# Classical-mechanical models without observable trajectories and the Dirac electron

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We construct non-Grassmann spinning-particle model which, by analogy with the quantum mechanics, does not admit the notion of a trajectory within the position space. The pseudo-classical character of the model allow us to to avoid the inconsistences arising in the quantum-mechanical interpretation of one-particle sector of the Dirac equation.

## I. INTRODUCTION

Non-abelian gauge groups play the crucial role in field theory as well as in the Standard model. In this work we observe that they imply the new possibilities when used in construction of finite-dimensional theories as well. We suggest and discuss the notion of pseudo-classical mechanics (pCM), the term which we refer to the models with the number of observable configuration-space variables less then the number of physical degrees of freedom. In other words, we consider the classical-mechanical models which, by analogy with the quantum mechanics, do not admit the notion of a trajectory within the position (i.e. configuration) space. So we expect that pCM turns out to be useful in description the quantum phenomena by (semi) classical methods [1-11]. Classical mechanics with such a strange property can be constructed on the basis of a singular Lagrangian with many-parametric group of local symmetries. As the examples of pCM we present the models invariant under non-abelian gauge group with two and three local parameters. The symmetries imply the functional ambiguity in solutions to equations of motion: besides the integration constants $c_i$, solution depends on an arbitrary functions $e_a(\tau)$, $x = f(\tau, c_i, e_a(\tau))$. According to the general theory of singular systems [12-14], the variables with ambiguous dynamics do not represent the observable quantities. So, when we are dealing with the locally-invariant theory, our first task is to find candidates for observables, which are variables with unambiguous dynamics. Equivalently, we can look for the gauge-invariant variables.

We start in Section 2 with a couple of toy models and show that, generally, it is impossible to construct the observables within the position variables only. It is worth noting that on the phase space there always is the well-defined notion of a trajectory [13]. In Section 3 we consider more realistic case, presenting the non-Grassmann model of the Dirac electron. In Section 4 we show how the pseudo-classical character of the model allows us to solve the problems arising [2, 3, 18] when we try to apply the methods of relativistic quantum mechanics to one-particle sector of the Dirac equation.

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## II. TOY MODELS

One of the local symmetries which will be presented in our models is the reparametrization invariance. So, we first outline the reparametrization invariant formulation of a relativistic particle.

The motion of a particle in the special-relativity theory can be described starting from the three-dimensional action $-mc\int dt\sqrt{c^2-(\frac{dx^i}{dt})^2}$. The problem here is that the Lorentz transformations, $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, act on the physical dynamical variables $x^i(t)$ in a higher nonlinear way. To improve this, we pass from the three-dimensional to four-dimensional formulation. Introducing the parametric representation $x^\mu(\tau) = (ct(\tau), x^i(\tau))$ for the trajectory $x^i(t)$, the particle can be described by the Lagrangian action

$$S = \int d\tau(\frac{1}{2e}(\dot{x}^\mu)^2 - \frac{e}{2}m^2c^2). \qquad (1)$$

The corresponding Hamiltonian action reads

$$S_H = \int d\tau p_\mu\dot{x}^\mu + p_e\dot{e} - \frac{1}{2}e(p^2+m^2c^2) - \lambda_e\pi_e, \qquad (2)$$

where $\lambda_e(\tau)$ stands for the Lagrangian multiplier of the primary constraint $\pi_e = 0$. Variation of the functional implies the Hamiltonian equations

$$\dot{e} = \lambda_e, \qquad \dot{\pi}_e = 0, \qquad \dot{x}^\mu = ep^\mu, \qquad \dot{p}_\mu = 0, \qquad (3)$$

as well as the constraints $\pi_e = 0$, $p^2 + m^2c^2 = 0$. We note that the variable $\lambda_e(\tau)$ cannot be determined neither with the constraints nor with the dynamical equations. As a consequence (see the first from Eqs. (3)), the variable $e$ turns out to be the arbitrary function as well. Since $e(\tau)$ enter into the equation for $x^\mu$, its general solution contains, besides the arbitrary integration constants, the arbitrary function $e(\tau)$. Hence the only unambiguous among the initial variables are $p^\mu$ and $\pi_e$, see Eqs. (3). $x^\mu$ has one-parameter ambiguity due to $e_2$.

