The incompleteness of the Schrödinger equation in classical limit

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Abstract

The Schrödinger equation with the nonlinear term $-b(\ln|\Psi|^2)\Psi$ is derived in the framework of the Dirac heuristics. The particle behaves classically in case the mass of it is infinite. The nonlinear term is crucial and involves new physical constant b. The constant can be measured by the same methods that were used in the case of the Casimir effect (Spaarnay, 1958; Tabor et al., 1969). Of course, the experimental procedure is based on well educated experimenters. The new experiments, different from the Zeilinger ones, are proposed, with Faraday simplicity, for the determination of this new very small constant b. The article is the extended and perfectionized version of the articles by author (Pardy, 1993; 1994; 2001).

1 Introduction

The non-Gödelian incompleteness of quantum mechanics was introduced by Einstein, Podolsky and Rosen (EPR) in his well known article (Einstein et al., 1935), where they argued that the description of physical reality provided by quantum mechanics was incomplete.

The EPR thought experiment involves a pair of particles prepared in an so called entangled state. Einstein, Podolsky, and Rosen pointed out that, in this state, if the position of the first particle were measured, the result of measuring the position of the second particle could be predicted. If, instead, the momentum of the first particle were measured, then the result of measuring the momentum of the second particle could be predicted. They argued that no action taken on the first particle could instantaneously affect the other, since this would involve information being transmitted faster than light, which is forbidden by the theory of relativity.

This contradicted the view associated with Niels Bohr and Werner Heisenberg, according to which a quantum particle does not have a definite value of a property like momentum until the measurement takes place.

While the EPR paradox is solved continuosly many years without rigorous result (Schnabel, 2022), there is the crucial incompleteness of the Schrödinger equation. Namely the nonexistence of the classical limit of this equation for the infinite mass of a particle. We here investigate this problem using the nonlinear Schrödinger equation with the logarithmic correction and the adequate new constant.

2 The nonlinear wave mechanics

Many authors have suggested that the quantum mechanics based on linear Schrödinger equation is only an approximation of some more nonlinear theory with the nonlinear Schrödinger equation. The motivation for considering the nonlinear equations is to get some more nonstandard solution in order to get the better understanding of the synergism of wave and particle.

The ambitious program to create nonlinear wave mechanics was elaborated by de Broglie (1960) and his group. Bialynicki-Birula and Mycielski (1976) considered the generalized Schrödinger equation with the additional term $F(|\Psi|^2)\Psi$ where F is some arbitrary function which they later specified to $-b(\ln|\Psi|^2)$, $b > 0$. The nonlinear term was selected by assuming the factorization of the wave function for the composed system.

The most attractive feature of the logarithmic nonlinearity is the existence of the lower energy bound and validity of Planck's relation $E = \hbar\omega$. At the same time the Born interpretation of the wave function cannot be changed. In this theory the estimation of b was given by the relation $b < 4 \times 10^{-10} eV$ following from the agreement between theory and the observed $2S - 2P$ Lamb shift in hydrogen. This implies an upper bound to the electron soliton spatial width of 10 μm.

Shimony (1979) proposed an experiment which is based on idea that a phase shift occurs when an absorber is moved from one point to another along the path of one of the coherent split beams in a neutron interferometer. In case of the logarithmic nonlinearity Shull at al. (1980) performed the experiment with a two-crystal interferometer. They searched for a phase shift when an attenuator was moved along the neutron propagation direction in one arm of the interferometer. A sheet of Cd, 0.086 mm thick, was used for the absorber.They obtained the upper bound on b of $3.4 \times 10^{-13} eV$ which is more than three orders of magnitude smaller than the bound estimated by Bialynicky-Birula and Mycielski (1976).

The best upper limit on b has been reported by Gähler, Klein and Zeilinger (1981) who has been searched for variations in the free space propagation of neutrons. 20 Å neutrons were diffracted from an abrupt highly absorbing knife edge at the object position. By comparing the experimental results with the solution to the ordinary Schrödinger equation they were able to get the limit $b < 3 \times 10^{-15} eV$, which corresponds to an alectron soliton width of 3 mm. The similar results was obtained by the same group from diffraction a 100 μm boron wire.

To our knowledge the Mössbauer effect was not used to determine the constant b although this effect allows to measure energy losses smaller than $10^{-15}eV$. Similarly the Josephson effect has been not applied for the determination of the constant b.

We see that the constant b is very small, nevertheless we cannot it neglect a priori, because we do not know its role in the future physics. The corresponding analog is the Planck constant which is also very small, however, it plays the fundamental role in physics.

