

An Illustration of Generalized Thermodynamics by Several Physical Examples

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Abstract

It has been shown recently that Bose Gase with weak pair (enough well) interaction is non ergodic system. But Bose Gase with weak pair interaction is so general system that it is evident that the majority of statistical mechanics systems are non ergodic too. It is also has been shown that it is possible to generalize the scheme of standard statistical mechanics and thermodynamics to take into account non ergodicity. This generalization is called a generalized thermodynamics. In some points this generalized thermodynamics coincide with standard equilibrium thermodynamics but some new specific results take place. It has been shown that this new generalized thermodynamics can be used to explain some physiological phenomena which take place in the living cell when the cell is exciting and dying.

In the present paper we try to illustrate some basic points of this generalized thermodynamics on some physical examples.

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1 Introduction

The purpose of this paper to give some examples on so called generalized thermodynamics recently developed in [1,2].

In [1] it has been shown that even most realistic systems of statistical mechanics (for example usual Bose Gase with weak (enough well) pair interaction) are non ergodic systems. These means that there exist non trivial first integrals of the system commuting with momenta and number particle operators. It has been recently shown [2] that it is possible to generalize the scheme of standard statistical mechanics and thermodynamics to take into account the existence of such non trivial first integrals. These generalization we call the generalized thermodynamics. It has been also shown that this generalized thermodynamics is very useful to explanation of some physiological phenomena which take place in the living cell when the cell is excited dying.

Let us describe the basic elements of the scheme of this generalized thermodynamics. The non ergodic theorem states that for a wide class of realistic systems of statistical mechanics there exists non trivial commuting (in involution) first integrals K_1, \dots, K_N , $N = 1, 2, \dots$ commuting with momenta and number particle operators. In purpose of simplicity we will talk about Hamiltonian instead of Hamiltonian, momenta and number particle operators. The starting point of generalized thermodynamics is a following expression for distribution function (density matrix in quantum case):

$$\rho(x) = \text{const} \delta(H(x) - E) \prod_{i=1}^N \delta(K_i(x) - K'_i), \quad (1)$$

where x is a point of phase space of the system and E, K'_1, \dots, K'_N are the observable values of energy and integrals K_1, \dots, K_N . The entropy corresponding to this distribution is defined as a logarithm of statistical weight. The statistical weigh, by definition, is a number of microscopic configuration of the system, corresponding to a given macroscopic state i.e.:

$$W(E, K'_1, \dots, K'_N) = \int d\Gamma_x \delta(H(x) - E) \prod_{i=1}^N \delta(K_i(x) - K'_i) \quad (2)$$

and

$$S(E, K'_1, \dots, K'_N) = \ln W(E, K'_1, \dots, K'_N), \quad (3)$$

where $d\Gamma_x$ is an element of phase volume. The expression (1), (2) and (3) are the generalization of standard microcanonical Gibbs distribution and its entropy to the case when there exists non trivial commuting first integrals K_1, \dots, K_N of the system.

Let us explain why we require pairwise commutativity of integrals K_1, \dots, K_N or why we require that $\forall i, j = 1, \dots, N (K_i, K_j) = 0$ where (\cdot, \cdot) denotes the commutator in quantum case or the Poisson bracket in classical case. We want to describe the state with definite values K'_1, \dots, K'_N of first integrals K_1, \dots, K_N . So the integrals $K_i, i = 1, \dots, N$ must be contemporaneously measurable. But in quantum mechanics this means that the integrals K_1, \dots, K_N must be pairwise commutative. The requirement that the integrals K_1, \dots, K_N must be in involution (in classical case) is clear now from the remark that the Poisson bracket is a classical analog of commutator.

Now let answer the question when generalized thermodynamics gives results which differs from results following from standard thermodynamics. I.e. let us answer the question when using of distribution (1) leads to results which differ from result obtained by using standard microcanonical Gibbs distribution.

For simplicity consider the case $N = 1$. The case of an arbitrary N can be considered by analogy to this case. For a fixed E $S(E, K')$ is a function of K' . There exist two typical cases of behavior of this function.

- 1) $S(E, K')$ (for fixed E) has a maximum in isolated point $K' = K''$.
- 2) $S(E, K')$ achieve a maximum at whole interval $K' \in [a, b]$ of nonzero length.

There arise a question, which observable values K' of integral K could be realized in nature. It is essentially to suppose that it could be realized such and only such values K' of K which corresponds to the maximum of entropy (for fixed energy). This is one of main point of generalized thermodynamics. It has been shown [2] that in case 2) generalized thermodynamics gives no more than standard equilibrium thermodynamics (K' is uniquely defined). But new interesting physics arise in case 2). Note that the situation when thermodynamical functions have a plateau as functions of their arguments is a typical situation in the theory of phase transitions.

Some general results of standard equilibrium thermodynamics take place in generalized thermodynamics [2]. For example a well-known relation

$$dE = TdS - PdV \tag{4}$$

takes place in new situation. Here E is an energy of the system, T is a

temperature of the system, V is a volume of the system, P is a pressure. But some new specific results take place for generalized thermodynamics. These new results have been used [2,3] for explanation of several physiological phenomena which takes place in biological cell when the cell is dying or moving from resting state into activated state.

In the present paper we give some new physical examples to illustrate some basic points of this generalized thermodynamics. The paper is composed as follows. In section 3 we describe some new derivation of the Gibbs distribution from the property of asymptotical factorization of correlations based on ideas of nonequilibrium renormalization theory. In section 3 we reformulate the Bardeen — Cooper — Schrieffer model on the language of the generalized thermodynamics. In section 5 we describe some derivation of the Boltzmann kinetic equation based on some ideas of N.N. Bogoliubov concerned with thermalization in oscillator interacting with thermostat. In section 6 we describe how to obtain non-trivial first integrals for the equations on order parameter of the system in many-phase domain in the theory of second order phase transitions. Section 6 is a conclusion.

2 Derivation of Gibbs distribution from the property of asymptotic space factorization of correlations.

In the present section we prove that all stationary translation-invariant states of Bose Gas with weak (enough good) pair interaction satisfying to the property of (enough fast) asymptotic space factorization of correlations are the Gibbs states (described by the Gibbs distribution).

But at first let us recall the main result of the paper [1] and demonstrate, how the non ergodic property follows from this result.

The main result of [1] can be formulated as follows:

Theorem 1. For Bose Gas with weak pair interaction with kernels from Schwartz space in the sense of formal power series on coupling constant there exists non-Gibbs functional $\langle \cdot \rangle$, commuting with the number of particle operator such that the correlators

$$\langle \Psi^\pm(t, x_1) \dots \Psi^\pm(t, x_n) \rangle$$

are translation invariant, do not depend on t and satisfy the weak cluster

property. Here Ψ^\pm are the secondary quantized wave function and its complex conjugated $\Psi^-(x) := \Psi(x)$ and the weak cluster property means the following

$$\begin{aligned} & \lim_{|a| \rightarrow \infty} \int_{R^{3n}} \langle \Psi^\pm(t, x_1 + \delta_1 e_1 a) \dots \Psi^\pm(t, x_n + \delta_n e_1 a) \rangle f(x_1, \dots, x_n) d^3 x_1 \dots d^3 x_n \\ &= \int_{R^{3n}} \langle \Psi^\pm(t, x_{i_1}) \dots \Psi^\pm(t, x_{i_k}) \rangle \langle \Psi^\pm(t, x_{i_k}) \dots \Psi^\pm(t, x_{i_n}) \rangle \times f(x_1, \dots, x_n) d^3 x_1 \dots d^3 x_n, \end{aligned}$$

there $\delta_i \in \{1, 0\}$, $i = 1, 2 \dots n$ and

$$\begin{aligned} & i_1 < i_2 < \dots < i_k, \\ & i_{k+1} < i_{k+2} < \dots < i_n, \\ & \{i_1, i_2, \dots, i_k\} = \{i = 1, 2 \dots n \mid \delta_i = 0\} \neq \emptyset, \\ & \{i_{k+1}, i_{k+2}, \dots, i_n\} = \{i = 1, 2 \dots n \mid \delta_i = 1\} \neq \emptyset. \end{aligned}$$

$f(x_1, \dots, x_n)$ is a test function (i.e. the function from the Schwartz space), e_1 is a unit vector parallel to the x -axis. About secondary quantization see for example [4, 5].

Let us prove that the existence of such functionals implies non-ergodic property of the system. The more accurate proof of this fact see in section 10 of [1]. Suppose that our system is ergodic, i.e. there are no first integrals of the system except energy. Then, the density matrix ρ of the system corresponding to the functional $\langle \cdot \rangle$ is a function of energy. We can represent this density matrix ρ as follows:

$$\rho = \sum c_\alpha \delta(H - E_\alpha),$$

where H is a Hamiltonian of the system and the sum can be continuous (integral). Let 1 be some enough large but finite subsystem of our system. Let 2 be a subsystem obtained from 1 by translation on the vector \vec{l} of sufficiently large length parallel to the x -axis. Let 12 be a union of the subsystems 1 and 2. Let ρ_1 , ρ_2 and ρ_{12} be the density matrix of the subsystems 1, 2 and 12 respectively. By the same method as the method used for the derivation of the Gibbs distribution we find:

$$\rho_{12} = \sum c_\alpha d_\alpha \frac{e^{-\frac{H_1}{T_\alpha}}}{Z_\alpha} \otimes \frac{e^{-\frac{H_2}{T_\alpha}}}{Z_\alpha}, \quad d_\alpha > 0 \forall \alpha$$

in the obvious notation. Here H_1 and H_2 be the Hamiltonians of subsystems 1 and 2 respectively. But the weak cluster property implies that

$$\rho_{12} = \rho_1 \otimes \rho_2.$$

Therefore all the coefficients c_α are equal to zero except one. We find that

$$\rho = c\delta(H - E_0)$$

for some constants c and E_0 . So each finite subsystem of our system can be described by Gibbs formula and we obtain a contradiction.

But the state (not necessary positive defined) mentioned in theorem 1 satisfies only to the weak cluster property i.e. to the property of asymptotic space factorization of correlations only "in one direction".

It follows from the proof of theorem 1 from [1] that we can achieve, that $\langle \cdot \rangle$ will satisfies to the property of asymptotic factorization of correlations "in two directions". But there arise principled difficulties if one try to prove that the state $\langle \cdot \rangle$ satisfy to cluster property in "all (three) directions".

It has been mentioned above that all stationary translation-invariant states satisfying to (enough fast) cluster property are the Gibbs states. Let us now recall the standard derivation of this fact. Then we will point out some problems connected with this derivation and present our new proof of this fact based on results of our renormalization theory of nonequilibrium (Keldysh) diagram technique.

Let us at last describe the standard derivation of Gibbs distribution (from cluster property) at classical level [6]. This derivation is based on the fact that there no exist additive first integrals of the system (linear) independent of Hamiltonian H , momenta \vec{P} and angular momenta \vec{M} . Let ρ be a stationary translaion-invariant distribution function of the system satisfying to the cluster property of the system. If the system is divided into two subsystems 1 and 2 cluster property implies that

$$\rho = \rho_1 \rho_2, \tag{5}$$

where ρ_1 and ρ_2 are distribution functions for subsystems 1 and 2 respectively. In other word the distributions for subsystems 1 and 2 are independent. Therefore $\ln \rho$ is an additive integral of motion and can be represented as linear function of H , \vec{P} , \vec{M} . In other words

$$\ln \rho = \alpha H + \vec{\beta} \vec{P} + \vec{\gamma} \vec{M} \tag{6}$$

for some real number α and real vectors $\vec{\beta}$ and $\vec{\gamma}$. Therefore

$$\rho = e^{\alpha H + \vec{\beta} \vec{P} + \vec{\gamma} \vec{M}}. \quad (7)$$

But we assumed that ρ is translation-invariant. Therefore $\vec{\gamma} = 0$ and

$$\rho = e^{\alpha H + \vec{\beta} \vec{P}} \quad (8)$$

The distribution (8) is a standard Gibbs distribution. But there are two problems with this derivation. The first one, this derivation is performed only at classical level. The second one, we could not find anywhere the proof of the fact that Hamiltonian H momenta \vec{P} and angular momenta \vec{M} are the complete linear independent set of additive first integrals.

