NO-COUNTERTERM APPROACH TO QUANTUM FIELD THEORY

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Abstract. We give a conjectural way for computing the S-matrix and the correlation functions in quantum field theory beyond perturbation theory. The basic idea seems universal and naively simple: to compute the physical quantities one should consider the functional differential Schrodinger equation (without normal orderings!), regularize it, consider the regularized evolution operator in the Fock space from $t = T_1$ to $t = T_2$, where the interval (T_1, T_2) contains the support of the interaction cutoff function, remove regularization (without adding counterterms!), and tend the interaction cutoff function to a constant.

We call this approach to QFT the No-Counterterm approach. We show how to compute the No-Counterterm perturbation series for the φ^4 model in $\mathbb{R}^{d+1}$. We give rough estimates which show that some summands of this perturbation series are finite without renormalization (in particular, one-loop integrals for $d = 3$ and all integrals for $d \geq 6$).

1. The main conjecture

In this paper we propose a conjectural way for exact computing of the S-matrix and the Green functions of quantum field theory. Recall that the Schrodinger functional differential equation reads

$$ih\frac{\partial\Psi}{\partial t} = \hat{H}(t)\Psi, \tag{1}$$

where $\Psi = \Psi(t, \varphi(\cdot))$ is the unknown "half-form" on the space of functions $\varphi(\mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_d)$, $\hat{H}(t) = H(t, \hat{\varphi}(\cdot), \hat{\pi}(\cdot))$ is the quantum Hamiltonian of the theory, and the operators $\hat{\varphi}(\mathbf{x}) = \varphi(\mathbf{x})$ and $\hat{\pi}(\mathbf{x}) = -ih\frac{\delta}{\delta\varphi(\mathbf{x})}$ satisfy the canonical commutation relations

$$[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')] = ih\delta(\mathbf{x} - \mathbf{x}'), \quad [\hat{\varphi}(\mathbf{x}), \hat{\varphi}(\mathbf{x}')] = [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')] = 0. \tag{2}$$

For the relativistically invariant generalization of the functional differential Schrodinger equation, see [1]. For example, for the scalar field

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with self-action in $\mathbb{R}^{d+1}$ the Hamiltonian reads

$$H(t,\varphi(\cdot),\pi(\cdot)) = H_0(\varphi(\cdot),\pi(\cdot)) + \int\left(\frac{1}{k!}g(t,\mathbf{x})\varphi(\mathbf{x})^k + j(t,\mathbf{x})\varphi(\mathbf{x})\right)d\mathbf{x},$$
$$H_0(\varphi(\cdot),\pi(\cdot)) = \int\frac{1}{2}\left(\pi(\mathbf{x})^2 + \sum_{j=1}^{d}\varphi_{x_j}(\mathbf{x})^2 + m^2\varphi(\mathbf{x})^2\right)d\mathbf{x}. \tag{3}$$

Here $g(t,\mathbf{x})$ (the interaction cutoff function) and $j(t,\mathbf{x})$ (the source) are smooth functions with compact support. For simplicity of exposition, below we restrict ourselves by this model. (One can see that equation (1,3) has no nonzero solutions if Ψ is a usual functional of $\varphi(\mathbf{x})$, see [1].)

Let us regularize the operators $\hat{\pi}(\mathbf{x})$ and $\hat{\varphi}(\mathbf{x})$, as in [2], as follows: consider the delta-like family of smooth functions with compact support $f_\Lambda(\mathbf{x}) \to \delta(\mathbf{x})$, where $\Lambda \to \infty$ is the regularization parameter (the ultraviolet regularization at small distances), and a family of smooth functions with increasing compact support $g_L(\mathbf{x}) \to 1$ as $L \to \infty$ (the infrared regularization at big distances), and put

$$\hat{\varphi}_{\Lambda,L}(\mathbf{x}) = g_L(\mathbf{x})\int f_\Lambda(\mathbf{x}-\mathbf{x}_1)\hat{\varphi}(\mathbf{x}_1)d\mathbf{x}_1,$$
$$\hat{\pi}_{\Lambda,L}(\mathbf{x}) = g_L(\mathbf{x})\int f_\Lambda(\mathbf{x}-\mathbf{x}_1)\hat{\pi}(\mathbf{x}_1)d\mathbf{x}_1. \tag{4}$$

Consider the regularized Schrodinger functional differential equation

$$ih\frac{\partial\Psi}{\partial t} = \hat{H}^{\Lambda,L}(t)\Psi, \tag{5}$$

where

$$\hat{H}^{\Lambda,L}(t) = H(t,\hat{\varphi}_{\Lambda,L}(\cdot),\hat{\pi}_{\Lambda,L}(\cdot)). \tag{6}$$

