Fractal Space-Time and the Dynamic Generation of Mass Scales in Field Theory

Ervin Goldfain

Photonics CoE, Welch Allyn Inc., Skaneateles Falls, NY 13153, USA

Abstract

As of today, the mechanism underlying the generation of mass scales in field theory remains elusive. Here we show how the concept of fractal space-time having minimal deviations from four-dimensionality (the so-called *minimal fractal manifold* defined through ε = 4 - D, with ε << 1) can naturally account for the onset of these scales. A counterintuitive outcome of this analysis is the deep link between the minimal fractal manifold and the holographic principle.

Key words: Fractal Space-time, Dimensional flow, Dimensional reduction, Electroweak scale, QCD scale, Holographic principle, Holographic bound.

1. Introduction and motivation

One of the many unsettled questions raised by field theory revolves around the vast hierarchy of scales in Nature [2-3, 32]. A large numerical disparity exists between the Planck scale (M_{Pl}), the electroweak scale (M_{EW}), the hadronization scale of quantum chromodynamics (Λ_{QCD}) and the cosmological constant scale ($\Lambda_{cc}^{1/4}$, with Λ_{cc} expressed as energy density in 3+1 dimensions). The goal of this work is to suggest that the answer to this question may lie in the fractal geometry of space-time near or above M_{EW}.

It has been long known that perturbative quantum field theory (QFT) cannot provide a complete description of Nature since its formalism entails divergences at both ends of the energy spectrum

[1-3]. For instance, many textbooks emphasize that the singular behavior of momentum integrals in the ultraviolet (UV) sector arises from the poorly understood space-time structure at short distances [2-3]. Lattice field models handle infinities through discretization of the space-time continuum on a grid of spacing $"\Delta"$. This procedure naturally bounds the maximal momentum allowed to propagate through the lattice, namely,

$$p \le p_{\max} \sim (2\Delta)^{-1} \tag{1}$$

The downside of lattice models is that they generally fail to be either gauge or Poincaré invariant [1-4]. Restoring formal consistency is further enabled via the Renormalization Group program (RG) [2-3, 15]. RG regulates the *n*-th order momentum integrals of the generic form

$$I_n(p) = \int dp\, f(p^{2n}) \tag{2}$$

by either inserting an arbitrary momentum cutoff $0 < \Lambda \sim \Delta^{-1} < \infty$ *or* by continuously "deforming" the four-dimensional space-time via the dimensional parameter $\varepsilon = 4 - D$, $\varepsilon << 1$. The resulting theory is free from divergences and operates with a finite number of redefined physical parameters. Restoring the continuum space-time limit is done at the end by taking the limit $\Lambda \to \infty$ or $\varepsilon \to 0$.

Regularization techniques employed in RG are not independent from each other. The connection between dimensional and UV cutoff regularizations ($\Lambda = \Lambda_{UV}$) is given by [13, 15, 18]

$$\log \frac{\Lambda_{UV}^2}{\mu^2} = \frac{2}{\varepsilon} - \gamma_E + \log 4\pi + \frac{5}{6} \tag{3}$$

Here, γ_E stands for the Euler constant and μ for the observation (or "sliding") scale. It is more convenient to present (3) is a slightly different form, that is,

$$\varepsilon \sim \frac{1}{\log(\frac{\Lambda_{UV}^2}{\mu^2})} \tag{4}$$

If the numerical disparity between μ and Λ_{UV} is large enough, one can reasonably approximate ε as in

$$\varepsilon \sim (\frac{\mu}{\Lambda_{UV}})^2 \tag{5a}$$

Following [18, 31, 35], the far infrared (IR) scale of field theory set by the cosmological constant ($\Lambda_{cc}^{1/4}$), the electroweak scale (M_{EW}) and the far UV scale fixed by the Planck mass ($\Lambda_{UV} = M_{Pl}$) satisfy the constraint

$$\boxed{\frac{\Lambda_{cc}^{1/4}}{M_{EW}} = \frac{M_{EW}}{\Lambda_{UV}} = O(\varepsilon)} \tag{5b}$$