Let us present now our derivation of Gibbs distribution (from the cluster property). This derivation is not rigorous too but we hope it is of some interest. Note that this derivation uses some basic ideas of Haag-Ruelle scattering theory [7].

But at first let us give some previous definitions.

Definition 1. Let $S(\mathbb{R}^3)$ be a Schwartz space of test functions (infinitely-differentiable functions decaying at infinity faster than any inverse polynomial with all its derivatives). The algebra of canonical commutative relations \mathcal{C} is an unital algebra generated by symbols $a^+(f)$ and $a(f)$ $f \in S(\mathbb{R}^3)$ satisfying the following canonical commutative relations:

- a) $a^+(f)$ is a linear functional of f ,
- b) $a(f)$ is an antilinear functional of f ,

$$\begin{aligned} [a(f), a(g)] &= [a^+(f), a^+(g)] = 0, \\ [a(f), a^+(g)] &= \langle f, g \rangle, \end{aligned}$$

where $\langle f, g \rangle$ is a standard scalar product in $L^2(\mathbb{R}^3)$,

$$\langle f, g \rangle := \int f^*(x)g(x)d^3x$$

Remark. We will widely use generalized "elements" of the algebra of canonical commutative relations $a(k)$, $a^+(k)$ defined according to the following relations

$$\begin{aligned} a(f) &= \int a(k)f(k)d^3k, \\ a^+(f) &= \int a^+(k)f^+(k)d^3k. \end{aligned} \quad (9)$$

$a(k)$, $a^+(k')$ are called the annihilation-creation operators and satisfy to the following canonical commutative relations:

$$\begin{aligned} [a^+(k), a^+(k')] &= [a(k), a(k')] = 0, \\ [a(k), a(k')] &= \delta(k - k'). \end{aligned} \tag{10}$$

Definition 2. The field operators $\Psi(x)$, $\Psi^+(x)$ are defined as follows

$$\begin{aligned} \Psi(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int a(k) e^{ikx} d^3k, \\ \Psi^+(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int a^+(k) e^{ikx} d^3k \end{aligned} \tag{11}$$

The rigorous definition of $\Psi(x)$ and $\Psi^+(x)$ could be obtained from this definition by using the notion of the Fourier transform of distributions.

Definition 3. Let ρ be a state on the algebra of canonical commutative relations (the algebra generated by smoothed secondary quantized wave functions) (CCR-algebra). We say that ρ is a Gauss state if we can calculate its values at elements of CCR-algebra by using the Wick (Bloch — De Dominicis) theorem through pair correlations.

Now let us recall some notions connected with nonequilibrium (Keldysh) diagram technique. Let ρ be an arbitrary state on the algebra of canonical commutative relations. Let us introduce the Green functions for the system (corresponding to this state).

$$\rho(T(\Psi_H^\pm(t_1, x_1), \dots, \Psi_H^\pm(t_n, x_n))).$$

Here t_1, \dots, t_n are times, symbol H near Ψ_H^\pm means here that Ψ_H^\pm are operators in Heizenberg representation and the symbol T is a symbol of chronological ordering.

We will consider the system describing by the following Hamiltonian

$$H = H_0 + \lambda V, \tag{12}$$

where H_0 is a free Hamiltonian

$$H_0 = \int \omega(k) a^+(k) a(k) d^3k, \tag{13}$$

$$\omega(k) = \frac{k^2}{2}. \quad (14)$$

$\lambda \in \mathbb{R}$ is a coupling constant and

$$V = \frac{1}{2} \int d^3x d^3x' \Psi^+(x) \Psi^+(x') \mathcal{V}(x-x') \Psi(x') \Psi(x), \quad (15)$$

where $\mathcal{V}(x)$ is an arbitrary test function from $S(\mathbb{R}^3)$. In nonequilibrium diagram technique we require the following representation for the Green functions:

$$\begin{aligned} \rho(T(\Psi_H^\pm(t_1, x_1), \dots, \Psi_H^\pm(t_n, x_n))) = \\ \rho_0(S^{-1}T(\Psi_0^\pm(t_1, x_1), \dots, \Psi_0^\pm(t_n, x_n)S)). \end{aligned} \quad (16)$$

The symbol 0 near Ψ^\pm means here that Ψ_0^\pm are operators in the Dirac representation (representation of interaction). The S -matrix has the form

$$S = T \exp\left(-i \int_{-\infty}^{+\infty} V_0(t) dt\right),$$

and

$$S^{-1} = \tilde{T} \exp\left(i \int_{-\infty}^{+\infty} V_0(t) dt\right).$$

Here T and \tilde{T} are symbols of the chronological and the antichronological ordering respectively. ρ_0 is some Gauss state defined by density function $n(k)$ as follows

$$\begin{aligned} \rho_0(a^+(k')a^+(k)) = \rho_0(a(k)a(k')) = 0, \\ \rho_0(a^+(k)a(k')) = n(k)\delta(k-k'). \end{aligned} \quad (17)$$

The state ρ_0 is called the asymptotical state.

Let us recall the basic elements of nonequilibrium diagram technique. The vertices coming from T -exponent are marked by symbol $-$. The vertices coming from \tilde{T} -exponent are marked by symbol $+$. There exist four types of

propagators

$$\begin{aligned}
G_0^{+-}(t_1 - t_2, x_1 - x_2) &:= \rho_0(\Psi(t_1, x_1)\Psi^+(t_2, x_2)), \\
G_0^{-+}(t_1 - t_2, x_1 - x_2) &:= \rho_0(\Psi^+(t_2, x_2)\Psi(t_1, x_1)), \\
G_0^{--}(t_1 - t_2, x_1 - x_2) &:= \rho_0(T(\Psi(t_1, x_1)\Psi^+(t_2, x_2))), \\
G_0^{++}(t_1 - t_2, x_1 - x_2) &:= \rho_0(\tilde{T}(\Psi(t_1, x_1)\Psi^+(t_2, x_2))).
\end{aligned}$$

Let us write the table of propagators

$$\begin{aligned}
G_0^{+-}(t, x) &= \int \frac{d^4k}{(2\pi)^4} (2\pi)\delta(\omega - \omega(k))(1 + n(k))e^{-i(\omega t - kx)}, \\
G_0^{-+}(t, x) &= \int \frac{d^4k}{(2\pi)^4} (2\pi)\delta(\omega - \omega(k))n(k)e^{-i(\omega t - kx)}, \\
G_0^{--}(t, x) &= i \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{1 + n(k)}{\omega - \omega(k) + i0} - \frac{n(k)}{\omega - \omega(k) - i0} \right\} e^{-i(\omega t - kx)}, \\
G_0^{++}(t, x) &= i \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{n(k)}{\omega - \omega(k) + i0} - \frac{1 + n(k)}{\omega - \omega(k) - i0} \right\} e^{-i(\omega t - kx)}.
\end{aligned}$$

It has been shown in [1] that (usually) there exists divergences in Keldysh diagram technique. A typical example of divergent diagrams is pictured at fig.1

fig. 1



The ovals represent the sum of one-particle irreducible diagrams. These diagrams are called chain diagrams. Let us suppose that all divergences of self-energy parts (ovals) are subtracted. The divergences arise from the fact that singular supports of propagators coincide.

For all $t_1, t_2 \in \mathbb{R}$ let us define the evolution operator $S(t_1, t_2)$ as follows:

$$S(t_1, t_2) = e^{it_1 H_0} e^{-i(t_1 - t_2)H} e^{-it_2 H_0}. \quad (18)$$

Note that, for example, that

$$S(0, -\infty) = T \exp\left(-i \int_{-\infty}^0 V_0(t) dt\right),$$

$$S(+\infty, 0) = T \exp(-i \int_0^{+\infty} V_0(t) dt) \quad (19)$$

One can prove that the following lemma holds.

Lemma 1. $\forall t \in \mathbb{R}$ the following equalities holds:

$$e^{-itH} S(0, -\infty) = S(0, -\infty) e^{-itH_0} \quad (20)$$

and

$$S(+\infty, 0) e^{-itH} = e^{-itH_0} S(+\infty, 0). \quad (21)$$

Now let us consider the equilibrium Green functions defined as follows

$$\frac{1}{Z} \text{tr}(T(\Psi_H^\pm(t_1, x_1) \dots \Psi_H^\pm(t_n, x_n)) e^{-\frac{H-\mu N}{T}}), \quad (22)$$

where

$$Z = \text{tr}(e^{-\frac{H-\mu N}{T}}), \quad (23)$$

$T > 0$ is a temperature, $\mu \in \mathbb{R}$ is a chemical potential and N is a number particle operator

$$N = \int a^+(k) a(k) d^3k \quad (24)$$

In other words we put in (16)

$$\rho(\cdot) = \frac{1}{Z} \text{tr}((\cdot) e^{-\frac{H-\mu T}{T}}). \quad (25)$$

It follows from the lemma 1 that the following lemma holds.

Lemma 2.

$$\begin{aligned} & \frac{1}{Z} \text{tr}(T(\Psi_H^\pm(t_1, x_1) \dots \Psi_H^\pm(t_n, x_n)) e^{-\frac{H-\mu N}{T}}) \\ &= \rho_0(S^{-1} T(\Psi_0^\pm(t_1, x_1), \dots, \Psi_0^\pm(t_n, x_n) S)), \end{aligned} \quad (26)$$

where ρ_0 is a Gauss state on the CCR-algebra defined as usual by its density function $n(k)$ of the form:

$$n(k) = \frac{1}{e^{\frac{\omega(k)-\mu}{T}} - 1} \quad (27)$$

It follows from physical reasonings that the equilibrium Green functions does not contain divergences. So we have the following lemma:

Lemma 3. If the asymptotical state ρ_0 in Keldysh diagram technique corresponds to the density function $n(k)$ of Bose-Einstein form

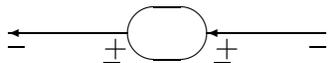
$$n(k) = \frac{1}{e^{\frac{\omega(k)-\mu}{T}} - 1} \quad (28)$$

then the Keldysh diagram technique does not contain divergences and corresponding green function are equilibrium

$$\begin{aligned} & \rho(T(\Psi_H^\pm(t_1, x_1), \dots, \Psi_H^\pm(t_n, x_n))) \\ &= \frac{1}{Z} \text{tr}(T(\Psi_H^\pm(t_1, x_1) \dots \Psi_H^\pm(t_n, x_n)) e^{-\frac{H-\mu N}{T}}), \end{aligned} \quad (29)$$

Now let us prove the fact that if the Keldysh diagram technique does not contain divergences then the corresponding Green functions are equilibrium, i.e. the asymptotical state ρ_0 has Bose-Einstein form. In order to do this let us analyze the divergences of Keldysh diagram technique in lowest possible order in λ . Such divergences may come only from the diagrams pictured at fig. 2.

fig. 2



where ovals means here the sum of all self-energy diagrams of lowest possible order in λ . It has been shown in [1] that these diagrams contain divergences for density function $n(k)$ of general form. It has been also shown in [1] that these diagrams could be subtracted by the following renormalization of the asymptotical state

$$\rho_0(\cdot) \rightarrow \frac{1}{Z} \rho_0(e^{-\int_{-\infty}^{+\infty} h_0(t) dt}(\cdot)),$$

where

$$h = \int h(k) a^+(k) a(k) d^3 k,$$

$h(k)$ is a real-valued function,

$$h_0(t) := e^{itH_0} h e^{-itH_0} \quad (30)$$

and

$$Z = \rho_0 \left(e^{-\int_{-\infty}^{+\infty} h(t) dt} \right)$$

for suitable $h = \int h(k) a^+(k) a(k) d^3k$.