The goal of this article is to give the new derivation of the logarithmic nonlinearity, to find the solution of the nonlinear Schrödinger equation of the one-dimensional case and to show that in the mass limit $m \to \infty$ we get exactly the delta-function behavior of the probability of finding the particle at point x. It means that there exists the classical motion of a particle with sufficient big mass. The nonlinearity of the Schrödinger equation also solves the colaps of the wave function and the Schrödinger cat paradox. We will start from the hydrodynamical formulation of quantum mechanics. The mathematical generalization of the Euler hydrodynamical equations leads automatically to the logarithmic term with $b > 0$. The article is the modified articles by author (Pardy, 1993; 1994; 2001).

3 The derivation of the nonlinear Schrödinger equation

We respect here the so called Dirac heuristic principle (Pais, 1986) according to which it is useful to postulate some mathematical requirement in order to get the true information about nature. While the mathematical assumption is intuitive, the consequences have the physical interpretation, or, in other words they are physically meaningful. In derivation of the logarithmic nonlinearity we use just the Dirac method.

According to Madelung (1926), Bohm and Vigier (1954), Wilhelm (1970), Rosen (1974) and others, the original Schrödinger equation can be transformed into the hydrodynamical system of equations by using the so called Madelung ansatz:

$$\Psi = \sqrt{n}e^{\frac{i}{\hbar}S}, \tag{1}$$

where n is interpreted as the density of particles and S is the classical action for $\hbar \to 0$. The mass density is defined by relation $\varrho = nm$ where m is mass of a particle.

It is well known that after insertion of the relation (1) into the original Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi, \tag{2}$$

where V is the potential energy, we get, after separating the real and imaginary parts, the following system of equations:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V = \frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}} \tag{3}$$

$$\frac{\partial n}{\partial t} + \mathrm{div}(n\mathbf{v}) = 0 \tag{4}$$

with

$$\mathbf{v} = \frac{\nabla S}{m}. \tag{5}$$

Equation (3) is the Hamilton-Jacobi equation with the additional term

$$V_q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}}, \tag{6}$$

which is called the quantum Bohm potential and equation (4) is the continuity equation.

After application of operator $\bigtriangledown$ on eq. (3), it can be cast into the Euler hydrodynamical equation of the form:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{m}\nabla(V + V_q). \tag{7}$$

It is evident that this equation is from the hydrodynamical point of view incomplete as a consequence of the missing term $-\varrho^{-1}\nabla p$ where p is hydrodynamical pressure. We use here this fact just as the crucial point for derivation of the nonlinear Schrödinger equation. We complete the eq. (7) by adding the pressure term and in such a way we get the total Euler equation in the form:

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla(V + V_q) - \nabla F, \tag{8}$$

where

$$\nabla F = \frac{1}{n}\nabla p. \tag{9}$$

The equation (8) can be obtained by the Madelung procedure from the following extended Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi + F\Psi \tag{10}$$

on the assumption that it is possible to determine F in term of the wave function. From the vector analysis follows that the necessary condition of the existence of F as the solution of the eq. (9) is rot grad F = 0, or,

$$\mathrm{rot}(n^{-1}\nabla p) = 0, \tag{11}$$

which enables to take the linear solution in the form

$$p = -bn = -b|\Psi|^2, \tag{12}$$

where b is some arbitrary constant. We do not consider the more general solution of eq. (11). Then, from eq. (9) i.e. grad $F = \mathbf{a}$ we have:

$$F = \int a_i\, dx_i = -b\int \frac{1}{n}\, dn = -b\ln|\Psi|^2, \tag{13}$$

where we have omitted the additive constant which plays no substantial role in the Schrödinger equation.

Now, we can write the generalized Schrödinger equation which corresponds to the complete Euler equation (8) in the following form:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi - b(\ln|\Psi|^2)\Psi. \tag{14}$$

Let us remark that the stochastic derivation of the nonlinear equation (14) was given by Lemos (1983). However, the author (Pardy) derivation is more attractive from the pedagogical view of point.

Let us approach the solving eq. (14).

4 The soliton-wave solution of the nonlinear Schrödinger equation

Let be $c, (\mathrm{Im}\, c = 0), v, k, \omega$ some parameters and let us insert function

$$\Psi(x,t) = cG(x - vt)e^{ikx - i\omega t} \tag{15}$$

into the one-dimensional equation (14) with $V = 0$. Putting the imaginary part of the new equation to zero, we get

$$v = \frac{\hbar k}{m} \tag{16}$$

and for function G we get the following nonlinear equation (the symbol $'$ denotes derivation with respect to $\xi = x - vt$):

$$G'' + AG + B(\ln G)G = 0, \tag{17}$$

where

$$A = \frac{2m}{\hbar}\omega - k^2 + \frac{2m}{\hbar^2}b\ln c^2 \tag{18}$$

$$B = \frac{4mb}{\hbar^2}. \tag{19}$$

After multiplication of eq. (17) by G' we get:

$$\frac{1}{2}\left[G'^2\right]' + \frac{A}{2}\left[G^2\right]' + B\left[\frac{G^2}{2}\ln G - \frac{G^2}{4}\right]' = 0, \tag{20}$$

or, after integration

$$G'^2 = -AG^2 - BG^2\ln G + \frac{B}{2}G^2 + const. \tag{21}$$

If we choose the solution in such a way that $G(\infty) = 0$ and $G'(\infty) = 0$, we get $const. = 0$ and after elementary operations we get the following differential equation to be solved:

$$\frac{dG}{G\sqrt{a - B\ln G}} = d\xi, \tag{22}$$

where

$$a = \frac{B}{2} - A. \tag{23}$$

Eq. (22) can be solved by the elementary integration and the result is

$$G = e^{\frac{a}{B}} e^{-\frac{B}{4}(\xi + d)^2}, \tag{24}$$

where d is some constant.

The corresponding soliton-wave function is evidently in the one-dimensional free particle case of the form:

$$\Psi(x,t) = ce^{\frac{a}{B}} e^{-\frac{B}{4}(x - vt + d)^2} e^{ikx - i\omega t}. \tag{25}$$

5 Normalization and the classical limit

It is not necessary to change the standard probability interpretation of the wave function. It means that the normalization condition in our one-dimensional case is

$$\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1. \tag{26}$$

Using the Gauss integral

$$\int_0^{\infty} e^{-\lambda^2 x^2} \, dx = \frac{\sqrt{\pi}}{2\lambda}, \tag{27}$$

we get with $\lambda = \left(\frac{B}{2}\right)^{\frac{1}{2}}$

$$c^2 e^{\frac{2a}{B}} = \left(\frac{B}{2\pi}\right)^{\frac{1}{2}} \tag{28}$$

and the density probability $\Psi^*\Psi = \delta_m(\xi)$ is of the form (with $d = 0$):

$$\delta_m(\xi) = \sqrt{\frac{m\alpha}{\pi}} e^{-\alpha m \xi^2} \quad ; \quad \alpha = \frac{2b}{\hbar^2}. \tag{29}$$

It may be easy to see that $\delta_m(\xi)$ is the delta-generating function and for $m \to \infty$ is just the Dirac δ-function.

It means that the motion of a particle with sufficiently big mass m is strongly localized and in other words it means that the motion of this particle is the classical one. Such behavior of a particle cannot be obtained in the standard quantum mechanics because the plane wave

$$e^{ikx-i\omega t} \tag{30}$$

corresponds to the free particle with no possibility of localization for $m \to \infty$.

Let us still remark that coefficient c^2 is real and positive number because it is a result of the solution of eq. (28) which can be transformed into equation ($x = c^2$)

$$x^{1-r} = const. \tag{31}$$

6 The principle of superposition

The principle of superposition is in nonlinear theory broken. If φ_1 and φ_2 are two different solution of the nonlinear Schrödinger equation then the linear combination $\varphi = a\varphi_1 + b\varphi_2$ where a and b are the arbitrary constants is not the solution of the same equation because of its nonlinearity. In other words the original principle of superposition of the standard quantum mechanics is broken. The consequence of the breaking of the principle of superposition is the resolution of the Schrödinger cat paradox (Glauber, 1986).

7 The determination of the constant b by experiment

After insertion of the function

$$\Psi(x,t) = exp - [i(E/\hbar)t]\phi(x) \tag{32}$$

into Eq. (14), we get, in the one-dimensional case,

$$\phi^{''} + k^2\phi = D\phi \ln|\phi|, \tag{33}$$

with

$$k^2 = \frac{2mE}{\hbar^2}, \quad D = -\frac{4mb}{\hbar^2}. \tag{34}$$

We suppose that the approximate solution of Eq. (33) is of the form

$$\phi(x) = A \sin \kappa x, \tag{35}$$

where A and κ are to be determined. For $|\sin\kappa x \cdot \ln|A|| \gg |\sin\kappa x \cdot \ln|\sin\kappa x||$, we get

$$(k^2 - \kappa^2) = D \ln|A| \tag{36}$$

and, from the boundary conditions $\phi(0) = \phi(L) = 0$ and from the normalization condition of the wave function on the space interval $(0, L)$, we get

$$\phi(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x. \tag{37}$$

The corresponding energies of the stationary states then are

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2 - b\lg\frac{2}{L}; \quad n = 1, 2, 3, ... \quad . \tag{38}$$

The last formula indicates that it is not possible to determine the constant b spectroscopically, because relation $\hbar\omega = E_n - E_m$ does not contain the constant b.