The regularized quantum Hamiltonian $\hat{H}^{\Lambda,L}(t)$ and the regularized free quantum Hamiltonian

$$\hat{H}_0^{\Lambda,L} = H_0(\hat{\varphi}_{\Lambda,L}(\cdot),\hat{\pi}_{\Lambda,L}(\cdot)) \tag{7}$$

are well-defined and regular operators in the Fock Hilbert space of functionals

$$\Psi(\varphi(\cdot)) = \Psi_0(\varphi(\cdot))\exp\left(-\frac{1}{2h}\int\tilde{\varphi}(\mathbf{p})\tilde{\varphi}(-\mathbf{p})\omega_{\mathbf{p}}d\mathbf{p}\right),$$

where $\mathbf{p} = (p_1,\ldots,p_d)$, $\tilde{\varphi}(\mathbf{p}) = \frac{1}{(2\pi)^{n/2}}\int e^{-i\mathbf{p}\mathbf{x}}\varphi(\mathbf{x})d\mathbf{x}$, $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2}$.

Denote by $U_{\Lambda,L}(T_1,T_2)$ the evolution unitary operator of equation (5) in the Fock space from $t = T_1$ to $t = T_2$, and choose the numbers

$-T_1, T_2$ so large that the supports of the functions $g(t, \mathbf{x})$ and $j(t, \mathbf{x})$ be contained in the interval (T_1, T_2). Denote

$$S_{\Lambda,L}(g(\cdot), j(\cdot)) = e^{iT_2 \hat{H}_0^{\Lambda,L}/h} U_{\Lambda,L}(T_1, T_2) e^{-iT_1 \hat{H}_0^{\Lambda,L}/h}. \tag{8}$$

Clearly, this unitary operator in the Fock space does not depend on T_1, T_2.

The Main Conjecture. *The strong limit*

$$S(g(\cdot), j(\cdot)) = \lim_{\Lambda, L \to \infty} S_{\Lambda,L}(g(\cdot), j(\cdot)) \tag{9}$$

is correctly defined modulo multiplication by a phase factor e^{ic}, for c a real number, and does not depend on the way of regularization (i. e., on the choice of the functions $f_\Lambda(\mathbf{x})$, $g_L(\mathbf{x})$). The strong limit

$$S(g, j(\cdot)) = \lim_{g(t,\mathbf{x}) \to g} S(g(\cdot), j(\cdot)) \tag{10}$$

exists and coincides with the generating functional for the operator Green functions, and the unitary operator

$$S(g) = S(g, j \equiv 0) \tag{11}$$

coincides with the physical S-matrix.

This Conjecture is partly a mathematical conjecture, and partly a conjectural physical law.

2. Discussion

In this Section we present heuristic arguments in favor of the Main Conjecture from §1, and discuss the mathematical and physical contents of this Conjecture.

Regarding the mathematical contents of the Conjecture, one can imagine that there exists a space of distribution "half-forms" (or "half-densities") Ψ on the Schwartz space of functions $\varphi(\mathbf{x})$, and that there exists a mathematical theory of functional differential equations (for example, like the Schrodinger functional differential equation above) with solutions in this space of half-forms. (It was the main aim of Dirac in his book [3] to construct a similar space for fermions.) The Fock spaces are parts of this space of half-forms. Then a surprising and mysterious fact which follows from the physical picture and which I do not understand, is that the result of evolution of the Schrodinger functional differential equation with the initial conditions in the Fock space at $t = T_1$, returns to the Fock space at $t = T_2$. It is clear that between $t = T_1$ and $t = T_2$ the vector Ψ leaves the Fock space. In [4,5] it is conjectured that under the evolution of the Schrodinger functional differential equation and its relativistically invariant generalization from the surfaces $t = const$ to curved space-like surfaces in space-time, the Fock space evolves into a family of Hilbert spaces parameterized by space-like surfaces, and the generalized Schrodinger equation yields an integrable flat connection in this family. Even for the free scalar field, it is proved in [6] that the result of evolution of the generalized Schrodinger equation from the surface $t = const$ to a curved space-like surface, for $d > 1$ leaves the Fock space. The fact that the solution of the functional differential Schrodinger equation returns to the Fock space, is mathematically confirmed by results of the theory of complex germ of Maslov and Shvedov ([2], cf. [7]), which state that the result of quasiclassical evolution of the functional differential Schrodinger equation along any classical trajectory in the phase space returns to the Fock space.