It is apparent from (4) or (5a) that the four-dimensional space-time continuum is recovered in either one of these limits:

a) $\Lambda_{UV} \to \infty$ and $0 < \mu << \Lambda_{UV}$,

b) $\Lambda_{UV} < \infty$ and $\mu \to 0$

However, both limits are disfavored by our current understanding of the far UV and the far IR boundaries of field theory (see e.g. [3]). Theory and experimental data alike tell us that the notions of infinite *or* zero energy are, strictly speaking, meaningless. This is to say that either infinite energies (point-like objects) or zero energy (infinite distance scales) are *unphysical idealizations*. Indeed, there is always a finite cutoff at both ends of either energy or energy density scale (far UV = Planck scale, far IR = finite radius of the observable Universe or the non-vanishing energy density of the vacuum set by cosmological constant). These observations are also consistent with the estimated infinitesimal (yet non-vanishing) photon mass, as highlighted in [23-24].

Reinforcing this viewpoint, some authors argue that the idea of smooth space-time stands in manifest conflict with the basic premises of quantum theory [11]. To confine an event within a region of extension Δ requires a momentum transfer on the order of Δ^{-1} which, in turn, generates a local gravitational field. If the density of momentum transfer is comparable in magnitude with the right hand side of Einstein's equation, the local curvature of space-time (~ R_0^{-2}) induced by this transfer is given by (in natural units, $\hbar = c = 1$)

$$R_0^{-2} \sim G_N \Delta^{-4} \tag{6}$$

However, collapse of the event within a short region of extent $\Delta = O(R_0)$ amounts to trapping outgoing light signals and preventing direct observation.

All these considerations invariably point to the following challenge: on the one hand, a continuum model of space-time on or above M_{EW} serves as an effective paradigm that is likely to fail at large probing energies. Yet on the other, any discrete model of space-time typically violates Poincaré or gauge symmetries. It seems only natural, in this context, to take a fresh look at (4) and (5) and appreciate the message it conveys: if either Λ_{UV} stays finite *or* ε << 1 is arbitrarily small but non-vanishing, space-time dimensionality becomes a non-integer arbitrarily close to four. Stated differently, in the neighborhood of M_{EW}, conventional space-time turns into a *minimal fractal manifold* (MFM) [13, 15-18].

On closer examination, this finding is hinted by a number of alternative theoretical arguments:

a) It is well known that the principle of *general covariance* lies at the core of classical relativistic field theory. This principle asserts that all physical laws must take the same mathematical form regardless of the coordinate system used by observers in arbitrary relative motion. In different words, general covariance means invariance of physical laws under all possible coordinate transformations and is typically formulated in terms of tensor fields. An implicit assumption of general covariance is that any coordinate transformation and its inverse are *smooth* functions that can be differentiated arbitrarily many times. However, as it is also known, there is a plethora of non-differentiable curves and surfaces in Nature, as repeatedly discovered since the introduction of fractal geometry in 1983 [29, 31]. The unavoidable conclusion is that relativistic field theory assigns a preferential status to differentiable transformations and the smooth geometry of space-time, which is at odds with the very spirit of general covariance.

b) On the mathematical front, significant effort was recently invested in the development of *q*-deformed Lie algebras, non-commutative field theory, quantum groups, fractional field theory and its relationship to the MFM [1, 5-6, 12, 27-28, 31]. It is instructive to note that all these contributions appear to be directly or indirectly related to fractal geometry [13, 31]. Moreover, the condition ε << 1, defined within the framework of MFM, is the sole sensible setting where fractal geometry asymptotically approaches all consistency requirements mandated by QFT and the Standard Model [16, 30].

c) Demanding that phenomena associated with gravitational collapse follow the postulates of quantum theory implies that the world is no longer four-dimensional near M_{Pl}. This statement has lately received considerable attention and forms the basis for *dimensional reduction* and for the *holographic principle* of Quantum Gravity theories [10, 25-28, 32-35]. If we accept that the four-dimensional continuum is an emergent property of the electroweak scale and below ($\mu <<$ M_{EW}), the holographic principle implies that space-time dimensionality evolves with the energy scale between M_{EW}, where ε << 1, and M_{Pl}, where space is expected to become two-dimensional ($\varepsilon = O(1)$) (27-28, 34-35).

Our paper is organized as follows: next section introduces the concept of holographic bound and derives the relationship involving the IR and UV cutoffs of field theory. Building on these premises, section 3 presents a comparison between mass scales estimated using our approach and their currently known values.

