Random walks in random environments without ellipticity

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June 2011

Abstract

We consider random walks in random environments on $\mathbb{Z}^d$. Under a transitivity hypothesis that is much weaker than the customary ellipticity condition, and assuming an absolutely continuous invariant measure on the space of the environments, we prove the ergodicity of the annealed process w.r.t. the dynamics "from the point of view of the particle". This implies in particular that the environment viewed from the particle is ergodic. An immediate application of this result is to bistochastic environments. In this case, assuming zero local drift as well (martingale condition), we also prove the quenched Invariance Principle.

MSC 2010: 60G50, 60K37, 37A50 (82C41, 37A20, 60G42, 60F17).

1 Introduction

In this note we investigate random walks in random environments (RWREs) on $\mathbb{Z}^d$, i.e., $\mathbb{Z}^d$-valued Markov chains defined by the transition matrix $p(\omega) = (p_{xy}(\omega))_{x,y\in\mathbb{Z}^d}$, where ω is a random parameter ranging in a complete probability space (Ω, Π). (In the remainder we will refer to either ω or $p(\omega)$ as the *environment*.)

Although the precise nature of Ω is irrelevant, a natural choice is $\Omega = (\mathcal{S}_{K,\gamma})^{\otimes\mathbb{Z}^d}$, where, for $K, \gamma > 0$, $\mathcal{S}_{K,\gamma}$ is the space of all probability distributions $\omega_o = (\omega_{oy})_{y\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$, such that $\omega_{oy} \le K|y|^{-d-\gamma}$. By tightness [B2], $\mathcal{S}_{K,\gamma}$ is compact in the weak-* topology, which can be metrized, e.g., by the total variation distance between two distributions. Hence, by Tychonoff's Theorem and a standard argument, Ω is also compact and metrizable. (This is important in case one needs to construct suitable invariant measures.) For an element $\omega = (\omega_x)_{x\in\mathbb{Z}^d} \in \Omega$, where $\omega_x = (\omega_{xy})_{y\in\mathbb{Z}^d} \in \mathcal{S}_{K,\gamma}$, $p(\omega)$ is defined by $p_{xy}(\omega) := \omega_{x,y-x}$.

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Ω is acted upon by $\mathbb{Z}^d$ via the group of Π-automorphisms $(\tau_z)_{z\in\mathbb{Z}^d}$, such that

$$p_{xy}(\tau_z\omega) = p_{x+z,y+z}(\omega). \tag{1.1}$$

(In the representation above, τ_z is defined as $(\tau_z\omega)_x := \omega_{x+z}$.) Because of this, there is no loss of generality in requiring that the walk always starts at 0. The random walk (RW) in the environment $p(\omega)$ is then be defined as the Markov chain $(X_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}^d$ whose law P_ω is uniquely determined by

$$P_\omega(X_0 = 0) = 1; \tag{1.2}$$
$$P_\omega(X_{n+1} = y \,|\, X_n = x) = p_{xy}(\omega). \tag{1.3}$$

The complete randomness of the problem is accounted for by the *annealed* (or *averaged*) law, which is defined on $(\mathbb{Z}^d)^{\mathbb{N}} \times \Omega$ via

$$\mathbb{P}(E \times B) := \int_B \Pi(d\omega)\, P_\omega(E), \tag{1.4}$$

where E is a Borel set of $(\mathbb{Z}^d)^{\mathbb{N}}$ (the latter being the space of the trajectories, where P_ω is defined) and B is a measurable set of Ω. A natural dynamics that can be defined on this process is the one induced by the map $\mathcal{F} : (\mathbb{Z}^d)^{\mathbb{N}} \times \Omega \longrightarrow (\mathbb{Z}^d)^{\mathbb{N}} \times \Omega$, given by

$$\mathcal{F}\left((x_n)_{n\in\mathbb{N}}, \omega\right) := \left((x_{n+1})_{n\in\mathbb{N}}, \tau_{x_1}\omega\right). \tag{1.5}$$

As is apparent, the first component of $\mathcal{F}$ updates the trajectory of the RW to the next time, while the second component updates the environment as seen by the random walker, or particle. This dynamics may thus be called 'the point of view of the particle' (PVP) for the annealed process.

