Stability of Discrete State-delayed Systems

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Abstract: - This paper is concerned with the robust stability of a class of uncertain discrete-time systems with time-varying delays and state saturation nonlinearities. The saturation nonlinearities are assumed to be the standard saturation function, and the delay is allowed to be time-varying with known bounds. By applying the Lyapunov stability theorem and the delay-fractioning approach, a suitable Lyapunov-Krasovskii functional and a non-negative scalar are constructed respectively. A delay-dependent criterion of the robust stability is given for the addressed systems. Finally, a numerical example is given to illustrate the effectiveness of the presented criterion.

Key-Words: - Robust stability, Time-varying delay, State saturation nonlinearities, Lyapunov-Krasovskii functional, Delay-fractioning approach

1 Introduction

Due to its important significance, the stability of dynamical systems has been widely studied over the past decades. On the other hand, in various engineering systems, time delays are introduced into the model of physical systems due to the inevitable reason such as measurement, transmission and transport lags, computational delays or un-modeled inertia of system components. The time delay is an important source of instability of the systems. So far, a large amount of results have been published for the stability and control of time-delay systems, see e.g. [1, 2]. Moreover, it is also well known that parameter uncertainties are inherent features of many physical systems. The uncertainties exist due to the variations in system parameters, modeling errors or some other ignored factors. Accordingly, the problems of analysis and synthesis have been addressed for uncertain systems [3-5].

Dynamical systems with state saturation nonlinearities exist commonly in neural networks, analogue circuits and control systems, and hence the stability analysis of such systems is highly nontrivial [6, 7]. Recently, many important results have been reported on this issue, see e.g. [8-14]. To be specific, the problems of the global asymptotic stability of the equilibrium have been investigated in [8, 9] for n-order discrete-time systems with state saturations and partial state saturation. An improved version of Ritzerfeld-Werter's criterion for the nonexistence of overflow oscillations in second-order state-space digital filters has been presented. A new zero-input limit cycle-free realizability condition has been given in [10] for a generalized overflow characteristic and a complete stability analysis has been proposed in [11] for a planar discrete-time linear system with saturation. In [12], a criterion of the global asymptotic stability has been presented for discrete-time systems with partial state saturation nonlinearities. Subsequently, the extension of this approach has been performed in [13] to a situation involving partial state saturation nonlinearities.

Recently, the problem of global asymptotic stability has been studied in [15] for uncertain discrete-time state-delayed systems with saturation nonlinearities. By using the linear matrix inequality (LMI) technique, a new criterion has been presented to guarantee the global asymptotic stability for the related systems. It has been shown that the presented result has improved the results in [8, 13]. However, it is worth pointing out that the delays have been assumed to be time-invariant in most relevant literature concerning the robust stability problem for the uncertain time-delay systems with state saturation nonlinearity. To the best of the authors' knowledge, there has been little work undertaken on the robust stability of uncertain discrete-time systems with time-varying delay and state saturation nonlinearity. Note that the delay-fractioning approach has been proved to be an effective way in [19-21] for addressing the time-delay. Hence, we will employ this approach to give a new stability criterion.

Motivated by the above discussions, in this paper, we aim to investigate the problem of robust stability for a class of uncertain discrete-time state-delayed systems with state saturation nonlinearity. Here, the delay is time-varying with known bounds and the parametric uncertainties are norm-bounded. By using the delay-fractioning approach and constructing an appropriate Lyapunov-Krasovskii functional, the delay-dependent robust stability condition is presented. It is shown that the proposed condition is in term of the solutions of the linear matrix inequalities (LMIs) which can be easily solved by using the standard numerical software. Finally, an illustrative example is given to demonstrate the effectiveness of the proposed results.

The rest of this paper is organized as follows. Section 2 briefly introduces the problem under consideration and gives some useful Lemmas. The criterion of robustly global asymptotic stability is given in Section 3. In Section 4, a numerical example is presented to illustrate the feasibility and effectiveness of the developed results. This paper is concluded in Section 5.