It also could be extracted from [1] that the divergences in considered diagrams exists if and only if $h \neq 0$. But it has been also shown in [1] that

$$h(p) = \frac{1 + 2n(p)}{2n(p)(1 + n(p))} St(p), \quad (31)$$

where $St(p)$ is a collision integral (in lowest possible order of perturbation theory) and $St(p) \equiv 0$ if and only if $n(p)$ has a Bose-Einstein form (see for example [8]). So we have proved the following

Lemma 4. If Keldysh diagram technique does not contain divergences the the asimptotical state has a Bose-Einstein form

$$\begin{aligned} \rho_0(a^+(k') a^+(k)) &= \rho_0(a(k) a(k')) = 0, \\ \rho_0(a^+(k) a(k')) &= n(k) \delta(k - k'), \\ n(k) &= \frac{1}{e^{\frac{\omega(k) - \mu}{T}} - 1} \end{aligned} \quad (32)$$

and the corresponding Green function are equilibrium.

Now let $\langle \cdot \rangle$ be some stationary translation invariant state on the CCR-algebra, commuting with the number particle operator. Suppose that $\langle \cdot \rangle$ satisfy to the asymptotic property of (enough fast) space factorization of correlations. Put by definition

$$W = S(0, -\infty). \quad (33)$$

Let $\langle \cdot \rangle_0$ be a state on the CCR-algebra such that

$$\langle \cdot \rangle = \langle W^{-1}(\cdot) W \rangle_0. \quad (34)$$

One has

$$\langle \cdot \rangle_0 = \langle W(\cdot) W^{-1} \rangle. \quad (35)$$

Let us formulate now our main lemma.

Lemma 5. If the state $\langle \cdot \rangle$ satisfies to the property of asymptotic (enough fast) factorization of correlations then $\langle \cdot \rangle_0$ is finite (does not contain divergences) and Gauss.

Proof. We have

$$W = \lim_{t \rightarrow -\infty} e^{+itH} e^{-itH_0} \quad (36)$$

and

$$W^{-1} = \lim_{t \rightarrow -\infty} e^{+itH_0} e^{-itH}. \quad (37)$$

These two equations could be simply established in the sense of formal series of perturbation theory and this enough for our aims.

Therefore we have

$$\begin{aligned} \langle \cdot \rangle_0 &= \langle W(\cdot)W^{-1} \rangle \\ &= \lim_{t_1 \rightarrow -\infty, t_2 \rightarrow -\infty} \langle e^{+it_1H} e^{-it_1H_0}(\cdot) e^{+it_2H_0} e^{-it_2H} \rangle \\ &= \lim_{t \rightarrow -\infty} \langle e^{+itH} e^{-itH_0}(\cdot) e^{+itH_0} e^{-itH} \rangle \\ &= \lim_{t \rightarrow -\infty} \langle e^{-itH_0}(\cdot) e^{+itH_0} \rangle. \end{aligned} \quad (38)$$

Last equality takes place because the state $\langle \cdot \rangle$ is a stationary state. In result

$$\langle \cdot \rangle_0 = \lim_{t \rightarrow -\infty} \langle e^{-itH_0}(\cdot) e^{+itH_0} \rangle. \quad (39)$$

Let us now introduce so-called Wightman functions (distributions) $W_{n,t}(x_1, \sigma_1, \dots, x_n, \sigma_n)$, $t \in \mathbb{R}$, $x_1, \dots, x_n \in \mathbb{R}^3$, $\sigma_1, \dots, \sigma_n \in \{+, -\}$ as follows:

$$W_{n,t}(x_1, \sigma_1, \dots, x_n, \sigma_n) := \langle e^{itH_0}(\Psi^{\sigma_1}(x_1) \dots \Psi^{\sigma_n}(x_n)) e^{-itH_0} \rangle. \quad (40)$$

Let us also introduce so-called truncated Wightman functions according W_t^T to following relations:

$$W_{n,t}(x_1, \sigma_1, \dots, x_n, \sigma_n) = \sum_{\pi \in \mathcal{P}} \prod_{P \in \pi} W_{|P|,t}^T(x_{i_1^P}, \sigma_{i_1^P}, \dots, x_{i_{|P|}^P}, \sigma_{i_{|P|}^P}). \quad (41)$$

Here $|A|$ is a number of elements of finite set A , \mathcal{P} is a set of all decompositions $\pi = \{P_1, \dots, P_{|\pi|}\}$ of the set $\{1, \dots, n\}$ into disjoint union of sets $P_1, \dots, P_{|\pi|}$.

For each $k = 1, \dots, |\pi|$ we put by definition $P_k = \{i_1^{P_k}, \dots, i_{|P_k|}^{P_k}\}$, $i_1^{P_k} < i_2^{P_k} < \dots < i_{|P_k|}^{P_k}$.

Note that the property of enough fast asymptotic space factorization of correlations by definition means that truncated Wightman functions $W_{n,t}^T(x_1, \sigma_1, \dots, x_n, \sigma_n)$ tends to zero enough fast if their relative arguments $\xi_1 = x_2 - x_1, \dots, \xi_{n-1} = x_n - x_1$ tends to infinity.

Now let us introduce the Fourier transformations of Wightman functions as follows:

$$\begin{aligned} & \tilde{W}_{n,t}(p_1, \sigma_1, \dots, p_n, \sigma_n) : \\ & = \int d^3x_1, \dots, d^3x_n e^{i(\sigma_1 p_1 x_1 + \dots + \sigma_n p_n x_n)} W_{n,t}(x_1, \sigma_1, \dots, x_n, \sigma_n) \end{aligned} \quad (42)$$

and

$$\begin{aligned} & \tilde{W}_{n,t}^T(p_1, \sigma_1, \dots, p_n, \sigma_n) : \\ & = \int d^3x_1, \dots, d^3x_n e^{i(\sigma_1 p_1 x_1 + \dots + \sigma_n p_n x_n)} W_{n,t}^T(x_1, \sigma_1, \dots, x_n, \sigma_n). \end{aligned} \quad (43)$$

one can prove that

$$\tilde{W}_{n,t}(p_1, \sigma_1, \dots, p_n, \sigma_n) = \sum_{\pi \in \mathcal{P}} \prod_{P \in \pi} \tilde{W}_{|P|,t}^T(p_{i_1^P}, \sigma_{i_1^P}, \dots, p_{i_{|P|}^P}, \sigma_{i_{|P|}^P}), \quad (44)$$

where notations are such that as in (41).

The truncated Wightman functions $W_{n,t}^T(x_1, \sigma_1, \dots, x_n, \sigma_n)$ are translation-invariant and the function of (enough) fast decay on relative arguments. Therefore their Fourier transforms $\tilde{W}_{n,t}^T$ could be represented as follows:

$$\tilde{W}_{n,t}(p_1, \sigma_1, \dots, p_n, \sigma_n) = \delta(p_1 \sigma_1 + \dots + p_n \sigma_n) \tilde{\mathcal{W}}_{n,t}^T(p_1, \sigma_1, \dots, p_n, \sigma_n), \quad (45)$$

where $\tilde{\mathcal{W}}_{n,t}^T$ is an enough smooth function of at most polynomial increment at infinity (with enough large number of derivatives).

It is obvious that $\forall n = 1, 2, \dots$ and $\forall t \in \mathbb{R}$

$$\tilde{W}_{n,t}^T(p_1, \sigma_1, \dots, p_n, \sigma_n) = e^{i(\sigma_1 \omega(p_1) + \dots + \sigma_n \omega(p_n))t} \tilde{W}_{n,0}^T(p_1, \sigma_1, \dots, p_n, \sigma_n). \quad (46)$$

Now, let us prove that for each $n = 3, 4, \dots$ $\tilde{W}_{n,t}(p_1, \sigma_1, \dots, p_n, \sigma_n)$ tends to zero as $t \rightarrow -\infty$ in the sense of distributions. This means that for each test function $f(p_1, \dots, p_n) \in S(\mathbb{R}^{3n})$

$$\int d^3p_1 \dots d^3p_n \tilde{W}_{n,t}^T(p_1, \sigma_1, \dots, p_n, \sigma_n) f(p_1, \dots, p_n) \rightarrow 0 \quad (47)$$

as $t \rightarrow -\infty$. It is obvious from remarks above that to prove this fact it is enough to prove that for each enough smooth function $g(x_1, \dots, x_n)$ of enough fast decaying at infinity

$$\int d^3 p_1 \dots d^3 p_n e^{i(\sigma_1 \omega(p_1) + \dots + \sigma_n \omega(p_n))t} \delta(p_1 \sigma_1 + \dots + p_n \sigma_n) g(p_1, \dots, p_n) \rightarrow 0 \quad (48)$$

as $t \rightarrow -\infty$.

Put by definition $L = \{(p_1, \dots, p_n) | \sigma_1 p_1 + \dots + \sigma_n p_n = 0\}$. Note that for each $n = 3, 4, \dots$ $\sigma_1 \omega_1(p) + \dots + \sigma_n \omega_n(p)$ is not identically equal to zero on L . This fact could be proved very simply but we omit this proof. So the limit equality (48) could be directly obtained by using the stationary phase method.

So, for each $n = 3, 4, \dots$ $\tilde{W}_{n,t}(p_1, \sigma_1, \dots, p_n, \sigma_n) \rightarrow 0$ as $t \rightarrow -\infty$ in the sense of distributions. From other hand, it is obvious that $W_{2,t}^T = W_{2,0}^T$. In result $\forall t \in \mathbb{R}$

$$W_{n,t}(x_1, \sigma_1, \dots, x_n, \sigma_n) = 0 \quad (49)$$

if n is odd and

$$\lim_{t \rightarrow -\infty} W_{n,t}(x_1, \sigma_1, \dots, x_n, \sigma_n) = \sum_{\tau \in \mathcal{T}} \prod_{T \in \tau} W_{2,0}(x_{i_T}, \sigma_{i_T}, x_{j_T}, \sigma_{j_T}), \quad (50)$$

if n is even (in the sense of distributions). Here \mathcal{T} is a set of all decomposition $\tau = \{T_1, \dots, T_{\frac{n}{2}}\}$ of the set $\{1, 2, \dots, n\}$ into disjoint union of pairs $\tau_k = \{i_{\tau_k}, j_{\tau_k}\}$, $i_{\tau_k} < j_{\tau_k}$, $k = 1, 2, \dots, \frac{n}{2}$.

But let us recall that

$$\lim_{t \rightarrow -\infty} W_{n,t}(x_1, \sigma_1, \dots, x_n, \sigma_n) = \langle \Psi^{\sigma_1}(x_1) \dots P S i^{\sigma_n}(x_n) \rangle_0. \quad (51)$$

in the sense of distributions. Therefore $\langle \cdot \rangle_0$ is finite and Gauss. Now let us formulate main theorem of present section

Theorem 2. let $\langle \cdot \rangle$ be some stationary translation invariant state on the CCR-algebra, commuting with the number particle operator. Suppose that $\langle \cdot \rangle$ satisfy to the asymptotic property of (enough fast) space factorization of correlations. Then

$$\rho(\cdot) = \frac{1}{Z} \text{tr}((\cdot) e^{-\frac{H - \mu N}{T}}) \quad (52)$$

for some $T > 0$ and $\mu \in \mathbb{R}$. Z is a statistical sum defined by equation (23). In other words the state $\langle \cdot \rangle$ is a Gibbsian state.

Proof. It follows from lemma 5 that

$$\begin{aligned} & \langle \cdot \rangle \\ &= \langle S^{-1}(0, -\infty)(\cdot)S(0, -\infty) \rangle_0. \end{aligned} \quad (53)$$

This implies that

$$\begin{aligned} & \langle T\Psi_H^\pm(t_1, x_1)\dots\Psi_H^\pm(t_n, x_n) \rangle \\ &= \langle S^{-1}T\Psi_0^\pm(t_1, x_1), \dots, \Psi_0^\pm(t_n, x_n)S \rangle_0. \end{aligned} \quad (54)$$

But one can prove (in the sense of formal series of perturbation theory) that the fact that $\langle \cdot \rangle$ finite implies that $\langle T\Psi_H^\pm(t_1, x_1)\dots\Psi_H^\pm(t_n, x_n) \rangle$ is finite. Therefore the Keldysh diagram technique for $\langle T\Psi_H^\pm(t_1, x_1)\dots\Psi_H^\pm(t_n, x_n) \rangle$ does not contain divergences. Therefore lemma 4 implies that

$$\begin{aligned} \langle a^+(k)a^+(k') \rangle_0 &= \langle a(k)a(k') \rangle_0 = 0, \\ \langle a(k)a^+(k') \rangle_0 &= \delta(k - k') \frac{1}{e^{\frac{\omega(k)-\mu}{T}} - 1} \end{aligned} \quad (55)$$

for some $T > 0$ and $\mu \in \mathbb{R}$. And it follows from lemma 2 that

$$\langle \cdot \rangle = \frac{1}{Z} \text{tr}((\cdot)e^{-\frac{H-\mu T}{T}}). \quad (56)$$

Therefore theorem is proved.

