)$ colored mod 9, $b - a$ has to be a multiple of 3, hence not invertible mod 9. With such a choice for $b - a$ there are only 3 colors in the coloring of $T(2, 12)$ ($0, 3, 6$, mod 9, in the case at issue). The set $\{0, 3, 6\}$ is closed under the operation $a * b = 2b - a$ mod 9. The same applies to the sets $\{1, 4, 7\}$ or $\{2, 5, 8\}$ mod 9. Therefore, it is impossible to increase the number of colors starting from one of these colorings of this diagram.

In Figure 6 we see that, with $R(4, 2)$ we may increase the minimum number of colors, with a judicious choice of the coloring. Specifically, with the coloring on the left we are not able to increase the number of colors (arguing as above). For the coloring on the right it is possible to increase the number of colors using the technique set forth in the proof of Theorem 1.1.

3 Proofs of Theorem 1.2 and of Corollary 1.2.

Proof (of Theorem 1.2).

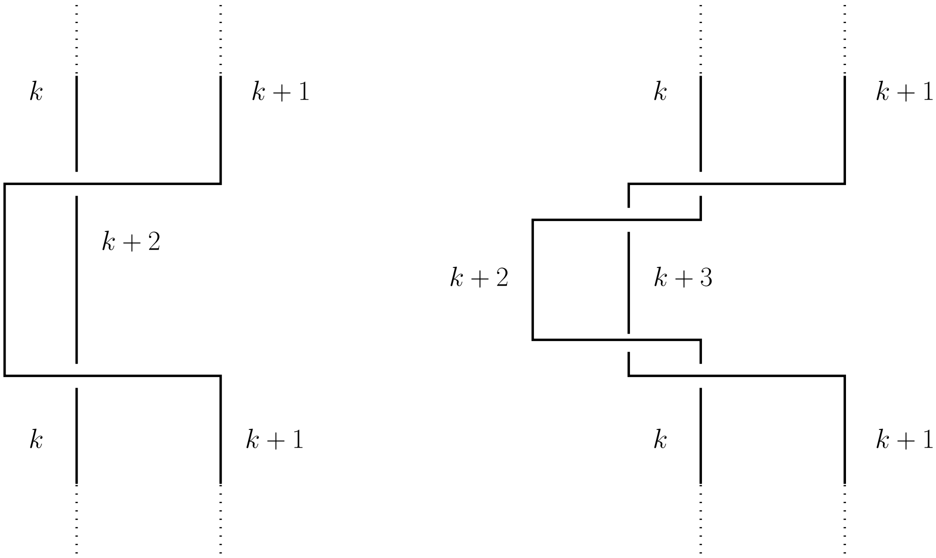

Figure 4: Obtaining a diagram involving colors $k, k+1, k+2, k+3$ (right-hand side) from a diagram involving colors $k, k+1, k+2$ (left-hand side).

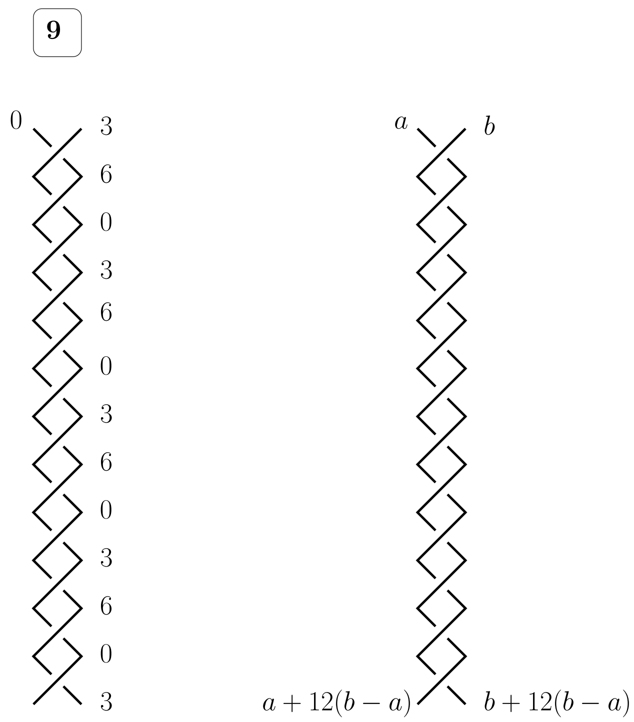

Figure 5: $T(2, 12)$ endowed with a non-trivial 9-coloring involving 3 colors (left-hand side). The propagation of colors down the $T(2, 12)$: $b - a$ has to be a multiple of 3 for a non-trivial 9-color to occur (right-hand side). Braid closure of each diagram is understood.

