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Bifurcations and pattern formation in particle physics: An introductory study

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Abstract – Quantum field theories lead in general to a large number of coupled nonlinear equations. Solving field equations in analytic form or through lattice-based computations is a difficult task that has been only partially successful. We argue that the theory of nonlinear dynamical systems offers valuable insights and a fresh approach to this challenge. It is suggested that universal transition to chaos in nonlinear dissipative systems provides novel answers to some of the open questions surrounding the standard model for particle physics.

Overview and motivation. – Quantum field theory (QFT) is a mature conceptual framework whose predictive power has been consistently proven in both high-energy physics and condensed-matter phenomena [1–3]. From a historical perspective, QFT represents a successful synthesis of quantum mechanics and special relativity and consists of several models. Among these, gauge theories play a leading role. The standard model (SM) is a subset of QFT whose gauge group structure includes the electroweak and strong interactions of all known elementary particles. SM is a robust theoretical framework; however, it contains some 20 adjustable parameters whose physical origin is presently unknown and whose numerical values are exclusively fixed by experiments.

Non-Abelian gauge theories are essentially nonlinear field models. Quantizing this class of models is a non-trivial effort and raises a series of theoretical challenges [4–6]. For example, no complete quantum version of classical gravity exists. Quantum chromodynamics (QCD) is considered a reliable field theory at short distances but because its coupling constant becomes large in the infrared sector, standard perturbative techniques do not apply. At present, there is no universal prescription for deriving and handling closed-form solutions of QCD field equations. This is in manifest contrast with quantum electrodynamics (QED) and the electroweak theory, where perturbative methods are applicable and analytic results possible. In general, dealing with closed-form solutions of field theories is seldom a practical alternative. For example, Heisenberg's non-perturbative quantization procedure [7,8] or Schwinger-Dyson formalism [9] lead to an infinite set of coupled differential equations which connect all orders of Green's functions. This system does not have analytic and uniquely determined solutions. In these instances, one seeks plausible assumptions that simplify the equations *or* employs suitable numerical techniques for approximation.

In its traditional form, one frequently cited shortcoming of QFT is its inherent limitation in dealing with the effect of highly unstable fluctuations *or* with a dynamics regime that is driven far away from equilibrium [10–12]. In general, pattern formation is possible in out-of-equilibrium physical systems that are *open* and *nonlinear* [13–15]. Within a *closed* system patterns may only survive as a transient and die out as a result of the relaxation towards equilibrium. It is for this reason that traditional QFT, with few notable exceptions, is largely unable to properly detect and characterize pattern formation. Recent years have shown that pattern formation is relevant to a wealth of applications ranging from reaction-diffusion processes, nonlinear optics, nanostructures and fluid mechanics to hot plasma, traffic models, epidemic spreading, transport in heterogeneous media and neural networks. [13,16,17]

Understanding non-equilibrium phenomena and pattern formation is still in its infancy. Progress in this field has benefited from tools that have been recently developed for nonlinear dynamics, bifurcation and stability theory [13,15,18–22]. Our goal here is to explore the far-from-equilibrium sector of field theory using some of these

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newly developed methods. The underlying motivation is that nonlinear dynamics brings novel insights and a practical alternative for the analysis of field equations.

The paper is organized as follows: working at a classical level, we start from a non-equilibrium "toy" model containing an Abelian gauge field coupled to a massless scalar field. The concept of universality and the emergence of the complex Ginzburg-Landau equation (CGLE) are discussed in the next section. Mass generation through period-doubling bifurcations of CGLE and the link between CGLE and the generalized exclusion statistics (GES) follow from these premises. Summary and concluding remarks are detailed in the last section.

Our contribution needs to be exclusively regarded as a preliminary research on the topic. It is neither fully rigorous nor comprehensive. We wish to convey a new qualitative view rather than an in-depth analysis of phenomena. Independent studies are required to confirm, expand or refute these tentative findings.