Remark. Note that we have supposed the following when we have proved the theorem 2 above. If the state $\langle \cdot \rangle$ is a finite state (does not contain the divergences) then the divergences does not exist in each order in coupling constant λ of the power decomposition of the state $\langle \cdot \rangle$ according to perturbation theory. We have not been prove this statement but it is essential to suppose that this statement holds for all states of the physical interest.

3 The Bardeen — Cooper — Schrieffer model on the language of generalized thermodynamics.

In this section we show how to formulate the Bardeen — Cooper — Schrieffer model of suoverconductivity on the language of the generalized thermodynamics. But at first let us recall several results on some model Hamiltonian in superconductive theory which belong to Bogoliubov — Zubarev — Tserkovikov [5,9].

Let us consider a model dynamical system described by the following Hamiltonian:

$$H = \sum_f T(f) a_f^+ a_f - \frac{1}{2V} \sum_{f', f} \lambda(f) \lambda(f') a_f^+ a_{-f}^+ a_{-f'} a_{f'}. \quad (57)$$

Here we use the following notations:

$$f = (p, s), \quad -f = (-p, -s). \quad (58)$$

p is a particle's momentum, $s = \pm 1$,

$$T(f) = \frac{p^2}{2m} - \mu, \quad \mu > 0. \quad (59)$$

$$\begin{aligned} \lambda(f) &= J\varepsilon(f) \text{ if } \left| \frac{p^2}{2} - \mu \right| < \Delta, \\ \lambda(f) &= 0 \text{ if } \left| \frac{p^2}{2} - \mu \right| > \Delta, \\ \varepsilon((p, s)) &= s, \end{aligned} \quad (60)$$

a_f, a_f^+ are usual fermion annihilation-creation operators.

We will also consider the Hamiltonian H with some additional terms which are the sources of creation and annihilation of pairs.

$$H_\nu = H - \frac{\nu}{2} \sum_f \lambda(f) \{a_{-f} a_f + a_f^+ a_{-f}^+\}, \quad \nu > 0. \quad (61)$$

Let C be an arbitrary complex number. Let us represent H as follows:

$$H = H_0(C) + H_1(C), \quad (62)$$

where

$$H_0(C) = \sum_f T(f) a_f^+ a_f - \frac{1}{2} \sum_f \lambda(f) \{C^* a_{-f} a_f + C a_f^+ a_{-f}^+\} + \frac{|C|^2}{2} V \quad (63)$$

and

$$H_1(C) = \frac{V}{2} \left(\frac{1}{V} \sum_f \lambda(f) a_f^+ a_{-f}^+ - C^* \right) \left(\frac{1}{V} \sum_f \lambda(f) a_{-f} a_f - C \right) \quad (64)$$

Note that $H_0(C)$ is a quadratic form on fermion annihilation-creation operators. So it may be diagonalized by means of canonical fermion Bogoliubov transformation. More precisely:

$$H_0(C) = \sum_f \sqrt{\lambda^+(f)|C|^2 + T^2(f)}(a_f^+ u_f + a_{-f} v_f^*)(a_f u_f + a_{-f}^+ v_f) + \frac{V}{2}(|C|^2 - \frac{1}{V} \sum_f [\sqrt{\lambda^2(f)|C|^2 + T^2(f)} - T(f)]), \quad (65)$$

where

$$u_f = \frac{1}{\sqrt{2}} \left(1 + \frac{T(f)}{\sqrt{\lambda^2(f)|C|^2 + T^2(f)}}\right)^{1/2}, \quad (66)$$

$$v_f = -\frac{\varepsilon(f)}{\sqrt{2}} \left(1 - \frac{T(f)}{\sqrt{\lambda^2(f)|C|^2 + T^2(f)}}\right)^{1/2} \frac{C}{|C|} \quad (67)$$

Note that $u(-f) = u(f)$ and $v(-f) = -v(f)$. Function u is a real-valued function and function v is a complex-valued function, and $u^2 + |v|^2 = 1$. These implies that the operators

$$\begin{aligned} \alpha_f &:= a_f u_f + a_{-f}^+ v_f, \\ \alpha_f^+ &:= a_f^+ u_f + a_{-f} v_f^* \end{aligned} \quad (68)$$

are the operators of the fermion type.

Put by definition:

$$J(C) := \frac{1}{2V} \sum_f \lambda(f) a_f^+ a_{-f} - C^*. \quad (69)$$

Now the expression for $H_1(C)$ could be rewritten as follows:

$$H_1(C) = -\frac{V}{2} J^*(C) J(C). \quad (70)$$

Let Γ be an arbitrary (low bounded) Hamiltonian. Denote by $\langle \cdot \rangle_{\Gamma, T}$ the averaging with respect to the Gibbs distribution corresponding to the Hamiltonian Γ and the temperature T :

$$\langle \cdot \rangle_{\Gamma, T} := \frac{\text{tr}((\cdot) e^{-\frac{\Gamma}{T}})}{\text{tr}(e^{-\frac{\Gamma}{T}})}. \quad (71)$$

Let us consider the following equation:

$$\langle J(C) \rangle_{H_0(C), T} = 0 \quad (72)$$

on the variable C . This equation is called the equation of compensation. Note that $H_0(C)$ is a quadratic form on creation-annihilation operators. Therefore the equation of compensation could be rewritten explicitly as follows [5]:

$$1 = \frac{1}{2V} \sum_f \frac{\lambda^2(f)}{\sqrt{\lambda^2(f)|C|^2 + T^2(f)}} \text{th} \left\{ \frac{\sqrt{\lambda^2(f)|C|^2 + T^2(f)}}{2T} \right\}. \quad (73)$$

In the limit $V \rightarrow \infty$ this equation will take the form:

$$1 = \frac{1}{(2\pi)^2} \int \frac{\lambda(p)d^3p}{\sqrt{\lambda^2(p)|C|^2 + T^2(p)}} \text{th} \left\{ \frac{\sqrt{\lambda^2(p)|C|^2 + T^2(p)}}{2T} \right\}. \quad (74)$$

It is well-known fact that last equation has a non-zero solution for all temperature $T < T_{cr}$ for some critical temperatures $T < T_{cr}$ for some temperature T_{cr} . We will consider only the case when $T < T_{cr}$. Let C_0 be an arbitrary solution of the equation (72) or equation (73). Note that if C_0 is a solution of equation (73) then $C_0 e^{i\phi}$ is a solution of equation (73) too for an arbitrary real number ϕ . Let C'_0 is a positive solution of the equation (1) or equation (73). Now let us formulate main results of Bogoliubov, Zubarev, Tserkovnikov [9] concerned with the system under consideration.

Let Γ be a Hamiltonian of some system contained in some volume V . Denote by $F_\Gamma(V, T)$ the free (Gibbsian) energy of this system corresponding to the temperature T . Then the following equalities hold

$$\lim_{V \rightarrow \infty} \frac{F_H(V, T)}{V} = \lim_{\nu \rightarrow \infty, \nu > 0} \lim_{V \rightarrow \infty} \frac{F_{H_\nu}(V, T)}{V} = \lim_{V \rightarrow \infty} \frac{F_{H(C_0)}(V, T)}{V}. \quad (75)$$

Now, let us consider quasi-averages

$$\begin{aligned} & \prec \dots a_{f_j}^+ \dots a_{f_s} \dots \succ_{H, T} := \\ & = \lim_{\nu \rightarrow 0, \nu > 0} \lim_{V \rightarrow \infty} \langle \dots a_{f_j}^+ \dots a_{f_s} \dots \rangle_{H_\nu, T} \end{aligned} \quad (76)$$

Here $\langle \dots a_{f_j}^+ \dots a_{f_s} \dots \rangle_{H_\nu, T}$ are fermion creation-annihilation operators in Heisenberg representation. The following equation holds [9,5]:

$$\begin{aligned} & \prec \dots a_{f_j}^+ \dots a_{f_s} \dots \succ_{H, T} \\ & = \lim_{V \rightarrow \infty} \langle \dots a_{f_j}^+ \dots a_{f_s} \dots \rangle_{H(C'_0), T}. \end{aligned} \quad (77)$$

Now let us consider the following Hamiltonian H_ν^φ instead of H_ν :

$$H_\nu^\varphi = -\frac{\nu}{2} \sum_f \lambda(f) \{a_{-f} a_f e^{i\varphi} + a_f^+ a_{-f}^+ e^{-i\varphi}\}, \quad (78)$$

where φ is an arbitrary. We will have again the equality (75), or more precisely:

$$\lim_{V \rightarrow \infty} \frac{F_H(V, T)}{V} = \lim_{\nu \rightarrow \infty, \nu > 0} \lim_{V \rightarrow \infty} \frac{F_{H_\nu^\varphi}(V, T)}{V} = \lim_{V \rightarrow \infty} \frac{F_{H(C_0)}(V, T)}{V}. \quad (79)$$

where we put now

$$\begin{aligned} & \prec \dots a_{f_j}^+ \dots a_{f_s} \dots \succ_{H, T} \\ &= \lim_{\nu \rightarrow 0, \nu > 0} \lim_{V \rightarrow \infty} \langle \dots a_{f_j}^+ \dots a_{f_s} \dots \rangle_{H_\nu^\varphi, T} \end{aligned} \quad (80)$$

and

$$C_0^\varphi := C_0 e^{i\varphi}. \quad (81)$$

It is necessary to note that the equation of compensation (73) is equal to a condition of the minimum of free energy $F_{H(C_0)}(V, T)$ on C at fixed energy T (see [5]). Recall that $F_{H_0(C)}$ is a free energy of the dynamical system which is described by the Hamiltonian $H_0(C)$ and contained in volume V at the temperature T . But the condition that C_0 corresponds to the minimum of free energy $F_{H_0(C_0)}$ is equivalent to the following condition: the entropy of our system $S_{H_0(C)}$ achieve the maximum at the point $C = C_0$ (as a function of C) at fixed energy E . Last remark admit us to suppose that thermodynamics of our model could be described in terms of our generalized thermodynamics and the parameter C here plays the role of the observable values of commuting integrals of the system.

Let us recall that one of the basics principles of Generalized thermodynamics states the following. The observable values of of the commuting first integrals (from microcanonical distribution) which may be realized in nature must corresponds to the maximum of the entropy (under fixed energy).

Let us start to reformulate thermodynamics of our model in terms of the generalized thermodynamics. It follows from the basics principles of statistical mechanics that one can use the microcanonical distribution:

$$\rho_E(C) := \text{const} \delta(H_0(C) - E) \quad (82)$$

instead of the canonical distribution

$$\rho_T = \frac{e^{-\frac{H_0(C)}{T}}}{\text{tr}(e^{-\frac{H_0(C)}{T}})}. \quad (83)$$

the entropy corresponding to the distribution has the form:

$$S(E, C) = \text{tr} \delta(H_0(C) - E). \quad (84)$$

Note that in equation (82) and (84) by symbol $\delta(x)$ we mean some regularization of δ -function $\Delta(x)$ defined as follow:

$$\begin{aligned} \Delta(x) &= 0, \text{ if } |x| > \frac{\Delta}{2} \\ \Delta(x) &= \frac{1}{\Delta}, \text{ if } |x| < \frac{\Delta}{2} \end{aligned} \quad (85)$$

where Δ is some small positive real number (asymptotical constant as $V \rightarrow \infty$).

Let us consider the following symplectic manifold \mathfrak{S}_2 with the canonically-conjugated variables S_R, S_I, Φ_R, Φ_I on it defined as follows.

