IDEAS OF E. CARTAN AND S. LIE IN MODERN GEOMETRY: G-STRUCTURES AND DIFFERENTIAL EQUATIONS. LECTURE 4

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Problem:
How to find invariants of singularities of a G-structure?

Simple example

Let $V = (v^1, v^2)$ be a vector field on $\mathbb{R}^2$. If $V(x_0) \neq 0$, then V restricted to a neighborhood $U(x_0)$ is equivalent to ∂_1, hence all nonvanishing vector fields are locally equivalent and there are no invariants.

If $V(x_0) = 0$, and $W(x_0) = 0$, and the matrix $||\partial_i V^j(x_0)||$ is not similar to the matrix $||\partial_i W^j(x_0)||$, then V and W are not locally equivalent and there arise invariants.

Generally, *non-regular points have their own invariants.*

1. Jets of smooth maps

Let M and N be smooth manifolds, $x \in M$ is a point. We will denote by $f : (M, x) \to N$ a smooth map which is defined in an open neighborhood U of x. By $D(f)$ we will denote the domain U.

Denote by $C^\infty((M, x), N)$ the set of all smooth maps $f : (M, x) \to N$. On the set $C^\infty((M, x), N)$ we introduce the equivalence relation: we say that f, g in $C^\infty((M, x), N)$ are equivalent at $x \in M$ ($f \sim_x g$) if

$$\exists W \in \mathcal{U}_x, \quad W \subset D(f) \cap D(g) \text{ such that } f|_W = g|_W. \tag{1}$$

Definition 1. A *germ of a map* $f \in C^\infty((M, x), N)$ at the point x is the equivalence class of $f \in C^\infty((M, x), N)$ with respect to $\sim_x$ denoted by $\langle f \rangle_x$.

We set

$$\mathcal{G}_x(M, N) = C^\infty((M, x), N)/\sim_x = \{\langle f \rangle_x \mid f \in C^\infty((M, x), N)\}. \tag{2}$$

If M_1, M_2, and M_3 are smooth manifolds, and $f_1 \in C^\infty((M_1, x_1), M_2)$, $f_2 \in C^\infty((M_2, x_2), M_3)$ with $f(x_1) = x_2$, then we can define the composition of germs as follows:

$$\langle f_2 \rangle_{x_2} \circ \langle f_1 \rangle_{x_1} = \langle f_2 \circ f_1 \rangle_{x_1} \tag{3}$$

Another equivalence relation $\sim_k$ on the set $C^\infty((M, x), N)$ is introduced as follows: for $f, g \in C^\infty_x(M, N)$ such that $f(x) = g(x)$ we take the coordinate systems (U, x^i) in a neighborhood U of x and (V, y^α) in a neighborhood V of y. Then we say that f and g are equivalent ($f \sim_k g$) if the Taylor series of the coordinate representations of f and g coincide up to the order k. One can prove that this equivalence relation does not depend on the choice of coordinate systems.

Definition 2. The equivalence class $j^k_x f$ of f is called *k-jet of the map f at the point x.*

The set of all k-jets of maps $f \in C^\infty((M,x),N)$ will be denoted by $J^k_x(M,N)$.

It is clear that if $f,g \in C^\infty((M,x),N)$ determine the same germ at x, that is if $f \sim_x g$, then $j^k_x f = j^k_x g$ for any k. Also, the composition of maps, or the composition of germs, define the composition of k-jets:

$$j^k_{f_1(x)}(f_2) \circ j^k_x(f_1) = j^k_x(f_2 \circ f_1). \tag{4}$$

For the set of all k-jets $J^k(M,N)$ one can define two natural projections:

$$\pi_0 : J^k(M,N) \to M, \quad j^k_x f \mapsto x, \tag{5}$$

$$\pi_1 : J^k(M,N) \to N, \quad j^k_x f \mapsto f(x). \tag{6}$$

The set $J^k(M,N)$, $\dim M = m$, $\dim N = n$ can be endowed by a manifold structure so that these projections are bundle projections. We set

$$J^k_{x,y}(M,N) = \left\{ j^k f \in J^k(M,N) \mid \pi_0(j^k f) = x, \pi_1(j^k f) = y \right\}. \tag{7}$$

The typical fiber of the bundle $(J^k(M,N), \pi_0, M)$ is the manifold $J^k_{0,0}(\mathbb{R}^m, \mathbb{R}^n) \times N$, and of the bundle $(J^k(M,N), \pi_1, N)$ is $J^k_{0,0}(\mathbb{R}^m, \mathbb{R}^n) \times M$.

The manifold $J^k(M,M)$ endowed with the projections π_0 and π_1, and the composition of k-jets is a groupoid.

If $\pi : E \to M$ is a fiber bundle, then the set of k-jets of sections of the bundle E is denoted by $J^k(E)$.

2. Bundle of germs of diffeomorphisms. Coframe bundle of kth order

2.1. **Bundle of germs of diffeomorphisms.** Let $\mathcal{D}(m)$ be the group of all germs of local diffeomorphisms of $\mathbb{R}^m$ at $0 \in \mathbb{R}^m$:

$$\mathcal{D}(m) = \{ \langle \varphi \rangle_0 \mid \varphi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0) \text{ is a local diffeomorphism} \}$$

endowed with the operation of composition of germs: $\langle \varphi_1 \rangle_0 \circ \langle \varphi_2 \rangle_0 = \langle \varphi_1 \circ \varphi_2 \rangle_0$.