2 Problem Formulation and Preliminaries

In this paper, we consider the following uncertain discrete-time systems with time-varying delay and state saturated nonlinearities:

$$x(k+1) = f((y(k)) = \left[f_1((y_1(k)), \cdots, f_n((y_n(k)) \right]^T \tag{1a}$$

$$\begin{aligned} y(k) &= \left[y_1(k), \cdots, y_n(k) \right]^T \\ &= (A + \Delta A)x(k) + (A_d + \Delta A_d)x(k - d(k)) \end{aligned} \tag{1b}$$

$$x(k) = \phi(k), k \in [-d_M, 0] \tag{1c}$$

where, $x(k) \in R^n$ is the state vector; $A, A_d \in R^{n\times n}$ are known matrices; $\Delta A, \Delta A_d \in R^{n\times n}$ are the unknown matrices representing parametric uncertainties in the state matrices; $d(k)$ is the positive integer for time delays; $\phi(k) \in R^n$ is the initial state value at time k; $f(y(k))$ is the vector saturation function.

The saturation functions $f_i(y_i(k))$ are defined as

$$f_i(y_i(k)) = \begin{cases} -1, & y_i(k) < -1 \\ y_i(k), & |y_i(k)| \le 1 \\ 1, & y_i(k) > 1 \end{cases}, \; i=1, 2, \dots, n \tag{1d}$$

The uncertain matrix satisfies the following condition:

$$\Delta A = HFE, \Delta A_d = HFE_d \tag{1e}$$

where H, E and E_d are known constant matrices with appropriate dimensions and F is an unknown matrix satisfying

$$F^T F \le I \tag{1f}$$

The time-varying delay $d(k)$ satisfies

$$d_m \le d(k) \le d_M \tag{1g}$$

where d_m and d_M are known positive integers representing the upper and lower bounds of delay $d(k)$. The lower bound of delay d_m can always be described by $d_m = \tau m$, where τ and m are positive integers.

To proceed, we introduce the following definition and lemmas that will be used in the proofs of the main results.

Definition 1[15] The zero solution of the system described by (1a)-(1g) is globally asymptotically stable if the following holds:

(i) it is stable in the sense of Lyapunov, i.e., for every $\varepsilon > 0$ there exists a $\delta > 0$ so that $\|x(k)\| < \varepsilon$ for all $k = 1, 2, \cdots$, whenever $\|\phi(\cdot)\| = \max_{k\in[-d_M,0]} \|\phi(k)\| < \delta$;

(ii) it is attractive, i.e., $\lim_{k\to\infty} x(k) = 0$.

Lemma 1[16] Let D, E, F and M be real matrices of appropriate dimensions with M satisfying $M = M^T$, then

$$M + DFE + E^T F^T D^T < 0 \tag{2a}$$

for all $F^T F \le I$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$M + \varepsilon^{-1} DD^T + \varepsilon E^T E < 0 \tag{2b}$$

Lemma 2[16] Let D, E, F and M be real matrices of appropriate dimensions with M satisfying $M = M^T$, and P is a symmetric positive definite matrix, then

$$\begin{aligned} &(M + DFE)^T P (M + DFE) \\ &\le M^T \left(P^{-1} + \varepsilon^{-1} DD^T \right)^{-1} M + \varepsilon E^T E \end{aligned} \tag{3}$$

for any scalar $\varepsilon > 0$

Lemma 3[17] Given constant matrices A, B and C of appropriate dimension, with A and C symmetrical, then

$$A + BC^{-1}B^T > 0 \text{ and } C > 0 \tag{4a}$$

if and only if

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0 \tag{4b}$$

or equivalently

$$\begin{bmatrix} C & B^T \\ B & A \end{bmatrix} > 0 \qquad (4c)$$

Lemma 4[15] Suppose matrix $C=(c_{ij})\in R^{n\times n}$ is characterized by

$$c_{ii} = \sum_{j=1, j\neq i}^{n} (\alpha_{ij}+\beta_{ij}),\ \ i=1,2,\cdots,n$$
$$c_{ij} = \alpha_{ij}-\beta_{ij},\ \ i,j=1,2,\cdots,n(i\neq j) \qquad (5a)$$
$$\alpha_{ij}>0,\beta_{ij}>0,\ \ i,j=1,2,\cdots,n$$