The ambiguity reflects the freedom in the choice of parametrization for the particle trajectory

$$\begin{array}{l}\tau\to\tau'=\tau+\alpha,\\ x^\mu(\tau)\to x'^\mu(\tau')=x^\mu(\tau),\\ e(\tau)\to e'(\tau')=(1+\dot{\alpha})e(\tau).\end{array} \quad \text{then} \quad \begin{array}{r}\delta x^\mu=\alpha\dot{x}^\mu,\\ \delta e=(\alpha e)^{\cdot}.\end{array} \qquad (4)$$

The action (1) turns out to be invariant under the reparametrizations.

By construction, the expression for the physical trajectory $x^i(t)$ is obtained resolving the equation $x^0 = x^0(\tau)$ with respect to $\tau$, $\tau = \tau(x^0)$, then $x^i(t) \equiv x^i(\tau(x^0))$. Using the expression

$$\frac{df}{dx^0} = \frac{\dot{f}(\tau)}{\dot{x}^0(\tau)}, \tag{5}$$

for derivative of a function given in the parametric form, we obtain

$$\frac{dx^i}{dt} = c\frac{\dot{x}^i}{\dot{x}^0} = c\frac{p^i}{\sqrt{\vec{p}^2 + m^2c^2}}, \tag{6}$$

Eq. (6) coincides with that of three-dimensional formulation. As it should be, the physical coordinate $x^i(t)$ has unambiguous evolution.

**Toy model which admits the position-space trajectories.** Consider the Lagrangian action

$$S = \int d\tau \frac{1}{2e_1}(\dot{x}^\mu - e_2 x^\mu)^2. \tag{7}$$

It is written on the configuration space $x^\mu$, $e_1$ and $e_2$, the Minkowski metric is $\eta^{\mu\nu} = (-,+,+,+)$. The action is invariant under the reparametrizations, $\delta x^\mu = \alpha \dot{x}^\mu$, $\delta e_1 = \dot{\alpha} e_1 - \alpha \dot{e}_1$, $\delta e_2 = (\alpha e_2)\dot{}$, as well as under the following transformations with the parameter $\beta(\tau)$

$$\delta x^\mu = \beta x^\mu, \qquad \delta e_1 = 2\beta e_1, \qquad \delta e_2 = \dot{\beta}. \tag{8}$$

The transformations form non-abelian group, $[\delta_\alpha, \delta_\beta] = \delta_{\tilde{\beta}}$, $\tilde{\beta} = -\alpha\dot{\beta}$. Each local symmetry removes two degrees of freedom [22], so the number of the configuration-space observables is equal to 2.

Hamiltonian of the theory (7) is

$$H = \frac{e_1}{2}p^2 + e_2(px). \tag{9}$$

It implies the Hamiltonian equations (in what follows, we omit equations for the auxiliary variables $e_k$, $\pi_{ek}$ which are not relevant to our discussion)

$$\dot{x}^\mu = e_1 p^\mu + e_2 x^\mu, \qquad \dot{p}^\mu = -e_2 p^\mu, \tag{10}$$

as well as the constraints

$$p^2 = 0, \qquad (px) = 0. \tag{11}$$

The equation for $x^\mu$ has two-parametric ambiguity due to $e_1$ and $e_2$, while that for $p^\mu$ has one-parametric ambiguity. We construct the variables

$$\tilde{x}^\mu = \frac{x^\mu}{\sqrt{x^2}}, \qquad \tilde{p}^\mu = \sqrt{x^2}p^\mu. \tag{12}$$

Their equations read

$$\dot{\tilde{x}}^\mu = e_1\frac{p^\mu}{\sqrt{x^2}}, \qquad \dot{\tilde{p}}^\mu = 0. \tag{13}$$

They can be compared with Eqs. (3). The constraints (11) acquires the form $\tilde{p}^2 = 0$, $(\tilde{p}\tilde{x}) = 0$. Note that $\tilde{x}$ is $\beta$-invariant variable. So the ambiguity presented in Eq. (13) is due to the reparametrization symmetry. In accordance with this observation, we assume that the functions $\tilde{x}^\mu(\tau)$ and $\tilde{p}^\mu(\tau)$ represent the reparametrization-invariant variables $\tilde{x}^i(t)$ and $\tilde{p}^\mu(t)$ in the parametric form. Their equations of motion read

$$\frac{d\tilde{x}^i}{dt} = c\frac{p^i}{p^0} \equiv c\frac{\tilde{p}^i}{\tilde{p}^0}, \qquad \frac{d\tilde{p}^\mu}{dt} = 0. \tag{14}$$

Since they are unambiguous, the variables $\tilde{x}^i(t)$ and $\tilde{p}^\mu(t)$ are candidates for the observables.