Another process of great importance, which is directly related to the above, is the so-called 'environment viewed from the particle' (EVP). It can be defined independently as the Markov chain $(\Omega_n)_{n\in\mathbb{N}}$ on Ω, with law $\mathcal{P}_\Pi$, such that:

$$\mathcal{P}_\Pi(\Omega_0 \in B) = \Pi(B); \tag{1.6}$$
$$\mathcal{P}_\Pi(\Omega_{n+1} = \omega \,|\, \Omega_n = \omega') = \sum_{y:\, \tau_y\omega' = \omega} p_{0y}(\omega'). \tag{1.7}$$

(The annoying notation whereby Ω_n denotes an element of Ω is not going to be used again in this note.)

Notational convention. Throughout the paper, the dependance of whichever quantity (e.g., p) on ω will not be explicitly indicated when there is no risk of confusion.

When studying the stochastic properties of a RWRE (say, the recurrence or transience of (X_n), the ergodicity of $\mathcal{P}_\Pi$, the CLT w.r.t. $\mathbb{P}$ or P_ω, etc.) an assumption that is almost always made is the *ellipticity* of the environment. We state two of the most common versions it may come in.

Definition 1.1 *A random environment* (Ω, Π) *is called* elliptic *if, for* Π*-a.a.* $\omega \in \Omega$ *and all* $e \in \mathbb{Z}^d$ *with* $|e| = 1$, $p_{0e} > 0$. *It is called* uniformly elliptic *if* $\exists \varepsilon > 0$ *such that, for* Π*-a.a.* $\omega \in \Omega$ *and all* $e \in \mathbb{Z}^d$ *with* $|e| = 1$, $p_{0e} \geq \varepsilon$.

To the author's knowledge, at least within the scope of non-ballistic RWREs, only certain results on RWs on percolation clusters [Be, SS, BB, MP] do not (and cannot) require ellipticity. In general, uniform ellipticity is one of the assumptions, though recent work focuses on non-uniformly elliptic systems, cf. [M, GZ, Sa] and references therein.

There are reasons to consider the ellipticity condition too strong in many cases; for example, uniform ellipticity is a deterministic condition in a probabilistic problem. More in general, both formulations of Definition 1.1 seem to point to a form of *transitivity* of the motion. So it seems natural to try to replace ellipticity with the simple hypothesis that the walker goes anywhere with positive probability; cf. (A3)-(A4) below. This is the starting point of this note.

It turns out that some of the results that are commonly looked for within the scope of RWRE can be proved using this weaker assumption only. The general idea seems to be that, once a Π-absolutely continuous measure on Ω has been found that is a steady state for the EVP process, the ellipticity condition is no longer needed.

A result that can be proved in this setup is the ergodicity of the PVP for the annealed process (Theorem 1.3) and thus of the EVP (Corollary 1.4). The simplest environments for which the above hypotheses hold are the bistochastic environments, for which Π is automatically invariant in the right sense. If we further assume that the RW is a martingale, then we can prove the Invariance Principle (IP) as well (Theorem 1.7).

Assumptions. The RWRE satisfies the following:

(A1) *Effective randomness.* Denote by δ_{xy} the Kronecker delta in $\mathbb{Z}^d$. Setting

$$\Omega_{\mathrm{det}} := \left\{ \omega \in \Omega \;\middle|\; \exists y_o \in \mathbb{Z}^d \text{ such that } p_{0y} = \delta_{y_o y}, \forall y \in \mathbb{Z}^d \right\},$$

then $\Pi(\Omega_{\mathrm{det}}) < 1$. In other words, the jumps at the origin are not almost surely deterministic.

