On Speeds in Special Relativity

Edward Kapuścik[1] and Rajmund Orlicki[1,2]
[1]Alfred Meissner Graduate School for Dental Engineering, Ustroń, Poland
[2] Silesian Medical University, Katowice, Poland

February 28, 2012

Abstract

It is shown that the commonly accepted fundamental paradigm of Special Relativity on the maximality of the speed of light is false. Tachyons, the superluminal objects, are allowed by Special Relativity equally well as subluminal particles are. However, for tachyons there do not exist rest frames and the notion of rest mass or rest energy for tachyons cannot be operationally defined. Instead of that the notion of energy in the reference frame where tachyons move with infinite speed is defined. The expressions for energy and momentum of superluminal objects are derived. The momentum vanishes for objects moving with infinite speed. This may explain the weakness of interaction of superluminal objects with other objects. The paper appeals for a change in teaching Special Relativity.

1 Introduction

The speed of light plays a distinguished role in Special Theory of Relativity. It is generally believed that, apart from its invariant character, the speed of light is also the maximal speed allowed by Special Relativity. This believe is continuously repeated by many prominent scientists. For example, Stephen Hawking in his *A Brief History of Time*[1] wrote "that nothing may travel faster than the speed of light". Next, more carefully, he stated that "any normal object is forever confined by relativity to move at speeds slower than the speed of light" without specification what the normal objects are. Below we shall show that by "normal" objects we should understand objects which possess rest frames.

The same or similar statements can be found in an endless number of textbooks and scientific articles. As a matter of fact, such statement is commonly treated as one of the fundamental paradigm of modern science. In spite of this, in the present paper, we shall show that this paradigm is false. The correct statement should be that "no object which posses rest frame (in particular, no observer) can move faster than the speed of light". But there exist objects (tachyons) which can move faster than light but never can be at rest. In this sense tachyons are not "normal" objects.

In the past there where many attempts to show that superluminal speeds can be incorporated into the framework of Special Relativity [2]. Unfortunately, all of them were purely formal without sufficient physical background and in one or another way used complex numbers in the course of reasoning. In addition, the derived in this way expressions for energy and momentum of tachyons are not correct.

Our discussion is rigorously based on Special Relativity and do not contain any elements of speculation. The goal of our paper is to rectify the erroneous treatment of the problem of speeds in Special Relativity. The paper has mainly a pedagogical character. It is needless to say that the presented result should be incorporated into the way of standard teaching Special Relativity.

2 Lorentz transformations and velocities of motion

Special Theory of Relativity is based on Lorentz transformations of spacetime coordinates $(\vec{x}, t)$ of the form [3]

$$t \to t' = \gamma \left(t + \frac{\vec{V} \cdot R\vec{x}}{c^2} \right), \tag{1}$$

$$\vec{x} \to \vec{x}' = R\vec{x} + (\gamma - 1)\frac{(\vec{V} \cdot R\vec{x})}{V^2}\vec{V} + \gamma\vec{V}t, \tag{2}$$

where

$$\gamma = \left(1 - \frac{V^2}{c^2} \right)^{-1/2} \tag{3}$$

is the famous Lorentz factor with $\vec{V}$ being the relative velocity of two observers tightly connected with two inertial reference frames in which the spacetime coordinates $(\vec{x}, t)$ and $(\vec{x'}, t')$ are used and c is the invariant speed of light. Clearly, due to the square root in γ the relative velocities of reference frames must respect the condition

$$\vec{V}^2 < c^2. \tag{4}$$

In two-dimensional spacetime formula (1) and (2) reduce to the more familiar ones

$$t^{'} = \gamma \left(t + \frac{Vx}{c^2} \right), \tag{5}$$

$$x^{'} = \gamma \left(x + Vt \right). \tag{6}$$

The restriction (4) is the basis for the customary statement that in Special Relativity no velocity can exceed the speed of light. It is hovewer clear that such conclusion is not justified because the velocities $\vec{V}$ in (1) and (2) (or (5) and (6)) refer to the velocities of reference frames but not to velocities of motions. The correct statement should therefore be that no reference frames (or, equivalently, no observers) can move faster than light.