then the scalar δ is a nonnegative number with

$$\delta=\sum_{i=1}^{n}2\left[y_i(k)-f_i\left(y_i(k)\right)\right]$$
$$\cdot\left[\sum_{j=1, j\neq i}^{n}\left((\alpha_{ij}+\beta_{ij})f_i\left(y_i(k)\right)+(\alpha_{ij}-\beta_{ij})f_j\left(y_j(k)\right)\right)\right]$$
$$=y(k)^T Cf\left(y(k)\right)+f\left(y(k)\right)^T C^T y(k)-f\left(y(k)\right)^T$$
$$\cdot(C+C^T)f\left(y(k)\right) \qquad (5b)$$

3 Main Result

In this section, by using the delay-fractioning approach and the LMI technique, the criterion of robustly global asymptotic stability is given for uncertain discrete-time systems with time-varying delay and state saturation nonlinearities.

Theorem 1 Consider system (1a)-(1g). If there exist symmetry positive definite matrices $P>0,Q>0,R>0,S>0$, the symmetry positive semi-definite matrix $M\geq 0,N\geq 0$ real matrices *X, Y, Z* with appropriate dimensions and matrix *C* satisfying (5a), and positive scalar $\varepsilon>0$ such that

$$\begin{bmatrix} \Pi_1+\Pi_2+\Pi_2^T+\Pi_3 +\Pi_3^T+\Pi_4+\Pi_5 & \sqrt{2h_2+1}C^T H \\ * & -\varepsilon I \end{bmatrix}<0 \qquad (6a)$$

$$\begin{bmatrix} M & X \\ X^T & P \end{bmatrix}\geq 0,\ \begin{bmatrix} N & Y \\ Y^T & P \end{bmatrix}\geq 0,\ \begin{bmatrix} N & Z \\ Z^T & P \end{bmatrix}\geq 0 \qquad (6b)$$

where

$h_1=d_M-d_m+1,\ h_2=d_M-d_m+\tau$,

$\Pi_1=\tau M+(d_M-d_m)N$,

$$\Pi_2=[X,Y,Z]\begin{bmatrix} I_{n\times n},-I_{n\times n},O_{n\times(m+2)n} \\ O_{n\times mn},I_{n\times n},-I_{n\times n},O_{n\times 2n} \\ O_{n\times(m+1)n},I_{n\times n},-I_{n\times n},O_{n\times n}\end{bmatrix},$$

$\Pi_3=2(h_2+1)(A\theta_1+A_d\theta_2)^T C\theta_4$,

$\Pi_4=W_R^T\bar{R}W_R$,

$$\Pi_5=\theta_1^T\left((2h_2-1)P+h_1Q+S\right)\theta_1-\theta_2^TQ\theta_2$$
$$-\theta_3^TS\theta_3+(2h_2+1)\theta_4^T(P-C-C^T)\theta_4$$
$$+\varepsilon(2h_2+1)(E\theta_1+E_d\theta_2)^T(E\theta_1+E_d\theta_2),$$

$$W_R=\begin{bmatrix} I_{mn\times mn},O_{mn\times n},O_{mn\times 3n} \\ O_{mn\times n},I_{mn\times mn},O_{mn\times 3n}\end{bmatrix},\ \bar{R}=\begin{bmatrix} R & \\ & -R\end{bmatrix},$$

$\theta_1=\left[I_{n\times n},O_{n\times(m+3)n}\right],\ \theta_2=\left[O_{n\times(m+1)n},I_{n\times n},O_{n\times 2n}\right]$,

$\theta_3=\left[O_{n\times(m+2)n},I_{n\times n},O_{n\times n}\right],\ \theta_4=\left[O_{n\times(m+3)n},I_{n\times n}\right]$,

then the zero solution of system (1a)-(1g) is globally asymptotically stable.