(A2) *Decaying transition probabilities.* There exist $K, \gamma > 0$ such that, almost surely,

$$p_{xy} \leq K |y - x|^{-d-\gamma}.$$

(A3) *Ergodicity.* There is a subgroup $\Gamma \subseteq \mathbb{Z}^d$ such that $(\Omega, \Pi, (\tau_z)_{z \in \Gamma})$ is ergodic. (In particular, the random environment is ergodic w.r.t. the whole group of translations.)

(A4) *Transitivity.* Let Γ be the same as in (A3). For Π-a.e. $\omega \in \Omega$, it holds that $\forall y \in \Gamma$, $\exists n = n(\omega, y)$ for which

$$p_{0y}^{(n)} := \sum_{x_1, \ldots, x_{n-1}} p_{0x_1} \, p_{x_1 x_2} \cdots p_{x_{n-1} y} > 0.$$

(A5) *Absolute continuity of steady state.* There exists a probability measure Π_* on Ω, absolutely continuous w.r.t. Π, such that the EVP process $\mathcal{P}_{\Pi_*}$, defined as in (1.6)-(1.7), is stationary.

Notice that there is a trade-off between (A3) and (A4): if Γ gets smaller, thus making (A3) stronger, then (A4) becomes easier to verify, and viceversa. In particular, for an i.i.d. environment (namely, the stochastic vectors $p_x = (p_{xy})_{y \in \mathbb{Z}^d}$ are i.i.d. in x), one only need verify transitivity w.r.t. $\Gamma = \mathbb{Z} y_o$, for some $y_o \in \mathbb{Z}^d$. In any event, $\Gamma = \mathbb{Z}^d$ is a reasonable choice for many applications.

Remark 1.2 (A5) could be replaced by the (slightly) weaker condition that the steady state Π_* is non-singular w.r.t. Π. The two conditions are actually equivalent in our case. In fact, it can be seen (cf. Section 2) that the evolution of Π in the EVP process is absolutely continuous w.r.t. Π, which implies that, if Π_* decomposes into an absolutely continuous measure and a singular measure, both of them are invariant for the process.

RESULTS. These are our main results:

Theorem 1.3 *Under assumptions* (A1)-(A5)*,*

(a) the measures Π *and* Π_* *are equivalent (i.e., mutually absolutely continuous);*

(b) if $\mathbb{P}_*$ *is the annealed law relative to* Π_* *(i.e., the measure defined by* (1.4) *with* Π_* *in lieu of* Π*), then* $\mathbb{P}_*$ *is stationary and ergodic for the dynamics induced by* $\mathcal{F}$ *on the annealed process.*

Corollary 1.4 *The EVP with initial state* Π_* *is ergodic.*

Another easy corollary of Theorem 1.3 concerns the ballisticity of the RW. In order to state it, we introduce the *mean displacement* (or *local drift*) at the origin, for the environment ω. This is the function $\mathcal{D} : \Omega \longrightarrow \mathbb{Z}^d$ given by

$$\mathcal{D}(\omega) := \sum_{y \in \mathbb{Z}^d} p_{0y}(\omega) \, y. \tag{1.8}$$

Corollary 1.5 *For* Π*-a.e. environment* ω*,* P_ω*-almost surely,*

$$\lim_{n \to \infty} \frac{X_n}{n} = \int_\Omega \Pi_*(d\omega') \, \mathcal{D}(\omega').$$

The above statements apply in their generality to a fairly large class of RWREs, e.g., [La, BK, KO, SS, BB, MP, M, BD, GZ, Sa], for which some results were already known by case-specific arguments. Moreover, in the case of bistochastic RWREs, stronger new results can be obtained, which we now present.

A *bistochastic environment* is defined by the condition that, Π-almost surely,

$$\forall y \in \mathbb{Z}^d, \qquad \sum_{x \in \mathbb{Z}^d} p_{xy} = 1. \tag{1.9}$$

It is easy to see, cf. Section 4, that, for a bistochastic random environment, $\mathcal{P}_\Pi$ is stationary, so (A5) is verified by Π itself.