Now let us consider

$$\mathcal{B}(M) = \{ \langle f \rangle_x \mid f : (M,x) \to (\mathbb{R}^n, 0) \text{ is a local diffeomorphism } \}$$

We have natural projection

$$\pi : \mathcal{B}(M) \to M, \quad \pi(\langle f \rangle_x) = x,$$

and the natural right action of $\mathcal{D}(m)$ on $\mathcal{B}(M)$:

$$\langle f \rangle_x \cdot \langle \varphi \rangle_0 = \langle \varphi^{-1} \circ f \rangle_x.$$

so $(\mathbb{B}(M), \pi, M)$ can be considered as a "principal fiber bundle".

Let $(U, u : U \to V \subset \mathbb{R}^m)$ be a coordinate map. This map determines the "trivialization" of the bundle $\mathcal{B}(M)$ over U. Let

$$t_a : \mathbb{R}^m \to \mathbb{R}^m, \quad t_a(v) = v + a$$

be the parallel translation of $\mathbb{R}^m$. Then

$$\mathcal{U} : \pi^{-1}(U) \to U \times \mathcal{D}(m), \quad \langle f \rangle_x \to \big(x, \langle f \circ u^{-1} \circ t_{u(x)} \rangle_0\big).$$

gives us the required trivialization. The inverse map is

$$\mathcal{U}^{-1} : U \times \mathcal{D}(m) \to \pi^{-1}(U) \quad (x, \langle \varphi \rangle_0) \to \langle t_{-u(x)} \circ u \circ f^{-1} \rangle.$$

Now assume that we have two coordinate systems (U, u) and $(U, \bar{u})$ on M. Then,

$$\bar{\mathcal{U}} \circ \mathcal{U}^{-1} : U \times \mathcal{D}(m) \to U \times \mathcal{D}(m), \quad (x, \varphi) \to (x, t_{-\bar{u}(x)} \circ \bar{u} \circ u^{-1} \circ t_{u(x)}).$$

so the gluing functions of the atlas of the "principal bundle" $(\mathcal{B}(M), \pi, M)$ constructed by an atlas (U_α, u_α) of the manifold M are

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathcal{D}(m), \quad g_{\beta\alpha}(x) = t_{-u_\beta(x)} \circ u_\beta \circ u_\alpha^{-1} \circ t_{u_\alpha(x)}.$$

Remark 1. In what follows we will use unordered multiindices. We denote by $\mathcal{I}(m)$ the set of all unordered multiindices $I = \{i_1 i_2 \dots i_k\}$, where $1 \leq i_l \leq m$, for all $l = \overline{1,k}$. The number k is called the length of the multiindex and is denoted by $|I|$. Also, we set $I_k(m) = \{I \in \mathcal{I}(m) \mid |I| = k\}$.

2.2. **Differential group of kth order.** The kth order *differential group* is the set of k-jets:

$$D^k(m) = \left\{ j^k_0(\varphi) \mid \varphi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0) \text{ is a local diffeomorphism } \right\}.$$

On the set $D^k(m)$ consider the operation

$$D^k(m) \times D^k(m) \to D^k(m), \quad j^k_0(\varphi) \cdot j^k_0(\psi) = j^k(\varphi \circ \psi), \tag{8}$$

then $(D^k(m), \cdot)$ is a group.

Denote $\varphi^k_I = \partial_I|_0 \, \varphi^k$. Then

$$\mathcal{C}^k : D^k(m) \to \mathbb{R}^N, \quad j^k_0(\varphi) \to \{\varphi^k_I\} \tag{9}$$

in a one-to-one map of $D^k(m)$ onto the open set in $\mathbb{R}^N$ determined by the inequality $\det \|\varphi^k_i\| \neq 0$. In this way we get globally defined coordinates on $D^k(m)$ which will be called *natural coordinates*. With respect to the natural coordinates the product (8) is written in terms of polynomials, therefore is a smooth map. Thus $D^k(m)$ is a Lie group.

2.3. **Bundle of kth order holonomic coframes.** For an m-dimensional manifold M consider the set

$$B^k(M) = \left\{ j^k_x f \mid f : (M, x) \to (\mathbb{R}^m, 0) \text{ is a local diffeomorphism } \right\}$$

whose elements are called *k-coframes* or *coframes of order k* of the manifold M. We have the projection

$$\pi^k : B^k(M) \to M, \quad \pi(j_x(f)) = x.$$

On the set $B^k(M)$ we have the right $D^k(m)$-action:

$$j^k_x(f) \cdot j^k_0(\varphi) = j^k_0(\varphi^{-1} \circ f).$$

and one can easily prove that this action is free and its orbits are the fibers of the projection π.

2.3.1. *Trivializing charts of $B^k(M)$. Gluing maps.* In what follows we set $t_a : \mathbb{R}^m \to \mathbb{R}^m$, $t_a(x) = x + a$, the parallel translation of $\mathbb{R}^m$ with respect to $a \in \mathbb{R}^m$.

Let $(U, u : U \to V \subset \mathbb{R}^m)$ be a coordinate chart on M. We have the one-to-one map

$$\mathcal{T}^k : (\pi^k)^{-1}(U) \to U \times D^k(m), \qquad j^k_x(f) \to \left(x, j^k_0(t_{-u(x)} \circ u \circ f^{-1})\right). \tag{10}$$

The map $\mathcal{T}^k$ is $D^k(m)$-equivariant because

$$\mathcal{T}^k(j^k_x f \cdot j^k_0\varphi) = \mathcal{T}^k\left(j^k_x(\varphi^{-1} \circ f)\right) = \left(x, j^k_0(t_{-u(x)} \circ u \circ f^{-1} \circ \varphi)\right) = \\ = \left(x, j^k_0(t_{-u(x)} \circ u \circ f^{-1}) \cdot j^k_0\varphi\right) = \left(x, j^k_0(t_{-u(x)} \circ u \circ f^{-1})\right) \cdot j^k_0\varphi.$$

Since $D^k(m)$ is a Lie group, the map $\mathcal{T}^k$ defines a trivializing chart for the map $\pi^k : B^k(M) \to M$.