By construction, $\tilde{x}^\mu$ obey the identity $\tilde{x}^\mu\tilde{x}_\mu = 1$. So only three of them can be taken as coordinates of the configuration space. Adding the variable $\sigma = \frac{1}{\sqrt{x^2}}$ to the set $\tilde{x}^i$, we obtain a coordinate system. As the two independent observables we can take the gauge-invariant variables $\tilde{x}^1(t)$ and $\tilde{x}^2(t)$. Hence, the present model admits the observable trajectories constructed within the position space, see Eq. (12).

To conclude with, we point out that Poisson brackets of the Lorentz-covariant observables (12) generate the non-commutative algebra

$$\{\tilde{x}^\mu, \tilde{p}^\nu\} = N^{\mu\nu}(\tilde{x}), \ \{\tilde{x}^\mu, \tilde{x}^\nu\} = 0, \ \{\tilde{p}^\mu, \tilde{p}^\nu\} = \tilde{p}^{[\mu}\tilde{x}^{\nu]} \tag{15}$$

Here and below we denote

$$N^{\mu\nu}(a) \equiv \eta^{\mu\nu} - \frac{a^\mu a^\nu}{a^2}. \tag{16}$$

**Toy model without the position-space trajectories.** Consider the following Lagrangian action written for the variables $x^\mu$, $\omega^\mu$, $e_1$ and $e_2$

$$S = \int d\tau \frac{1}{2(e_1 - e_2^2)}\left[(Dx)^2 + 2e_2(Dx\dot{\omega}) + e_1\dot{\omega}^2\right] - \frac{e_1}{2}m^2c^2 + \frac{1}{2}\omega^2. \tag{17}$$

We have denoted $Dx^\mu \equiv \dot{x}^\mu - e_2\omega^\mu$. The non-abelian gauge group is composed by reparametrizations as well as by the following transformations with the parameter $\beta(\tau)$:

$$\delta x^\mu = \frac{\beta}{e_1 - e_2^2}(Dx^\mu + e_2\dot{\omega}^\mu), \qquad \delta e_1 = \dot{\beta}. \tag{18}$$

This implies that the number of physical degrees of freedom on configuration (phase) space is equal to 6 (12).

Denoting conjugate momenta of $x$, $\omega$ by $p$, $\pi$, the Hamiltonian of the theory (17) reads

$$H = \frac{1}{2}\pi^2 - \frac{1}{2}\omega^2 + \frac{1}{2}e_1(p^2 + m^2c^2) + e_2 p_\mu(\omega^\mu - \pi^\mu) \tag{19}$$

It implies the Hamiltonian equations

$$\dot{x}^\mu = e_1 p^\mu + e_2(\omega^\mu - \pi^\mu), \quad \dot{p}^\mu = 0,$$

$$\dot{\omega}^\mu = \pi^\mu - e_2 p^\mu, \quad \dot{\pi}^\mu = \omega^\mu - e_2 p^\mu. \tag{20}$$

as well as the first-class constraints

$$p^2 + m^2c^2 = 0, \qquad (p, \omega - \pi) = 0. \tag{21}$$

The equation for $x$ has two-parametric ambiguity due to $e_1$ and $e_2$, while those for $\omega$ and $\pi$ have one-parametric ambiguity.

Taking into account the first-class constraints, we could expect 6 observable dynamical variables on the configuration space. However, it is easy to see that any configuration-space quantity $a^\mu(x,\omega)$ with one-parametric ambiguity is proportional to $\omega^\mu$. Similarly to the previous model, this can be used to construct only three unambiguous dynamical variables. As the six-dimensional configuration space can not be spanned with the unambiguous variables, the model represents an example of pseudo-classical mechanics.