Variables S_R, S_I run the whole real line, but variables Φ_R, Φ_I are defined modulo the substitutions:

$$\begin{aligned} \Phi_R &\mapsto \Phi_R + 2\pi n, \\ \Phi_I &\mapsto \Phi_I + 2\pi n, \end{aligned} \quad (86)$$

where $n \in \mathbb{Z}$. The Poisson brackets of variables S_R, S_I, Φ_R, Φ_I by definition has the form:

$$\begin{aligned} (S_R, S_I) &= (\Phi_R, \Phi_I) = 0, \\ (S_R, \Phi_I) &= (S_I, \Phi_R) = 0, \\ (S_R, \Phi_R) &= (S_I, \Phi_I) = 1. \end{aligned} \quad (87)$$

Let \mathfrak{S}_1 be a (non-commutative) phase space of the dynamical system, described by the Hamiltonian (57). Let \mathfrak{S} be a direct product of the (non-commutative)spaces \mathfrak{S}_1 and \mathfrak{S}_2 , $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2$. Now let us consider the dynamical system on the space \mathfrak{S} which described by Hamiltonian $H_0(S_R + S_I)$.

It is evident that the dynamical variables S_R and S_I are the commuting first integrals with respect to the Hamiltonian $H_0(S_R + iS_I)$. The generalized micro-canonical distribution corresponding to the integral $H_0(S_R + iS_I)$ and the integrals S_R and S_I could be written as follows:

$$S^\ominus(E, C) := \text{tr}_{\mathfrak{E}^1} \int d\Phi_R \dots dS_I \delta(H_0(S_R + iS_I) - E) \\ \times \delta(S_R - \mathfrak{R}C) \delta(S_I - \mathfrak{I}C) = S(E, C) + c. \quad (88)$$

Here $S(E, C)$ is defined by equation (84), c is some real constant which does not depend on E and C and symbol $\text{tr}_{\mathfrak{E}^1}$ means trace over the Hilbert space of our model dynamical system described by the Hamiltonian (57). Recall that one of the basics principles of our generalized thermodynamics state that the observable value of C (which may be realized in nature) must corresponds to the maximum of $S^\ominus(E, C)$ (at fixed energy E). We have proved that this condition is equivalent to the equation of compensation (72). Therefore, we have formulated the thermodynamics of our model in terms of generalized thermodynamics. Note also that all nonzero solutions of equation of compensation form some one-dimensional manifold. So the entropy $S^\ominus(E, C)$ (at fixed energy) has one-dimensional plateau.

4 On some new method of derivation of Boltzmann kinetic equation.

In the present section we consider Bose gas with weak pair enough good interaction. For this system we derive usual kinetic Boltzmann equation in the limit of weak interaction by some new method. We consider a single mode of Bose gas (corresponding to some momentum k as an oscillator interacting with other modes as an oscillator interacting with thermostat. To study the evolution of this fixed mode interacting with other modes (considered as thermostat) we will use the so called statistical perturbation theory [10].

One of the interesting consequence from non-ergoding theorem states the following. To obtain an irreversible macroscopic evolution from the reversible microscopic evolution one need to take into account the behavior of the system at its boundary. In other words it is impossible to prove that the thermalization takes place in the system if we neglect by the role which plays the environment of the system in this process. The derivation of kinetic equa-

tions presented in this section gives us a new point of view at the role which plays the environment of the system in the process of its thermalization.

So let us consider a model dynamical system, described by the following Hamiltonian,

$$H = H_0 + \lambda V, \quad (89)$$

where H_0 is a free Hamiltonian

$$H_0 = \int \left(\frac{k^2}{2} - \mu \right) a^+(k) a(k) d^3k \quad (90)$$

and

$$H_1 = \frac{\lambda}{2} \int d^3p_1 d^3p_2 d^3q_1 d^3q_2 v(p_1, p_2 | q_1, q_2) \times \delta(p_1 + p_2 - q_1 - q_2) a^+(p_1) a^+(p_2) a(q_2) a(q_1) \quad (91)$$

Here $a^+(k)$, $a(k)$ are usual Bose creation-annihilation operators (described in section 2), $\mu \in \mathbb{R}$ is a chemical potential $\lambda \in \mathbb{R}$ is a coupling constant. $v(p_1, p_2 | q_1, q_2)$ is a test function from the Schwartz space such that

$$v(p_1, p_2 | q_1, q_2) = v^*(q_1, q_2 | p_1, p_2). \quad (92)$$

Note that the Hamiltonian which describes Bose gas with weak (enough good) is a Hamiltonian of such form.

Now let us briefly describe the representation of quasi-particles in statistical mechanics. We will use this representation below when we try to establish the form of the density operator for non-equilibrium state.

Let us consider Bose Gas contained in macroscopic volume V . Let us formulate the main physical assumption which lays in the fundament of quasi-particles representation. The classification of the energetic levels of interacting particle could be described in the same manner as a classification of energetic levels of the system of non interacting particles. Therefore the energetic levels of the system could be described by means of real valued pointwise positive function $n(p)$ of momentum $p \in \mathbb{R}^3$. One can interpret the density function $n(p)$ as a density of quasi-particles with momentum p . In other words $n(p)d^3p$ is a number of particles with momenta from infinitesimally small volume (in momenta space) d^3p with the center at p . Let $E[n]$

be an energy of an energetic level corresponding to the function $n(p)$. Let $\varepsilon(p)$ be a functional derivation of $E[n]$ with respect $n(p)$, i.e.

$$\varepsilon(p, [n]) = \frac{\delta E[n]}{\delta n(p)}. \quad (93)$$

One can interpret $\varepsilon(p)$ as a dispersion law of quasi-particles. We have supposed that the classification of energetic levels of interacting system is the same as a classification of energetic levels of non-interacting system. Therefore the entropy of the system corresponding to the density function $n(p)$ has the form:

$$S[n] = \int d^3p [(1 + n(p)) \ln(1 + n(p)) - n(p) \ln n(p)]. \quad (94)$$

Note that the dynamics of (isolated) interacting system differs from the dynamics of non-interacting systems only in the following/ The dispersion law for $\varepsilon(p, [n])$ for interacting system depends of the density function n .

Therefore for all piece-wise continuous functions $f(p)$ the quantities

$$K[f] = \int \hat{n}(p) f(p) d^3p \quad (95)$$

are commuting integrals of motion. Here, by definition

$$\hat{n}(p) := \alpha^+(p)\alpha(p) \quad (96)$$

and $\alpha^+(p)$ and $\alpha(p)$ are formally defined operators of quasi-particles, satisfying to the Bose-Einstein statistics.

Note that last observation is the main idea of the proof of non-ergodic theorem. More precisely for the systems which contains of finite number of particles (quantum fields) wave operator W (defined in section 2) maps dynamics of free particles into dynamics of interacting particles. Therefore it is essential try to find the construction analogues to the construction of wave operators to build the quasi-particles representation. But the series of perturbation theory obtained by such a way contain (secular) divergences. One of the authors comes to non-ergodic theorem when tried to renormalize such divergences.

Let $\{\mathcal{O}_i | i = 1, 2, 3, \dots\}$ is a decomposition of \mathbb{R}^3 into a set of domains such that $\forall i \neq j \mathcal{O}_i \cap \mathcal{O}_j \subset \partial\mathcal{O}_i \cup \partial\mathcal{O}_j$, where ∂U means the boundary of the set

U . Suppose that $\forall i = 1, 2, 3 \dots N$ and $\forall p, p' \in \mathcal{O}_i$ the difference $p - p'$ is very small. $\forall i = 1, 2, 3 \dots$ put by definition

$$K_i = \int_{\mathcal{O}_i} \hat{n}(p) d^3p \quad (97)$$

Now let us find the form of distribution function $n(p)$ corresponding to some (generally) equilibrium state (in the sense of generalized microcanonical distribution) which corresponds to fixed values K'_i of integrals K_i , $i = 1, 2, 3, \dots$

It is evident that (generally) equilibrium distribution function $n(p)$ corresponds to (relative) maximum of the entropy $S(n)$ under the following constrains:

$$K_i[n] := \int d^3p n(p) = K'_i, \quad i = 1, 2, 3 \dots \quad (98)$$

and

$$E[n] = E. \quad (99)$$

According to the method of undefined lagrange multipliers we reduce this problem to the solution of the following system of equations

$$\frac{\delta}{\delta n(p)} \left\{ - \sum_{i=1}^{\infty} G_i \left[\left(1 + \frac{N_i}{G_i}\right) \ln \left(1 + \frac{N_i}{G_i}\right) - \frac{N_i}{G_i} \ln \frac{N_i}{G_i} \right] - \sum_{i=1}^N \mu_i N_i + E[n] \right\} = 0, \quad (100)$$

where

$$G_i := \int_{\mathcal{O}_i} d^3p \quad (101)$$

and

$$N_i = \int_{\mathcal{O}_i} n(p) d^3p, \quad i = 1, 2, 3 \dots \quad (102)$$

This system of equation could be immediately solved and one finds

$$n(p) = \frac{1}{e^{\frac{\varepsilon(p,[n])-\mu(p)}{T}} - 1}, \quad (103)$$

where

$$\mu(p) = \mu_i \in \mathbb{R}, \text{ if } p \in \mathcal{O}_i \quad (104)$$

and T some positive number

In other words if one moves from generalized microcanonical distribution to the generalized microcanonical distribution this is equivalent to the following. The chemical potential μ becomes to the function of momenta p .

Note that in spite of volume V is macroscopic it is finite. Therefore the set of all possible momenta is discrete $\{p_i | i = 1, 2, 3, \dots\}$. Let by definition n_i be a number of particles with momentum p_i .

$\forall i = 1, 2, 3, \dots$ the cells \mathcal{O}_i corresponds to the approximately equal frequencies. Therefore each two states corresponding to the same numbers N_i , $i = 1, 2, 3, \dots$ corresponds to approximately the same energy. Therefore all the states corresponding to the same numbers N_i $i = 1, 2, 3, \dots$ have approximately the same probabilities.

One can easily find the form of the distribution function $\rho_i(n_i)$ by using the method of the most probable distribution. Omitting the well-known reasoning one finds

$$\rho_i(n_i) = \frac{e^{-\frac{\varepsilon(p_i,[n])n_i - \mu_i n_i}{T}}}{Z_i}, \quad (105)$$

where Z_i is a real positive number such that

$$\sum_{n=0}^{\infty} \rho_i(n) = 1. \quad (106)$$

Therefore the (generalized) equilibrium state of quasiparticles could be described by means the Gauss state on the algebra of canonical commutative relations generated by $\alpha(p)$, $\alpha^+(p)$ of the form

$$\begin{aligned} \langle \alpha(p)\alpha(p') \rangle &= \langle \alpha^+(p)\alpha^+(p') \rangle = 0, \\ \langle \alpha^+(p)\alpha(p') \rangle &= \delta(p - p')(1 + n(p)), \\ n(p) &= \delta(p - p') \frac{1}{e^{-\frac{\varepsilon(p,[n])-\mu(p)}{T}} - 1}, \end{aligned} \quad (107)$$

$$\mu(p) = \mu_i, \text{ if } p \in \mathcal{O}_i. \quad (108)$$

So, let us consider a single mode of momenta k_i interacting with other modes of Bose gas as with a thermostat. To analyze this system we will use so-called statistical perturbation theory [10]. Bogoliubov [10] has considered the following system. He has considered the Hamiltonian system of n degrees of freedom described by the Hamiltonian $H(p, q)$ in the field of external force $\varepsilon f(t)$, $\varepsilon \in \mathbb{R}$. So, total Hamiltonian of the system has the form

$$\Gamma = H + \varepsilon f(t)P, \quad (109)$$

where $P(p, q)$ is some function of canonically-conjugated of momenta and coordinates. We also suppose that external force $f(t)$ is the function of time t of the form:

$$f(t) = \sum_{\nu} a_{\nu} \cos(\nu t + \varphi_{\nu}). \quad (110)$$

Here the phases φ_{ν} are independent random quantities uniformly distributed on circle. We also suppose that the spectrum of frequencies is almost continuous so we can the sums of the form

$$\sum_{\nu} F(\nu) a_{\nu}^2 \quad (111)$$

for continuous functions $F(\nu)$ replace by the integral

$$\int_0^{+\infty} F(\nu) I(\nu) d\nu. \quad (112)$$

Let D_t denote the probability density of momenta and coordinates at fixed time moment t under the condition that phases φ_{ν} take some defined values. The probability density of momenta and coordinates in usual sense (for arbitrary values of phases φ_{ν}) could be obtained from D_t by means of averaging over the phases φ_{ν} . In other words

$$D_0 = \rho_0. \quad (113)$$

The evolution of D_t in times could be described by means well known Liouville equation

$$\frac{\partial D_t}{\partial t} = (H, D_t) + \varepsilon f(t)(P, D_t), \quad (114)$$

where (A, B) denotes usual Poisson Brackets defined by the formula

$$(A, B) = \sum_{i=1}^n \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right). \quad (115)$$