It is known that a rational link has either one or two components and that its coloring system of equations is equivalent to a single equation of the form

$$D \times (b - a) = 0$$

where D is the determinant of the rational link and a, b are appropriate arcs of the diagram we are using (see [8], for example). We will now prove that we can choose these arcs to be the bridges when we

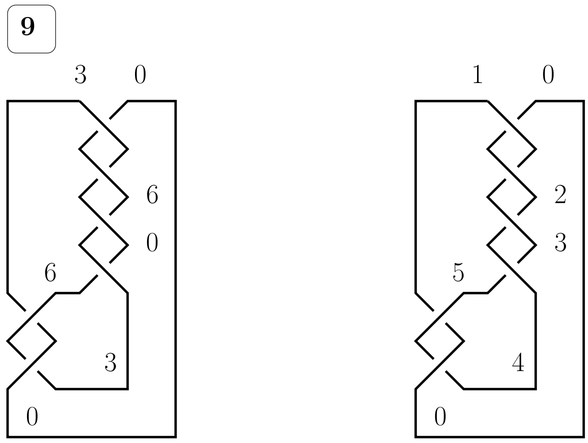

Figure 6: $R(4,2)$ endowed with two essentially distinct non-trivial 9-colorings.

represent a rational knot by a 2-bridge diagram. Referring to Figure 7, and having fixed an otherwise arbitrary orientation of our 2-bridge link, let us start by writing down the expression for the outgoing under-arc from a crossing where b_0 is the over-arc and b_1 is the under-arc, $2b_0 - b_1$. Now this arc meets a crossing where b_1 is the over-arc so now the outgoing under-arc is $2b_1 - (2b_0 - b_1) = 3b_1 - 2b_0$. And so on and so forth. It is easy to check by induction the at the $2k$-th crossing the outgoing under-arc is $(2k+1)b_1 - (2k)b_0 = 2k(b_1 - b_0) + b_1$ whereas at the $(2k+1)$-th crossing the outgoing under-arc is $(2k+2)b_0 - (2k+1)b_1 = (2k+1)(b_0 - b_1) + b_0$. If we are dealing with a 2-component link after a fashion the outgoing under-arc will be the original under-arc, b_1, whereas if the we are dealing with a 1-component link, this outgoing under-arc will be the original over-arc, b_0. In either case, the equation one obtains is of the form

$$\text{linear polynomial on the number of crossings traversed } \times (b_0 - b_1) = 0$$

where "linear polynomial on the number of crossings traversed" has to be, modulo sign, the determinant of the rational link under study. This proves that when coloring modulo the determinant of the rational knot under study, assigning any two distinct integers (modulo the determinant) to the bridges of a presentation of the rational knots yields a non-trivial coloring.

In the sequel we will then use 0 and 1 for the colors on the bridges of our 2-bridge diagrams to obtain non-trivial colorings, modulo the determinant of the 2-bridge link under study.

Consider a rational link with determinant p schematically represented in Figure 7 by a 2-bridge diagram. As shown above, we obtain a non-trivial coloring modulo p by assigning $b_0 = 0, b_1 = 1$. Let

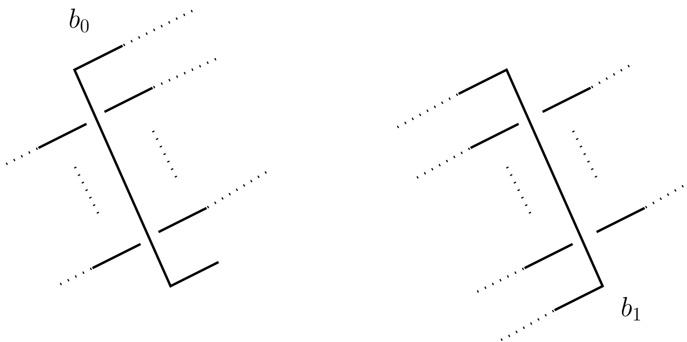

Figure 7: A 2-bridge diagram endowed with a coloring where the bridges bear colors b_0 and b_1.