On the phase space we can construct various variables with one-parametric ambiguity due to $e_2$

$$\tilde{x}^\mu = x^\mu - \frac{(px)}{p^2}p^\mu, \qquad \dot{\tilde{x}}^\mu = e_2(\omega^\mu - \pi^\mu), \tag{22}$$

$$\tilde{p}^\mu = p^\mu, \qquad \dot{\tilde{p}}^\mu = 0, \tag{23}$$

$$\tilde{\omega}^\mu = \frac{(\omega+\pi, p)}{2p^2}(\omega^\mu - \pi^\mu), \quad \dot{\tilde{\omega}}^\mu = -e_2(\omega^\mu - \pi^\mu), \tag{24}$$

$$\tilde{\pi}^\mu = \frac{\omega^\mu + \pi^\mu}{\sqrt{(\omega+\pi)^2}}, \qquad \dot{\tilde{\pi}}^\mu = -\frac{2e_2 N^{\mu\nu}(\omega+\pi)p_\nu}{\sqrt{(\omega+\pi)^2}}. \tag{25}$$

$$J^{\mu\nu} = \omega^\mu\pi^\nu - \omega^\nu\pi^\mu, \qquad \dot{J}^{\mu\nu} = e_2 p^{[\mu}(\omega - \pi)^{\nu]}. \tag{26}$$

The constraints (21) acquires the form $\tilde{p}^2 + m^2 = 0$, $(\tilde{p}\tilde{\omega}) = 0$. The new variables are invariants of the $\beta$-transformation[1]. So the ambiguity presented in Eqs. (22)-(26) is due to the reparametrization symmetry. Similarly to the case of relativistic particle, we assume that the functions $\tilde{x}^\mu(\tau)$, $\tilde{p}^\mu(\tau)$, $\tilde{\omega}^\mu(\tau)$, $\tilde{\pi}^\mu(\tau)$ and $J^{\mu\nu}(\tau)$ represent the physical variables $\tilde{x}^i(t)$, $\tilde{p}^\mu(t), \ldots$ in the parametric form. According to Eq. (5), dynamics of the physical variables is unambiguous.

Poisson-bracket algebra of the Lorentz-covariant observables is highly noncommutative, the nonvanishing brackets are

$$\{\tilde{x}^\mu, \tilde{x}^\nu\} = \frac{p^{[\mu}\tilde{x}^{\nu]}}{p^2}, \qquad \{\tilde{x}^\mu, p^\nu\} = N^{\mu\nu}(p),$$
$$\{\tilde{\omega}^\mu, \tilde{\omega}^\nu\} = \frac{p^{[\mu}\tilde{\omega}^{\nu]}}{p^2}, \quad \{\tilde{\omega}^\mu, \tilde{\pi}^\nu\} = \frac{(p\tilde{\pi})}{p^2}N^{\mu\nu}(\tilde{\pi}),$$
$$\{\tilde{x}^\mu, \tilde{\omega}^\nu\} = \frac{\tilde{\pi}^\mu\tilde{\omega}^\nu}{(p\tilde{\pi})} - \frac{p^\mu\tilde{\omega}^\nu}{p^2},$$
$$\{J^{\mu\nu}, J^{\alpha\beta}\} = \eta^{\mu\alpha}J^{\nu\beta} - \eta^{\mu\beta}J^{\nu\alpha} - \eta^{\nu\alpha}J^{\mu\beta} + \eta^{\nu\beta}J^{\mu\alpha},$$
$$\{J^{\mu\nu}, \tilde{\omega}^\alpha\} = \eta^{\alpha[\mu}\tilde{\omega}^{\nu]} + \frac{p^{[\mu}\tilde{\pi}^{\nu]}\tilde{\omega}^\alpha}{(p\tilde{\pi})},$$
$$\{J^{\mu\nu}, \tilde{\pi}^\alpha\} = \eta^{\alpha[\mu}\tilde{\pi}^{\nu]}. \tag{27}$$

The set of 12 independent observables of the phase-space can be selected as follows. We parameterize the initial space by the coordinates $x^0$, $\tilde{x}^i$, $\tilde{p}^\mu$, $\tilde{\omega}^\mu$, $S^i \equiv \epsilon^{ijk}J_{jk}$ and $\gamma = \sqrt{(\omega+\pi)^2}$. Dynamics of the theory is restricted on the surface $\tilde{p}^2 + m^2 = 0$, $(\tilde{p}\tilde{\omega}) = 0$ which is invariant under the action of gauge group[2]. The surface can be parameterized by $x^0$, $\tilde{x}^i$, $\tilde{p}^i$, $\tilde{\omega}^i$, $S^i$ and $\gamma$. The corresponding dynamical variables $\tilde{x}^i(t)$, $\tilde{p}^i(t)$, $\tilde{\omega}^i(t)$ and $S^i(t)$ have unambiguous dynamics. Hence we can take them as the independent observables.