Corollary 1.6 *For a bistochastic RWRE verifying* (A1)-(A4)*, the annealed process (with law $\mathbb{P}$ and dynamics $\mathcal{F}$) and the EVP (with initial state Π) are stationary and ergodic.*

Suppose further that the RW has *zero local drift*, i.e., $\mathcal{D} \equiv 0$, cf. (1.8). By the invariance of Π, this is the same as:

$$\forall x \in \mathbb{Z}^d, \qquad \sum_{y \in \mathbb{Z}^d} p_{xy}\,(y - x) = 0, \tag{1.10}$$

for Π-a.e. ω. This means, of course, that (X_n) is a martingale. (Examples of bistochastic martingales may be found, for instance, in Appendix A of [L2].) In this case, assumption (A1) might well be dropped because, if $\omega \in \Omega_{\mathrm{det}}$, i.e., $\forall y \in \mathbb{Z}^d$, $p_{0y} = \delta_{y_o y}$, then (1.10) implies that $y_o = 0$, which, in view of (A4), entails that $p_{0y}^{(n)} = \delta_{0y}$, $\forall n \in \mathbb{Z}_+$. Therefore, (A4) implies that $\Pi(\Omega_{\mathrm{det}}) = 0$, and (A1) is automatically verified.

In the martingale case, subject to a natural extra condition on the variance of the jumps, we can prove the *quenched IP*. We state it in the form of a theorem as soon as we have established some notation. Given (X_n), define the continuous trajectory $R_n : [0, 1] \longrightarrow \mathbb{R}^d$ via the following: For $k = 0, 1, \ldots, n-1$ and $t \in [k/n, (k+1)/n]$,

$$R_n(t) := \frac{X_k + (nt - k)(X_{k+1} - X_k)}{\sqrt{n}}. \tag{1.11}$$

(Evidently, the graph of R_n is the polyline joining the points $(k/n, X_k/\sqrt{n})$, for $k = 0, \ldots, n$.) The above can be regarded as a stochastic process relative to either P_ω (the *quenched trajectory*) or $\mathbb{P}$ (the *annealed trajectory*). Then we have:

Theorem 1.7 *Assume* (A2)-(A4)*, with $\gamma > 2$, and* (1.10)*. There exists a symmetric $d \times d$ matrix C such that, for Π-a.e. $\omega \in \Omega$, the quenched trajectory R_n, relative to P_ω, converges to the d-dimensional Brownian motion with drift 0 and diffusion matrix C. The convergence is intended in the weak-* sense in $C([0,1])$ endowed with the sup norm.*

Since the diffusion matrix, an expression for which is given later in (4.13), does not depend on ω, one immediately gets the *annealed IP*.

Corollary 1.8 *The annealed trajectory converges to the same Brownian motion as in Theorem 1.7.*

A consequence of this is the almost sure recurrence in dimension one and two.

Corollary 1.9 *If $d \le 2$ then, $\mathbb{P}$-almost surely (equivalently, P_ω-almost surely for Π-a.e. $\omega \in \Omega$),*

$$\liminf_{n\to\infty} |X_n| = 0.$$

The proofs of Theorems 1.3 and 1.7 are based on a convenient representation of the RWRE as a probability-preserving dynamical system which, roughly speaking, is a "measured family" of one-dimensional Markov maps. Each map embodies the dynamics of one random jump, and thus contains only local information. We will see that this dynamical system is isomorphic to the annealed process. In any case, Section 2 should convince the reader that it is natural to call this object 'the dynamical system for the point of view of the particle'; in short, *PVP dynamical system*.