C be the set of colors (mod p) of the current non-trivial p-coloring of the 2-bridge diagram of R. We remark that, with the exception of the bridges, each under-arc is an under-arc of both bridges.

Suppose color i is missing from this diagram i.e., $i \notin C$ (note that $0 \neq i \neq 1$).

Figure 8: If $-i \in C$ then $i \in C$ (left); if $-i+2 \in C$ then $i \in C$ (right).

Then (mod p) $-i$ and $2-i$ are not in C for otherwise i would also be in C (see Figure 8). But if $-i$ and $2-i$ are not in C, then $i+2$ and $i-2$ are also not in C (see Figure 9). Therefore, by induction if

Figure 9: If $i-2 \in C$ then $-i+2 \in C$ (left); if $i+2 \in C$ then $-i \in C$ (right).

$i \notin C$ then $i+2, i-2 \notin C$, $i+4, i-4 \notin C$, and so on and so forth. If p is odd, then colors 0 and 1 will also not be present which is absurd. If p is even at least one of them will not present, which is also absurd. This completes the proof.

■

Proof (of Corollary 1.2).

We keep the notation and terminology of the statement of the Corollary.

Consider a non-trivial p-coloring of a 2-bridge diagram of the rational knot R. We will first prove that the bridges cannot bear the same color. Assume to the contrary and without loss of generality, suppose this same color born by the bridges is 0. Since the p-coloring is non-trivial then one of the under-arcs bears color $i \neq 0$. Starting from this under-arc and going under a bridge, the next under-arc is colored $-i \neq 0$. Hence, each under-arc is colored either $i \neq 0$ or $-i \neq 0$. Since the bridges are also under-arcs at some crossings then the bridges would also have to be colored $i \neq 0$ or $-i \neq 0$, which is absurd. Therefore, the bridges bear distinct colors when the 2-bridge diagram is endowed with a non-trivial p-coloring.

If the non-trivial p-coloring endows the bridges with colors 0 and 1, then we can use the proof of the Theorem to conclude that this coloring uses all colors available.

Consider now a non-trivial p-coloring where the bridges bear distinct colors $\{b_0, b_1\} \neq \{0, 1\}$ modulo p. We are going to map this p-coloring generated by the colors b_0, b_1 on the bridges, to a p-coloring generated by the colors $0, 1$ on the bridges, and prove that this mapping is bijective, thus proving that the original coloring generated by colors b_0, b_1 uses all colors available. Subtract each color of the coloring generated by bridges bearing b_0, b_1 by color b_0. Multiply each color of the coloring so obtained by $(b_1 - b_0)^{-1}$ modulo p. The p-coloring now obtained is a non-trivial p-coloring with colors $0, 1$ on the bridges. Thus, it uses all colors available, as we saw before. To obtain the colors of the original coloring (the c_i's), from the colors of this coloring (the c'_i's), the expression is:

$$c_i = (b_1 - b_0)c'_i + b_0$$

If $c_i = c_j$ then

$$(b_1 - b_0)c'_i + b_0 = (b_1 - b_0)c'_j + b_0 \qquad \Leftrightarrow \qquad (b_1 - b_0)c'_i = (b_1 - b_0)c'_j \qquad \Leftrightarrow \qquad c'_i = c'_j$$

The proof is complete.

■

4 Proof of Theorem 1.3.

Proof (of Theorem 1.3).

For this proof we will refer to Figures 10 and 11. In these Figures the braid closure of the braids there depicted is understood. The boxed expressions on the top left of these Figures stand for the moduli with respect to which the colorings of the diagrams are being done.

