

# Emergence of Quantum Mechanics from Iterated Maps

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## *Abstract*

Iterated maps are deterministic models of dynamical systems in discrete time. A key feature of these models is the concept of *invariant density* associated with the asymptotic onset of stationarity. Drawing from the minimal fractality of spacetime near the Fermi scale, we show here that invariant density enables a step-by-step derivation of Quantum Mechanics from iterated maps.

**Key words:** Quantum Mechanics, iterated maps, invariant density, invariant measure, ergodicity, minimal fractal manifold.

## 1. Iterated maps and invariant density functions

Refs. [3-9] discuss at length the physical meaning of the *minimal fractal manifold* (MFM), a spacetime continuum characterized by arbitrarily small and scale-dependent deviations from four dimensions ( $\varepsilon = 4 - D \ll 1$ ). MFM is conjectured to develop in out-of-equilibrium conditions near the ultraviolet scale  $\Lambda_{UV}$  and flow towards equilibrium below the Fermi scale ( $M_{EW} < \Lambda_{UV}$ ). The main point of the MFM is that the dimensional deviation  $\varepsilon$  runs with the energy scale as in  $\varepsilon = \varepsilon(\mu)$ , where  $\mu = O(M_{EW}/\Lambda_{UV})$ . There are reasons to believe that dimensional fluctuations driven by  $\varepsilon(\mu)$  near the limit of four-dimensional spacetime ( $\varepsilon = 0$ ) are asymptotically compatible with the phenomenology of effective field theory, in general, and the Standard Model of particle physics in particular [3-9].

The goal of this paragraph is to outline the description of  $\varepsilon(\mu)$  in terms of iterated maps and invariant density functions. It complements our earlier work on the emergence of Planck's constant from iterated maps [10].

The evolution of  $\varepsilon(\mu)$  in a generic  $N$ -dimensional phase space  $X$  is defined by the first-order differential equation [1-2]

$$\frac{d\varepsilon}{d\mu} = \beta(\varepsilon) \quad (1)$$

whose map analog is given by

$$\varepsilon_{n+1} = \beta(\varepsilon_n) \quad (2)$$

Here,  $n$  is the map iteration index while

$$\varepsilon_n = (\varepsilon_n^{(1)}, \varepsilon_n^{(2)}, \dots, \varepsilon_n^{(N)}) \quad (3)$$

is a vector in  $X$  and

$$\beta = (\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(N)}) \quad (4)$$

a vector-valued function. One starts with  $\varepsilon_0$  and iterate it step by step using (2). The sequence of iterates  $\varepsilon_0, \varepsilon_1, \dots$  forms a trajectory (orbit) in  $X$ . A periodic orbit of length  $L = 1$  defines a fixed point of the map and satisfies the condition

$$\varepsilon^* = \beta(\varepsilon^*) = 0 \Rightarrow D = 4 \quad (5)$$

$$\beta^L(\varepsilon) = \beta(\beta(\dots\beta(\varepsilon))) , \quad L - \text{ times} \quad (6)$$

Although the iterates of (2) are deterministic events, a useful concept for the analysis of (2) is the *probability distribution of iterates*. Let us partition  $X$  into an array of disjoint cells indexed by the subscript  $i = 1, 2, \dots, R$ , where  $R$  is the total number of cells. Let the number of iterates located in cell  $i$  be  $n_i$ . The relative frequencies (or weights) associated with a large number of iterations  $n \gg 1$  is given by

$$p_i = \frac{n_i}{n} = \frac{n_i}{\sum_i n_i} \quad (7)$$

Definition (7) enables bridging the gap between the theory of iterated maps and classical statistical physics [1-2].

Assuming unbounded precision, the numerical value of  $\varepsilon$  includes an infinite string of digits. A reasonable approximation is obtained by truncating the string to  $M \gg 1$  digits according to

$$\varepsilon = \eta_1 \eta_2 \dots \eta_{M-1} \eta_M \quad (8)$$

where each unit  $\eta_j$ ,  $j = 1, 2, \dots, M$  is the quartet of binary pairs as in

$$\eta_j = (00, 01, 10, 11) \quad (9)$$

The probability measure associated with the initial density  $\rho_0(\varepsilon)$  assumes the form

$$\mu_0(A) = \int_A d\varepsilon \rho_0(\varepsilon) \quad (10)$$

It is important to note that, unlike statistical physics, the initial density  $\rho_0(\varepsilon)$  *does not reflect* the statistical uncertainty of choosing an initial condition. Rather, the initial

density follows from the inherent numerical approximation of  $\varepsilon$  as expressed by (8) and (9). In general, one can state that the density  $\rho(\varepsilon)$  at any iteration stage quantifies the *rounding error* in the estimation of  $\varepsilon$ , a process that can be symbolically presented as

$$\rho(\varepsilon) \Rightarrow \varepsilon \approx \eta_1, \eta_2, \dots, \eta_T, \quad T \leq M \quad (11)$$