2. The holographic bound

Consider an effective QFT confined to a space-time region with characteristic length scale L and assume that the theory makes valid predictions up to an UV cutoff scale Λ_{UV} >> L^{-1}. It can be shown that the entropy associated with this effective QFT takes the form [10]

$$S \sim \Lambda_{UV}^3 L^3 \tag{7}$$

To understand the significance of (7), consider an ensemble of fermions living on a periodic space lattice with characteristic size L and period Λ_{UV}^{-1}. One finds that (7) simply follows from counting the number of occupied states for this system, which turns out to be $N = 2^{(L\Lambda_{UV})^3}$ [10]. The holographic principle stipulates that (7) must not exceed the corresponding black hole entropy S_{BH}, that is,

$$L^3\Lambda_{UV}^3 \leq S_{BH} = \frac{A_{BH}}{4l_{Pl}^2} = \pi R^2 M_{Pl}^2 \tag{8}$$

in which A_{BH} is the area of the spherical event horizon of radius R. Introducing a new reference length scale Δ defined as

$$\frac{\Delta}{\pi} = \frac{L^3}{\pi R^2} \tag{9}$$

leads to the condition

$$\Delta \leq \pi \Lambda_{UV}^{-3} M_{Pl}^2 \tag{10}$$

On the other hand, since the maximum energy density in a QFT bounded by the UV cutoff is Λ_{UV}^4, the holography bound (8) leads to [7-8]

$$\Lambda_{UV}^4 \sim \frac{(\pi^{-1}\Delta)\, M_{Pl}^2}{(\pi^{-1}\Delta)^3} = \pi^2 \frac{M_{Pl}^2}{\Delta^2} \Rightarrow \Lambda_{UV}^2 \sim \pi \frac{M_{Pl}}{\Delta} \tag{11}$$

Since the IR cutoff is fixed by $\Lambda_{IR} = \Delta^{-1}$, combined use of (10) and (11) yields the scaling behavior

$$\boxed{\frac{\Lambda_{IR}}{\Lambda_{UV}} \sim \frac{\Lambda_{UV}}{\pi M_{Pl}}} \tag{12}$$

Although conventional wisdom suggests that the Standard Model retains its validity all the way up in the far UV sector of particle physics, there are indications that it may break at a scale that is at least an order of magnitude lower than M_{Pl}, that is, $\Lambda'_{UV} < M_{Pl}$ [14]. Relation (12) may be conveniently reformulated at $\Lambda'_{UV} > \Lambda_{UV}$ as in

$$\frac{\Lambda_{UV}}{\pi M_{Pl}} = \frac{\Lambda_{UV}}{\pi \Lambda'_{UV}} \frac{\Lambda'_{UV}}{M_{Pl}} \tag{13}$$

such that

$$\frac{M_{Pl}}{\Lambda'_{UV}} \frac{\Lambda_{IR}}{\Lambda_{UV}} \sim \frac{\Lambda_{UV}}{\pi \Lambda'_{UV}} \tag{14}$$

or

$$\boxed{\frac{\Lambda'_{IR}}{\Lambda_{UV}} \sim \frac{\Lambda_{UV}}{\pi \Lambda'_{UV}}} \tag{15}$$

in which $\Lambda'_{IR} > \Lambda_{IR}$ is a new IR scale given by

$$\Lambda'_{IR} = \frac{M_{Pl} \Lambda_{IR}}{\Lambda'_{UV}} \tag{16}$$

A glance at (5b), (12) and (15) reveals deep similarities between the holographic principle and the minimal fractal manifold (MFM). All represent scaling relations mixing and constraining largely separated mass scales. They form the basis for numerical estimates derived in the next section.

3. Numerical estimates

Tab. 1 displays currently known values for the representative scales of QFT and classical field theory. The electroweak scale (M_{EW}) is set by the vacuum expectation value of the Higgs boson, the far UV scale is set by either Planck mass (M_{Pl}) or the unification scale (M_{GUT}). The near UV cutoff is assumed to be close to the so-called Cohen-Kaplan threshold ($\Lambda_{CK} \sim 10^2$ TeV), according to [7-8, 19-21].