Regarding the physical contents of the Conjecture, one should prove that for the renormalizable theories, the Taylor series of the S-matrix $S(g, j(\cdot))$ at $g = 0$, $j \equiv 0$ coincides with the renormalized perturbation series for the generating functional of operator Green functions of the theory, since these renormalized perturbation series are well checked by experiment. Let us sketch a plan of such a proof.

In the book [8] by Bogolubov and Shirkov, the renormalized perturbation series for the S-matrix and the Green functions are constructed as the limit as $g(t, \mathbf{x}) \to g = const$ of a more general object, the renormalized perturbation series $\tilde{S}(g(\cdot), j(\cdot))$ with non-constant interaction cutoff function $g(t, \mathbf{x})$. This object is almost uniquely (up to the change of parameters m, $g(x)$) characterized by the properties of unitarity, causality, Lorentz invariance, and the correspondence principle stating that the coefficient before $g(x)$ in $\tilde{S}(g(\cdot), 0)$ coincides with the normally ordered interaction Lagrangian. After taking the limit $g(x) \to g$, the parameters are fixed uniquely by conditions on the Green functions of the theory (e. g., for the φ^4 theory in $\mathbb{R}^{3+1}$, the condition that the two-point Green function $G^{(2)}(p_1, p_2)$ has poles at $p_i^2 = m^2$, and the four-point one-particle irreducible Green function $\Gamma^{(4)}(p_1, p_2, p_3, p_4)$ equals g at the point $p_1 = p_2 = p_3 = p_4 = 0$).

Note that the conditions of unitarity and causality are fulfilled for any evolution operator (or limit of evolution operators) of unitary evolution differential equations with $g(t, \mathbf{x})$, $j(t, \mathbf{x})$ as coefficients. Hence our operator $S(g(\cdot), j(\cdot))$ and its Taylor series at 0 satisfy these conditions. Regarding Lorentz invariance for $S(g(\cdot), j(\cdot))$, it follows from the fact that the Schrodinger functional differential equation (and its regularizations) admit a relativistically invariant generalization, the generalized Schrodinger equation [1] which forms an integrable flat connection over the family of space-like surfaces. Finally, the correspondence principle, say, for the φ^4 model in $\mathbb{R}^{3+1}$ is an easy direct computation. Therefore, the Taylor series of our $S(g(\cdot), j(\cdot))$ coincides with one of Bogolubov S-matrices $\tilde{S}(g(\cdot), j(\cdot))$. The remaining check of parameters as $g(x) \to g$ should not be a difficult task. We are so sure that we obtain the right result due to our final argument which is the inner conceptual simplicity of the theory.

Finally, note that regarding computational part of our approach, it yields an algorithm of computation different from the renormalization in the Feynman diagram technique. This is seen, for example, already on the φ^4 model (see below). However, this part of our investigation is not finished yet, so we leave it as a challenging problem, especially for physically interesting theories such as Yang–Mills theory or quantum gravity.

For further problems closely related to this paper, see [9].

3. No-Counterterm perturbation series for the φ^4 model: the setup

Traditional perturbation series for the φ^4 model is obtained by renormalization of the expression

$$T \exp \int g \, :\varphi(x)^4: /4! \, dx, \tag{12}$$

where dots denote the normal ordering, and $\varphi(x)$ is the free scalar field, $x \in \mathbb{R}^{d+1}$. It is easy to see that the perturbation series for the S-matrix in the No-Counterterm approach is obtained by developing the expression

$$T \exp \int g\varphi(x)^4/4! dx \tag{13}$$

(without normal orderings) into power series with respect to g. This means that we first regularize the operator

$$\varphi(x)^4 \to \operatorname{reg} \varphi(x)^4 = \varphi_{\Lambda,L}(x)^4,$$

where $\Lambda \to \infty$ is the parameter of the ultraviolet regularization at small distances (and large momenta), and $L \to \infty$ is the parameter of the infrared regularization at large distances (and small momenta). Next, we develop the regularized integral (13) into series over powers of g, and finally we omit the regularization.