Given the map  $\beta$ , one wishes to study the evolution of the ensemble of trajectories corresponding to an ensemble of initial values  $\varepsilon_0$ . Let  $\mu_n$  denote the probability distribution of iterates after  $n$  iterations. The probability measure of finding an iterate  $\varepsilon_n$  in the subset of the phase space  $A \subset X$  amounts to [1]

$$\mu_n(A) = \int_A d\varepsilon \rho_n(\varepsilon) \quad (12)$$

By definition, an invariant probability measure stays unchanged upon the action of the map  $\beta$ , which means that it satisfies the requirement

$$\mu_{n+1}(A) = \mu_n(A) \quad (13)$$

It can be shown that, based on (12) to (13), the corresponding invariant density  $\rho$  complies with the condition

$$\int_A d\varepsilon \rho(\varepsilon) = \int_{\beta^{-1}(A)} d\varepsilon \rho(\varepsilon) \quad (14)$$

where  $\beta^{-1}(A)$  denotes the set of all points that are mapped onto  $A$  by one iteration step.

The ensemble expectation value of an arbitrary test function (or operator)  $Q(\varepsilon)$  with respect to the invariant density  $\rho$  is given by [1]

$$\langle Q \rangle = \int_X d\varepsilon \rho(\varepsilon) Q(\varepsilon) \quad (15)$$

*Ergodicity* demands the identity of the ensemble average with the time average, where the latter is supplied by

$$\bar{Q} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{n=0}^{N-1} Q(\varepsilon_n) \Rightarrow \langle Q \rangle = \bar{Q} \quad (16)$$

A remarkable property of ergodic maps is *mixing*. The map  $\beta$  is called “mixing” if the initial smooth density  $\rho_0(\varepsilon)$  converges to the invariant density  $\rho(\varepsilon)$  as in [1]

$$\lim_{n \rightarrow \infty} \rho_n(\varepsilon) = \rho(\varepsilon) \quad (17)$$

It is apparent that (17) is automatically fulfilled if the map (2) ends up on the attractor  $\varepsilon^* = 0$  ( $D = 4$ ), where the dimensional flow  $\varepsilon = \varepsilon(\mu)$  settles down.

Mixing may be also defined in terms of *correlation functions* (CF). The CF for any two integrable test functions  $\varphi_1, \varphi_2$  takes the form

$$C(\varphi_1, \varphi_2; n) = \lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \varphi_1(\varepsilon_{k+n}) \varphi_2(\varepsilon_k) - \langle \varphi_1 \rangle \langle \varphi_2 \rangle \quad (18)$$

The map is considered mixing if

$$\lim_{n \rightarrow \infty} C(\varphi_1, \varphi_2; n) = 0 \quad (19)$$

which occurs when  $\varphi_1$  and  $\varphi_2$  are *statistically independent*. Mixing implies ergodicity, but the reverse is not true.

## **2. Quantum Mechanics from invariant density functions**

The behavior of iterated maps previously outlined hints to an unforeseen connection between the invariant density  $\rho(\varepsilon)$  in close proximity to  $\varepsilon^* = 0$  and the probability density of quantum states. In particular, a straightforward conjecture is that

$$\rho(\varepsilon) \Leftrightarrow |\psi(x)|^2 \quad (20)$$

where  $\psi(x)$  is the square-integrable wavefunction of Quantum Mechanics (QM). To unveil this connection, we cast the invariant density in the form

$$\rho(\varepsilon) = \rho_c(\varepsilon)\rho_c^*(\varepsilon) = \left| \sqrt{\rho(\varepsilon)} \exp[i\theta(\varepsilon)] \right|^2 \quad (21)$$

where  $\rho_c(\varepsilon) = \sqrt{\rho(\varepsilon)} \exp[i\theta(\varepsilon)]$  represents the complex-valued amplitude of  $\rho(\varepsilon)$ .