In fact, from formula (1) and (2) it follows that the velocities of motion of point particles transform according to the rule

$$\vec{v}(t) = \frac{d\vec{x}(t)}{dt} \rightarrow \vec{v}\,'(t^{'}) = \frac{d\vec{x}^{'}}{dt'} = \frac{R\vec{v(t)} + (\gamma - 1)\frac{(\vec{V}\cdot R\vec{v}(t))}{\vec{V}^2}\vec{V} + \gamma\vec{V}}{\gamma\left[1 + \frac{(\vec{V}\cdot R\vec{v}(t))}{c^2}\right]}, \tag{7}$$

where we follow the good habit to use lower case letters for quantities which may vary in time and capital letters for quantities which are constant in time [4].

Again, in two-dimensional spacetime we have the more familiar simpler rule

$$v^{'}(t^{'}) = \frac{v(t) + V}{1 + \frac{Vv(t)}{c^2}}. \tag{8}$$

It is seen that contrary to the transformation rule for the spacetime coordinates which contains the factor γ restricting the relative velocities $\vec{V}$ of reference frames, the transformation rule for velocities (7) or (8) does not contain any factor which may restrict the magnitude of the velocities of motion $\vec{v}(t)$.

For a mass point which is at rest in the unprimed reference frame we have $\vec{v}(t) = 0$ and it follows from (7) that in the primed reference frame the mass point moves with constant velocity $\vec{v}^{'}(t^{'}) = \vec{V}$. The restriction (4) implies then that mass points for which there exist rest frames cannot move faster than light. These are therefore the Hawking's "normal" objects.

The situation however is quite different for objects for which there is no rest frame. Such objects cannot move with velocities less than the speed of light because for such velocities using Lorentz transformation we always can find a reference frame in which the velocity is zero and this would mean that the rest reference would exist.

An example of objects without rest frame are provided by photons which in all reference frame move with the same speed c.

For other objects without rest frames the velocities cannot be restricted from above because then apart from the speed of light there would exist also a second invariant velocity. The only logical way out is to admit that for objects without rest frames there exist reference frames in which their velocity is infinite. Assuming that in the unprimed reference frame the velocity $|\vec{v}(t)| \rightarrow \infty$ from (7) we get that in the primed reference frame the velocity $\vec{W}$ of the same object is given by

$$\vec{W} = \frac{c^2}{\vec{V}^2}\vec{V} \tag{9}$$

and from (4) it follows now that

$$\vec{W}^2 > c^2. \tag{10}$$

This clearly shows that in the framework of Special Relativity the statement "that nothing may travel faster than the speed of light" is not justified. However, we must agree that the faster than light objects, called tachyons[3], are not quite "normal" because for them there are no rest frames of reference. Instead of that, in some reference frames the tachyons may travel with infinite speed. Contrary to Galilean physics, the infinite speed of tachyons is not an invariant speed because in any other reference frame according to (9) it is finite and greater than the speed of light.

Reversing the relation (9) we get

$$\vec{V} = \frac{c^2}{\vec{W}^2}\vec{W} \tag{11}$$

and substituting this into formulas (1) and(2) we get a new form of the Lorentz transformations [5]

$$t' = \gamma\left(t + \frac{\vec{W}\cdot R\vec{x}}{\vec{W}^2}\right) \tag{12}$$

$$\vec{x}' = R\vec{x} + (\gamma - 1)\frac{\vec{W}\cdot R\vec{x}}{\vec{W}^2}\ \vec{W} + \gamma\frac{c^2}{\vec{W}^2}\vec{W}t, \tag{13}$$

where γ now is given by

$$\gamma = \left(1 - \frac{c^2}{\vec{W}^2}\right)^{-1/2}. \tag{14}$$

It is clear that the restriction (10) must be satisfied.

In two-dimensional spacetime these formula simplify significantly. The transformations (12) and (13) imply also that the transformation rule for the velocities of motion takes now the form

$$\vec{v}(t') = \frac{R\vec{v}(t) + (\gamma - 1)\frac{\vec{W}\cdot R\vec{v}(t)}{\vec{W}^2}\ \vec{W} + \gamma\frac{c^2}{\vec{W}^2}\vec{W}}{\gamma\left[1 + \frac{\vec{W}\cdot R\vec{v}(t)}{\vec{W}^2}\right]}. \tag{15}$$

Again, we may put in this formula either $\vec{v}(t) = \vec{0}$ for particles having rest frames or take the limit $|\vec{v}(t)| \to \infty$ for superluminal objects. In both cases we get the previous results.

From (12) and (13) it is seen that for $R = I$ and $|\vec{W}| \to \infty$ these transformations are the identity transformation.