Therefore, for each atlas (U_α, u_α), we construct the atlas of trivializing charts $(U_\alpha, \mathcal{T}^k_\alpha)$. Find the gluing maps for this atlas.

Assume that (U_α, u_α), (U_β, u_β) are two coordinate systems on M, and $U_\alpha \cap U_\beta \neq \emptyset$. Then

$$j^k_0(t_{-u_\beta(x)} \circ u_\beta \circ f^{-1}) = j^k_0(t_{-u_\beta(x)} \circ u_\beta \circ u_\alpha^{-1} \circ t_{u_\alpha(x)}) \cdot j^k_0(t_{-u_\alpha(x)} \circ u_\alpha \circ f^{-1})$$

Therefore, the gluing maps are

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to D^k(m), \quad g_{\beta\alpha}(x) = j^k_0(t_{-u_\beta(x)} \circ u_\beta \circ u_\alpha^{-1} \circ t_{u_\alpha(x)}) \tag{11}$$

Since the gluing functions are smooth, we conclude that $\pi^k : B^k(M) \to M$ is a $D^k(m)$-principal bundle over M which is called *the bundle of k-coframes of M* or *the bundle of coframes of order k of M*.

For a coordinate chart (U, u) there is defined a section

$$s : U \to B^k(M), \quad s(x) = j^k_x u, \tag{12}$$

which is called the *natural k-coframe field* associated with a coordinate chart (U_α, u_α).

2.3.2. *Natural coordinates on $B^k(M)$.* If (U, u) is a coordinate chart on M, and $\mathcal{T}^k$ is the corresponding trivialization of $B^k(M)$. Then

$$(u \times \mathcal{C}^k) \circ \mathcal{T}^k : (\pi^k)^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^N \tag{13}$$

gives *natural local coordinates on $B^k(M)$*.

The section $s : U \to B^k(M)$ (12) is written with respect to this coordinate system as follows:

$$s(u^k) = (u^k, \delta^k_i, 0). \tag{14}$$

Remark 2. We have the natural projections $\pi^k_l : B^k(M) \to B^l(M)$, $k \geq l$, which are, in turn, principal fiber bundles with the group H^k_l which is the kernel of the natural homomorphism $D^k(m) \to D^l(m)$.

2.4. **Case $k = 1$.** The Lie group $D^1(m) \cong GL(m)$ and $B^1(M) = B(M)$ is the coframe bundle of M.

2.5. **Case $k = 2$.**

2.5.1. *The group $D^2(m)$.* Elements of the group $D^2(m)$ are the 2-jets of germs $\varphi \in \mathcal{D}(m)$. The coordinate system (9) in this case is

$$j_0^2\varphi \longrightarrow (\varphi_i^k, \varphi_{ij}^k), \text{ where } \varphi_i^k = \frac{\partial \varphi^k}{\partial u^i}(0), \quad \varphi_{ij}^k = \frac{\partial^2 \varphi^k}{\partial u^i \partial u^j}(0). \tag{15}$$

Here u^i are coordinates on $\mathbb{R}^m$, and it is clear that $\varphi_{ij}^k = \varphi_{ji}^k$. From this follows that $\dim D^2(m) = m^2 + m^2(m+1)/2$.

Now, if $j_0^2\varphi \to (\varphi_i^k, \varphi_{ij}^k)$, $j_0^2\psi \to (\psi_i^k, \psi_{ij}^k)$, and

$$j_0^2\psi \cdot j_0^2\varphi = j_0^2(\psi \circ \varphi) \to (\eta_i^k, \eta_{ij}^k),$$

by the chain rule we get that

$$\eta_i^k = \psi_s^k \varphi_i^s, \quad \eta_{ij}^k = \psi_{pq}^k \varphi_i^p \varphi_j^q + \psi_s^k \varphi_{ij}^s \tag{16}$$

These formulas express the product in the group $D^2(m)$ in terms of the natural coordinates $(\varphi_i^k, \varphi_{ij}^k)$.

2.5.2. *The bundle $B^2(M)$.* The elements of $B^2(M)$ are the 2-jets of local diffeomorphisms $f : (M, x) \to (\mathbb{R}^m, 0)$. The natural coordinates (14) in this case can be found as follows. Let (U, u) be a coordinate chart on M. Then, for any $j_x^2 f$ with $x \in U$, the diffeomorphism

$$f \circ u^{-1} : (\mathbb{R}^m, u(x)) \to (\mathbb{R}^m, 0)$$

can be written as $w^k = f^k(u^i)$, where w^k are standard coordinates on $\mathbb{R}^m$, and $f^k(u^i(x)) = 0$. Then take the inverse diffeomorphism $u^k = \widetilde{f}^k(w^i)$, and the local diffeomorphism $t_{-u(x)} \circ u \circ f^{-1}$ has the form $\widetilde{f}^k(w^i) - u^k(x)$. Therefore, the natural coordinates of $j_x^k f$ induced by a coordinate chart (U, u) on M are (u^k, u_i^k, u_{ij}^k), where

$$u_i^k = \frac{\partial \widetilde{f}^k}{\partial w^i}(0), \quad u_{ij}^k = \frac{\partial^2 \widetilde{f}^k}{\partial w^i \partial w^j}(0). \tag{17}$$