A "toy" model in non-equilibrium field theory. – As mentioned earlier, nonlinear field theories amount to a large set of coupled differential equations that are difficult to solve or manage through numerical approximations. The universal nature of nonlinear dynamics near the threshold of the primary instability (see, *e.g.*, [16]) suggests a shortcut route. One can start from a plausible "toy" model and generalize results to more realistic theories. One example of such a "toy" model of classical field theory describes an Abelian gauge field $a_\mu(x,t)$ in interaction with a complex massless scalar field $\varphi=\varphi_1+i\varphi_2=\varphi(x,t)$. The Lagrangian density reads [23]

$$\mathrm{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+|D_\mu\varphi|^2. \tag{1}$$

Here, $\mu,\nu=0,1,2,3$ denote the space-time index, $x=(x_1,x_2,x_3)$ the spatial coordinate, $F^{\mu\nu}$ the gauge field tensor, e the coupling constant and

$$D_\mu=\partial_\mu+iea_\mu \tag{2}$$

the operator of covariant differentiation. If we take $\varphi_1\gg\varphi_2$ for simplicity, field equations derived from (1) are given by

$$\begin{aligned}&D^\mu(D_\mu\varphi)=0,\\&\partial^\nu F_{\mu\nu}=2e^2a_\mu\varphi^2.\end{aligned} \tag{3}$$

Developing (3) yields

$$\begin{aligned}&\Box\varphi=-ie(\varphi\partial^\mu a_\mu+a_\mu\partial^\mu\varphi-a^\mu\partial_\mu\varphi)-e^2a^\mu a_\mu\varphi,\\&\Box a_\mu=\partial^\nu\partial_\mu a_\nu-2e^2\varphi^2a_\mu,\end{aligned} \tag{4}$$

where $\Box=\partial^2/\partial t^2-\nabla^2$ is the d'Alembert operator. To further streamline the derivation and highlight the basic argument, we proceed by assuming that the gauge field satisfies

$$\partial_\mu a_i=0 \quad \text{for} \quad i=1,2,3. \tag{5}$$

If a_0 denotes the temporal component of the gauge field, the system (4) can be brought to the generic form of a coupled system of partial differential equations,

$$\begin{aligned}\partial_0\varphi&=\eta,\\\partial_0\eta&=f(\eta,\varphi,a_0,\xi,\nabla\eta,\ldots),\\\partial_0a_0&=\xi,\\\partial_0\xi&=g(\eta,\varphi,a_0,\xi,\nabla\eta,\ldots),\end{aligned} \tag{6}$$

in which $f(\ldots)$ and $g(\ldots)$ are time-evolution functions and $\partial_0=\partial/\partial t$. (6) may be presented in vector form as

$$\partial_0\mathbf{u}=U(\mathbf{u},\nabla\mathbf{u},\ldots), \tag{7}$$

where $\mathbf{u}=(\eta,\varphi,\xi,a_0)$. We next posit that transition to non-equilibrium in (7) is controlled by a small external parameter $\varepsilon\ll1$. This parameter is continuously adjustable and measures the departure from equilibrium ($\varepsilon_c=0$). Accordingly, (7) becomes

$$\partial_0\mathbf{u}=U(\mathbf{u},\nabla\mathbf{u},\ldots,\varepsilon). \tag{8}$$

The physical content of ε depends on the context of the problem at hand. In open systems ε encodes the combined effect of environmental and internal fluctuations [24]. Critical behavior in continuous dimension identifies ε with the Wilson-Fisher parameter of the regularization program ($\varepsilon=4-d$) [25,26]. In models involving fractional dynamics, ε characterizes the range of nonlocal interactions in space or the extent of temporal memory [22,27–29].

Universality and CGLE. – Non-equilibrium processes such as (8) display remarkable universality. Regardless of the specific application, macroscopic patterns that develop near the threshold of a dynamic instability are robust and largely insensitive to microscopic fluctuations [13,16,17].