1. View Figure 10. We believe this Figure clarifies the existence of a $(2k+1)$-coloring using all colors available, as stated. Now for the issue of color uniformity. Consider vertical lines which meet the diagram along vertical line segments. There are two sorts of such vertical lines. Along one of these we see the sequence $0, 1, 2, \dots, 2k$; along the other one we see the sequence $1, 2, 3, \dots, 2k, 0$. There is an even number of such vertical lines, on half of them the sequences begin at 0, on the other half of them they begin at 1. The occurrences of the colors is then uniform: each color occurs $2k+1$ times in the diagram.

   It has been brought to our attention that this non-trivial coloring is essentially the same as found on page 6 of the pre-print [1].

2. View Figure 11. Again, we believe this Figure clarifies the existence of a $(2k+1)$-coloring using all colors available, as stated. Now for the issue of color uniformity. Consider the sectors between two consecutive bridges. In each of these sectors we find the sequence of colors (from left to right) $1, 2, 3, 4, \dots, 2k, 0$ or $0, 2k, 2k-1, \dots, 3, 2, 1$. Now 0's and 1's occur on arcs belonging to two such consecutive sectors; the remaining colors occur on arcs which belong to only one of these sectors. Thus colors $2, 3, 4, \dots, 2k-1, 2k$ occur each $2l$ times. Colors 0 and 1 occur each l times. In the case $\gcd(2k+1, 2l) = 1$ (the knot case) note that with this diagram it is not possible to have a $(2k+1)$-coloring using all colors available and with a uniform distribution of the colors. In fact, this diagram has $2k \times 2l$ crossings (thus arcs) and this number is not divisible by $2k+1$ if $\gcd(2k+1, 2l) = 1$.

3. This can be viewed as a particular case of 1. above. For a given factor f, of the determinant, $2kl$, consider Figure 10 again and think of the numbers assigned to the arcs as multiples of $2kl/f$; in particular the label $2k$ in the lower part of Figure 10 should be understood as $f-1$. So interpreted, these colors constitute a non-trivial $2kl$-coloring of $T(2k, 2l)$.

   There are $(2k-1) \times 2l$ crossings in the diagram. Since $2kl$ does not divide $(2k-1) \times 2l$, non-trivial $2kl$-colorings using all colors available cannot exhibit a uniform distribution. We leave the final details for the reader.

■

5 Proof of Theorem 1.4 and Corollary 1.4.

Proof (of Theorem 1.4).

We keep the notation and terminology of the statement of the Theorem.

Consider a diagram of L along with a p-coloring of it which uses $\mathrm{mincol}_p L$ colors. Without loss of generality this colored diagram includes a crossing colored as depicted in Figure 2. We call this colored diagram D_0. We consider the colors this diagram uses, taking representatives from the set $\{0, 1, 2, \dots, p-1\}$, and order them into an increasing sequence.

$$C = (c_0, c_1, c_2, \dots, c_{-1+\mathrm{mincol}_p L}) \qquad 0 \leq c_i < c_{i+1} \leq p-1.$$

Of course,

$$c_0 = 0, \qquad c_1 = 1, \qquad c_2 = 2.$$

We do not have any information on the other colors of this coloring.

Either C is already using all the colors available (e.g., when $p = 3$) or there exists a least index m, $2 \leq m < -1 + mincol_p L$ such that $1 + c_m \notin C$.

We will now construct a diagram of L containing all the colors already in D_0 plus $1 + c_m$.

We start from diagram D_0 endowed with the minimal coloring and including a crossing as depicted in Figure 2. Then, either D_0 includes a crossing with colors as depicted in Figure 12 (left-hand side), in which case we apply the technique employed in the proof of Theorem 1.1, obtaining the right-hand side of Figure 12 thus ensuring that our diagram now includes color $1 + c_m$.

Otherwise, if a crossing as depicted in the right-hand side of Figure 12 is not part of D_0, then, if need be, we go back to the crossing as depicted in Figure 2 and iterating the aforementioned technique, we eventually obtain a region of the new diagram looking like the right-hand side of Figure 12. This new diagram is endowed with a p-coloring which includes color $1 + c_m$ and all the other colors that D_0 had.