The derivatives of $\widetilde{f}$ at 0 can be expressed in terms of the derivatives of f at $u^k(x)$. If we denote

$$f_i^k = \frac{\partial f^k}{\partial u^i}(u(x)), f_{ij}^k = \frac{\partial^2 f^k}{\partial u^i \partial u^j}(u(x)).$$

then

$$u_i^k = \widetilde{f}_i^k, \quad u_{ij}^k = -\widetilde{f}_s^k f_{lm}^s \widetilde{f}_i^l \widetilde{f}_j^m.$$

With respect to the natural coordinate system the $D^2(m)$-action is written as follows:

$$(u^k, u_i^k, u_{ij}^k) \cdot (\varphi_i^k, \varphi_{ij}^k) = (u^k, u_s^k \varphi_i^s, u_{pq}^k \varphi_i^p \varphi_j^q + u_s^k \varphi_{ij}^s). \tag{18}$$

Let us express the gluing maps in terms of the natural coordinates. If (U, u) and (U', u') are coordinate charts on M sucht that $U \cap U' \neq \emptyset$, then from (11) it follows that the corresponding gluing map is

$$g : U \cap U' \to D^2(m), g(x) = j_0^2(t_{-u'(x)} \circ u' \circ u^{-1} \circ t_{u(x)})$$

If the coordinate change $u' \circ u^{-1}$ is written as $v^k = v^k(u^i)$, then we have to take derivatives at 0 of the map $v^k(u^i + u^i(x)) - v^k(u^i(x))$, which are equal to the derivatives of the functions v^k at $u^k(x)$. Therefore,

$$g : U \cap U' \to D^2(m), \quad g(x) = \left(\frac{\partial v^k}{\partial u^i}(u(x)), \frac{\partial v^k}{\partial u^i \partial u^j}(u(x)) \right). \tag{19}$$

Therefore, *the bundle $B^2(M) \to M$ is the $D^2(m)$-principal bundle with gluing maps* (19).

3. First prolongation of a G-structure

3.1. **First prolongation of an integrable G-structure.** Let $P(M) \to M$ be an integrable G-structure, that is a subbundle of $B(M)$ such that there exists an atlas $\mathcal{A} = (U_\alpha, u_\alpha)$ such that the natural coframes of the atlas are sections of $P(M)$, or equivalently, the coordinate change $u^{k'} = u^{k'}(u^i)$ has the property that $\|\frac{\partial u^{k'}}{\partial u^k}\| \in G$.

In this case, we can specify the set $\mathcal{B}_G$ of local diffeomorphisms $f : (M, x) \to (\mathbb{R}^m, 0)$ such that for each coordinate map u of the atlas $\mathcal{A}$, the local diffeomorphism $f \circ u^{-1}$ has the Jacobi matrix at in G at all points of its domain. It is clear that

$$P(M) = \left\{ j^1_x f \mid \|\frac{\partial (f \circ u^{-1})^k}{\partial u^i}|_{u(x)}\| \in G \right\} \tag{20}$$

and consider

$$P^1(M) = \left\{ j^2_x f \mid f \in \mathcal{B}_G \right\} \tag{21}$$

Remark 3. Note that if $j^2_x f$ is an element of $P^1(M)$, then the Jacobi matrix $\| \frac{\partial (f \circ u^{-1})^k}{\partial u^i}\Big|_{u(x)} \|$ is an element of G. However, the converse is not true because, by definition, the Jacobi matrix belongs to G for each point which implies conditions on the second derivatives of $f \circ u^{-1}$ (see below).

In the same manner introduce the set $\mathcal{D}_G$ of local diffeomorphisms $\varphi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ whose Jacobi matrices are elements of G at all points of their domains. Consider the Lie subgroup of $D^2(m)$:

$$G^1 = \left\{ j^2_0 \varphi \in D^2(m) \mid \varphi \in \mathcal{D}_G \right\} \tag{22}$$

The Lie subgroup $G^1 \subset D^2(m)$ is called the *first holonomic prolongation of the group G*.

The subset $P^1(M)$ is a submanifold of $B^2(M)$, and is the total space of a principal subbundle of $B^2(M) \to M$ with the subgroup $G^1 \subset D^2(m)$. This subbundle is called the *first holonomic prolongation of the integrable G-structure P*.

It is also clear that, for the atlas $\mathcal{A}$ the gluing map (19) takes values in the subgroup G^1. Therefore, *an integrable G-structure defines reduction of the principal bundle $B^2(M)$ to the structure group $G^1 \subset D^2(m)$*.

3.2. **Algebraic structure of the Lie group G^1.** We have the surjective Lie group morphism

$$p^1 : G^1 \to G, \quad j^2_0 \varphi \mapsto j^1_0 \varphi. \tag{23}$$

Let us find the kernel of p^1. Let $\varphi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ be an element of $\mathcal{D}_G$. Then, the map $g(u^k) = \|\frac{\partial \varphi^k}{\partial u^i}\|$ takes values in G, and $g(0) = I \in G$. Therefore,

$$\left.\frac{\partial^2 \varphi}{\partial u^i \partial u^j}\right|_0 = \left.\frac{\partial g_i^k}{\partial u^j}\right|_0$$

is a linear map $t : \mathbb{R}^m \cong T_0\mathbb{R}^m \to \mathfrak{g}(G) \subset \mathfrak{gl}(m)$ but with property that $t(u)v = t(v)u$. In other words, elements of $\ker p^1$ are tensors of type $(2,1)$ on $\mathbb{R}^m$ such that $t_{ij}^k \in \mathfrak{g}$ for each i, and $t_{ij}^k = t_{ji}^k$. The vector space of such tensors is called the *first prolongation of the Lie algebra* $\mathfrak{g}$ and is denoted by $\mathfrak{g}^1$. From this follows that $\ker p^1$ is a commutative Lie group.