Proof. By using the delay fractioning approach in [19], construct the following Lyapunov-Krasovskii functional

$$V(k)=\sum_{i=1}^{4}V_i(k) \qquad (7)$$

where

$$V_1(k)=x(k)^T Px(k)$$
$$V_2(k)=\sum_{j=k-d(k)}^{k-1}x^T(j)Qx(j)+\sum_{j=-d_M+1}^{-d_m}\sum_{l=j+k}^{k-1}x^T(l)Qx(l)$$
$$V_3(k)=\sum_{j=k-\tau}^{k-1}\xi^T(j)R\xi(j)+\sum_{j=k-d_M}^{k-1}x^T(j)Sx(j)$$
$$V_4(k)=\sum_{j=-\tau}^{-1}\sum_{l=k+j}^{k-1}\eta^T(l)P\eta(l)+\sum_{j=-d_M}^{-d_m-1}\sum_{l=k+j}^{k-1}\eta^T(l)P\eta(l)$$
$$\eta(l)=x(l+1)-x(l)$$
$$\varsigma^T(j)=\left[x^T(j),x^T(j-\tau),\cdots,x^T(j-(m-1)\tau)\right]$$

with $P>0,Q>0,R>0,S>0$ being matrices to be determined.

Denote $x_k=x(k),\ y_k=y(k),\ \eta_l=\eta(l),\ \varsigma_j=\varsigma(j),\ d_k=d(k)$. Calculate the difference of *V*(*k*) along systems (1a)-(1g).

$$\Delta V(k)=V(k+1)-V(k)=\sum_{i=1}^{4}\Delta V_i(k) \qquad (8)$$

where

$$\Delta V_1(k) = f(y_k)^T Pf(y_k) - x_k^T Px_k \le f(y_k)^T (P - C - C^T) f(y_k) + 2y_k^T Cf(y_k) - x_k^T Px_k \quad (9)$$

$$\Delta V_2(k) = \sum_{j=k+1-d_{k+1}}^{k} x_j^T Qx_j + \sum_{j=-d_M+1}^{-d_m} \sum_{l=k+1+j}^{k} x_l^T Qx_l - \sum_{j=k-d_k}^{k-1} x_j^T Qx_j - \sum_{j=-d_M+1}^{-d_m} \sum_{l=k+j}^{k-1} x_l^T Qx_l \le h_1 x_k^T Qx_k - x_{k-d_k}^T Qx_{k-d_k} \quad (10)$$

$$\Delta V_3(k) = \sum_{l=k+1-\tau}^{k} \varsigma_l^T R\varsigma_l + \sum_{l=k+1-d_M}^{k} x_l^T Sx_l - \sum_{l=k-\tau}^{k-1} \varsigma_l^T R\varsigma_l - \sum_{l=k-d_M}^{k-1} x_l^T Sx_l = \varsigma_k^T R\varsigma_k - \varsigma_{k-\tau}^T R\varsigma_{k-\tau} + x_k^T Sx_k - x_{k-d_M}^T Sx_{k-d_M} \quad (11)$$

$$\Delta V_4(k) = \sum_{j=-\tau}^{-1} \sum_{l=k+1+j}^{k} \eta_l^T P\eta_l + \sum_{j=-d_M}^{-d_m-1} \sum_{l=k+1+j}^{k} \eta_l^T P\eta_l - \sum_{j=-\tau}^{-1} \sum_{l=k+j}^{k-1} \eta_l^T P\eta_l - \sum_{j=-d_M}^{-d_m-1} \sum_{l=k+j}^{k-1} \eta_l^T P\eta_l = h_2 \eta_k^T P\eta_k - \sum_{j=k-\tau}^{k-1} \eta_j^T P\eta_j - \sum_{j=k-d_M}^{k-d_m-1} \eta_j^T P\eta_j \quad (12)$$

