MATHEMATICAL DEFINITION OF QUANTUM FIELD THEORY ON A MANIFOLD

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ABSTRACT. We give a mathematical definition of quantum field theory on a manifold, and definition of quantization of classical field theory given by a variational principle.

To the memory of I. M. Gelfand

1. Introduction

In this note we give a definition of quantum field theory (QFT) on a space-time being a manifold M. Such definition is necessary for unification of QFT with general relativity. Our definition is almost directly motivated by the definition of dynamical evolution on space-like surfaces in QFT on $M = \mathbb{R}^{3+1}$ given in our previous paper [1]. The only essential difference is that we impose the additional condition that the Hilbert spaces in question be representations of canonical commutation relations, if the theory is quantization of a classical field theory. This condition seems reasonable. Classification of unitary representations of canonical commutation relations can be found, for example, in the book [2] (in the bosonic case).

2. Definition of QFT on a manifold

2.1. Let M be a (pseudo-Riemannian) manifold of dimension D, and let G be a Lie group acting on M. By definition, a QFT on M assigns

a) to each (space-like) closed connected co-oriented hypersurface C in M (of codimension 1, below we call them simply surfaces) a Hilbert space $\mathcal{H}_C$, and

b) to each closed co-oriented surface C in M with the connected components $C_1, \ldots, C_n$ it assigns the space $\mathcal{H}_C \stackrel{\text{def}}{=} \mathcal{H}_{C_1} \otimes \ldots \otimes \mathcal{H}_{C_n}$. Here $\otimes$ means bounded tensor product of Banach spaces, so that for two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\overline{\mathcal{H}_1} \otimes \mathcal{H}_2$ (bar means complex conjugation) is identified with the space $Hom(\mathcal{H}_1, \mathcal{H}_2)$ of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$;

c) to each manifold N of the same dimension D with the boundary ∂N and a topological type of smooth mappings $N \to M$ which isomorphically map ∂N to a surface C in a compatible way with co-orientation, it assigns a vector $\Psi_N \in \mathcal{H}_C$, so that the following conditions hold.

(i) Change of co-orientation of C corresponds to complex conjugation of $\mathcal{H}_C$.

(ii) If N is the union of two open submanifolds N_1, N_2 with the common boundary C_1, so that $\partial N_1 = C \sqcup C_1$ and $\partial N_2 = C_1 \sqcup C'$, then $\Psi_N \in \overline{\mathcal{H}}_C \otimes \mathcal{H}_{C'}$ is obtained from

$$\Psi_{N_1} \otimes \Psi_{N_2} \in \overline{\mathcal{H}}_C \otimes \mathcal{H}_{C_1} \otimes \overline{\mathcal{H}}_{C_1} \otimes \mathcal{H}_{C'}$$

by contraction $\overline{\mathcal{H}}_C \otimes \mathcal{H}_{C_1} \otimes \overline{\mathcal{H}}_{C_1} \otimes \mathcal{H}_{C'} \to \overline{\mathcal{H}}_C \otimes \mathcal{H}_{C'}$.

Corollary. If we identify $\overline{\mathcal{H}}_C \otimes \mathcal{H}_{C_1}$ with $Hom(\mathcal{H}_C, \mathcal{H}_{C_1})$, then Ψ_{N_1} is a unitary operator from $\mathcal{H}_C$ to $\mathcal{H}_{C_1}$, and its composition with $\Psi_{N_2} : \mathcal{H}_{C_1} \to \mathcal{H}_{C'}$ equals $\Psi_N : \mathcal{H}_C \to \mathcal{H}_{C'}$.

(iii) Ψ_N smoothly depends on C; hence the bundle with fiber $\mathcal{H}_C$ over the infinite dimensional manifold of surfaces C carries a canonical integrable flat connection ∇.

All these data should be compatible with the action of the group G in the obvious sense.