Since one is familiar with the language of harmonic oscillations, we are interested in the simplest bifurcation in the dynamics of $\mathbf{u}(x,t)$ that creates oscillatory behavior. This is known as a Hopf bifurcation and represents the simplest transition that leads from a focus point to a periodic behavior. As the bifurcation point is approached, the focus point becomes unstable and gives rise to a harmonic limit cycle. CGLE is a *universal* model that holds for all pattern-forming systems undergoing a Hopf bifurcation [13,16]. The theory of the reduction to CGLE from generic systems of autonomous nonlinear equations such as (8) has been developed by several authors. The derivation of CGLE for a $(1+1)$-dimensional system starts from the ansatz

$$\mathbf{u}(x,t)=\mathbf{u}_0+A(\tilde{x},\tilde{t})\exp[i(k_cx-\Omega_ct]\mathbf{u}_1+\text{c.c.}, \tag{9}$$

where $\tilde{x}$, $\tilde{t}$ represent slow variables and k_c, Ω_c are critical values in wave number and frequency spaces. Replacing in (8), dropping the tildes and expanding in power series of the small parameter $\tilde{\varepsilon} = \varepsilon - \varepsilon_c$ leads to CGLE in its standard form

$$\partial_t A = A + (1 + ic_1)\nabla^2 A - (1 - ic_3)\left|A\right|^2 A. \tag{10}$$

Here,

$$A(x,t) = \rho(x,t)\exp[-i\Phi(x,t)] \tag{11}$$

is a complex-valued amplitude defining the slow modulation in space and time of the underlying periodic pattern. The real parameters c_1, c_3 denote the linear and nonlinear dispersion parameters, respectively. The limit $c_1, c_3 \to 0$ corresponds to the real Ginzburg-Landau equation, whereas $c_1^{-1},\ c_3^{-1} \to 0$ recovers the nonlinear Schrödinger equation.

Higgs-free generations of particle masses. – Among the simplest coherent structures generated by CGLE are plane-wave solutions having the form [13,16]

$$\begin{aligned} A(x,t) &= A_0 \exp[-i(qx + mt)] + \text{c.c.}, \\ A_0 &= \sqrt{1-q^2}. \end{aligned} \tag{12}$$

The frequency m satisfies the dispersion equation

$$m_q = c_1 q^2 - c_3(1-q^2) \tag{13}$$

and $q \in [-1,1]$ represents the phase gradient of the complex amplitude (12),

$$q = -\nabla\left|\Phi\right| \tag{14}$$

Linear stability analysis of (12) reveals that plane waves having a wave number larger than the so-called Eckhaus threshold

$$q_E = \sqrt{\frac{1 - c_1 c_3}{2(1+c_3^2) + 1 - c_1 c_3}} \tag{15}$$

are unstable with respect to the long-wavelength modulation. In particular, a vanishing Eckhaus threshold leads to the Benjamin-Feir-Newell (BFN) instability criterion (A.1)

$$c_1 c_3 = 1. \tag{16}$$

The dispersion equation (13) has two complementary limits: $q = \pm 1$ ($A_0 = 0$) and $q = 0$ ($A_0 = \pm 1$). Arguments presented in appendix A suggest a natural identification of these two modes with fermion and electroweak gauge boson fields, respectively. Equation (14) implies that fermions have a non-vanishing and uniform phase gradient $\nabla\Phi \neq 0$, whereas gauge bosons have a uniform phase and a vanishing phase gradient $\nabla\Phi = 0$. Although we have started from a massless model, from (13) and (16) it follows that both these modes acquire non-vanishing masses. In non-dimensional form and near the BFN instability, the two sets of masses are

$$\begin{aligned} m_\pm &= c_1, \\ m_0 &= -c_3, \end{aligned} \tag{17a}$$

such that

$$m_\pm = \left|m_0\right|^{-1}. \tag{17b}$$

Table 1: Actual *vs.* predicted mass scaling ratios for $\bar{\delta} = 3.9$.