The exposition is organized as follows: In Section 2 we introduce the dynamical system and find a suitable invariant measure for it. In Section 3 we prove its ergodicity, which is equivalent to Theorem 1.3 and implies its corollaries. In Section 4 we consider the bistochastic environments, establishing Theorem 1.7 and the other results.

Acknowledgments. I am grateful to Alessandra Bianchi, Stefano Olla, Firas Rassoul-Agha, Luc Rey-Bellet, and Vladas Sidoravicius for useful discussions. This work was partially supported by the FIRB-"Futuro in Ricerca" Project RBFR08UH60 (MIUR, Italy).

2 The PVP dynamical system

Let us fix an enumeration $(d_i)_{i\in\mathbb{Z}_+}$ of $\mathbb{Z}^d$. For $\omega \in \Omega$ and $i \in \mathbb{Z}_+$, we define

$$q_i(\omega) := p_{0d_i}(\omega) \tag{2.1}$$

Certainly, $\sum_i q_i(\omega) = 1$. We then set $a_0(\omega) := 0$ and, recursively on i,

$$a_i(\omega) := a_{i-1}(\omega) + q_i(\omega) \tag{2.2}$$
$$I_i(\omega) := [a_{i-1}(\omega)\,,\, a_i(\omega)). \tag{2.3}$$

Clearly, $\{I_i(\omega)\}_{i\in\mathbb{Z}_+}$ is a partition of $I := [0,1)$. For $(s,\omega) \in I \times \Omega$, denote by $i(s,\omega)$ the unique i such that $s \in I_i(\omega)$. We define the function $\phi : I \times \Omega \longrightarrow I$ via

$$\phi(s,\omega) := \frac{s - a_{i(s,\omega)}(\omega)}{q_{i(s,\omega)}(\omega)}. \tag{2.4}$$

(The definition above is well-posed because, if i is such that $q_i(\omega) = 0$, there is no s such that $i(s,\omega) = i$.) By construction, $\phi(\cdot,\omega)$ is the perfect Markov map $I \longrightarrow I$ relative to the partition $\{I_i(\omega)\}$. Finally, we denote $D(s,\omega) := d_{i(s,\omega)}$.

The main technical tool of this paper is the map $T : \mathcal{M} \longrightarrow \mathcal{M}$, defined on $\mathcal{M} := I \times \Omega$ by

$$T(s,\omega) := \left(\phi(s,\omega), \tau_{D(s,\omega)}\omega\right). \tag{2.5}$$

We endow $\mathcal{M}$ with either the probability measure $\mu := m \otimes \Pi$ or $\mu_* := m \otimes \Pi_*$, where m is the Lebesgue measure on I.

What this dynamical system has to do with our RWRE is presently explained. Let us recall the notational convention whereby the dependance on ω is not always indicated. Fix $\omega \in \Omega$ and consider a random $s \in I$ w.r.t. m. We have that $D(s,\omega) = d_i$ if and only if $s \in I_i$, and this occurs with probability $m(I_i) = q_i$. In terms of our RW, this is exactly the probability that a particle placed in the origin of $\mathbb{Z}^d$, endowed with the environment $p(\omega)$, jumps by a quantity d_i. Then, back to the dynamical system, condition the measure m to I_i. Calling $(s_1,\omega_1) := T(s,\omega)$, we see that, upon conditioning, s_1 ranges in I with law m. Therefore, in a sense, the variable s (which we may call the *internal variable*) has "refreshed" itself. Furthermore, $\omega_1 = \tau_{D(s,\omega)}\omega = \tau_{d_i}\omega$ is the translation of ω in the opposite direction to d_i, cf. (1.1). Hence we can imagine that we have reset the system to a new initial condition (s_1,ω_1), corresponding to the particle sitting in $0 \in \mathbb{Z}^d$ and subject to the environment $p(\omega_1)$. Applying the same reasoning to $(s_2,\omega_2) := T(s_1,\omega_1)$, and so on, shows that we are following the motion "from the point of view of the particle". We thus call the above the 'PVP dynamical system'.