2.2. **Definition of quantization of a classical field theory.** Consider a G-invariant classical field theory on M given by the action functional

$$I = \int L(x, \varphi(x), d\varphi(x))dx, \tag{1}$$

where L is the Lagrangian depending on points $x \in M$, fields $\varphi(x)$ (we omit the indices of fields), and their first derivatives $d\varphi(x)$. Then the Euler–Lagrange equations can be written in the covariant Hamiltonian form, as it is described, for example, in [3,4]:

$$\frac{\delta \Phi}{\delta x^j(s)} = \{H^j(s), \Phi\}, \tag{2}$$

where $x(s) = (x^j(s))$ is a parameterization of the surface C, x^j are local coordinates on M, $\Phi = \Phi(x^j(\cdot); \varphi(\cdot), \pi(\cdot))$ is a functional of fields $\varphi(s)$ and canonically conjugate variables $\pi(s)$, which changes together with the surface $x = x(s)$; $H^j(s) = H^j(x(s), x_{s^k}(s), \varphi(s), \varphi_{s^k}(s), \pi(s))$ are the covariant Hamiltonian densities, and $\{,\}$ is the standard Poisson bracket. Then a QFT on M depending on a parameter $h \neq 0$ is said to be a *quantization* of this classical field theory if the following additional conditions hold:

(iv) each space $\mathcal{H}_C$ corresponding to a connected surface C is an irreducible unitary representation (in the sense of [2]) of the canonical commutation relations between the variables $\hat{\varphi}(s)$, $\hat{\pi}(s)$:

$$[\hat{\varphi}(s),\hat{\varphi}(s')]=[\hat{\pi}(s),\hat{\pi}(s')]=0,\quad [\hat{\varphi}(s),\hat{\pi}(s')]=ih\delta(s-s'), \tag{3}$$

where $[,]$ is the supercommutator;

(v) Consider the flat integrable connection on the bundle $End(\mathcal{H}_C)=Hom(\mathcal{H}_C,\mathcal{H}_C)$ induced from ∇; denote it by ∇_1. Then in local coordinates x^j on M, and for local parameterizations $x=x(s)$ of surfaces C, the connection ∇_1 up to $O(h)$ coincides with the differential operator

$$\begin{aligned}\nabla_{1,\frac{\delta}{\delta x^j(s)}}(A)&=\frac{\delta}{\delta x^j(s)}A-\frac{1}{ih}[H^j(\hat{\varphi}(s),\hat{\pi}(s)),A]\mod O(h)\\ &\equiv\frac{\delta}{\delta x^j(s)}A-\{H^j(s),A\},\end{aligned} \tag{4}$$

where the operators $\hat{\varphi}(s),\hat{\pi}(s)$ are put in the Hamiltonian density in their natural order (note that the covariant Schrodinger functional differential equation in all standard cases does not contain terms like $\hat{\varphi}(s)\hat{\pi}(s)$ which depend on the order of operators); $A=A(x(\cdot);\,\hat{\varphi}(\cdot),\,\hat{\pi}(\cdot))$ is a regular expression, i. e. a polynomial expression of smoothed operators $\int f(s)\hat{\varphi}(s)ds$ and $\int g(s)\hat{\pi}(s)ds$ for some smooth functions $f(s)$, $g(s)$.

(vi) For any smooth density $j(x)$ on M with compact support, called *source*, and for each co-orientation of the surfaces C, the connection

$$\nabla_j=\nabla+\frac{1}{ih}\int_C j(x(s))\hat{\varphi}(s) \tag{5}$$

on the bundle $\mathcal{H}_C$ is also flat.

The latter condition is necessary for construction of the Green functions $\langle\varphi(x_1)\ldots\varphi(x_n)\rangle$, as in [1].

References

[1] A. V. Stoyanovsky, Quantization on space-like surfaces, http://arxiv.org/abs/0909.4918 [math-ph].

[2] I. M. Gelfand, N. Ya. Vilenkin, Generalized functions, vol. 4. Some applications of harmonic analysis. Equipped Hilbert spaces. Fizmatlit, Moscow, 1961 (in Russian).

[3] A. V. Stoyanovsky, Introduction to the mathematical principles of quantum field theory, Editorial URSS, Moscow, 2007 (in Russian).

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