Thus we have the exact sequence of Lie groups

$$0 \to \mathfrak{g}^1 \to G^1 \to G \to e,$$

This sequence admits a splitting: for any $\|g_i^k\| \in G$ we take the diffeomorphism $\varphi^k(u^i) = g_i^k u^i$, which is evidently lies in $\mathcal{D}_G$, and set $s(j_0^1\varphi) = j_0^2\varphi$.

According to the group theory, the exact splitting sequence (3.2) determines the right action of G on $\mathfrak{g}^1$: $R_g t = s(g^{-1}) \cdot t \cdot s(g)$, and G^1 is the extension of G by the commutative group $\mathfrak{g}^1$ with respect to the action R. This means that we have the group isomorphism

$$G^1 \to G \times \mathfrak{g}^1, \quad g^1 \mapsto (p^1(g^1), s\left(p^1((g^1)^{-1})\right) g^1, \tag{24}$$

therefore

$$G^1 \cong G \times \mathfrak{g}^1, \text{ and } (g_1, t_1) \cdot (g_2, t_2) = (g_1 g_2, R(g_2)t_1 + t_2). \tag{25}$$

Let us express this representation of G^1 in terms of the canonical coordinates. We have $p^1(\varphi_i^k, \varphi_{ij}^k) = \varphi_i^k$, and $s(\varphi_i^k) = (\varphi_i^k, 0)$. Hence follows that the isomorphism (24) is

$$(\varphi_i^k, \varphi_{ij}^k) \to (g_i^k, t_{ij}^k) \text{ with } g_i^k = \varphi_i^k,\ t_{ij}^k = \widetilde{\varphi}_s^k \varphi_{ij}^s.$$

Thus we get *algebraic coordinates* (g_i^k, t_{ij}^k) on G^1 which are adopted to the algebraic structure of G^1.

The right action R with respect to the canonical coordinates is written as follows (we use (16)):

$$(\widetilde{\varphi}_i^k, 0) \cdot (\delta_i^k, \varphi_{ij}^k) \cdot (\varphi_i^k, 0) = (\delta_i^k, \widetilde{\varphi}_s^k \psi_{lm}^s \varphi_i^l \varphi_j^m).$$

At the same time, the algebraic coordinates of the elements $(\varphi_i^k, 0)$ and $(\delta_i^k, 0)$ are the same, this means that they are $(g_i^k = \varphi_i^k, 0)$ and $(\delta_i^k, t_{ij}^k = \varphi_{ij}^k)$. Therefore, with respect to the algebraic coordinates the product of the group G^1 looks like (see (25)):

$$(g_i^k, t_{ij}^k) \cdot (h_i^k, q_{ij}^k) = (g_s^k h_i^s, \widetilde{h}_s^k t_{pq}^s h_i^p h_j^q + q_{ij}^k). \tag{26}$$

3.3. **Algebraic coordinates on $B^2(M)$. Description of $P^1(M)$ with respect to algebraic coordinates.** One can easily see that $D^2(m) = (GL(m))^1$. Therefore, we can consider the algebraic coordinates on $D^2(m)$ which give rise to the *algebraic coordinates on the total space* $B^2(M)$. Namely, if (U, u) is a coordinate chart on M, and (u^k, u_i^k, u_{ij}^k) are the corresponding natural coordinates on $(\pi^2)^{-1}(U)$, for the *algebraic coordinates* on $(\pi^2)^{-1}(U)$ we take

$$(u^k, p_i^k, p_{ij}^k) \text{ where } p_i^k = u_i^k, p_{ij}^k = \widetilde{u}_s^k u_{ij}^s. \tag{27}$$

In fact, we change coordinates on the second factor of $U \times D^2(m) \cong (\pi^2)^{-1}(U)$.

With respect to the algebraic coordinates the first prolongation P^1 of an integrable G-structure P is described in the following way:

$$P^1\big|_U = \left\{(u^k, p^k_i, p^k_{ij}) \mid \|p^k_i\| \in G, \|p^k_{ij}\| \in \mathfrak{g}^1\right\}. \tag{28}$$

If we have two coordinate charts of the atlas $\mathcal{A}$ on M and $v^k = v^k(u^i)$ is the coordinate change, the corresponding gluing map (19) of trivializing charts of P^1 is written with respect to the algebraic coordinates as follows:

$$g : U \cap U' \to G^1, \quad g(x) = \left(\frac{\partial v^k}{\partial u^i}(u(x)), \frac{\partial u^k}{\partial v^s}(v(x))\frac{\partial v^s}{\partial u^i \partial u^j}(u(x))\right). \tag{29}$$

3.4. $P^1(M)$ **in terms of** $P(M)$.

4. First prolongation of G-structure and associated bundles.

Let us now consider the constructions of the previous section for cases $k = 1$ and $k = 2$.

The case $k = 1$ is rather simple. We have $D^1(m) \cong GL(m)$ and $B^1(M)$ is the coframe bundle of M, because to each $j^1_x f$ we can put in correspondence the coframe $\{f^* du^i\}$ at $x \in M$.