Parameter ratio	Behavior	Actual	Predicted
m_u/m_c	$\bar{\delta}^{-4}$	3.365×10^{-3}	4.323×10^{-3}
m_c/m_t	$\bar{\delta}^{-4}$	3.689×10^{-3}	4.323×10^{-3}
m_d/m_s	$\bar{\delta}^{-2}$	0.052	0.066
m_s/m_b	$\bar{\delta}^{-2}$	0.028	0.066
m_e/m_μ	$\bar{\delta}^{-4}$	4.745×10^{-3}	4.323×10^{-3}
m_μ/m_τ	$\bar{\delta}^{-2}$	0.061	0.066
M_W/M_Z	$\left(1-\bar{\delta}^{-1}\right)^{1/2}$	0.8823	0.8623

It is known that plane-wave solutions consist of both positive and negative frequencies. Because mass is positive definite, in what follows we are limiting the discussion to the cases $c_1 > 0$ and $c_3 < 0$.

The Feigenbaum-Sharkovskii-Magnitskii (FSM) paradigm. – The FSM paradigm of universal transition to chaos in nonlinear dissipative systems is briefly detailed in appendix B. Extensive numerical data [20,21] show that both parameters of linear and nonlinear dispersion c_1, c_3 of (17a) are distributed in a geometric progression, that is

$$\begin{aligned} c_{1,n} &= c_{1,\infty} + K_1\bar{\delta}^{-n}, \\ c_{3,n} &= c_{3,\infty} + K_2\bar{\sigma}^{-n}, \end{aligned} \tag{18}$$

where $\bar{\delta}$, $\bar{\sigma}$ are scaling constants and $n = 1,2,3\ldots$ represents the number of tori accumulated through bifurcations. Since $K_1, K_2, c_{1,\infty}$ and $c_{3,\infty}$ are independent of n, they can be both absorbed into a redefinition of masses. We have, accordingly,

$$\begin{aligned} m_n^* &= \frac{1}{K_1}(m_{\pm,n} - c_{1,\infty}), \\ M_n &= \frac{1}{K_2}(m_{0,n} - c_{3,\infty}). \end{aligned} \tag{19}$$

The ratios of two arbitrary masses in the bifurcation sequence take the form

$$\begin{aligned} \frac{m_n^*}{m_{n+p}^*} &= \bar{\delta}^p, \\ \frac{M_n}{M_{n+p}} &= \bar{\sigma}^p, \end{aligned} \tag{20}$$

in which $p = 2^k,\ k = 1,2,3\ldots$. Based on (17) it can be concluded that, near the BFN instability, the two scaling constants are linked to each other.

Analysis of the Renormalization Group flow for the real Ginzburg-Landau equation leads to the following relationship between $\bar{\delta}$ and $\bar{\sigma}$ [28]:

$$1 - \left(\frac{M_1}{M_2}\right)^2 = 1 - \left(\bar{\sigma}^1\right)^2 \approx \frac{1}{\bar{\delta}}, \tag{21}$$

where $M_1 = M_W,\ M_2 = M_Z$ are vector boson masses. Table 1 shows a side-by-side comparison between predictions inferred from (20) and experiment, where $\bar{\delta} = 3.9$

Table 2: Actual values of elementary particle masses.

Parameter	Value	Units
m_u	2.12	MeV
m_d	4.22	MeV
m_s	80.90	MeV
m_c	630	MeV
m_b	2847	MeV
m_t	170,800	MeV
M_W	80.46	GeV
M_Z	91.19	GeV

represents the numerical value of the scaling constant that best fits laboratory data [30]. Actual values of particle masses, computed at the reference scale given by the mass of the top quark [31], are listed in table 2. Note that the choice of the mass scale is completely arbitrary since (20) involves ratios of consecutive masses.