In any case, it should be clear that the stochastic process $(X_n)_{n\in\mathbb{N}}$, with $X_0 \equiv 0$ and, for $n \geq 1$,

$$X_n(s,\omega) := \sum_{k=0}^{n-1} D \circ T^k(s,\omega). \tag{2.6}$$

is precisely the RW in the environment $p(\omega)$, provided that ω is regarded as a fixed parameter. To emphasize this point, we occasionally write $X_{n,\omega}(s) := X_n(s,\omega)$. $(X_{n,\omega})$ is called the 'quenched trajectory', and it is a Markov chain. If both s and ω are considered random, w.r.t. μ, then (2.6) defines the 'annealed trajectory'. This is not a Markov chain and, by the definition of μ and (1.4), it is none other than the RWRE of Section 1 with law $\mathbb{P}$. For a formal relation between the annealed process and the PVP dynamical system see Proposition 3.3.

Proposition 2.1 *The measure μ_* is preserved by T.*

We need the following lemma:

Lemma 2.2 *For every measurable set $B \subseteq \Omega$,*

$$\Pi_*(B) = \sum_{i \in \mathbb{Z}_+} \int_{\tau_{-d_i}(B)} \Pi_*(d\omega')\, q_i(\omega').$$

Proof of Lemma 2.2. Via (2.1) and (A2), we observe that the series

$$\sum_{i \in \mathbb{Z}_+} q_i(\omega') = \sum_{x \in \mathbb{Z}^d} p_{0x}(\omega'), \tag{2.7}$$

is uniformly bounded in ω'. Thus, as we will do more than once presently, it is correct to interchange the above summation with an integration over Ω, if it is relative to a probability measure.

Using again (2.1), the transition kernel of the EVP process (1.7) can be written as

$$\mathcal{K}(\,\cdot\,|\,\omega') = \sum_{i \in \mathbb{Z}_+} q_i(\omega')\, \delta_{\tau_{d_i}\omega'}, \tag{2.8}$$

where the Dirac delta in the r.h.s. is thought of as a measure. In other words, for a measurable set B,

$$\mathcal{K}(B\,|\,\omega') = \int_\Omega \mathcal{K}(d\omega\,|\,\omega')\, 1_B(\omega) = \sum_{i \in \mathbb{Z}_+} q_i(\omega')\, 1_{\tau_{-d_i(B)}}(\omega'). \tag{2.9}$$

Thus, the hypothesis on Π_* from (A5), namely,

$$\Pi_*(B) = \int_\Omega \Pi_*(d\omega')\, \mathcal{K}(B\,|\,\omega'), \tag{2.10}$$

reads precisely as in the statement of the lemma. Q.E.D.

Proof of Proposition 2.1. It is enough to prove that $\mu_*(T^{-1}A) = \mu_*(A)$ for all sets of the type $A = [b, c) \times B$, where B is a measurable set of Ω.

By direct inspection of the map (2.5), we can write $T^{-1}A = \bigcup_{i \in \mathbb{Z}_+} A'_i$, where

$$A'_i := \{(s', \omega') \mid \omega' \in \tau_{-d_i}(B),\ s' \in [a_i(\omega') + q_i(\omega')b\,,\ a_i(\omega') + q_i(\omega')c)\}\,; \tag{2.11}$$

cf. (2.2), (2.4). Thus,

$$\mu_*(A'_i) = \int_{\tau_{-d_i}(B)} \Pi_*(d\omega')\, q_i(\omega')(c-b). \tag{2.12}$$

The sets A'_i are pairwise disjoint because, by construction, they belong to different level sets of the function D. Therefore, by Lemma 2.2,

$$\mu_*(T^{-1}A) = (c-b) \sum_{i \in \mathbb{Z}_+} \int_{\tau_{-d_i}(B)} \Pi_*(d\omega')\, q_i(\omega') = (c-b)\, \Pi_*(B) = \mu_*(A), \tag{2.13}$$

which is what we wanted to prove. Q.E.D.