## III. NON-GRASSMANN MECHANICAL MODEL OF THE DIRAC ELECTRON

As a more realistic example of pCM, we discuss the spinning-particle model suggested in the recent work [15]. The configuration space of the model consist of the dynamical variables $Q^\alpha(\tau) = (x^\mu, \omega^\nu, \omega^5)$ as well as the auxiliary variables $e_l$, $l = 1,2,3,4$. $x^\mu$ are coordinates of the Minkowski space with the metric $\eta^{\mu\nu} = (-,+,+,+)$. The spin-space $\omega^A = (\omega^\mu, \omega^5)$ is equipped with $SO(2,3)$-metric $\eta^{AB} = (-,+,+,+,-)$. Consider the Poincare-invariant Lagrangian

$$L = \frac{1}{2}G_{\alpha\beta}\dot{Q}^\alpha\dot{Q}^\beta - \frac{e_4}{2}\omega^A\omega_A - \frac{e_l}{2}a_l. \tag{28}$$

It has been denoted $a_1 = m^2c^2, a_2 = mc\hbar$, and $a_3$, $a_4$ are real numbers. In what follows, we discuss the free theory. Interaction with an external electromagnetic field will be discussed at the end of Section 4. The kinetic term looks like that of a free particle moving on the curved nine-dimensional space with the metric

$$G_{\alpha\beta} = \begin{pmatrix} e_3 G_{\mu\nu} & -e_2\omega^5 G_{\mu\nu} & \frac{e_2}{A}\omega_\mu \\ -e_2\omega^5 G_{\mu\nu} & e_1 G_{\mu\nu} + \frac{e_2^2\omega_\mu\omega_\nu}{e_3 A} & -\frac{e_2^2\omega^5}{e_3 A}\omega_\mu \\ \frac{e_2}{A}\omega_\nu & -\frac{e_2^2\omega^5}{e_3 A}\omega_\nu & -\frac{B}{e_3 A} \end{pmatrix} \tag{29}$$

We have denoted $G_{\mu\nu} = \frac{1}{B}[\eta_{\mu\nu} - \frac{e_2^2\omega_\mu\omega_\nu}{A}]$, $B = e_1e_3 - e_2^2(\omega^5)^2$ and $A = B + e_2^2(\omega^\mu)^2$.

We introduce the abbreviation

$$Dx^\mu \equiv \dot{x}^\mu - \frac{e_2}{e_3}(\omega^5\dot{\omega}^\mu - \omega^\mu\dot{\omega}^5), \tag{30}$$

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[1] Due to the identities $\tilde{x}^\mu\tilde{p}_\mu = 0$, $\tilde{\pi}^\mu\tilde{\pi}_\mu = 1$ and $\epsilon^{ijk}J_{ij}J_{0k}$, not all among them are independent.

[2] We use the phase-space form of reparametrizations, $\delta\tilde{\omega}^\mu = -\alpha e_2(\omega^\mu - \pi^\mu)$, $\delta p^\mu = 0$, see [15] for the details.

then manifest form of the Lagrangian (28) is

$$L = \frac{e_3}{2} G_{\mu\nu} Dx^\mu Dx^\nu + \frac{1}{2e_3}\dot{\omega}^A \dot{\omega}_A - \frac{e_4}{2}\omega^A\omega_A - \frac{e_l}{2}a_l. \quad (31)$$

The Lagrangian is invariant under a three-parametric group of local symmetries. One of them is the reparametrization symmetry. Besides, there are two more symmetries with the local parameters $\beta(\tau)$, $\gamma(\tau)$

$$\delta_\beta x^\mu = \beta p^\mu, \quad \delta_\beta e_1 = \dot{\beta}; \quad (32)$$

$$\delta_\gamma \omega^A = \gamma e_3 \pi^A, \quad \delta_\gamma \pi^A = -\gamma e_4 \omega^A, \\ \delta_\gamma e_3 = (\gamma e_3)\dot{}, \quad \delta_\gamma e_4 = (\gamma e_4)\dot{}. \quad (33)$$