3 Energy - momentum four vectors

In physical applications the notions of energy and momentum play a central role. Special Relativity assumes that energy and momentum under Lorentz transformations behave as components of a covariant fourvector. It is however easier to consider first the components of a contravariant four vector because under Lorentz transformations they transform exactly in the same way as the spacetime coordinates do. Therefore for the contravariant components of any fourvector $P^\mu = (P^0, \vec{\mathbf{P}})$ according to (1) and (2) we have the transformation rule

$$P^{0\,\prime} = \gamma \left(P^0 + \frac{\vec{V}\cdot R\vec{\mathbf{P}}}{c^2} \right), \tag{16}$$

$$\vec{\mathbf{P}}\,' = R\vec{\mathbf{P}} + (\gamma - 1)\frac{(\vec{V}\cdot R\vec{\mathbf{P}})}{V^2}\vec{V} + \gamma\vec{V}P^0. \tag{17}$$

Taking into account that both energy and momentum are form invariant functions of the velocity (this means that $P^{0\,\prime}(\vec{v}) = P^0(\vec{v})$ and $\vec{P}\,'(\vec{v}) = \vec{P}(\vec{v})$) we arrive to the following set of functional equations for energy and momentum

$$P^0(\vec{v}\,'(t')) = \gamma \left(P^0(\vec{v}(t)) + \frac{\vec{V}\cdot R\vec{\mathbf{P}}(\vec{v}(t))}{c^2} \right), \tag{18}$$

$$\vec{\mathbf{P}}(\vec{v}'(t')) = R\vec{\mathbf{P}}(\vec{v}(t)) + (\gamma - 1)\frac{\left(\vec{V}\cdot R\vec{\mathbf{P}}(\vec{v}(t))\right)}{V^2}\vec{V} + \gamma\vec{V}P^0(\vec{v}(t)), \tag{19}$$

where $\vec{v}\,'(t')$ is given by (7).

In the rest reference frame of subluminal particles we may put $\vec{v}(t) = \vec{0}$ and get the solution of these equations in the form

$$P^0(\vec{V}) = \gamma \left(P^0(\vec{0}) + \frac{\vec{V}\cdot R\vec{\mathbf{P}}(\vec{0})}{c^2} \right), \tag{20}$$

$$\vec{\mathbf{P}}(\vec{V}) = R\vec{\mathbf{P}}(\vec{0}) + (\gamma - 1)\frac{\left(\vec{V}\cdot R\vec{\mathbf{P}}(\vec{0})\right)}{V^2}\vec{V} + \gamma\vec{V}P^0(\vec{0}), \tag{21}$$

because for $\vec{v}(t) = \vec{0}$ from (7) we get $\vec{v}\,'(t') = \vec{V}$. Here $P^0(\vec{0})$ and $\vec{\mathbf{P}}(\vec{0})$ are the integration constant for the functional equations (18) and (19). The energy is an even function of the velocity while momentum is an odd function. This means that we have to put $\vec{\mathbf{P}}(\vec{0}) = \vec{0}$. Moreover, energy is the time component of the covariant fourvector, and therefore

$$E(\vec{V}) = g_{00}P^0(\vec{V}) = c^2 P^0(\vec{V}), \tag{22}$$

where g_{00} is the time component of the metric tensor which due to our use of the coordinate time instead of the variable $x^0 = ct$ is equal to c^2. Comparing (20) and (22) and taking into account that $E(\vec{0}) = Mc^2$ we must put $P^0(\vec{0}) = M$, where M is the mass of the considered particle. As a result we get the standard expressions for energy and momentum in the form

$$E = \gamma M c^2, \qquad \vec{p} = \gamma M \vec{V}. \tag{23}$$

For superluminal objects we have to take in (18) and (19) the limit $|\vec{v}(t)| \to \infty$. As a result we get

$$P^0(\vec{W}) = \gamma \left(P^0(\vec{\infty}) + \frac{\vec{W}\cdot R\vec{\mathbf{P}}(\vec{\infty})}{\vec{W}^2} \right), \tag{24}$$

$$\vec{\mathbf{P}}(\vec{W}) = R\vec{\mathbf{P}}(\vec{\infty}) + (\gamma - 1)\frac{\left(\vec{W}\cdot R\vec{\mathbf{P}}(\vec{\infty})\right)}{W^2}\vec{W} + \gamma\frac{c^2 P^0(\vec{\infty})}{\vec{W}^2}\vec{W}, \tag{25}$$

where on the right hand side we have used (11) to have the same variable $\vec{W}$ on both side of these relations. The γ factor is now given by