Let us introduce the following one-parameter group of operators T_t acting on dynamical variables according to the formula

$$T_t F(p, q) = F(p_t, q_t), \quad (116)$$

where (p_t, q_t) are the solution of canonical Hamiltonian equations

$$\begin{aligned} \frac{dp_t}{dt} &= - \frac{\partial H(p_t, q_t)}{\partial q_t}, \\ \frac{dq_t}{dt} &= \frac{\partial H(p_t, q_t)}{\partial p_t}, \end{aligned} \quad (117)$$

corresponding to the following initial conditions

$$\begin{aligned} p_0 &= p, \\ q_0 &= q. \end{aligned} \quad (118)$$

In these assumptions and denotations in the limit of small coupling constant ε it is derived [Bogoliubov, 1945] the following equation on ρ_t

$$\frac{\partial \rho_t}{\partial t} = (H, \rho_t) + \varepsilon^2 \int_0^t \Delta(t - \tau) (P, (T_{t-\tau}(P, \rho_\tau))) d\tau, \quad (119)$$

where

$$\Delta(\tau) = \frac{1}{2} \int_0^{+\infty} I(\nu) \cos(\nu\tau) d\nu. \quad (120)$$

The equation (119) could be essentially generalized to the case of quantum mechanics. It is enough to replace the Poisson brackets by the commutator.

Note that the random force $f(t)$ here is an example of random processes. Recall that the random process is the set (T, f_t) , where $T = (S, \Sigma, P)$ is a Kolmogorov triple (S is some set, Σ is a σ -algebra, P is a probability measure

on S defined on sets from Σ). f_t is a map which to an arbitrary $t \in \mathbb{R}$ assigns a random quantity i.e. a measurable (with respect to Σ) function on S . Note that the equation (119) could be rewritten as follows

$$\frac{\partial \rho_t}{\partial t} = (H, \rho_t) + \varepsilon^2 \int_0^t \overline{f(t)t(\tau)}(P, (T_{t-\tau}(P, \rho_t))) d\tau, \quad (121)$$

where $\bar{\xi}$ means the mathematical expectation of ξ i.e.

$$\bar{\xi} := \int_S \xi(x) dP(x). \quad (122)$$

Note that how it follows from the derivation of equation (119) [Bogoliubov, 194] it is not necessary that $f(t)$ has the form described previously. It is enough that $f(t)$ has the form

$$f(t) = \sum_{\nu} \frac{1}{2} (f_{\nu} e^{i\nu t} + f_{\nu}^* e^{-i\nu t}), \quad (123)$$

where f_{ν} are random quantities such that $\forall \mu, \nu$,

$$\overline{f_{\mu}} = \overline{f_{\mu}^*} = 0, \quad (124)$$

$$\overline{f_{\mu} f_{\nu}} = \overline{f_{\mu}^* f_{\nu}^*} = 0, \quad (125)$$

and

$$\overline{f_{\nu} f_{\mu}} = \delta_{\mu\nu} I_{\nu}. \quad (126)$$

The spectrum of frequencies $\{\nu\}$ is supposed almost continuous such that one can replace the sums

$$\sum_{\nu} I_{\nu} F(\nu) \quad (127)$$

for continuous functions $F(\nu)$ by the integral

$$\int_0^{+\infty} I(\nu) F(\nu) d\nu. \quad (128)$$

Now note that the generalization of a Kolmogorov triple to noncommutative (quantum) case is a pair consisting of an unital \star -algebra \mathcal{A} and a state $\langle \cdot \rangle$ on it. By definition state is a normalized positively defined linear functional on the algebra. Normalization condition means that $\langle \mathbf{1} \rangle = 1$, where $\mathbf{1}$ is a unite of \mathcal{A} . Positively definiteness means that $\forall a \in \mathcal{A} \langle aa^\star \rangle \geq 0$.

The algebra \mathcal{A} plays the role of the space of measurable functions on (nonexisting) noncommutative space. The state $\langle \cdot \rangle$ plays the role of integration on this state. The involution \star plays the role of pointwise complex conjugation.

By definition the noncommutative random process is a map which to an arbitrary $t \in \mathbb{R}$ assigns an element a_t of \mathcal{A} . We say that noncommutative random process $\{a_t | t \in \mathbb{R}\}$ is a real-valued process if $\forall t \in \mathbb{R} a_t = a_t^\star$.

The equation (119) could be generalized to the case of noncommutative (quantum) systems. More precisely let total Hamiltonian of the system has the form

$$H = H_0 + \varepsilon(f(t)P + f^+(t)P^+), \quad (129)$$

where $f(t)$ is a noncommutative random process of the form

$$f(t) = \sum_{\nu} f_{\nu} e^{i\nu t}, \quad (130)$$

$\forall \mu,$

$$\langle f_{\mu} \rangle = \langle f_{\mu}^{\star} \rangle = 0, \quad (131)$$

and $\forall \mu, \nu$

$$\langle f_{\mu} f_{\nu} \rangle = \langle f_{\mu}^{\star} f_{\nu}^{\star} \rangle = 0, \quad (132)$$

$$\begin{aligned} \langle f_{\mu}^{\star} f_{\nu} \rangle &= I_{\nu} \delta_{\mu\nu}, \\ \langle f_{\mu} f_{\nu}^{\star} \rangle &= J_{\nu} \delta_{\mu\nu}. \end{aligned} \quad (133)$$

The spectrum of external force $f(t)$ is assumed almost continuous such that we can replace the sums

$$\begin{aligned} \sum_{\nu} F(\nu) I_{\nu}, \\ \sum_{\nu} F(\nu) J_{\nu} \end{aligned} \quad (134)$$

by the integrals

$$\begin{aligned} & \int_0^{+\infty} F(\nu)I(\nu)d\nu, \\ & \int_0^{+\infty} F(\nu)J(\nu)d\nu \end{aligned} \quad (135)$$

respectively. In this case, instead of (119) we will have

$$\begin{aligned} & \frac{\partial \rho_t}{\partial t} = [H, \rho_t] \\ & + \varepsilon^2 \int_0^t \langle [f^*(t)P^*, f(\tau)T_{\tau-t}([P, \rho_t])] \rangle \\ & + \varepsilon^2 \int_0^t \langle [f(t)P, f^*(\tau)T_{\tau-t}([P^*, \rho_t])] \rangle. \end{aligned} \quad (136)$$

We suppose that our system is contained in some cube of volume V with edges parallel to the coordinate axis. We suppose that the length of the edges of this cube is equal to L . Therefore $V = L^3$. Put by definition

$$\mathfrak{K} = \{k = (k_x, k_y, k_z) \mid \frac{k_x}{2\pi L}, \frac{k_y}{2\pi L}, \frac{k_z}{2\pi L} \in \mathbb{Z}\}. \quad (137)$$

We need to use the discrete momentum representation for finite volume instead of continuous momentum representation for infinite volume. Instead of the Hamiltonian (89,90, 91) we need to use the following Hamiltonian in finite volume:

$$H^V = H_0^V + H_1^V, \quad (138)$$

where

$$H_0^V = \sum_{k \in \mathfrak{K}} \omega(k) a^+(k) a(k), \quad (139)$$

and

$$\omega(k) = \frac{k^2}{2}. \quad (140)$$

$$H_1^V = \frac{\lambda(2\pi)^3}{2V} \sum_{p_1, p_2, q_1, q_2 \in \mathfrak{K}} \Delta(p_1 + p_2 - q_1 - q_2) v(p_1, p_2 | q_1, q_2) \times a_{p_1}^+ a_{p_2}^+ a_{q_2} a_{q_1}, \quad (141)$$

where

$$\begin{aligned} \Delta(p) &= 1, \text{ if } p = 0 \\ \Delta(p) &= 0, \text{ if } p \neq 0, \end{aligned} \quad (142)$$

and a_p^+ and a_p are creation and annihilation operators in discrete momentum representation.

Now let us consider a single mode corresponding to some fixed momentum $k \in \mathfrak{K}$. Let us study how the state of this mode changes while time changes from t to $t + dt$ for infinitely small dt . To study how the density matrix ρ_t of this mode changes at time interval $[t, t + dt]$ one can suppose that the thermostat evolves at this interval according to free dynamics. So one can suppose that the mode corresponding to the momentum k affected by external (noncommutative) force defined by this thermostat. In other words we must take

$$\Gamma = H + \lambda(f(t)P + f^*(t)P^+), \quad (143)$$

where, by definition

$$\begin{aligned} H &= \omega(k)a^+(k)a(k), \\ P &= a_k. \end{aligned} \quad (144)$$

External forces $f(t)$ are the elements of the algebra \mathcal{A} generated by operators a_p^+ , a_p , $p \neq k$. External force $f(t)$ has the form

$$f(t) = \frac{(2\pi)^3}{V} \sum_{p_1, p_2, q_1 \in \mathfrak{K} \setminus \{k\}} \Delta(p_1 + p_2 - q_1 - k) v(p_1, p_2 | q_1, k) \times a_{p_1}^+ a_{p_2}^+ a_{q_1} e^{i(\omega(p_1) + \omega(p_2) - \omega(q_1))t} \quad (145)$$

To obtain the differential equation for the density matrix for our fixed mode we will average over the state $\langle \cdot \rangle$ on the algebra \mathcal{A} which is a Gauss state and is defined by the following its pair correlator

$$\begin{aligned} \langle a_p a_{p'} \rangle &= \langle a_p^+ a_{p'}^+ \rangle = 0, \\ \langle a_p^+ a_{p'} \rangle &= \Delta(p - p') n(p). \end{aligned} \quad (146)$$

Note that external force $f(t)$ has the form

$$f(t) = \sum_{\nu} f_{\nu} e^{i\nu t}, \quad (147)$$

where

$$f_{\nu} = \frac{(2\pi)^3}{V} \sum_{p_1, p_2, q_1 \in \mathcal{R} \setminus \{k\}} \Delta(p_1 + p_2 - q_1 - k) \\ \times \Delta(\omega(p_1) + \omega(p_2) - \omega(q_1) - \nu) v(p_1, p_2 | q_1, q_2) a_{p_1}^+ a_{p_2}^+ a_{q_2}. \quad (148)$$

Note that the averaged pair products of f_{ν} , f_{ν}^* satisfy to the equations (132,133). Let ρ_t be a density matrix corresponding to our fixed single mode with momentum k . Put by definition $\langle \cdot \rangle = \text{tr}(\cdot \rho_t)$.