Relations (20) and (21) imply a straightforward one-to-one correspondence written as

$$\boxed{\rho_c(\varepsilon) \Leftrightarrow \psi(x)} \quad (22a)$$

$$\boxed{\rho_c^*(\varepsilon) \Leftrightarrow \psi^*(x)} \quad (22b)$$

A critical observation is now in order. Recall that, by (11), the density  $\rho(\varepsilon)$  reflects the rounding error in the estimation of  $\varepsilon$ . In addition to the rounding error, one must consider that any measurement process involves a *finite sampling resolution*  $\Delta$ . For a given  $\Delta$ , there is an infinite number of rounded  $\varepsilon$  contained in  $\Delta$  and an invariant density  $\rho_\Delta(\varepsilon)$  that can symbolically presented as

$$\rho_\Delta(\varepsilon) \Rightarrow \varepsilon \in [\varepsilon_{\min}, \varepsilon_{\max}] \subset \Delta \quad (23)$$

It follows from these considerations that (22) is to be interpreted as an *infinite superposition* of complex amplitudes imposed by  $\Delta$ , as embodied in the expansion postulate of QM

$$\psi = \sum_s C_s \psi_s \quad (24)$$

Turning next to (15), it is apparent that it represents the analogue of operator average in QM, namely

$$\langle Q \rangle = \int dx \psi^*(x) Q \psi(x) \quad (25)$$

A surprising interpretation emerges from (18) - (19). For any arbitrarily small yet non-vanishing deviation  $\varepsilon > \varepsilon^* = 0$ , the correlation function of two operators  $Q_1, Q_2$  is never vanishing, which – under properly defined conditions - may lie behind the physics of *quantum entanglement*. This is to say that there are circumstances where quantum operators remain statistical dependent, regardless of the observation scale  $\mu$ .

We close by showing how the quantum generators of translations and rotations, as well as the time-dependent Schrödinger equation, emerge from (20) – (22). To this end, consider a scaling transformation of (22) written in the form

$$\psi' = U \psi \quad (26)$$

Since (20) is scale-invariant for  $\varepsilon \rightarrow \varepsilon^* = 0$ , the operator  $U$  must be unitary

$$U^\dagger U = 1 \quad (27)$$

which means

$$\psi'^* \psi = \psi'^* U^\dagger U \psi = \psi'^* \psi \quad (28)$$

Transformation (26) is therefore norm-conserving

$$|\psi'| = |\psi| = (\psi^\dagger \psi)^{1/2} \quad (29)$$

The infinitesimal counterpart of (26) is described by ( $\xi \ll 1$ )

$$U = \exp(i\theta) = \exp(-i\xi G) \approx 1 - i\xi G \quad (30)$$

or

$$\psi' = (1 - i\xi G)\psi \quad (31)$$

where  $G$  stands for the generator of (26). It was shown in [11] that an infinitesimal transformation involving a dilation, rotation or translation is operationally equivalent to an infinitesimal scale-invariant (self-similar) transformation. Assuming that (26) refers to a translation ( $x' = x - \delta x$ ) with  $\delta x \ll x$ , (31) turns into

$$\psi'(x') = \psi(x - \delta x) \approx \psi(x) - \delta x \frac{\partial \psi(x)}{\partial x}, \quad x = (x_\mu) \quad (32)$$

which yields the generator of translations in the form

$$G_\mu = -i \frac{\partial}{\partial x_\mu} \quad (33)$$

Including the reduced Planck's constant in (33) leads to the standard momentum operator



$$P_\mu = \hbar G_\mu \quad (34)$$

Consider now rotations about the  $z$  - axis which transform vectors  $\mathbf{V} = (V_x, V_y)$  according to the matrix equation

$$\begin{pmatrix} \bar{V}_x \\ \bar{V}_y \end{pmatrix} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} \quad (35)$$