To perform this procedure, note first that we have

$$\varphi_{\Lambda,L}(x)^4/4! =: \varphi_{\Lambda,L}(x)^4 : /4! + C_{\Lambda,L} : \varphi_{\Lambda,L}(x)^2 : /2 + const, \tag{14}$$

where $C_{\Lambda,L}$ and *const* are certain divergent constants. The latter constant can be neglected, since we are interested in the expression only modulo an overall phase factor (see §1). Now the regularized integral (13) can be developed into series by usual Feynman diagram techniques, using (14) (see, for example, [8]). The quadratic term in (14) means that we change the propagator as follows:

$$\begin{aligned}\frac{1}{p^2-m^2+i\varepsilon} &\to \frac{1}{p^2-m^2+i\varepsilon} + \frac{1}{p^2-m^2+i\varepsilon} gC_{\Lambda,L} \frac{1}{p^2-m^2+i\varepsilon} \\ &+ \frac{1}{p^2-m^2+i\varepsilon} gC_{\Lambda,L} \frac{1}{p^2-m^2+i\varepsilon} gC_{\Lambda,L} \frac{1}{p^2-m^2+i\varepsilon} + \dots .\end{aligned} \tag{15}$$

This sum of a geometric progression converges, for g small enough, to the new propagator

$$\frac{1}{p^2-m^2+i\varepsilon-gC_{\Lambda,L}}. \tag{16}$$

In the next Section the regularized integrals corresponding to Feynman diagrams with this new propagator are tested to converge as $\Lambda, L\to\infty$.

4. Rough estimates

We consider the ultraviolet cutoff regularization ($d=3$)

$$\begin{aligned}&\mathrm{reg}\, f(p) = \mathrm{reg}\, f(p_0,\mathbf{p}) = \mathrm{reg}\, f(p_0,p_1,p_2,p_3) \\ &= 0 \text{ if } |p_i|\ge\Lambda \text{ for some } i, 0\le i\le 3.\end{aligned} \tag{17}$$

If the mass $m>0$, then we shall not need the infrared regularization at all.

The computation shows that (for any time t)

$$\begin{aligned}C_{\Lambda,L} &= 6\,\mathrm{reg}\int[\varphi_-(t,\mathbf{p}),\varphi_+(t,\mathbf{p}')]d\mathbf{p}d\mathbf{p}' \\ &= \mathrm{reg}\int\frac{6h\delta(\mathbf{p}+\mathbf{p}')}{2\sqrt{\mathbf{p}^2+m^2}}d\mathbf{p}d\mathbf{p}' \sim 3h\Lambda^2.\end{aligned} \tag{18}$$

Substituting this into the propagator, one obtains for the one-loop "fish" diagram the following expression:

$$\mathrm{reg}\int\frac{1}{(p^2-m^2+i\varepsilon-3gh\Lambda^2)((k-p)^2-m^2+i\varepsilon-3gh\Lambda^2)}dp. \tag{19}$$

(Here k is the sum of ingoing 4-momenta of the diagram.) Let us divide each of the two brackets in the denominator by Λ^2. Then the integrand becomes ~ 1, and the integration domain is a 4-cube of size Λ. Hence the whole integral is $\sim \Lambda^{-4}\Lambda^4 \sim 1$, and it is finite.

However, for the simplest two-loop diagram with two outgoing edges we have the integral

$$\begin{aligned} \text{reg} \int & \frac{1}{(p^2 - m^2 + i\varepsilon - 3gh\Lambda^2)(q^2 - m^2 + i\varepsilon - 3gh\Lambda^2)} \\ & \times \frac{1}{(k-p-q)^2 - m^2 + i\varepsilon - 3gh\Lambda^2} dpdq, \end{aligned} \tag{20}$$

and the same argument shows that the integral diverges as $\Lambda^{-6}\Lambda^8 \sim \Lambda^2$.

5. Conclusion

Thus, if we believe into the No-Counterterm Conjecture, we should conclude that the estimate above is too rough for the two-loop diagram. Otherwise, if all the estimates above are correct, we see that for $d = 3$ the "No-Counterterm approach" is not valid and requires counterterms, as well as the traditional approach.

It seems that our estimate is correct if considered as an upper bound for the integral. As a lower bound it can be incorrect.

For a general φ^4 diagram in $(d+1)$-dimensional space-time, the same argument as above gives the following estimate of the diagram integral. Denote by E_i (E_e) the number of internal (respectively external) edges of the diagram, by L the number of independent loops, by V the number of vertices. Assume $d > 2$. Then the following Theorem holds:

Theorem. *The integral is no greater than* $O(\Lambda^m)$*, where*

$$\begin{aligned} m &= -(d-1)E_i + (d+1)L \\ &= (d+1)(L - E_i) + 2E_i \\ &= (d+1)(1 - V) + 2E_i \quad (\textit{since } V - E_i + L = 1) \\ &= (d+1)(1-V) + 4V - E_e \quad (\textit{since } 4V = 2E_i + E_e) \\ &= d + 1 - (d-3)V - E_e. \end{aligned} \tag{21}$$

Therefore, if the sign of m *is negative, then the limit of the integral is zero.*

At least, for $d = 3$, $L = 1$ and for $d \geq 6$ all the diagrams seemingly converge.

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