Suppose that $\langle \cdot \rangle$ be a Gauss state on the algebra \mathcal{A} . Equation (136) implies that $\forall t > 0$ the state $\langle \cdot \rangle_{t+dt}$ is a Gauss state if the state $\langle \cdot \rangle_t$ is a Gauss state for infinitely small positive dt . Therefore if $\langle \cdot \rangle_t$ is a gauss state for all positive t if $\langle \cdot \rangle_0$ is a Gauss state (in the limit of weak pair interaction). Let $\hat{n}_p = a_p^+ a_p$ be an operator of the particle number corresponding to the momentum p . Put by definition $n_t(p) = \langle \hat{n}_p \rangle_t$. Equation (136) implies that (in the limit of weak pair interaction)

$$\begin{aligned} \frac{d}{dt} n_t(k) &= -\lambda^2 \int_0^t \langle \langle [a_k^+ f^*(t), a_k f(\tau)] \rangle_t \rangle e^{i\omega(k)(t-\tau)} d\tau \\ &\quad + \lambda^2 \int_0^t \langle \langle [a_k f(t), a_k^+ f^*(\tau)] \rangle_t \rangle e^{-i\omega(k)(t-\tau)} d\tau \\ &= -\lambda^2 \int_0^t \{n_t(k) \langle f^*(t) f(\tau) \rangle - (1 + n_t(k)) \langle f(\tau) f^*(t) \rangle\} e^{i\omega(k)(t-\tau)} d\tau \\ &\quad + \lambda^2 \int_0^t \{(1 + n_t(k)) \langle f(t) f^*(\tau) \rangle - n_t(k) \langle f^*(\tau) f(t) \rangle\} e^{-i\omega(k)(t-\tau)} d\tau \\ &= -\lambda^2 n_k(t) \int_{-t}^t \langle f^*(0) f(\tau) \rangle e^{-i\omega(k)\tau} d\tau \\ &\quad + \lambda^2 (1 + n_k(t)) \int_{-t}^t \langle f(0) f^*(\tau) \rangle e^{i\omega(k)\tau} d\tau. \quad (149) \end{aligned}$$

Therefore

$$\begin{aligned}
& \frac{d}{dt} n_t(k) \\
= & -\frac{\lambda^2(2\pi)^6}{V^2} \sum_{p_1, p_2, q_1 \in \mathfrak{K} \setminus \{k\}} |v(p_1, p_2 | q_1, k)| n_t(q_1) n_t(k) (1 + n_t(p_1)) (1 + n_t(p_2)) \\
& \times \Delta(p_1 + p_2 - q_1 - k) \int_{-t}^t e^{i(\omega(p_1) + \omega(p_2) - \omega(q_1) - \omega(k))\tau} d\tau \\
& + \frac{\lambda^2(2\pi)^6}{V^2} \sum_{p_1, p_2, q_1 \in \mathfrak{K} \setminus \{k\}} |v(p_1, p_2 | q_1, k)| (1 + n_t(q_1)) (1 + n_t(k)) n_t(p_1) n_t(p_2) \\
& \times \Delta(p_1 + p_2 - q_1 - k) \int_{-t}^t e^{-i(\omega(p_1) + \omega(p_2) - \omega(q_1) - \omega(k))\tau} d\tau \quad (150)
\end{aligned}$$

But volume V is macroscopic. So we can replace the sums over \mathfrak{K} by integrals. In result

$$\begin{aligned}
\frac{d}{dt} n_t(k) = & (2\pi)\lambda^2 \int d^2 p_1 d^3 p_2 d^3 q_1 \delta(p_1 + p_2 - q_1 - k) \\
& \delta(\omega(p_1) + \omega(p_2) - \omega(q_1) - \omega(k)) \times \{(1 + n_t(q_1))(1 + n_t(k)) n_t(p_1) n_t(p_2) \\
& - n_t(q_1) n_t(k) (1 + n_t(p_1)) (1 + n_t(p_2))\}. \quad (151)
\end{aligned}$$

So we have obtain the usual kinetic equation for Bose gas in the limit of weak interaction.

5 Integrals for non-linear partial differential equations.

Solitary perturbations from generalized thermodynamics. Let us consider some system of statistical mechanics such that there second-order phase transition takes place at some temperature T_{cr} . Suppose that this transition concerning with some symmetry breaking. As an example of such system one can take some superfluid or superconductive system. Let $\langle \cdot \rangle$ be (generally) equilibrium translation-invariant stationary state corresponding to the temperature $T < T_{cr}$. Suppose that $\langle \vec{P} \rangle = 0$, where $\langle \vec{P} \rangle$ is a momentum operator. Let \mathcal{K} be some inertial reference system which moves

with velocity v . Let $U \subset \mathbb{R}^3$ be some domains of microscopical sizes tightly connected with reference system \mathcal{K} . Domain U by definition is macroscopic. Therefore, according to the non-ergodic theorem [1] there exist integrals K_1, \dots, K_N in involution for the part of our system contained in U . According to our generalized thermodynamics, the distribution function for the part of our system, contained in U has the form:

$$\rho(x) = \text{const} \delta(H(x) - E) \prod_{i=1}^N \delta(K'_i(x) - K'_i). \quad (152)$$

Now let us show, that the entropy of the system contained in U :

$$S(E, K'_1, \dots, K'_N) = \ln \int d\Gamma_x \delta(H(x) - E) \prod_{i=1}^N \delta(K'_i(x) - K'_i) \prod_{i=1}^N \delta(K_i(x) - K'_i) \quad (153)$$

achieve a maximum at some $D > 0$ dimensional manifold M as a function of integrals K_1, \dots, K_N at fixed energy E . Indeed suppose that the entropy achieve a maximum (at fixed energy) at isolated point. In this case the (generalized) microcanonical distribution gives us the same results as standard microcanonical distribution

$$\rho(x) = \text{const} \delta(H(x) - E). \quad (154)$$

But for this distribution all the symmetries of the Hamiltonian $H(x)$ are preserved. Therefore the entropy $S(E, K'_1, \dots, K'_N)$ achieve a maximum at some $M > 0$ dimensional manifold as function of K'_1, \dots, K'_N .

Therefore we can chose the observable values K'_1, \dots, K'_N of the integrals K_1, \dots, K_N such that (the intensive) properties of the system will be qualitatively different from the analogues properties of other part of the system (outside of U) and values K'_1, \dots, K'_N corresponds to the maximum of the entropy. According to our generalized thermodynamics this state is stable. For example, for superfluid gases, the the phase of condensate wave function inside U may differ from the phase of condensate wave function outside of U . If we return into initial (static) reference system we obtain a non-trivial perturbation of the state $\langle \cdot \rangle$ which moves with constant velocity without dissipation. Such perturbation we will call thermodynamical solitons.

An example of thermodynamical soliton. The Abrikosov Vortices. As an example of thermodynamical soliton we consider the situation

which arise in the superfluidity theory if one studies the condensate wave function [11].

Let $\hat{\Psi}(x)$ and $\Psi(x)$ be secondary quantized wave function and its hermitian-conjugated. Let $\hat{\Xi}$ and $\hat{\Xi}^+(x)$ be the parts of the function $\hat{\Psi}(x)$ and $\hat{\Psi}^+(x)$ which change by one number of condensate particles and preserve others quantum numbers. In other words, by definition:

$$\begin{aligned}\hat{\Xi}(x)|N+1, m\rangle &= \Sigma(x)|N, m\rangle, \\ \hat{\Xi}^+(x)|N, m\rangle &= \Sigma^*(x)|N+1, m\rangle\end{aligned}\tag{155}$$

Here

$$\Xi(x) := \langle N, m | \hat{\Xi}(x) | N+1, m \rangle,\tag{156}$$

and $\Xi^*(x)$ is a complex-conjugated to $\Xi(x)$ function. N is a number of condensate particle m is a number of other quantum numbers of the system. Let us represent $\hat{\Psi}(x)$ and $\Psi(x)$ as follows:

$$\begin{aligned}\hat{\Psi}(x) &= \Xi(x) + \hat{\Psi}'(x), \\ \hat{\Psi}^+(x) &= \Xi^+(x) + \hat{\Psi}'^+(x).\end{aligned}\tag{157}$$

In thermodynamical limit $N \rightarrow \infty$ the difference between the states $|N, m\rangle$ and $|N+1, m\rangle$ disappears at all. Therefore the operators $\hat{\Xi}(x)$ and $\hat{\Xi}^+(x)$ becomes the operators which commutes to each other and commutes with $\hat{\Psi}'(x)$ and $\hat{\Psi}'^+(x)$. In other words $\hat{\Xi}(x)$ and $\hat{\Xi}^+(x)$ becomes the classical variables. Now let us write the (partial) differential equation on condensate wave function $\Xi(x)$. Let us consider a weakly non-ideal Bose gas at absolute zero of temperature. Almost all particles in such a gas are in condensate state. In terms of secondary-quantized wave function this means that the over-condensate part $\hat{\Psi}'(x)$ is small with respect to the condensate wave function $\hat{\Xi}(x)$. If we neglect by small over-condensate part $\hat{\Psi}'(x)$ the condensate wave function will satisfy to the same Schrodinger equation as the equation which take place for the operator $\hat{\Psi}(x)$. I.e.

$$\begin{aligned}i\frac{\partial}{\partial t}\Xi(x, t) &= \left(-\frac{1}{2m}\nabla^2 + \mu\right)\Xi(x, t) \\ &+ \Xi(x, t) \int |\Xi(x', t)|^2 U(x-x') d^3x'.\end{aligned}\tag{158}$$

Here $U(x)$ is an interaction potential $\mu = nU_0$, n is a density of particles in gas and $U_0 := \int U(x) d^3x$. This equation is called the Pitaevskii equation.

We will show now that the Pitaevskii equation will precise in the limit when the interaction constant tends to zero. Let us suppose that the interaction potential $U(x)$ depends on small positive parameter λ as follows

$$U(x|\lambda) := \lambda\lambda^{3/2}U(x\sqrt{\lambda}). \quad (159)$$

For example if $U(x) = \delta(x)$ then $U(x, \lambda) = \lambda\delta(x)$. Let us introduce new condensate wave function:

$$\Sigma'(x, t) = \Sigma(x\sqrt{\lambda}, t\lambda). \quad (160)$$

In the limit $\lambda \rightarrow 0$, $\Sigma'(x, t)$ will satisfy precisely to the Pitaevskii equation (158). The fact that the Pitaevskii equation will be satisfied precisely will follows from the fact that all particles will be in condensate in limit $\lambda \rightarrow 0$ and the over-condensate wave function will be equal to zero in the limit $\lambda \rightarrow 0$.

The (partial) differential equation for the order parameter of the system has a lot of non trivial first integrals. In this subsection we consider only super fluid systems. These systems are described by the condensate wave function $\Xi(x)$ defined on \mathbb{R}^3 . The condensate wave function play the role of order parameter for the system. In the limit $\lambda \rightarrow 0$ the condensate wave function $\Xi(x)$ and its complex conjugated $\Xi^*(x)$ are the canonically-conjugated coordinates on some (infinitely-dimensional) phase space. The Poisson brackets of $\Xi(x)$ and $\Xi^*(x)$ has the form:

$$(\Xi(x), \Xi^*(x')) = \delta(x - x'). \quad (161)$$

The evolution of the condensate wave function $\Xi(x)$ is a Hamiltonian evolution with respect to the Poisson brackets (161). The corresponding Hamiltonian (for an arbitrary temperature) is $F(\Xi(x), \Xi^+(x))|T$, where $F(\Xi(x), \Xi^+(x))|T$ is a free (Gibbsian) energy of the system under the condition of fixed condensate wave function $\Xi(x)$. Now we will introduce the assumption of so called asymptotically completeness and prove that the equation for order parameter has a lot of first integrals in involution under this assumption.

Asymptotical completeness. Let $\Xi(x, t)$ be a solution of the Pitaevskii equation (158) which is localized (in some essential non rigorous sense) in some finite domain of the space at $t = 0$. We ssaume that this solution splits as $t \rightarrow \infty$ into the set of separated thermodynamical solitons described in the beginning of this section.

So let us suppose that the asymptotical completeness takes place. In this case first integrals for the system, defined by the Hamiltonian $F(\Xi(x), \Xi^+(x)|T)$ (for the temperature $T = 0$) could be described as follows.

Let L be a cubic lattice in \mathbb{R}^3 . Let $\{C_i | i = 1, 2, 3, \dots\}$ be the set of all elementary cubes of L . We suppose that the sizes of cubes C_i are macroscopic. We will also denote by C_i ($\forall i = 1, 2, 3, \dots$) the part of our system contained in the cube C_i . $\forall i = 1, 2, \dots$ system C_i is macroscopic. So Let $\{K_i^j\}$, $j = N$ be some set of its first integrals in involution. We have shown [2] that number N could be chosen as large as needed if each system C_i is enough large. Let $\forall \vec{a} \in \mathbb{R}^3$ $T_a : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a map defined as follows:

$$T_a : x \mapsto x + \vec{a}. \quad (162)$$

$\forall \vec{a} \in \mathbb{R}^3$ put by definition $C(\vec{a}) = T_{\vec{a}}C$.