CGLE and generalized exclusion statistics. – Dispersion relation (13) indicates that plane-wave solutions of CGLE interpolate between gauge boson states ($q=0$) and fermion states ($q=\pm 1$). From (13) and (14) it follows that the *spin* associated with an arbitrary mixed state is given by[1]

$$\sigma = 1 - \frac{(\nabla\Phi)^2}{2}. \tag{22}$$

From this standpoint, CGLE is remarkably similar to the framework describing quantum fractional statistics in condensed-matter physics. In what follows we briefly discuss this analogy. The generalized exclusion statistics (GES) is motivated by the properties of quasi-particles occurring in the fractional quantum Hall effect [32,33]. Consider a thermodynamic ensemble of N identical particles. Let d represent the dimension of the one-particle Hilbert space obtained by fixing the coordinates of the remaining $N-1$ particles. The statistics of a particle is defined by the so-called Haldane's parameter g,

$$g = -\frac{\partial d(N)}{\partial N} \approx -\frac{d(N+\Delta N)-d(N)}{\Delta N}. \tag{23}$$

Because any given state can be populated by any number of bosons, $d(N+\Delta N)=d(N)$ and hence $g=0$. By contrast, the Pauli exclusion principle restricts fermions to $g=1$. Quasi-particles with mixed statistics are characterized by an intermediate value of g and are said to satisfy a generalized exclusion principle. In this case, it can be shown that thermodynamic quantities such energy, heat capacity or entropy can be expressed in factorized form. In particular, the energy of the quasi-particle ensemble is given by

$$E(g) = gE(1) + (1-g)E(0). \tag{24}$$

[1]Strictly speaking, spin is a concept that is valid only in a quantum or semi-quantum context. Since our analysis is carried out at the classical level, (22) is meant to simply denote a numerical attribute of plane waves dependent on the wave number q.

Table 3: Comparison between CGLE and GES.

CGLE	GES
$q = -\dfrac{\partial\lvert\Phi\rvert}{\partial x}$	$g = -\dfrac{\partial d}{\partial N}$
$m_q = c_1 q^2 - c_3(1-q^2)$	$E_g = gE(1) + (1-g)E(0)$

An example of this type of objects is offered by *anyons*, quasi-particles that exist in two dimensions and carry fractional charges. When two particles of a system of bosons are exchanged, the phase of the system remains unchanged, whereas for a system of fermions it changes by exactly π. Exchanging two anyons results in a phase factor that falls between zero and π. Anyons play a key role in the fractional quantum Hall effect and high-temperature superconductivity [32,33].

A short comparison between plane-wave solutions of CGLE and GES is included in table 3.

Summary and conclusions. – This brief report has been motivated by recent advances in nonlinear dynamics and complexity theory. Exploiting the universal theory of transition to chaos in nonlinear dissipative systems, we have found that:

a) particles acquire mass as plane-wave solutions of CGLE, without reference to the hypothetical Higgs scalar or to a particular symmetry breaking mechanism. As of today, the reality of the Higgs doublet and nature of electroweak symmetry breaking are issues that remain unsettled.

b) Starting from a basic model of Abelian gauge bosons in interaction with scalar fields, CGLE leads to a natural separation of heavy *non-relativistic modes* ($q=0$) from light *relativistic modes* of maximal group velocity ($q=\pm 1$). The most straightforward interpretation of this result is that the first group of modes corresponds to electroweak gauge bosons and the second group to fermions.

c) A direct connection may be set up between GES in condensed-matter physics and the dispersion relation (13) corresponding to $q\neq 1$. Although different in methodology and content, both GES and CGLE point out that fractional quantum statistics and non-equilibrium field theory enable a *dynamic unification* of gauge bosons and fermions as particles with arbitrary spin. This is in contrast with super-symmetry and related models (see, *e.g.*, [34]) which are based on extended symmetry groups and pay virtually no attention to nonlinear dynamics of underlying fields.