Note that

$$\begin{aligned}\eta_k^T P\eta_k &= (x_{k+1} - x_k)^T P(x_{k+1} - x_k) \le 4y_k^T Cf(y_k) - 2f(y_k)(C + C^T - P) f(y_k) + 2x_k^T Px_k \\ &= 4\left[(Ax_k + A_d x_{k-d_k}) + HF(Ex_k + E_d x_{k-d_k})\right]^T \cdot Cf(y_k) - 2f(y_k)(C + C^T - P) f(y_k) + 2x_k^T Px_k \\ &\le 4(Ax_k + A_d x_{k-d_k})^T Cf(y_k) + 2\varepsilon^{-1} f(y_k)^T C^T H \cdot H^T Cf(y_k) + 2\varepsilon(Ex_k + E_d x_{k-d_k})^T (Ex_k + E_d x_{k-d_k}) \\ &\quad - 2f(y_k)^T (C + C^T - P) f(y_k) + 2x_k^T Px_k \quad (13)\end{aligned}$$

Hence, from (9)~(13), we have

$$\begin{aligned}\Delta V(k) &\le x_k^T \left[(2h_2 - 1)P + h_1 Q + S\right] x_k - x_{k-d_k}^T Qx_{k-d_k} - x_{k-d_M}^T Sx_{k-d_M} + \varsigma_k^T R\varsigma_k - \varsigma_{k-\tau}^T R\varsigma_{k-\tau} + (2h_2 + 1) \\ &\quad \cdot \Big[-f(y_k)^T (C + C^T - P) f(y_k) + \varepsilon^{-1} f(y_k)^T C^T H \cdot H^T Cf(y_k) + \varepsilon(Ex_k + E_d x_{k-d_k})^T (Ex_k + E_d x_{k-d_k}) + \\ &\quad (Ax_k + A_d x_{k-d_k})^T Cf(y_k) + f(y_k)^T C^T (Ax_k + A_d x_{k-d_k})\Big] \\ &\quad - \sum_{j=k-\tau}^{k-1} \eta_j^T P\eta_j - \sum_{j=k-d_k}^{k-d_m-1} \eta_j^T P\eta_j - \sum_{j=k-d_M}^{k-d_k-1} \eta_j^T P\eta_j \quad (14)\end{aligned}$$

Let

$$\xi^T = [\varsigma_k^T, x_{k-d_m}^T, x_{k-d_k}^T, x_{k-d_M}^T, f(y_k)^T] \in R^{(m+4)n}$$

Then, it is easy to see that

$$x_k = \theta_1 \xi_k, x_{k-d_k} = \theta_2 \xi_k, x_{k-d_M} = \theta_3 \xi_k, f(y_k) = \theta_4 \xi_k$$

Thus

$$\begin{aligned}\Delta V(k) &\le \xi_k^T \Big[\theta_1^T \left((2h_2 - 1)P + h_1 Q + S\right)\theta_1 - \theta_2^T Q\theta_2 - \theta_3^T S\theta_3 + W_R^T \bar{R} W_R + (2h_2 + 1)\Big(\theta_4^T (P - C - C^T \\ &\quad + \varepsilon^{-1} C^T HH^T C)\theta_4 + \varepsilon(E\theta_1 + E_d\theta_2)^T (E\theta_1 + E_d\theta_2) + (A\theta_1 + A_d\theta_2)^T C\theta_4 + \theta_4^T C^T (A\theta_1 + A_d\theta_2)\Big)\Big]\xi_k \\ &\quad - \sum_{j=k-\tau}^{k-1} \eta_j^T P\eta_j - \sum_{j=k-d_k}^{k-d_m-1} \eta_j^T P\eta_j - \sum_{j=k-d_M}^{k-d_k-1} \eta_j^T P\eta_j \quad (15)\end{aligned}$$