Here $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$, $\pi_A = \frac{\partial L}{\partial \dot{\omega}^A}$. While the Lagrangian has a complicated form, this leads to transparent Hamiltonian

$$H = \frac{e_1}{2}(p^2 + m^2c^2) + \frac{e_2}{2}(p_\mu J^{5\mu} + mc\hbar) + \\ \frac{e_3}{2}(\pi^A\pi_A + a_3) + \frac{e_4}{2}(\omega^A\omega_A + a_4) + \lambda_{ea}\pi_{ea}, \quad (34)$$

where

$$J^{5\mu} = 2(\omega^5\pi^\mu - \omega^\mu\pi^5). \quad (35)$$

If we omit the spin-space coordinates, $\omega^A = \pi_A = 0$, the Hamiltonian reduces to that of the spinless particle, see (2).

The Hamiltonian implies the constraints

$$\omega^A\omega_A + a_4 = 0, \qquad \pi^A\omega_A = 0; \quad (36)$$

$$p^2 + m^2c^2 = 0, \qquad \pi^A\pi_A + a_3 = 0; \quad (37)$$

$$p_\mu J^{5\mu} + mc\hbar = 0. \quad (38)$$

The first one states that configuration space of spin is anti-de Sitter space. The constraints (36) form the second-class pair while those of Eqs. (37), (38) are the first-class constraints. The constraint (38), being imposed on the state vector, leads to the Dirac equation[3], $(\gamma^\mu\hat{p}_\mu + mc)\Psi = 0$ (see [15] for the details).

Hamiltonian equations of the theory read

$$\dot{x}^\mu = e_1 p^\mu + \frac{1}{2}e_2 J^{5\mu}, \qquad \dot{p}^\mu = 0; \\ \dot{\omega}^\mu = e_3\pi^\mu + e_2\omega^5 p^\mu, \qquad \dot{\pi}^\mu = e_2\pi^5 p^\mu - \frac{a_3}{a_4}e_3\omega^\mu; \\ \dot{\omega}^5 = e_3\pi^5 + e_2(p\omega), \qquad \dot{\pi}^5 = e_2(p\pi) - \frac{a_3}{a_4}e_3\omega^5. \quad (39)$$

The only unambiguous variable is $p^\mu$. The configuration space variables $x^\mu$, $\omega^A$ have two-parametric ambiguity.

Let us compute the total number of physical degrees of freedom. Omitting the auxiliary variables and the corresponding constraints, we have 18 phase-space variables $x^\mu$, $p_\mu$, $\omega^A$, $\pi_A$ subject to the constraints (36)- (38). Taking into account that each second-class constraint rules out one variable, whereas each first-class constraint rules out two variables, the number of physical degrees of freedom on the phase space is $18-(2+2\times3) = 10$. Hence we could expect five observables on the configuration space. However, using the configuration-space variables only, it is impossible to construct five unambiguous quantities. Thus, once again we have an example of pseudo-classical mechanics.

Let us discuss physical sector of the phase space. Short inspection of the equations of motion allow us to construct the Lorentz-covariant variables with one-parameter ambiguity. They are the five-dimensional angular-momentum tensor[4] $J^{AB} = 2\omega^{[A}\pi^{B]}$, this obeys

$$\dot{J}^{5\mu} = -e_2 J^{\mu\nu}p_\nu, \qquad \dot{J}^{\mu\nu} = -e_2 J^{5[\mu}p^{\nu]}; \quad (40)$$

as well as the position variable

$$\tilde{x}^\mu = x^\mu + \frac{1}{2p^2}J^{\mu\nu}p_\nu, \qquad \dot{\tilde{x}}^\mu = \tilde{e}p^\mu. \quad (41)$$

It has been denoted $\tilde{e} \equiv e_1 + \frac{\hbar e_2}{2mc}$. So the reparametrization-invariant variable $\tilde{x}^i(t)$ has deterministic evolution:

$$\frac{d\tilde{x}^i}{dt} = c\frac{\dot{\tilde{x}}^i}{\dot{\tilde{x}}^0} = c\frac{p^i}{p^0}. \quad (42)$$

As the classical four-dimensional spin vector, we take the Pauli-Lubanski vector which has no precession in the free theory

$$S^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}p_\nu J_{\alpha\beta}, \qquad \dot{S}^\mu = 0 \quad (43)$$

In the rest frame $p^\mu = (mc, 0, 0, 0)$, it reduces to the three-dimensional rotation generator, $S^0 = 0$, $S^i = \frac{1}{2}mc\epsilon^{ijk}S_{jk}$, as is expected in the non-relativistic limit.