We close this section with two short remarks: 1) the approach developed here is based on *classical* field theory. Needless to say, a realistic model cannot ignore the quantum nature of fields evolving in four-dimensional space-time. However, as previously pointed out, future results are not expected to substantially deviate from these initial findings because of *universality arguments* related to nonlinear dynamics of (8) and CGLE (see, *e.g.*, [13,18,20]); 2) although our approach bypasses the conventional Higgs mechanism, it still remains compatible with it. The standard model asserts that particle masses are generated through electroweak symmetry breaking and are attributed to the Yukawa couplings of the fermions (g_f) to the Higgs condensate (v_{H^0}). The ratio of two arbitrary fermion masses in the spectrum is given by

$$\frac{m_f}{m_{f'}} = \frac{g_f(\mathrm{v}_{H^0})}{g_{f'}(\mathrm{v}_{H^0})} = \frac{g_f}{g_{f'}}. \tag{25}$$

It follows that the mass hierarchy shown in table 1 may be simply interpreted as reflecting the hierarchy of the corresponding Yukawa couplings.

Future research may be focused on a deeper understanding of pattern formation and its ramifications in the realm of SM and beyond. Of key interest is the emergence of novel states in the TeV range of particle physics. This probing energy will become accessible in the near future at the large hadron collider and other accelerator sites [35].

Appendix A. – The two dispersion parameters of CGLE are subject to the following dynamic constraints [13,16,17]:

a) the Benjamin-Feir-Newell (BFN) criterion states that stability becomes borderline for

$$c_1 c_3 = 1; \tag{A.1}$$

b) using (13), the group velocity of the plane-wave solutions is given by

$$\mathrm{v}_g = 2q(c_1 + c_3). \tag{A.2}$$

Compliance with relativity bounds (A.2) to a constant that represents the normalized value of light speed *in vacuo*. It is clear that $q = 0$ represents a *slow mode* (massive gauge boson), while $q = \pm 1$ describes the *fastest mode* (relativistic fermions). Masses associated with these modes are supplied by (17). From the BFN criterion it follows that the borderline value of the normalization constant $Q = \frac{\mathrm{V}_{g,\max}}{2}$ can be determined from

$$\begin{aligned} c_1 &= \frac{Q \pm \sqrt{Q^2 - 4}}{2} \Rightarrow Q \geqslant 2, \\ c_3 &= \frac{1}{c_1}. \end{aligned} \tag{A.3}$$

Equations (A.1) and (A.2) imply that, close to the border of BFN instability, gauge boson and fermion masses scale as dual entities. This finding is consistent with the behavior of the last entry in table 1.

Appendix B. – Consider the following boundary value problem for CGLE in $1+1$ space-time dimensions [20,21,36]:

$$\begin{aligned} &\partial_t A = A + (1 + ic_1)\partial_x^2 A - (1 - ic_3)\,|A|^2\, A, \\ &\partial_x A(0,t) = \partial_x A(L,t) = 0, \quad A(x,0) = A_0(x), \\ &0 \leqslant x \leqslant L,\ 0 \leqslant t \leqslant \infty. \end{aligned} \tag{B.1}$$

This model can be reduced to a three-dimensional system of nonlinear ordinary differential equations with the help of the Galerkin few-modes approximation:

$$A(x,t) \approx \sqrt{\xi(t)}\exp[i\theta_1(t)] + \sqrt{\eta(t)}\exp[i\theta_2(t)]\cos\left(\frac{\pi}{L}x\right) \tag{B.2}$$

in which

$$\begin{aligned} \partial_t \xi &= f_1(\xi, \eta, \theta, c_1, c_3, L), \\ \partial_t \eta &= f_2(\xi, \eta, \theta, c_1, c_3, L), \\ \partial_t \theta &= f_3(\xi, \eta, \theta, c_1, c_2, L) \end{aligned} \tag{B.3}$$

with $\theta(t) = \theta_2(t) - \theta_1(t)$. It can be shown that the transition to chaos in (B.3) occurs through a *sequential cascade of bifurcations* in three separate stages. This cascade starts with the Feigenbaum scenario of period-doubling bifurcations of stable cycles, followed by the Sharkovskii subharmonic cascade and ending with the Magnitskii cascade of stable homoclinic cycles.