For any matrices X, Y, Z with appropriate dimensions, we have

$$\begin{aligned}2\xi_k^T X\left(x_k - x_{k-\tau} - \sum_{j=k-\tau}^{k-1} \eta_j\right) &= 0 \\ 2\xi_k^T Y\left(x_{k-d_m} - x_{k-d_k} - \sum_{j=k-d_k}^{k-d_m-1} \eta_j\right) &= 0 \quad (16) \\ 2\xi_k^T Z\left(x_{k-d_k} - x_{k-d_M} - \sum_{j=k-d_M}^{k-d_k-1} \eta_j\right) &= 0\end{aligned}$$

On the other hand, for any symmetry positive semi-definite matrix $M \ge 0, N \ge 0$ with appropriate dimensions, the following equations always hold

$$\begin{aligned}\tau\xi_k^T M\xi_k - \sum_{j=k-\tau}^{k-1} \xi_k^T M\xi_k &= 0 \\ (d_M - d_m)\xi_k^T N\xi_k - \sum_{j=k-d_M}^{k-d_m-1} \xi_k^T N\xi_k &= 0\end{aligned} \quad (17)$$

It follows from (16)-(17) that

$$\Delta V(k) \le \xi_k^T \left[\Pi_1 + \Pi_2 + \Pi_2^T + \Pi_3 + \Pi_3^T + \Pi_4 + \Pi_5 + \varepsilon^{-1}(2h_2 + 1)\theta_4^T C^T HH^T C\theta_4\right]\xi_k$$

$$-\sum_{j=k-\tau}^{k-1}\xi(k,j)^T\begin{bmatrix}M & X\\ X^T & P\end{bmatrix}\xi(k,j)$$
$$-\sum_{j=k-d_M}^{k-d_k-1}\xi(k,j)^T\begin{bmatrix}N & Z\\ Z^T & P\end{bmatrix}\xi(k,j)$$
$$-\sum_{j=k-d_k}^{k-d_m-1}\xi(k,j)^T\begin{bmatrix}N & Y\\ Y^T & P\end{bmatrix}\xi(k,j)$$

Noting (6b), we know that $\Delta V(k)$ is negative definite if the following inequality

$$\Lambda=\Pi_1+\Pi_2+\Pi_2^T+\Pi_3+\Pi_3^T+\Pi_4+\Pi_5$$
$$+\varepsilon^{-1}(2h_2+1)\theta_4^TC^THH^TC\theta_4<0$$

holds. By using Lemmas 1 and 3, $\Lambda<0$ is equivalent to (6a). This completes the proof of this theorem.

Remark 1: Up to know, the problem of the robustly global asymptotic stability has been studied for a class of uncertain discrete-time systems with time-varying delay and state saturation nonlinearities. By using the delay-fractioning approach, an appropriate Lyapunov-Krasovskii functional has been introduced to deal with the time-varying delay. By employing the LMI technique, a new stability criterion has been given to guarantee the robustly global asymptotic stability of the addressed system.

As a special case, if there is no saturation nonlinearities in system (1a), we have the following system:

$$x_{k+1}=(A+\Delta A)x_k+(A_d+\Delta A_d)x_{k-d_k}=y_k \quad (18)$$

In the following corollary, a sufficient condition is proposed to ensure the robustly global asymptotic stability of system (18).

Corollary 1 Consider the system (18). If there exist the symmetry positive definite matrices $P>0$, $Q>0$, $R>0$, $S>0$, the symmetry positive semi-definite matrix $M\geq 0, N\geq 0$ real matrices *X, Y, Z,* and positive scalar $\varepsilon>0$ such that

$$\begin{bmatrix}W_{11} & W_{12}^T & W_{12}^T\\ * & -P-\varepsilon HH^T & 0\\ * & * & -\varepsilon I\end{bmatrix}<0 \quad (19a)$$