Scale	Name	Magnitude
$\Lambda_{IR} = \Lambda_{cc}^{1/4}$	Cosmological constant scale	$\leq \sim 10^{-3}$ eV
$\Lambda'_{IR} = \Lambda_{QCD}$	QCD scale	~ 200 MeV
$\Lambda_{UV} = M_{EW}$	EW scale	~ 246 GeV
$\Lambda'_{UV} = \Lambda_{CK}$	UV cutoff	$\sim 10^2$ TeV
M_{GUT}	GUT scale	$\sim 10^{16}$ GeV
M_{Pl}	Planck scale	$\sim 10^{19}$ GeV

Tab. 1: The spectrum of mass scales in field theory

Tab. 2 shows numerical results. We find that:

a) the cosmological constant scale is consistent with its experimentally determined value and with the scale of neutrino masses [34].

b) the near IR scale is consistent with the QCD scale (Λ_{QCD}). This conclusion may shed light into the long-standing problem of the QCD mass gap as well as on the non-perturbative properties of strongly coupled field theory [9, 22, 36].

Mass scale	Estimated	Units	Comments
$\Lambda_{IR} = \Lambda_{cc}^{1/4}$	$\sim 1.6\times10^{-6}$	eV	from M_{Pl}
$\Lambda_{IR} = \Lambda_{cc}^{1/4}$	$\sim 1.9\times10^{-3}$	eV	from M_{GUT}
$\Lambda'_{IR} = \Lambda_{QCD}$	~ 193	MeV	from Λ_{CK}

Tab 2: Estimated values of the cosmological constant and QCD scales (assuming the electroweak scale $M_{EW} \approx 246$ GeV and the Cohen-Kaplan cutoff $\Lambda_{CK} \approx 10^2$ TeV)

The hierarchy of mass scales derived in our work can be conveniently summarized in the diagram below:

"Far" cutoffs

"Near" cutoffs

$$\Lambda_{cc}^{1/4} \quad \Lambda_{QCD} \quad M_{EW} \quad \Lambda_{CK} \quad M_{Pl} \tag{17}$$

References

1) Wachter H., "Analysis on *q*-deformed quantum spaces", *International Journal of Modern Physics* A **22**(01), (2012) and http://arxiv.org/pdf/math-ph/0604028.pdf

2) Schwartz M. D., "Quantum Field Theory and the Standard Model", *Cambridge University Press*, 2013.

3) Duncan A., "Conceptual Framework of Quantum Field Theory", *Oxford University Press*, 2012.

4) See e.g. Wiese J. U., "An Introduction to Lattice Field Theory", available at http://www.itp.uni-hannover.de/saalburg/Lectures/wiese.pdf

5) Herrmann, R., "Common aspects of *q*-deformed Lie algebras and fractional calculus", *Physica* A, **389**(21), (2010), pp. 4613-4622 and http://arxiv.org/pdf/1007.1084v1.pdf

6) Herrmann, R., "*Fractional Calculus: an Introduction for Physicists*", World Scientific, Singapore, 2011.

7) Carmona, J. M. *et al.*, "Infrared and ultraviolet cutoffs of quantum field theory", *Phys. Rev.* ***D*** **65**, (2002), 025006, http://arxiv.org/pdf/hep-th/0012028v2.pdf

8) Cohen, A. G. *et al.*, "Effective Field Theory, Black Holes and the Cosmological Constant", *Phys. Rev. Lett.* **82**: 4971-4974, (1999), http://arxiv.org/pdf/hep-th/9803132v2.pdf

9) Brambilla, N. *et al.* ,"QCD and strongly coupled gauge theories: challenges and perspectives" http://arxiv.org/pdf/1404.3723v2.pdf

10) Lemets O.A., Yerokhin D.A., "Solution to the Basic Cosmological Problems by Using the Holographic Principle" http://www.ccsem.infn.it/issp2013/newtalent/poster_yerokhin.pdf

11) Grosse H., Wulkenhaar R., "Regularization and Renormalization of Quantum Field Theories on Non-Commutative Spaces", *Journal of Nonlinear Math. Phys.*, 11, (2004), 9-20 Bialowieza XXI, XXII.

12) Tao Y., "The Validity of Dimensional Regularization Method on Fractal Spacetime", *Journal of Applied Mathematics*, Hindawi, (2013), ID 308691, http://dx.doi.org/10.1155/2013/308691

13) Goldfain E., "Emergence of the Electroweak Scale from Fractal Spacetime", *Prespacetime Journal*, **4**(9), (2013), pp. 923-926.

14) Goldfain E., "Dynamic Instability of the Standard Model and the Fine Tuning Problem", *Prespacetime Journal*, **12**(12), (2012), pp.1175-1181.