REFERENCES

[1] Mandl F. and Shaw G., *Quantum Field Theory* (John Wiley & Sons) 1993.
[2] Zinn-Justin J., *Quantum Field Theory and Critical Phenomena* (Clarendon Press) 2002.
[3] Amit D. J. and Martin-Mayor V., *Field Theory, the Renormalization Group and Critical Phenomena* (World Scientific) 2005.
[4] Greiner W. and Reinhardt J., *Field Quantization* (Springer-Verlag) 1993.
[5] Segal I. E., *Phys. Scr.*, **24** (1981) 827.
[6] Volkov M. K. and Pervushin V. N., *Sov. Phys. Usp.*, **20** (1977) 89.
[7] Heisenberg W., *Introduction to the Unified Field Theory of Elementary Particles* (Interscience Publishers) 1966.
[8] Dzhunushaliev V. and Singleton D., *Phys. Rev. D*, **65** (2002) 125007.
[9] Miranskii V. A., *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific) 1993.
[10] Wilczek F., *Rev. Mod. Phys.*, **71** (1999) S85.
[11] Goldfain E., *Int. J. Nonlinear Sci. Numer. Simul.*, **6** (2005) 223.
[12] Goldfain E., *Commun. Nonlinear Sci. Numer. Simul.*, **13** (2008) 1397.
[13] Cross M. C. and Hohenberg P. C., *Rev. Mod. Phys.*, **65** (1993) 851.
[14] Tsoy E. N. *et al.*, *Phys. Rev. E*, **73** (2006) 036621.

[15] Farias R. S. L. *et al.*, *Nucl. Phys. A*, **782** (2007) 33.
[16] Aranson I. S. and Kramer L., *Rev. Mod. Phys.*, **74** (2002) 99.
[17] Gollub J. P. and Langer J. S., *Rev. Mod. Phys.*, **71** (1999) S396.
[18] Hinrichsen H., *Physica A*, **369** (2006) 1.
[19] Jona-Lasinio G. and Mitter P. K., *Commun. Math. Phys.*, **101** (1985) 409.
[20] Magnitskii N. A. and Sidorov E., *New Methods for Chaotic Dynamics* (World Scientific) 2007.
[21] Magnitskii N. A., *Commun. Nonlinear Sci. Numer. Simul.*, **13** (2007) 416.
[22] Zaslavsky G. M., *Hamiltonian Chaos and Fractional Dynamics* (Oxford University Press) 2005.
[23] Ryder L., *Quantum Field Theory* (Cambridge University Press) 1996.
[24] Kristensen K. and Moloney N. R., *Complexity and Criticality* (Imperial College Press) 2005.
[25] Ballhausen H. *et al.*, *Phys. Lett. B*, **582** (2004) 144.
[26] Ballhausen H., *Renormalization Group Flow Equations and Critical Phenomena in Continuous Dimension and at Finite Temperature*, Doctoral Thesis, Faculty of Physics and Astronomy, Ruprecht-Karls-University Heidelberg, 2003.
[27] Goldfain E., *Commun. Nonlinear Sci. Numer. Simul.*, **13** (2008) 666.
[28] Goldfain E., *Int. J. Nonlinear Sci.*, **3** (2007) 170.
[29] Goldfain E., *Chaos Solitons Fractals*, **28** (2006) 913.
[30] Goldfain E., *Int. J. Bifurcat. Chaos*, **18** (2008) 1.
[31] Particle Data Group, `http://pdg.lbl.gov/2005/reviews/quarks_q000.pdf`., 2005.
[32] Haldane F. D. M., *Phys. Rev. Lett.*, **67** (1991) 937.
[33] Nayak C. and Wilczek F., *Phys. Rev. Lett.*, **73** (1994) 2740.
[34] Wess J. and Bagger J., *Supersymmetry and Supergravity* (Princeton University Press) 1992.
[35] Large Hadron Collider: Periodic updates on LHC may be located at `http://public.web.cern.ch/Public/Content/Chapters/AboutCERN/CERNFuture/WhatLHC/WhatLHC-en.html`.
[36] Akhromeeva T. S. *et al.*, *Nonstationary Structures and Diffusion Chaos* (Nauka-Moscow) 1992.