So we will consider in details the case $k = 2$.

4.1. **First prolongation of the Lie subgroup** $G \subset GL(m)$**.** Now let us consider the set $\widetilde{D}^2(m) = (\varphi^k_i, \varphi^k_{ij})$, where $\|\varphi^k_i\|$ is an invertible matrix, and φ^k_{ij} are not necessarily symmetric with respect to lower indices. Then, the set $\widetilde{D}$ endowed with the operation $*$ given by (16) is a Lie group called the *nonholonomic differential group of second order*. We will call it *first prolongation of the group* $GL(m)$ and denote by $GL^{(1)}(m)$.

On the group $GL^{(1)}(m)$ we can introduce another coordinate system:

$$g^k_j = \varphi^k_j, \quad a^k_{ij} = \widetilde{\varphi}^k_s \varphi^s_{ij} \tag{30}$$

Then, with respect to these coordinates, by (16), we see that the product $*$ can be written as follows:

$$(g^k_i, a^k_{ij}) * (h^k_i, b^k_{ij}) = (g^k_s h^s_i, \widetilde{h}^k_s a^s_{pq} h^p_i h^q_j + b^k_{ij}). \tag{31}$$

These formulas can be written in a matrix form. To do this, we consider a^k_{ij} as a map $a : \mathbb{R}^m \to \mathfrak{gl}(m)$, $w^k \mapsto a^k_{ij} w^j$. Then, (31) takes the form

$$(g, a) * (h, b) = (gh, adh^{-1} a \circ h + b). \tag{32}$$

Therefore, the group

$$GL^{(1)} \cong (GL(m) \times Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{gl}(m)), *),$$

where $*$ is defined in (32).

Remark 4. The vector space $Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{gl}(m))$ is a right $GL(m)$-module with respect to the action

$$\forall g \in GL(m), \quad a \in Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{gl}(m)), \quad g \cdot a = adg^{-1} ag. \tag{33}$$

The group $(GL(m) \times Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{gl}(m)), *)$ is the extension of the group $GL(m)$ with the right $GL(m)$-module $Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{gl}(m)), *)$. This fact motivates the definition of first prolongation for a subgroup $G \subset GL(m)$.

Remark 5. The same considerations can be done for the holonomic jet group $D^2(m)$.

4.1.1. *First prolongation of a Lie subgroup $G \subset GL(m)$.* The considerations of the previous subsection motivate

Definition 3. Let $G \subset GL(m)$ be a Lie subgroup. Then the *first prolongation of G* is the group

$$G^{(1)} = G \times Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G)) \tag{34}$$

with product

$$(g_1, a_1)(g_2, a_2) = (g_1 g_2, adg_2^{-1} a_1 g_2 + a_2).$$

Remark 6. In this case also the vector space $Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G))$ is a right G-module with respect to the action

$$\forall g \in G, \quad a \in Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G)), \quad g \cdot a = adg^{-1} ag. \tag{35}$$

The group $(G \times Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G)), *)$ is the extension of the group G with the right G-module $Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G)), *)$.

Therefore we have the short exact sequence of Lie groups:

$$0 \to Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G)) \to G^{(1)} \xrightarrow{\pi} G.$$

The Lie homomorphism $\pi : G^{(1)} \to G$ comes from the natural projection of 2-jets onto 1-jets.

4.1.2. *Another coordinate system on $B^2(M)$.* Now recall that on $D^2(m)$ we can take another coordinate system (g_i^k, a_{ij}^k) (see 30). With respect to the coordinates (g_i^k, a_{ij}^k), the $D^2(m)$-action on $B^2(M)$ is written as

$$h_i^k = \widetilde{g}_s^k f_i^s, \quad h_{ij}^k = \widetilde{g}_s^k f_{ij}^s - a_{lm}^k \widetilde{g}_t^l \widetilde{g}_r^m f_i^t f_j^r. \tag{36}$$

Let us find the expression for a trivialization

$$(\pi^2)^{-1}(U) \to U \times D^2(m)$$

with respect to the coordinates (g, a). For this purpose we take a section $s(x^i) = (x^i, \delta_i^k, 0)$ of $B^2(M)$ over U, in fact this is the natural frame of order 2, this means that it consists of 2-jets of the coordinate functions. Then any point $b^2 = (x^i, f_i^k, f_{ij}^k)$ can be written as $b^2 = s(x) \cdot (g, a)$, and the trivialization is given by

$$b^2 = (x^i, f_i^k, f_{ij}^k) \leftrightarrow (x, (g, a)). \tag{37}$$

Using (36), and the coordinate expression $s(x)$, we get

$$f_i^k = \widetilde{g}_i^k, \quad f_{ij}^k = -a_{lm}^k \widetilde{g}_i^l \widetilde{g}_j^m. \tag{38}$$

Now let us write the gluing functions with respect to the coordinates (p_i^k, p_{ij}^k) on $B^2(M)$ and (g_i^k, a_{ij}^k). To do this we use the coordinate change (40) on $B^2(M)$ and (30) on the group $D^2(M)$. We have

$$p_i^k = \widetilde{f}_i^k, \quad p_{ij}^k = -f_{lm}^k \widetilde{f}_i^l \widetilde{f}_j^m$$
$$\bar{x}_i^k = g_i^k, \quad \bar{x}_{ij}^k = g_s^k a_{ij}^s$$

and so,

$$f_i^k = \widetilde{p}_i^k, \quad f_{ij}^k = -p_{lm}^k \widetilde{p}_i^l \widetilde{p}_j^m \tag{39}$$

$$(x^k, f^k_i, f^k_{ij}) \to (x^k, (p^k_i, p^k_{ij})), \text{ where} \\ p^k_i = \widetilde{f}^k_i, \quad p^k_{ij} = -f^k_{lm}\widetilde{f}^l_i\widetilde{f}^m_j. \tag{40}$$