15) Goldfain E., "Fractal Spacetime as Underlying Structure of the Standard Model", *Quantum Matter*, **3**(3), (2014), pp. 256-263.

16) Goldfain E., "Multifractal Theory and Physics of the Standard Model", *Prespacetime Journal*, **5**(7), (2014), pp. 554-565.

17) Goldfain E., "Fractal Propagators and the Asymptotic Sectors of Quantum Field Theory", *Prespacetime Journal*, **5**(8), (2014), pp. 712-719.

18) Goldfain E., "Fractional Field Theory and Physics Beyond the Standard Model", *Prespacetime Journal*, **3**(5), (2012), pp. 435-438.

19) Cheng H-C *et al*., "Electroweak Symmetry Breaking and Extra Dimensions", *Nucl. Phys.* **B**589, (2000), pp. 249-268 and http://arxiv.org/pdf/hep-ph/9912343.pdf

20) Kitano R. *et al.*, "Unified origin of baryons and dark matter", *Physics Letters* **B**669, (2008), 145 and http://arxiv.org/pdf/0807.4313.pdf

21) Chankowski P. H. *et al*., "Electroweak symmetry breaking in supersymmetric models with heavy scalar superpartners", *Phys. Lett.* **B**598, (2004), pp. 252-262 and http://arxiv.org/pdf/hep-ph/0407242.pdf

22) Goldfain E., "On the asymptotic transition to complexity in quantum chromodynamics", *Comm. Nonlin. Sci. Numer. Simul*. **14**(4), (2009), pp. 1431-1438.

23) Heeck, J., "How stable is the photon?", Phys. Rev. Lett. 111, (2013), 021801 and http://arxiv.org/abs/1304.2821

24) Goldhaber A. S., Nieto M.M., "Photon and Graviton Mass Limits", http://arxiv.org/pdf/0809.1003v5.pdf

25) Horowitz G. T., Polchinski J., "Gauge/gravity duality" http://arxiv.org/pdf/gr-qc/0602037v3.pdf

26) Klebanov I. R, "TASI lectures Introduction to AdS/CFT correspondence" http://arxiv.org/pdf/hep-th/0009139.pdf

27) Calcagni G., Nardelli G., "Symmetries and propagator in multi-fractional scalar field theory*", Phys. Rev. D* **87**, 2012, 085008, http://arxiv.org/pdf/1210.2754v3.pdf

28) Calcagni G., "Geometry and field theory in multifractional spacetime", *JHEP* **01**(2012) 065 and http://arxiv.org/pdf/1107.5041v4.pdf

29) Goldfain E., "Fractal Space-time as Tentative Solution for the Cosmological and Coincidence Problems", *Prespacetime Journal* **4**(8), (2013), pp. 742-747.

30) Goldfain E., "Limitations of Perturbative Renormalization and the Challenges of the Standard Model", *Prespacetime Journal* **5**(1), (2013), pp. 1-7.

31) Goldfain E., "Fractional Field Theory and High-Energy Physics: New Developments" in *Horizons in World Physics*, 279, Nova Science Publishers, 2013, pp. 69-92.

32) ‘t Hooft G., “The Holographic Principle”, http://arxiv.org/pdf/hep-th/0003004v2.pdf

33) ‘t Hooft G., “Dimensional Reduction in Quantum Gravity”, http://arxiv.org/pdf/gr-qc/9310026.pdf

34) D. Stojkovic, “Vanishing dimensions: Review”, 2014, http://arxiv.org/pdf/1406.2696v1.pdf

35) N. Afshordi, D. Stojkovic, “Emergent Spacetime in Stochastically Evolving Dimensions”, 2014, http://arxiv.org/pdf/1405.3297v1.pdf

34) Goldfain E., “Dynamics of Neutrino Oscillations and the Cosmological Constant Problem”, *Prespacetime Journal* **2**(3), (2011), pp.442-446.

35) Goldfain E., “On a Natural Solution for the Hierarchy Problem Using Dimensional Regularization”, *Prespacetime Journal*, **2**(3), 2011, pp. 437-439.

36) Goldfain E., “Chaos in Quantum Chromodynamics and the Hadron Spectrum”, *Electronic Journal of Theoretical Physics*, **7**(23), (2010), pp. 75-84.