We let this new diagram be denoted D_0, we collect all its distinct colors into an increasing sequence C and we decide whether C already uses all colors available or not. If it does not we call m the least index such that $3 \leq m < mincol_p L$ and $1 + c_m \notin C$. We then use the technique described in the preceding paragraph to obtain a diagram of L endowed with $m + 2$ colors.

Finally an inductive step will complete the proof. We leave the details to the reader. ■

Proof (of Corollary 1.3).
Omitted.

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Proof (of Corollary 1.4).

We keep the notation and terminology of the statement of the Corollary.

If it were possible to endow D_n with a p-coloring with less than n colors then, by undoing the type II Reidemeister moves from which D_n was constructed starting from a diagram with a minimal number of colorings, we would obtain a diagram with less colors than the assumed minimum number of colors. This is absurd thus completing the proof of the Corollary. ■

6 Final Remarks.

In [3], the Kauffman-Harary conjecture on alternating knots of prime determinant was set forth. Given an odd prime p and an alternating knot of determinant p, any non-trivial p-coloring of a reduced alternating diagram of this knot will assign different colors to different arcs. Let us call this the KH (Kauffman-Harary) property: a diagram D endowed with a coloring which assigns different colors to different arcs is said to have the KH property. We will say the diagram satisfies the KH property mod p when it is not clear from context which modulus is meant or to stress the modulus under study.

With this terminology the conjecture above reads: given an odd prime p along with an alternating knot of determinant p, any reduced alternating diagram of this knot possesses the KH property, mod p.

This conjecture has been proven to be true in [13]. From this conjecture we see that from a reduced alternating diagram endowed with a non-trivial coloring modulo an odd prime, we can, in principle, evolve either to maximize or to minimize the number of colors. In this respect we pose the following questions.

1. Which alternating knots of prime determinant already present the minimum number of colors on reduced alternating diagrams non-trivially colored modulo their determinants?

2. Same as above with "minimum" replaced for "maximum". We expect that the complete list consists of all knots with $det(K) = 3$ and torus knots $T(2, 2k + 1)$ $(k \geq 2)$.

3. Concerning non-alternating link diagrams with prime determinant, which of them (as diagrams) possess the KH property? Concerning non-alternating knots with prime determinant, which of their minimal reduced diagrams posses the KH property? Which non-alternating knots have at least one minimal reduced diagram which has the KH property? Consider these same questions when the determinant is not necessarily prime.

We aim to look into these issues in future work.

7 Acknowledgements

S.J. thanks for support of the Serbian Ministry of Science (Grant No. 174012).

P.L. acknowledges support from FCT (Fundação para a Ciência e a Tecnologia), Portugal, through project number PTDC/MAT/101503/2008, "New Geometry and Topology". P.L. also thanks the School of Mathematical Sciences at the University of Nottingham for hospitality during his stay there where the present article was written.