We point out that the second term in Eq. (41) has the structure typical for non-commutative extensions of the usual mechanics, see [23].

## IV. CONCLUSION. PSEUDO-CLASSICAL MECHANICS AND THE CLASSICAL LIMIT OF THE DIRAC EQUATION

Although the true understanding of spin is achieved in the framework of quantum electrodynamics, a lot of

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[3] The present model implies both the Dirac equation and the mass-shell condition $p^2+m^2c^2=0$. The model without the mass-shell condition have been discussed in [16, 17]. This shows the same undesirable properties as those of Dirac equation in the classical limit [2, 3].

[4] Note that the constraints (36) and (37) fix the value of Casimir operators of $SO(2,3)$ group. Besides, they guarantee that $J^{5\mu}$ is the time-like vector $(J^{5\mu})^2 = -4(a_3(\omega^5)^2 + a_4(\pi^5)^2) < 0$, for positive values of $a_3$, $a_4$, see [15] for the details.

efforts has been spent in attempts to construct the mechanical model of a spinning electron, see [1-11, 15-17] and references therein. The Dirac spinor $\Psi$ can be used to construct the four-dimensional current vector, $\bar{\Psi}\gamma^\mu\Psi$, which preserves for solutions to the Dirac equation, $\partial_\mu(\bar{\Psi}\gamma^\mu\Psi) = 0$. Hence its null-component, $\Psi^\dagger\Psi \geq 0$, admits the probabilistic interpretation, and we expect that one-particle sector of the Dirac equation could be described in the framework of relativistic quantum mechanics (RQM).

However, it is well known that adopting the RQM interpretation, we arrive at the rather strange and controversial picture [2, 3, 18]. To remind this, we use the Dirac matrices $\alpha^i$ and $\beta$, to represent the Dirac equation in the form of the Schrödinger one

$$i\hbar\partial_t\Psi = \hat{H}\Psi, \qquad \hat{H} = c\alpha^i\hat{p}_i + mc^2\beta. \tag{44}$$

Then $\hat{H}$ may be interpreted as the Hamiltonian. If we pass from the Schrödinger to Heisenberg picture, time derivative of an operator $a$ is $i\hbar\dot{a} = [a, H]$. For the basic operators of the Dirac theory we obtain

$$\dot{x}_i = c\alpha_i, \qquad i\hbar\dot{\alpha}_i = 2(cp_i - H\alpha_i), \qquad \dot{p}_i = 0. \tag{45}$$

Below we enumerate the inconsistences arising in RQM interpretation of these equations and show how our pCM allow us to avoid them.

*The wrong balance of the number of degrees of freedom.* Assuming $x$ as the position operator, the first equation from (45) implies that the operator $c\alpha^i$ represents velocity of the particle. Then the physical meaning of the operator $p^i$ became rather obscure in both the semiclassical and the RQM framework. Using the quantum-field-theory arguments, Foldy and Wouthuysen motivated [6] that the basic operator $x$ appeared in the Dirac equation does not correspond to the observable quantity. Then they constructed the position operator $\mathbf{X}^i$ with reasonable properties.

Our model supports the Foldy-Wouthuysen suggestion. Indeed, we observe that the variable $x^\mu$ is not gauge-invariant quantity in our model, so it is not an observable. The only observable variable of position is $\tilde{x}^i(t)$. According to Eq. (42), $p^i$ determines its velocity. $\tilde{x}^\mu(\tau)$ written in Eq. (41) represents the Lorentz-covariant analog of the operator $\mathbf{X}^i$ in the classical theory. We also point out that the Foldy-Wouthuysen transition $x \to \mathbf{X}$ corresponds in pCM to the transition from gauge-non-invariant to gauge-invariant variable.

*Zitterbewegung.* The equations (45) can be solved, with the result for $x^i(t)$ being [2, 3] $x^i = a^i + bp^it + c^i\exp(-\frac{2iH}{\hbar}t)$. The last term on the r.h.s. of this equation states that the free electron experiences rapid oscillations with higher frequency $\frac{2H}{\hbar} \sim \frac{2mc^2}{\hbar}$. It is often assumed that *Zitterbewegung* represents the physically observable motion of a real particle [18]. The analogous systems that are described by a Dirac-type equation and simulate *Zitterbewegung* are under intensive study in different physical set-ups, including graphene, trapped ions, photonic lattices and ultracold atoms, see [19] and references therein.