$$\gamma = \left(1 - \frac{c^2}{\vec{W}^2}\right)^{-1/2}. \tag{26}$$

To have the momentum as an odd function of the velocity $\vec{W}$ we must take $\vec{P}(\infty) = 0$. The component $P^0(\vec{\infty})$ is related to the energy E_∞ of an object moving with infinite velocity by the same relation (22) as before because it is dictated by the general relation between contravariant and covariant components of fourvectors. Contrary to the case of subluminal motions we cannot however express E_∞ in terms of some mass because for the superluminal objects there is no rest frame and the mass of such objects cannot be operationally defined. As a result we get the following expressions for the components of the energy-momentum fourvector of superluminal object moving with velocity $\vec{W}$

$$E(\vec{W}) = \gamma E_\infty = \frac{E_\infty}{\sqrt{1 - \frac{c^2}{\vec{W}^2}}}, \tag{27}$$

$$\vec{P}(\vec{W}) = \gamma\frac{E_\infty}{\vec{W}^2}\vec{W} = \frac{E_\infty}{\vec{W}^2\sqrt{1 - \frac{c^2}{\vec{W}^2}}}\vec{W}. \tag{28}$$

It is interesting to note that momentum of tachyons moving with infinite speeds vanishes. This implies that tachyons with infinite speeds cannot interact with other objects because they have no momentum to transfer it to that objects.

It is worth to note that our formula (27) and (28), the first time derived in [5],[6], differ from the customary incorrect expressions for energy and momentum for tachyons

$$E(\vec{W}) = \frac{Mc^2}{\sqrt{\frac{\vec{W}^2}{c^2} - 1}}, \tag{29}$$

$$\vec{p}(\vec{W}) = \frac{M\vec{W}}{\sqrt{\frac{\vec{W}^2}{c^2} - 1}}, \tag{30}$$

which were obtained by a formal introduction of an imaginary mass to the standard formula (23) of subluminal Special Relativity[2].

It is also worth to observe that our energy-momentum four vector with components given by (27) and (28) is a time-like four-vector because

$$E^2(\vec{W}) - c^2\vec{p}^2(\vec{W} = E_\infty^2 > 0. \tag{31}$$

This fact removes all troubles in construction of quantum field theory for tachyons [7] based up to now on the spacelike energy-momentum fourvector provided by the incorrect expressions (29) and (30).

4 Conclusions

The main result of the present paper consists in the elementary proof that Special Relativity based on Lorentz transformations does not exclude superluminal speeds. As a matter of fact, Special Relativity divides the energy-momentum space into two sectors in which subluminal motions and superluminal motions take place, correspondingly. The light cone is an impenetrable barier between these two sectors.

It is rather hopeless to expect that the superluminal objects will be found in macrophysical domain. In microphysics the possibility of creation of superluminal objects as a result of complicated quantum mechanical processes in dielectrics was recently announced in Ref.[8]. In view of the present paper the existence of such objects does not contradict Special Relativity provided they are created from the very beginning with superluminal speeds. It must also be stressed that the superluminal speeds do not violate the existing relativistic kinematical calculations because in microphysics the basic characteristics of objects are the frequencies and wave vectors of the quantum mechanical vawes which due to the de Broglie ansatz are related to energies and momenta of quantum objects and not directly to their velocities. The velocities always are determined from classical relations of energy and momentum to velocities. In this sens the velocities are auxiliary quantities which help to visualize quantum processes.

It is astonishing that in the centennial history of Special Relativity the problem of superluminal speeds was not correctly treated. In view of the present paper it is necessary to change the way in which Special Relativity is taught.

References

[1] S. W. Hawking, *A Brief History of Time*, Bantam Books, 1988

[2] G. Feinberg, Phys. Rev. **159**, 1089 (1967)

[3] J. D. Jackson, *Classical Electrodynamics*, New York, John Willey and Sons, (1998)

[4] See for example, F. W. Sears, M. W. Zemansky and H. D. Young, *College Physics*, Addison-Wesley Publishing Company, 1991

[5] E. Kapuścik, Condens. Matter Phys. **13**, 43102 (2010), arXiv:1010.5886v1[physics.gen-ph]

[6] E. Kapuścik, Physics of Atomic Nuclei, **74**, 919 (2011)

[7] S. Twareque, Phys. Rev. **D7**, 1668 (1973)

[8] E.Kapuścik and R. Orlicki, Ann.Phys. (Berlin) **523**, 235-238 (2011)