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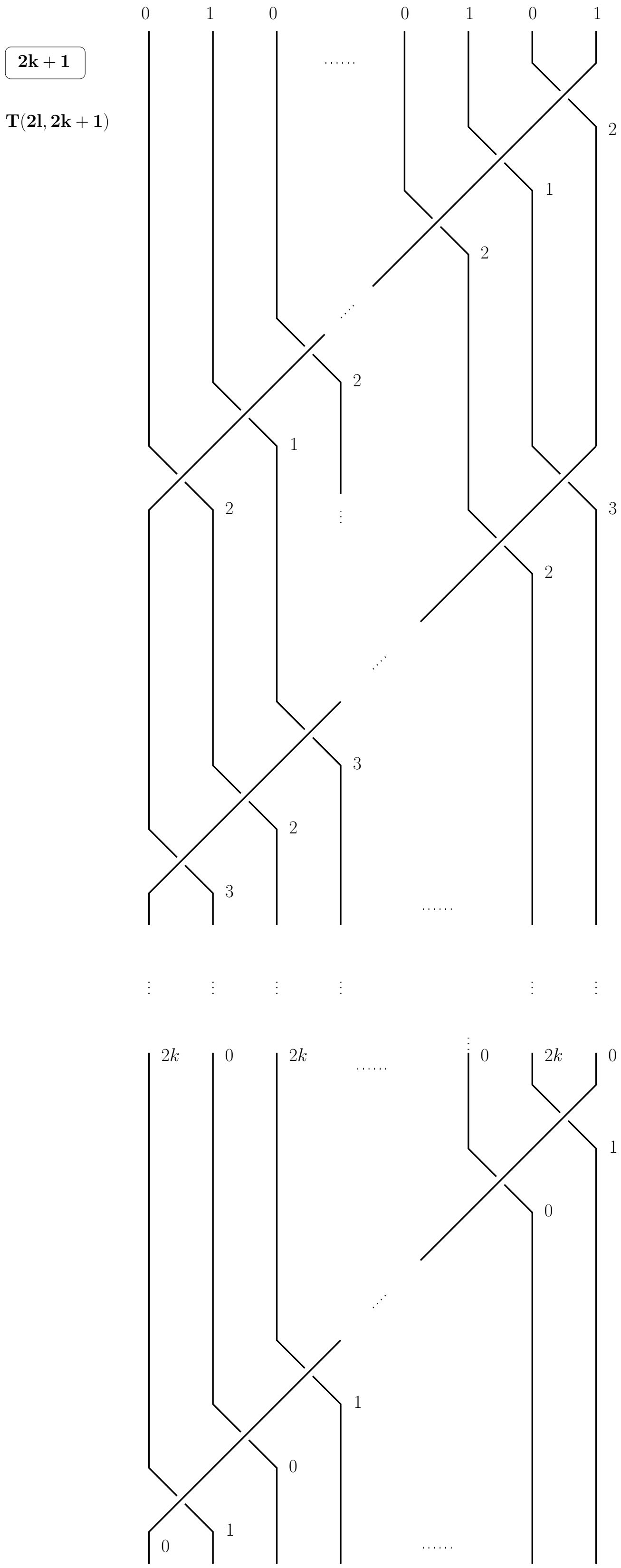

Figure 10: The “canonical” non-trivial $(2k+1)$-coloring of a $T(2l, 2k+1)$ using all colors available: for any positive integer l this Figure stands for a non-trivial $(2k+1)$-coloring.