Let us introduce a convenient notation that will be used throughout the paper: For $(s, \omega) \in \mathcal{M}$ and $k \in \mathbb{N}$, denote

$$(s_k, \omega_k) := T^k(s, \omega). \tag{2.14}$$

Proposition 2.3 *The measures μ_* and μ are equivalent or, which is the same, the measures Π_* and Π are equivalent.*

Proof. Denote $\Omega_* := \operatorname{supp} \Pi_* \subseteq \Omega$ and $\Omega_*^c := \Omega \setminus \Omega_*$. By (A5), $\Pi(\Omega_*^c) < 1$. Suppose, by way of contradiction, that $\Pi(\Omega_*^c) > 0$ as well.

By (A3), for Π-a.e. $\omega \in \Omega_*$, there exists $y_\omega \in \Gamma$ such that

$$\tau_{y_\omega} \omega \in \Omega_*^c. \tag{2.15}$$

To each such ω (excluding at most a Π-null set) we apply (A4) and its interpretation in terms of the PVP dynamical system: there exists a positive integer $n = n(\omega, y_\omega)$ such that

$$J_\omega := \{ s \in I \mid X_n(s, \omega) = y_\omega \} \tag{2.16}$$

has measure $p^{(n)}_{0 y_\omega} > 0$. By definition, $\forall s \in J_\omega$,

$$\begin{aligned} T^n(s, \omega) &= (s_n, \tau_{D(s_{n-1}, \omega_{n-1})} \circ \cdots \circ \tau_{D(s,\omega)} \, \omega) \\ &= (s_n, \tau_{X_n(s,\omega)} \, \omega) \\ &= (s_n, \tau_{y_\omega} \omega) \in I \times \Omega_*^c, \end{aligned} \tag{2.17}$$

having used (2.14), (2.6), (2.16) and finally (2.15). If we define

$$A := \bigcup_{\omega \in \Omega_*} J_\omega \times \{\omega\}, \tag{2.18}$$

we have $\mu(A) > 0$ and, via (2.17),

$$A \subset \bigcup_{n \geq 1} T^{-n}(I \times \Omega_*^c). \tag{2.19}$$

The definition of Ω_*^c implies that $\mu_*(I \times \Omega_*^c) = 0$, hence, since μ_* is T-invariant, $\mu_*(A) = 0$. Finally, since μ and μ_* are equivalent on $I \times \Omega_* \supset A$, $\mu(A) = 0$, which contradicts a previous statement. Q.E.D.

3 Ergodicity

In this section we will prove the ergodicity of $(\mathcal{M}, \mu_*, T)$. For this, we need to introduce some more notation and establish a few lemmas.

Given a positive integer n and a multi-index $\mathbf{i} := (i_0, i_1, ..., i_{n-1}) \in \mathbb{Z}_+^n$, we set

$$I_{\mathbf{i}}(\omega) := \left\{ s \in I \;\middle|\; D \circ T^k(s,\omega) = d_{i_k},\ \forall k = 0, \ldots, n-1 \right\}. \tag{3.1}$$

For $n = 1$ this reduces to definition (2.3). It is easy to ascertain that $\{I_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}_+^n}$ is a partition of I into countably many (possibly empty) right-open intervals, each of which corresponds to one of the realizations of the RW $(X_{k,\omega})_{k=0}^n$ (relative to the environment ω) in such a way that $m(I_{\mathbf{i}})$ is the probability of the corresponding realization. In analogy with the previous notation, we denote by $\mathbf{i}_n(s,\omega)$ the index of the element of the partition which contains s.

Furthermore, let us call *horizontal fiber* of $\mathcal{M}$ any segment of the type $I_\omega := I \times \{\omega\}$, and indicate with m_ω the Lebesgue measure on it. Lastly, we denote by $I_{\omega,\mathbf{i}}$ the subset of I_ω corresponding to $I_{\mathbf{i}}(\omega)$ via the natural isomorphism $I \cong I_\omega$.