Let \vec{a} be an arbitrary vector from \mathbb{R}^3 . Denote by $K_i^j(a)$, $j = 1, 2, 3, \dots$ the integrals of the system $C_i(\vec{a})$ obtained from K_i^j by translation on vector \vec{a} in obvious sense.

Let $\forall \vec{a} \in \mathbb{R}^3$ $\mathcal{K}_i^j(\vec{a}) = \langle K_i^j(\vec{a}) \rangle_{\Xi, \Xi^*, T=0}$. Here $\langle \cdot \rangle_{\Xi, \Xi^*, T}$ by definition is an averaging by relative Gibbs distribution corresponding to fixed $\Xi(x)$ and Ξ^* and the temperature T .

Now let $\Xi(x)$ be some localized in the space function. Let $\Xi(x, t)$ be a solution of Pitaevskii equation such that $\Xi(x, 0) = \Xi(x)$. Recall that Pitaevskii equations are Hamiltonian equations with respect to Hamiltonian $F(\Xi(x), \Xi^+(x)|T)$. According to the asymptotic completeness assumption if chose enough large time t we can assume that $\Xi(x, t)$ is a set of space separated thermodynamical solitons such that for each $i = 1, 2, 3, \dots$ each cube C_i has nontrivial intersection at most with one of such solitons. $\forall j = 1, 2, \dots, N$ put by definition:

$$\mathcal{K}_j(\Xi, \Xi^*) = \int_{C_1} d^3\vec{a} \sum_{i=1}^{\infty} \mathcal{K}_i^j(\vec{a}(\Xi(x, t), \Xi^*(x, t))). \quad (163)$$

It is evident that this definition do not depend on t (if t is enough large) and $\mathcal{K}_j(\Xi, \Xi^*)$ are integrals of motion.

Theorem. Integrals $\mathcal{K}_j(\Xi, \Xi^*)$, $j = 1, 2, \dots, N$ are in involution i.e. commutes to each other.

Proof. Suppose that the time t is chosen as above. We have:

$$\begin{aligned}
& (\mathcal{K}_m(\Xi, \Xi^*), \mathcal{K}_n(\Xi, \Xi^*)) \\
&= \int \int_{C_1 \times C_1} d^3 \vec{a}_1 d^3 \vec{a}_2 \sum_{i,j=1,2,\dots} (\mathcal{K}_m^i(\Xi(t), \Xi^*(t)), \mathcal{K}_n^j(\Xi(t), \Xi^*(t))) \\
& \quad = \int \int_{C_1 \times C_1} d^3 \vec{a}_1 d^3 \vec{a}_2 \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3 x_1 d^3 x_2 \\
& \times \sum_{i,j=1,2,\dots} \frac{\delta \mathcal{K}_m^i(\Xi(t), \Xi^*(t))}{\delta \Xi(x_1)}, \frac{\delta \mathcal{K}_n^j(\Xi(t), \Xi^*(t))}{\delta \Xi^*(x_2)} \times (\Xi(x_1), \Xi^*(x_2)) \\
& \quad + \int \int_{C_1 \times C_1} d^3 \vec{a}_1 d^3 \vec{a}_2 \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3 x_1 d^3 x_2 \\
& \times \sum_{i,j=1,2,\dots} \frac{\delta \mathcal{K}_m^i(\Xi(t), \Xi^*(t))}{\delta \Xi^*(x_1)}, \frac{\delta \mathcal{K}_n^j(\Xi(t), \Xi^*(t))}{\delta \Xi(x_2)} \times (\Xi^*(x_1), \Xi(x_2)). \quad (164)
\end{aligned}$$

Therefore to prove that the Poisson Bracket $(\mathcal{K}_m(\Xi, \Xi^*), \mathcal{K}_n(\Xi, \Xi^*))$ is equal to zero it is enough to prove that the following equalities holds:

$$\frac{\delta \mathcal{K}_m^i(\Xi, \Xi)}{\delta \Xi(x)} = \frac{\delta \mathcal{K}_m^i(\Xi, \Xi^*)}{\delta \Xi^*(x)} = 0 \quad (165)$$

for all $i = 1, 2, 3, \dots, j = 1, \dots, N, \vec{a}, x \in \mathbb{R}^3$. Let us prove second of these equalities. Second of these equalities could be proved by analogy.

Fix a cube $C_i(\vec{a})$. Suppose that the soliton (at most one) which has nontrivial intersection with $C_i(\vec{a})$ has zero velocity. In the opposite case we can chose an inertial system such that the soliton has zero velocity in this new system. Then we just perform all reasoning below in this moving system.

The (non-commutative) phase space of the system $C_i(\vec{a})$ could be represented as a direct product of two non-commutative phase spaces $\mathfrak{S}_1 \times \mathfrak{S}_2$ where the phase space \mathfrak{S}_1 is a classical space and $\Xi(x)$ and $\Xi^*(x)$ are the set of canonically-conjugated coordinates on it. And the space \mathfrak{S}_2 roughly speaking corresponds to over-condansate particles. Let \mathcal{H} be a Hilbert space corresponding \mathfrak{S}_1 . Hamiltonian H and the integrals K_j of the system $C_i(\vec{a})$ are the functionals of $\Xi(x)$ and $\Xi^*(x)$ which take values in the space of operators acting in \mathcal{H} . We have

$$\mathcal{K}_m^i(\Xi, \Xi^*) = \frac{\text{tr}(K_m(\Xi, \Xi^*) \exp(-\frac{H(\Xi, \Xi^*)}{T}))}{\text{tr}(\exp(-\frac{H(\Xi, \Xi^*)}{T}))}. \quad (166)$$

Here trace is taken over the space $\mathcal{H}_i(\vec{a})$. We have

$$\begin{aligned}
\frac{\delta \mathcal{K}_m^i(\Xi, \Xi)}{\delta \Xi(x)} &= \frac{\text{tr}\left\{\frac{\delta K_m(\Xi, \Xi^*)}{\delta \Xi(x)} \exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right\}}{\text{tr}\left(\exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right)} \\
+ \int_0^1 ds &\frac{\left\{\text{tr} K_m(\Xi, \Xi^*) \exp\left(-s\frac{H(\Xi, \Xi^*)}{T}\right) \frac{\delta H(\Xi, \Xi^*)}{\delta \Xi(x)} \exp\left(s\frac{H(\Xi, \Xi^*)}{T}\right) \exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right\}}{\text{tr}\left(\exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right)} \\
&- \frac{\text{tr}\left\{K_m(\Xi, \Xi^*) \exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right\}}{\text{tr}\left(\exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right)} \frac{\text{tr}\left\{\frac{\delta H(\Xi, \Xi^*)}{\delta \Xi(x)} \exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right\}}{\text{tr}\left(\frac{\delta H(\Xi, \Xi^*)}{\delta \Xi(x)} \exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right)} \quad (167)
\end{aligned}$$

Let us show that each of three term in right hand side of this equation is equal to zero. Let us consider the third term. It is evident that this term is proportional to $\dot{\Xi}(x, t)$. But the soliton which has nontrivial intersection with $C_i(\vec{a})$ moves with zero velocity. So, this term is equal to zero. Let us consider first term:

$$\frac{\text{tr}\left\{\frac{\delta K_m(\Xi, \Xi^*)}{\delta \Xi(x)} \exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right\}}{\text{tr}\left(\exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right)}. \quad (168)$$

This term is proportional to $\langle \Xi'(x, t) \rangle$, where dash means the derivative corresponding to the flow, which generates by $K_m(\Xi, \Xi^*)$. The averaging here is an averaging over generalized microcanonical distribution. The flow generated by $K_m(\Xi, \Xi^*)$ preserve this microcanonical distribution. Therefore $\langle \Xi'(x, t) \rangle = 0$.

At last let us consider second term. Let us represent it as follows:

$$\sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^1 \left(-\frac{s}{T}\right)^n \frac{\left\{\text{tr} K_m(\Xi, \Xi^*) [H(\Xi, \Xi^*), \dots, [H(\Xi, \Xi^*), \frac{\delta H(\Xi, \Xi^*)}{\delta \Xi(x)}] \dots] \exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right\}}{\text{tr}\left(\exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right)} \quad (169)$$

Note that the iterated commutator here remains finite if the volume of $C_i(\vec{a})$ tends to infinity. Note that the averaging in last formula is an averaging over the generalized microcanonical distribution:

$$\rho = \text{const} \delta(H - E) \prod_{j=1}^N \delta(K_j - K'_j) \quad (170)$$

In result in each term of series (169) the integral $K_m m(\Xi, \Xi^*)$ could be replaced by its observable value K'_m . In result the second term is equal to

$$\frac{K'_m \operatorname{tr}\left(\frac{\delta H(\Xi, \Xi^*)}{\delta \Xi(x)} \exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right)}{\operatorname{tr}\left(\exp\left(-\frac{H(\Xi, \Xi^*)}{T}\right)\right)} \quad (171)$$

One can prove that this term is equal to zero by the same method as has been used for third term. Therefore the theorem is proved.

Programm — Hypothesis. So we have proved that the equation for condensate wave function (hamiltonian with respect to $F(\Xi, \Xi^*|T)$) admit a lot of independent first integrals in involution (in limit $\lambda = 0$, $T = 0$). In the limit $\lambda = 0$, $T = 0$ the condensate wave function (after suitable rescaling of space coordinates and time) satisfies to Pitaevskii equation. Pitaevskii equation is a non linear Schrodinger equation. In $1D$ case the nonlinear Pitaevskii equation is an completely integrable system. But we have proved that in $3D$ case nonlinear Schrodinger equation (under the assumption of asymptotical completeness) admit a lot of independent integrals in involution. Therefore there arise the following program-hypothesis of constructing (not necessary completely) integrable systems. One can take some system of statistical mechanics such that there exists a phase transition of second order at some temperature concerning with some symmetry breaking. Then one may find some small parameter η for this system and write some (partial) differential equation for the order parameter in the limit $\lambda = 0$ by some exact asymptotical method. Obtained equation will be a hamiltonian system which admit a lot of independent commuting first integrals in involution.

6 Conclusion.

In the present paper we have considered some physical examples to illustrate basics principles of the generalized thermodynamics, developed in [2]. I am very grateful to A.V. Koshelkin for valuable critical comments on this article and very useful discussions.

References

- [1] Prokhorenko, D.V. *On Non Ergodic Property of Bose Gas with Weak Pair Interaction*, Journal of Physical Mathematics, 2009, vol 1, Ar-

ticle ID S090701, 1-31, math-ph/0904.2868.

- [2] D.V. Prokhorenko, V.V. Matveev, *The significance of non-ergodic property of statistical mechanics systems for understanding resting state of a living cell*, British Journal of Mathematics and Computer Science, 2011; 1(2): 26-86.
- [3] D.V. Prokhorenko, V.V. Matveev, *Thermodynamics of living cell* 2011.
- [4] F.A. Berezin, *The Method of Secondary Quantization*, New York: Academic Press, 1966.
- [5] Bogoliubov, N.N., Bogoliubov N.N. (jr.) *Introduction to Quantum Statistical Mechanics*, Nauka, Moscow, 1984.
- [6] L.D. Landau, E.M. Lifshitz, *Theoretical Physics, vol. 5, Statistical Physics*, Nauka-Fizmatlit, Moscow, 1995.
- [7] R. Jost, *The General Theory of Quantized Fields*, Providence: American Mathematical Society, 1965.
- [8] E.M. Lifshitz and L.P. Pitaevskii, *Physical Kinetics*, Nauka, Moscow, 1979.
- [9] N.N. Bogoliubov, *Physica*, 1960. V. 26. Suppl. P.S1: Proc. Inter. Congress on Many-Particle Problems. Utrecht, June. 1960; JINR Preprint E-568. Dubna, 1960.
- [10] N.N. Bogolyubov, *Statistical perurbation theory*, acc. MSU. Issue 77. Physics. 1945. Book 3. P. 74-100; N.N. Bogolyubov, Selected university lectures, State University Publishers, 2009.
- [11] Lifshitz, E.M. and Pitaevskii, L.P. (1978). *Theoretical Physics, vol. 9, Statistical Physics, part 2*, Nauka, Moscow.