Using these formulas, and (36), we can write the $D^2(m)$-action with respect to the coordinates (x^i, p^k_i, p^k_{ij}). If $(x^i, (p^k_i, p^k_{ij})) \cdot (g, a) = (x^i, (q^k_i, q^k_{ij}))$, then

$$q^k_i = p^k_s g^s_i \quad q^k_{ij} = \widetilde{g}^k_s p^s_{lm} g^l_i g^m_j + a^k_{ij}. \tag{41}$$

So, as it should be, at the second argument we get the product of elements of the group $D^2(m)$ (cf. (31)). Hence, first of all from we get that

$$\widetilde{\bar{p}}^k_s g^s_i = \widetilde{p}^k_i, \text{ and hence } \bar{p}^k_i = \widetilde{g}^k_s p^s_i. \tag{42}$$

Now, from with (39), we get

$$-\bar{p}^k_{rt}\widetilde{\bar{p}}^r_l\widetilde{\bar{p}}^t_m g^l_i g^m_j + \widetilde{\bar{p}}^k_s g^s_t a^t_{ij} = f^k_{ij},$$

hence follows

$$-\bar{p}^k_{lm}\widetilde{p}^l_i\widetilde{p}^m_j + \widetilde{p}^k_s a^s_{ij} = f^k_{ij} = -p^k_{lm}\widetilde{p}^l_i\widetilde{p}^m_j.$$

Finally, we obtain

$$\bar{p}^k_{lm} = p^k_{lm} + \widetilde{p}^k_t a^t_{lm} p^l_i p^m_j.$$

As the result, we get the following theorem.

Theorem 1.

Note that

$$x^k_i = \widetilde{\bar{x}}^k_i, \tag{43}$$

With this notation, we have

$$\bar{f}^k_i = \tag{44}$$

5. Prolongation of G-structure

5.1. **First prolongation of integrable G-structure.** Let $P(M, G)$ be an integrable G-structure, this means that there exists an atlas (U_α, u_α) such that $\left\{\frac{\partial}{\partial u_\alpha}\right\}$ are the sections of P.

A first prolongation of $P(M, G)$ is the subbundle in $B^2(M)$ with the total space

$$P^1(M) = \left\{ j^2_x f \mid \left(\frac{\partial (f \circ u^{-1})^k}{\partial u^i}|_{u(x)} \right) \in G \right\} \tag{45}$$

and the structure group $G^1 \subset D^2(m)$ (the holonomic prolongation of G):

$$G^1 = \left\{ j^2_0\varphi \in D^2(m) \mid \left(\frac{\partial \varphi^k}{\partial u^i} \right) \in G \right\}. \tag{46}$$

The product in G^1 is induced by the chain rule: if $j^k_0\varphi = (\varphi^k_i, \varphi^k_{i_1 i_2}, \dots, \varphi^k_{i_1,\dots,i_k})$ $j^k_0\psi = (\psi^k_i, \psi^k_{i_1 i_2}, \dots, \psi^k_{i_1,\dots,i_k})$, then

$$\eta^k_i = \psi^k_s\varphi^s_i, \quad \eta^k_{ij} = \psi^k_{pq}\varphi^p_i\varphi^q_j + \psi^k_s\varphi^s_{ij}, \dots \tag{47}$$

5.2. **Prolongation of G-structure.**

5.2.1. *Nonholonomic jet bundle.* A k-th order nonholonomic jet is a "nonsymmetric" Teylor series and is defined by coordinates:

$$f_0^j = \left(f^j, f_i^j, f_{i_1 i_2}^j, \dots, f_{i_1, i_2, \dots, i_k}^j\right), \tag{48}$$

where $f_{i_1 \dots i_j}^j$ are not supposed to be symmetric with respect to lower indices.

5.3. **Nonholonomic differential group.** The nonholonomic k-th order differential group is $D^{(k)}(m) = \{\varphi_0^k = \left(\varphi^j, \varphi_i^j, \varphi_{i_1 i_2}^j, \dots, \varphi_{i_1, i_2, \dots, i_k}^j\right) \mid \det\left(\varphi_i^j\right) \neq 0\}$,

and the product is also "induced by the chain rule": if $\eta_0^k = \psi_0^k \cdot \varphi_0^k$ and $\varphi_0^k = (\varphi_i^j, \varphi_{i_1 i_2}^j, \dots, \varphi_{i_1, \dots, i_k}^j)$ $\psi_0^k = (\psi_i^j, \psi_{i_1 i_2}^j, \dots, \psi_{i_1, \dots, i_k}^j)$, then $\eta_i^j = \psi_s^j \varphi_i^s$, $\eta_{i_1 i_2}^j = \psi_{p_1 p_2}^j \varphi_{i_1}^{p_1} \varphi_{i_2}^{p_2} + \psi_s^j \varphi_{i_1 i_2}^s$, ...

It is clear that $D^k(m)$ is a Lie subgroup of $D^{(k)}(m)$, so we have the left $D^k(m)$-action on $D^{(k)}(m)$.

A *nonholonomic k-th order jet bundle* is the locally trivial bundle over M with the fiber $D^{(k)}(m)$ associated with the $D^k(m)$-principal bundle $B^k(M)$ with respect to the left $D^k(m)$-action on $D^{(k)}(m)$.