$$\begin{bmatrix}M & X\\ X^T & P\end{bmatrix}\geq 0,\ \begin{bmatrix}N & Y\\ Y^T & P\end{bmatrix}\geq 0,\ \begin{bmatrix}N & Z\\ Z^T & P\end{bmatrix}\geq 0 \quad (19b)$$

where

$$W_{11}=\Pi_1+\Pi_2'+\Pi_2'^T+\Pi_4'+\Pi_5',$$
$$W_{12}=\sqrt{2h_2+1}\left(A\theta_1'+A_d\theta_2'\right),$$
$$W_{13}=\sqrt{2h_2+1}\left(E\theta_1'+E_d\theta_2'\right),$$

and $h_1,h_2,\Pi_1,\bar{R}$ are same as in the Theorem 1, and

$$\Pi_2'=[X,Y,Z]\begin{bmatrix}I_{n\times n},-I_{n\times n},O_{n\times(m+1)n}\\ O_{n\times mn},I_{n\times n},-I_{n\times n},O_{n\times n}\\ O_{n\times(m+1)n},I_{n\times n},-I_{n\times n}\end{bmatrix}$$
$$\Pi_4'=W_R'^T\bar{R}W_R'$$
$$\Pi_5'=\theta_1'^T\left((2h_2-1)P+h_1Q+S\right)\theta_1'-\theta_2'^TQ\theta_2'$$
$$-\theta_3'^TS\theta_3'+\varepsilon(2h_2+1)(E\theta_1'+E_d\theta_2')^T(E\theta_1'+E_d\theta_2')$$
$$W_R'=\begin{bmatrix}I_{mn\times mn},O_{mn\times n},O_{mn\times 2n}\\ O_{mn\times n},I_{mn\times mn},O_{mn\times 2n}\end{bmatrix}$$
$$\theta_1'=\left[I_{n\times n},O_{n\times(m+2)n}\right],\ \theta_2'=\left[O_{n\times(m+1)n},I_{n\times n},O_{n\times n}\right]$$
$$\theta_3=\left[O_{n\times(m+2)n},I_{n\times n}\right]$$

then the zero solution of system(18) is globally asymptotically stable.

Proof. Constructing the Lyapunov-Krasovskii functional as in (7), we have

$$\Delta V_1(k)=y_k^TPy_k-x_k^TPx_k$$

and $\Delta V_2(k),\Delta V_3(k),\Delta V_4(k)$ are same as in the Theorem 1. Note that

$$\eta_k^TP\eta_k=(y_k-x_k)^TP(y_k-x_k)\leq 2y_k^TPy_k+2x_k^TPx_k$$
$$y_k^TPy_k=\left((A+\Delta A)x_k+(A_d+\Delta A_d)x_{k-d_k}\right)^TP$$
$$\cdot\left((A+\Delta A)x_k+(A_d+\Delta A_d)x_{k-d_k}\right)$$

then

$$\Delta V(k)\leq x_k^T\left[(2h_2-1)P+h_1Q+S+(2h_2+1)\right.$$
$$\left.\cdot(A+\Delta A)^TP(A+\Delta A)\right]x_k+x_{k-d_k}^T\left[-Q+(2h_2+1)\right.$$
$$\left.\cdot(A_d+\Delta A_d)^TP(A_d+\Delta A_d)\right]x_{k-d_k}+2(2h_2+1)x_k^T$$
$$(A+\Delta A)^TP\,(A_d+\Delta A_d)\,x_{k-d_k}-x_{k-d_M}^TSx_{k-d_M}$$
$$+\varsigma_k^TR\varsigma_k-\varsigma_{k-\tau}^TR\varsigma_{k-\tau}-\sum_{j=k-\tau}^{k-1}\eta_j^TP\eta_j-\sum_{j=k-d_k}^{k-d_m-1}\eta_j^TP\eta_j$$
$$-\sum_{j=k-d_M}^{k-d_k-1}\eta_j^TP\eta_j$$