Our model prohibits the undesirable *Zitterbewegung*, since this represents dynamics of the unobservable variable $x^\mu$. The observable variable $\tilde{x}^i$ moves along a straight line, see Eq. (42).

*Velocity of an electron.* Since the velocity operator $c\alpha^i$ has eigenvalues $\pm c$, we conclude that a measurement of a component of the velocity of a free electron is certain to lead to the result $\pm c$.

In our model, the conjugate momentum $p^\mu$ determines velocity of the physical coordinate $\tilde{x}^\mu$, see Eqs. (41), (42). Then the mass-shell condition (37) guarantees that the particle cannot exceed the speed of light.

*Bargmann-Michel-Telegdi (BMT) equations.* In their seminal work [7], Bargmann, Michel and Telegdi suggested the relativistic equations for the classical trajectories and spin precession in uniform fields. The equations practically exactly reproduces the spin dynamics of polarized beams and agreed with the calculations based on the Dirac theory. While the canonical quantization of BMT model is not known, there is certain relation between the two schemes. Namely, the first term of the WKB solution to the Dirac equation can be used to construct the quantities which obey the BMT equations [20, 21].

We have seen above that performing the canonical quantization of our model in the initial variables we arrive at the Dirac equation. Now we show that physical variables of the model obey the BMT equations.

We take Hamiltonian of interacting theory in the form

$$\begin{aligned} H = \frac{e_1}{2}(\mathcal{P}^2 + \frac{e}{2c}F_{\mu\nu}J^{\mu\nu} + m^2c^2) + \\ \frac{e_2}{2}(\mathcal{P}_\mu J^{5\mu} + mc\hbar) + \\ \frac{e_3}{2}(\pi^A\pi_A + a_3) + \frac{e_4}{2}(\omega^A\omega_A + a_4) + \lambda_{ea}\pi_{ea}, \end{aligned} \tag{46}$$

where $\mathcal{P}_\mu = p_\mu + \frac{e}{c}A_\mu$ is the mechanical momentum. This does not break the local symmetries presented in the model for the case of uniform electric and magnetic fields. However, the interaction deforms the physical sector: unambiguous variables of the free theory no longer remain unambiguous in the interacting theory. In particular, $\tilde{x}^\mu$, $p^\mu$ and $S^\mu$ have now the two-parametric ambiguity. Up to the order $O(\hbar^2)$, the quantities with one-parametric ambiguity turn out to be

$$\begin{aligned} &\mathbf{P}_\mu = \mathcal{P}^\mu - \frac{e}{2c\mathcal{P}^2}(FJ\mathcal{P})_\mu, \quad &&\dot{\mathbf{P}}_\mu = -\frac{e}{c}\tilde{e}F_{\mu\nu}\mathbf{P}^\nu; \\ &\mathbf{x}^\mu = x^\mu + \frac{1}{2\mathbf{P}^2}J^{\mu\nu}\mathbf{P}_\nu, &&\dot{\mathbf{x}}^\mu = \tilde{e}\mathbf{P}^\mu; \\ &\mathbf{S}^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\mathbf{P}_\nu J_{\alpha\beta}, &&\dot{\mathbf{S}}^\mu = -\frac{e}{c}\tilde{e}F^\mu{}_\nu\mathbf{S}^\nu. \end{aligned} \tag{47}$$

It has been denoted $\tilde{e} = e_1 - \frac{mc\hbar}{2\mathcal{P}^2}e_2$. The corresponding reparametrization-invariant variables obey the BMT equations with $g = 2$

$$\frac{d}{dt}\mathbf{x}^i = c\frac{\mathbf{P}^i}{\mathbf{P}^0}, \quad \frac{d}{dt}\mathbf{P}^i = -\frac{e}{\mathbf{P}^0}F^{i\nu}\mathbf{P}_\nu,$$

$$\frac{d}{dt}\mathbf{S}^\mu = -\frac{e}{\mathbf{P}^0}F^{\mu\nu}\mathbf{S}_\nu. \qquad (48)$$

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