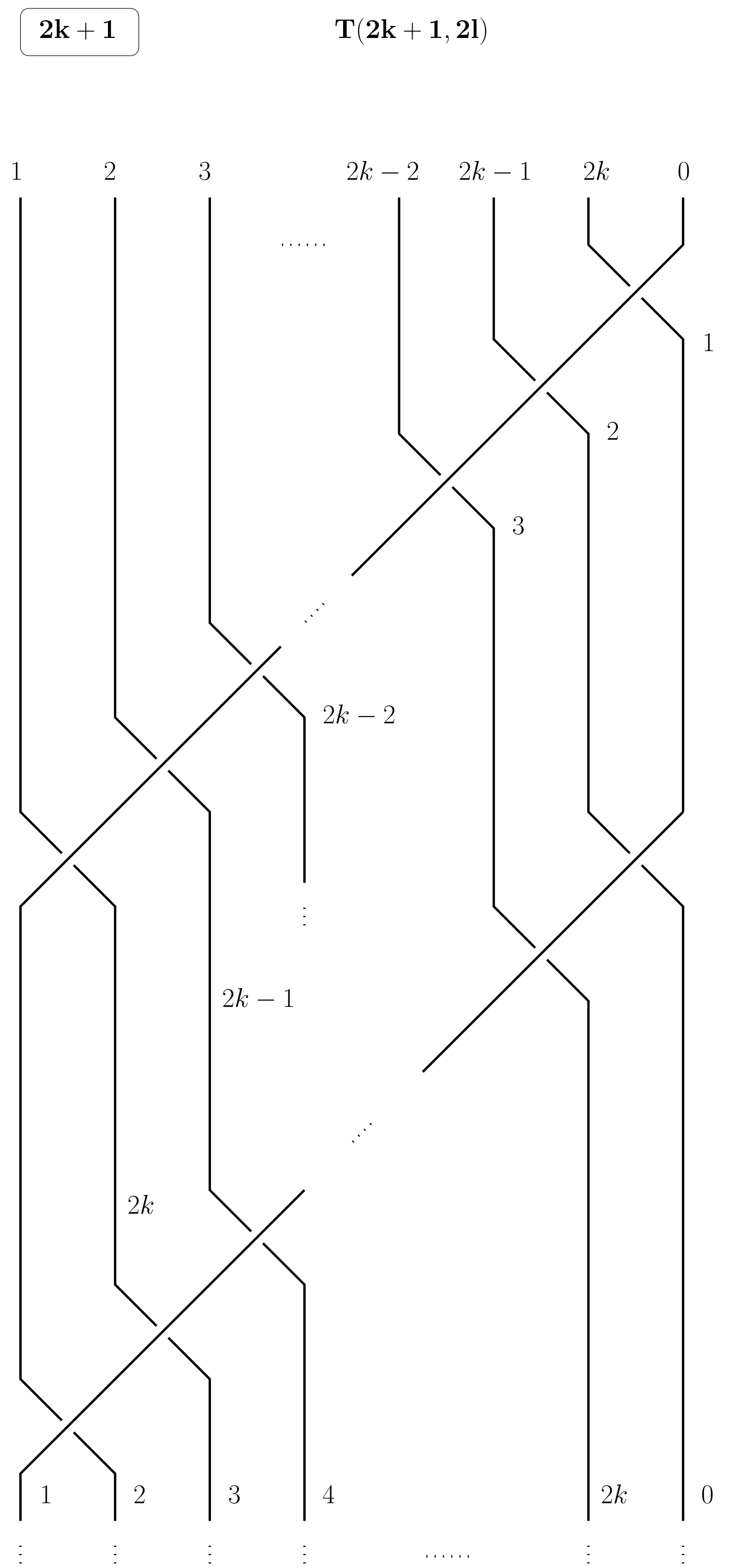

Figure 11: The "canonical" non-trivial $(2k+1)$-coloring of a $T(2k+1, 2l)$ using all colors available: for any positive integer l this Figure stands for a non-trivial $(2k+1)$-coloring.

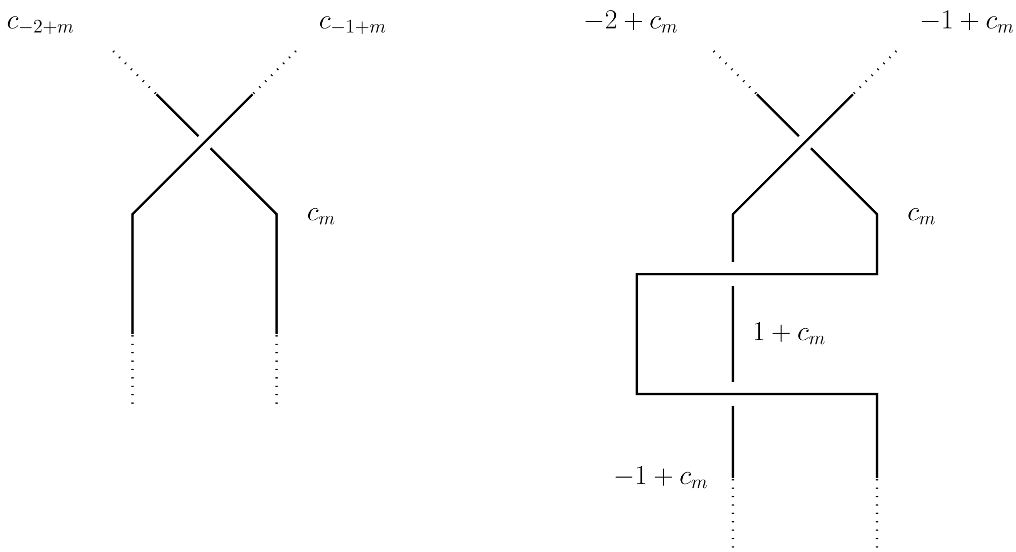

Figure 12: Note that $c_{-2+m} = -2 + c_m$, $c_{-1+m} = -1 + c_m$.