On the other hand, the force between boundaries, due to the the existence of the particle in the box, is $F = -\partial E/\partial L$, i.e.,

$$F_n = \frac{\hbar^2\pi^2}{mL^3}n^2 - \frac{b}{L} \tag{39}$$

for one particle in the box, and NF_n for the N noninteracting particles in the box. It can in principal be measured by the same methods that were used in the case of the Casimir effect (Spaarnay, 1958; Tabor et al., 1969). Of course, the difficulties will be greater than in the case of the Casimir effect.

The other possibility for the measurement of E_n, one which is here considered for the first time, is to consider the following experiment: Two rods with square cross section, are given, the near ends being at the distance L apart and forming a potential box of width L. Suppose particles are impinging on the rods in the locality of the boundary of the gap and are reflected. It is evident that the resonance absorption of particles by the gap occurs for a velocity that is determined by the equation

$$\frac{1}{2}mv_n^2 = E_n, \tag{40}$$

where v_n is the velocity perpendicular to the plane of the gap. If the source of particles is fixed to the rotating disk, then the velocity of the emitted particles can be continuously changed in order to get the resonance velocity, just as in case of the Mössbauer experiment (Mössbauer, 1958) , and in such a way that it gives the possibility for determination of the constant b.

For $\hbar$= 1.05 $\times$ 10^{-34} J, $L = 10^{-7}$ m, m = 1,67 $\times$ 10^{-27}kg, b = 3.3 $\times$ 10^{-15}eV = 3.3 $\times$ 1.6 $\times$ 10^{-34}J, n = 10, , we get E_{10} = 3,25 $\times$ 10^{-25}J - 8.87 $\times$ 10^{-33}J.

In this calculation we have used the results of the interferometric search for the nonlinear term in the Schrödinger equation of Shull et al. (Shull et al., 1980) and Gähler et al. (Gähler et al., 1981), who got the upper limit of b of $\leq$ 3.3 $\times$ 10^{-15}eV.

8 Discussion

We have seen that the introduction of the logarithmic nonlinearity in the Schrödinger equation was logically supported by the fact that the nonlinear Schrödinger equation gives results which are physically meaningful. We have obtained the correct mass limit of the wave function.

The further strong point of the nonlinear Schrödinger equation (14) is the result (16) which is equivalent to the famous de Broglie relation

$$\lambda = \frac{h}{p} \tag{41}$$

because of $\lambda = 2\pi/k = 2\pi(\hbar/mv) = 2\pi(h/2\pi)(1/p)$ and it means that de Broglie relation is involved in this form of the nonlinear quantum mechanics.

The nonlinear equation (14) has also the normalized plane-wave solution

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} e^{ikx - i\omega t}. \tag{42}$$

After insertion of eq. (33) into eq. (14), we get the following dispersion relation:

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + b\ln(2\pi), \tag{43}$$

from which the relations follows:

$$\hbar\omega = b\ln(2\pi); \quad k = 0 \tag{44}$$

and

$$k = \pm i\sqrt{\frac{2m}{\hbar^2} b\ln(2\pi)}; \quad \omega = 0. \tag{45}$$

It is no easy to give the physical interpretation of eqs. (44) and (45) and so we cannot say that the plane-solution of the nonlinear Schrödinger equation is physically meaningful. Only the soliton-wave solution of the nonlinear Schrödinger equation can be taken as relevant. Only this solution is suitable for the physical verification. The possible new tests of the nonlinear quantum mechanics are discussed in the author article (Pardy, 1994).

The generalization to the motion of particle in the electromagnetic field with potentials $\varphi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ can be performed by the standard transformation

$$\frac{\hbar}{i}\nabla \to \frac{\hbar}{i}\nabla - \left(\frac{e}{c}\right)\mathbf{A}(\mathbf{x},t) \tag{46}$$

and adding the scalar potential energy $\varphi(x,t)$ in the Schrödinger equation for the free particles. According to Bialynicky-Birula et al. (1976), the solution of the equation in this case can be taken in the form

$$\Psi(\mathbf{x},t) = e^{\frac{i}{\hbar}S} G(\mathbf{x} - \mathbf{u}(t)), \tag{47}$$

where function G is necessary to determine. In the similar form the problem was yet solved (Barut, 1990).

Kamesberger and Zeilinger (1998) have given the numerical solution of the original Schrödinger equation and this equation with the nonlinear term $-b(\ln|\Psi|^2)\Psi$ in order to visualize the spreading of the diffractive waves. When comparing the evolution patterns of the nonlinear case with the linear one, one notices that the maxims are more pronounced in the nonlinear solution. It can be understood as a mechanism compressing the wave maxims spatially.

In the quantitative comparison of the both cases this enhancement of the maxims and minims can be seen very clearly.

Although we have given reasons for the introducing of the nonlinear Schrödinger equation it is obvious that only the crucial experiments can establish the physical and not only logical necessity of such equation. In case that the nonlinear Schrödinger equation will be confirmed by experiment, then it can be expected that it will influence other parts of theoretical physics.

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