Denote $\xi'^T=[\varsigma_k^T,x_{k-d_m}^T,x_{k-d_k}^T,x_{k-d_M}^T]\in R^{(m+3)n}$, then, it yields

$$x_k=\theta_1'\xi_k',x_{k-d_k}=\theta_2'\xi_k',x_{k-d_M}=\theta_3'\xi_k'$$

Therefore,

$$\Delta V(k) \le \xi_k'^T \{ \theta_1'^T [(2h_2-1)P + h_1 Q + S + (2h_2+1)(A+\Delta A)^T P (A+\Delta A)] \theta_1' + \theta_2'^T [-Q + (2h_2+1) \cdot (A_d+\Delta A_d)^T P (A_d+\Delta A_d)] \theta_2' + (2h_2+1)\theta_1'^T \cdot (A+\Delta A)^T P (A_d+\Delta A_d) \theta_2' + (2h_2+1)\theta_2'^T (A_d + \Delta A_d)^T P (A+\Delta A) \theta_1' - \theta_3'^T S \theta_3' + W_R'^T \bar{R} W_R' \} \xi_k' - \sum_{j=k-\tau}^{k-1} \eta_j^T P \eta_j - \sum_{j=k-d_k}^{k-d_m-1} \eta_j^T P \eta_j - \sum_{j=k-d_M}^{k-d_k-1} \eta_j^T P \eta_j$$

By using Lemmas 2 and the method as in the Theorem 1, the proof of this corollary is complete.

4 A Numerical Example

In this section, a numerical example is given to illustrate the usefulness of the proposed results.

Example 1. Consider the system (1a) –(1g) with the following parameters:

$$d_M = 3, d_m = 1, \tau = 1, m = 1$$

$$A = \begin{bmatrix} 1.01 & -2.5 \\ 0.1 & 0 \end{bmatrix}, A_d = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.002 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, E = \begin{bmatrix} 0.01 & 0 \end{bmatrix}, E_d = \begin{bmatrix} 0 & 0.01 \end{bmatrix}$$

By using the Matlab LMI Toolbox, it turns out that LMI (6a)-(6b) is feasible. We obtain the following values:

$$P = \begin{bmatrix} 14.87 & 0 \\ 0 & 14.87 \end{bmatrix}, Q = \begin{bmatrix} 26.76 & 0 \\ 0 & 22.3 \end{bmatrix}$$

$$R = \begin{bmatrix} 22.3 & 0 \\ 0 & 17.84 \end{bmatrix}, S = \begin{bmatrix} 22.3 & 0 \\ 0 & 22.3 \end{bmatrix}$$

$$X = \begin{bmatrix} O_{2\times4}, X_1^T, O_{2\times4} \end{bmatrix}^T, X_1^T = \begin{bmatrix} 0 & 0 \\ -4.46 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} -44.6 & 0 & -8.92 & -40.14 & -4.62 & -4.46 & 4.46 & 104.15 & -8.92 & 4.46 \\ -4.46 & 0 & -360.91 & -4.46 & 4.46 & -492.88 & -4.46 & 8.94 & 0 & -8.92 \end{bmatrix} = -Z$$

$$C = \begin{bmatrix} 35.68 & 8.92 \\ 0 & 44.6 \end{bmatrix}, \varepsilon = 22.3$$

Thus, according to Theorem 1, the system under consideration is robust globally asymptotical stable which confirms the feasibility of the proposed stability criterion.

5 Conclusion

In this paper, a new criterion of the robust stability has been given for a class of uncertain discrete-time systems with time-varying delay and state saturation nonlinearities. The norm-bounded parametric uncertainties have been considered. The delay-dependent stability criterion has been given by integrating the delay-fractioning approach and the LMI technique. It is worth mentioning that the proposed results are less conservative than the existing results if there is no saturation nonlinearity in the addressed system. Also, both the constructed Lyapunov-Krasovskii functional based on the delay fractioning approach and the nonnegative scalar δ are important. It has been shown that the proposed scheme can be easily checked by using the standard numerical software. Finally, a numerical example has been given to illustrate the feasibility of the proposed results.

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