5.4. **First prolongation of arbitrary G-structure.** Let $P(M, G)$ be a G-structure. A *first prolongation of* $P(M, G)$ is the subbundle in $B^{(2)}(M)$ with the total space

$$P^{(1)}(M) = \left\{f_0^2 \in B^{(2)}(M) \mid \left(f_i^j\right) \in G\right\} \tag{49}$$

and the structure group $G^{(1)} \subset D^{(2)}(m)$ (the nonholonomic prolongation of G):

$$G^{(1)} = \left\{\varphi_0^2 \in D^2(m) \mid \left(\varphi_i^k\right) \in G\right\}. \tag{50}$$

6. First prolongation of G-structure in terms of coframe bundle

The first prolongation $P^{(1)}$ of P can be expressed in terms of P in the following ways:

$$P^{(1)} = \{p^1 : \mathbb{R}^m \to T_p P | \theta_p \alpha = 1_{\mathbb{R}^m}\}, \tag{51}$$

or

$$P^{(1)} = \{\omega : T_p P \to \mathfrak{g} | \omega \sigma_p = 1_{\mathfrak{g}}\} \tag{52}$$

or

$$P^{(1)} = \{H_p \mid H_p \oplus V_p = T_p P\}. \tag{53}$$

We will mainly use the first representation (51), but note that the third representation (52) says that, geometrically, P^1 consists of tangent subspaces transversal to vertical subspaces, i. e, of connections.

The projection $\pi_0^1 : P^1 \to P$, is defined in terms of (51) as follows $(p^1 : \mathbb{R}^m \to T_p P) \mapsto b$.

Theorem 2 (Algebraic structure of $G^{(1)}$)**.** *$G^{(1)}$ is isomorphic to the extension of G via the G-module $\mathcal{L}(\mathbb{R}^n, \mathfrak{g})$:*

$$G^{(1)} = G \times \mathcal{L}(\mathbb{R}^n, \mathfrak{g}), \quad (g_1, a_1) * (g_2, a_2) = (g_1 g_2, ad g_2^{-1} a_1 + a_2). \tag{54}$$

Action of $G^{(1)}$ on $P^{(1)}$ is described in the following way: for $b^1 : \mathbb{R}^m \to T_bP$, and $g^1 = (g, a) \in G^1$:

(55) $$R^{(1)}_{g^1} b^1 = dR_g(b^1 \circ g) + \sigma_{pg} \circ a \circ g$$

7. First prolongation of equivariant map

7.1. **First prolongation of a G-space.** Let V be a manifold, then the first prolongation of V is

(56) $$V^{(1)} = \left\{ v^1 : \mathbb{R}^m \to T_vV \mid v \in V \right\},$$

Let $\rho : G \times V \to V$ be a left action, then the first prolongation of ρ is

(57) $$\rho^{(1)} : G^{(1)} \times V^{(1)} \to V^{(1)}, \quad \rho^{(1)}(g^1, v^1) = [dL_g \circ v^1 + \sigma_{gv} \circ A] \circ g^{-1}.$$

7.2. **Prolongation of equivariant map.** $f : P \to V$ is an equivariant map.

Prolongation of f is $f^{(1)} : P^{(1)} \to V^{(1)}$, $f^{(1)}(b^1) = df_{\pi^1(b^1)} \circ b^1$.

An equivariant map $f : P \to V$ determines a section $s : M \to E$, where $\pi_E : E \to M$ is a bundle with standard fiber V associated with P.

$f^1 : P^1 \to V^1$ maps $b^1 \in P^1$ to the coordinates of $(s(\pi(b)), (\nabla s)(\pi(b)))$ with respect to b, where ∇ corresponds to b^1.

Example 1 (Simple example: vector field on $\mathbb{R}^n$)**.** V is a vector field on $\mathbb{R}^m$. The corresponding equivariant map is

(58) $$f : B(\mathbb{R}^m) \to \mathbb{R}^m, b = \{\eta^a\}, f(b) = \{\eta^a(V(\pi(b)))\}.$$

Then the first prolongation of f is defined as follows. If $b^{(1)} : e_i \in \mathbb{R}^m \to \frac{\partial}{\partial x^i} + \Gamma^k_{ij} \frac{\partial}{\partial x^k_j}$, then

(59) $$f^{(1)}(b^{(1)}) = (V, \nabla(\omega)V) = (V^i, \partial_j V^i + \Gamma^k_{js} V^s).$$

Action of G^1 is written as follows

(60) $$(g, a)(V, \nabla(\omega)V) = (\tilde{g}^k_i V^i, \tilde{g}^k_s \nabla_m V^s g^m_i + \tilde{g}^k_s a^s_{mj} V^j g^m_i),$$

At x_0 such that $V(x_0) = 0$, we get the action

(61) $$(g, a)(0, \nabla(\omega)V) = (0, \tilde{g}^k_s \nabla_m V^s g^m_i) = (0, \tilde{g}^k_s \partial_m V^s g^m_i)$$

Therefore, the prolonged action coincides with the action of the group $GL(n)$ on the vector space of $n \times n$-matrices by conjugation. The invariants of this action are well known, for example, these are the trace and the determinant. Therefore, to find the invariants of a zero x_0 of a vector field V we have to find the matrix $[\partial_i V^j(x_0)]$ and then write the invariants of this matrix under the conjugation, for example, one of them is $\det[\partial_i V^j(x_0)]$.

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