An infinitesimal rotation of angle  $\delta\varphi = \xi \ll 1$  is equivalent to an infinitesimal scaling operation defined by

$$\begin{pmatrix} \bar{V}_x \\ \bar{V}_y \end{pmatrix} = \begin{pmatrix} 1 & -\delta\varphi \\ \delta\varphi & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} V_x - V_y \delta\varphi \\ V_y + V_x \delta\varphi \end{pmatrix} \quad (36)$$

and so

$$\delta V_x = -V_y \delta\varphi = -V_y \delta\lambda \quad (37)$$

$$\delta V_y = V_x \delta\varphi = V_x \delta\lambda \quad (38)$$

It is known that expanding a generic function containing a two-component vector gives

$$F(V_x + \delta V_x, V_y + \delta V_y) = F(V_x, V_y) + \delta V_x \frac{\partial F}{\partial V_x} + \delta V_y \frac{\partial F}{\partial V_y} \quad (39)$$

which can be presented as

$$F(\varphi + \delta\varphi) = F(\varphi) + \delta\varphi \left( -V_y \frac{\partial}{\partial V_x} + V_x \frac{\partial}{\partial V_y} \right) \quad (40)$$

It follows from (40) that the generator of planar rotations can be written as

$$G = -i \left( V_x \frac{\partial}{\partial V_y} - V_y \frac{\partial}{\partial V_x} \right) \quad (41)$$

When  $\mathbf{V}$  represents the position vector  $\mathbf{r} = (x, y)$ , (41) renders the angular momentum about the  $z$  axis in the familiar form

$$L_z = xP_y - yP_x \quad (1.1)$$

Finally, let us apply an infinitesimal time translation  $t' = t - \delta t$  in (26) given by

$$\psi'(t') = U\psi(t) = \left(1 - \frac{i}{\hbar} H \delta t\right) \psi(t) \quad (42)$$

where

$$\psi'(t') = \psi(t) - \delta t \frac{\partial \psi}{\partial t} \quad (43)$$

Inspection of (42) and (43) reveals the time-dependent Schrödinger equation

$$H\psi = i\hbar \frac{\partial \psi}{\partial t} \quad (44)$$

For additional details on (26) – (44), the reader is directed to a couple of well-written introductory texts on the role of symmetry and invariance in physics [12-13].

## **References**

1. Beck C. and Schlögl F., “Thermodynamics of chaotic systems—an introduction”, Cambridge Univ. Press, 1993.
2. McCauley J. L., “Chaos, Dynamics and Fractals-an algorithmic approach to deterministic chaos”, Cambridge Univ. Press, 1993.
3. Available at the following sites:  
<http://www.aracneeditrice.it/aracneweb/index.php/pubblicazione.html?item=9788854889972>  
[https://www.researchgate.net/publication/278849474\\_Introduction\\_to\\_Fractional\\_Field\\_Theory\\_consolidated\\_version](https://www.researchgate.net/publication/278849474_Introduction_to_Fractional_Field_Theory_consolidated_version)
4. Available at the following site:  
[https://www.researchgate.net/publication/228625050\\_Chaotic\\_Dynamics\\_of\\_the\\_Re-normalization\\_Group\\_Flow\\_and\\_Standard\\_Model\\_Parameters\\_Intl](https://www.researchgate.net/publication/228625050_Chaotic_Dynamics_of_the_Re-normalization_Group_Flow_and_Standard_Model_Parameters_Intl)
5. Available at the following site:  
[https://www.academia.edu/38752380/Multifractal\\_Geometry\\_and\\_Standard\\_Model\\_Symmetries](https://www.academia.edu/38752380/Multifractal_Geometry_and_Standard_Model_Symmetries)
6. <https://www.prespacetime.com/index.php/pst/article/view/1619/1546>
7. <https://www.prespacetime.com/index.php/pst/article/view/1653/1575>
8. <https://www.prespacetime.com/index.php/pst/article/view/1648/1569>
9. <https://www.prespacetime.com/index.php/pst/article/view/1656/1570>
10. Available at the following site  
[https://www.researchgate.net/publication/341817622\\_Emergence\\_of\\_Planck's\\_Constant\\_from\\_Iterated\\_Maps#fullTextFileContent](https://www.researchgate.net/publication/341817622_Emergence_of_Planck's_Constant_from_Iterated_Maps#fullTextFileContent)

[11] Available at the following site

[https://www.researchgate.net/publication/342876637\\_Self-similarity\\_and\\_the\\_Foundation\\_of\\_Effective\\_Field\\_Theory#fullTextFileContent](https://www.researchgate.net/publication/342876637_Self-similarity_and_the_Foundation_of_Effective_Field_Theory#fullTextFileContent)

[12] Stenger V. J., “The comprehensible cosmos”, Prometheus Books, 2006.

[13] Schwichtenberg J., “Physics from Symmetry”, Springer, 2018.