Lemma 3.1 *Recalling the definition of* Ω_{det} *from* (A1)*,*

$$\mu_* \left(\bigcap_{n=0}^{\infty} T^{-n} \left(I \times \Omega_{\mathrm{det}} \right) \right) = 0.$$

In other words, almost no orbits of T *stay confined to* $I \times \Omega_{\mathrm{det}}$*.*

Proof. $\omega \in \Omega_{\mathrm{det}}$ if and only if $\{I_i(\omega)\}_{i \in \mathbb{Z}_+}$ is the trivial partition of I, whence $T_{|I_\omega}(s,\omega) = s$. Denote by A the set that is being measured in the statement of the lemma. Recalling notation (2.14), $(s,\omega) \in A$ if and only if $\omega_n \in \Omega_{\mathrm{det}}$, $\forall n \in \mathbb{N}$. Equivalently, $\{I_{\mathbf{i}}(\omega)\}_{\mathbf{i} \in \mathbb{Z}_+^n}$ is the trivial partition of I and $s_n = s$, for all n. This means that the T-orbit of (s,ω) does not depend on s and the quenched trajectory of the RW (starting at 0) is completely deterministic in the environment ω. In particular, A is in the from $A = I \times B$. The completeness of Π implies that B is measurable, as in Lemma A.1 of [L1] (cf. Lemma 3.4 of [L2]).

We want to show that $\mu_*(A) = 0$, that is, $\Pi_*(B) = 0$, that is, by Proposition 2.3, $\Pi(B) = 0$. By absurd, assume the contrary. Also denote $\Omega_{\mathrm{det}}^c := \Omega \setminus \Omega_{\mathrm{det}}$. By (A1), $\Pi(\Omega_{\mathrm{det}}^c) > 0$.

We reason along the same lines as Proposition 2.3. Take $\omega \in B$ that is typical in the sense of both (A3) and (A4): there exist $y \in \Gamma$ such that $\tau_y \omega \in \Omega_{\mathrm{det}}^c$, and $n \in \mathbb{Z}_+$ such that $p_{0y}^{(n)} > 0$. Since the trajectory is completely deterministic, $p_{0y}^{(n)} = 1$ and, $\forall s \in I$, $\omega_n = \tau_y \omega$; cf. (2.17). Thus $\omega_n \in \Omega_{\mathrm{det}}^c$, contradicting the fact that $(s,\omega) \in A$, or $\omega \in B$. Q.E.D.

Lemma 3.2 *For a.a.* $(s,\omega) \in \mathcal{M}$*,* $m(I_{\mathbf{i}_n(s,\omega)})$ *vanishes exponentially fast, as* $n \to \infty$*.*

Proof. Set

$$f(s,\omega) := \log q_{i(s,\omega)}^{-1}(\omega) = -\log m(I_{i(s,\omega)}(\omega)). \tag{3.2}$$

Clearly $f(s,\omega) \geq 0$, with $f(s,\omega) = 0$ if and only if $\{I_i(\omega)\}$ is the trivial partition of I, if and only if $\omega \in \Omega_{\mathrm{det}}$. The Birkhoff average of f,

$$f^+(s,\omega) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(s_k,\omega_k), \tag{3.3}$$

is non-negative as well. Set $A := \{(s,\omega) \mid f^+(s,\omega) = 0\}$. As a level set of an invariant function, A is invariant mod μ_*. We claim that $\mu_*(A) = 0$. If not, we can apply one of the assertions of Birkhoff's Theorem to the measure-preserving dynamical system $(A, \mu_*, T_{|A})$ and conclude that $\int_A d\mu_*\, f = \int_A d\mu_*\, f^+ = 0$. Therefore, $f(s,\omega) = 0$, for a.a. $(s,\omega) \in A$. In other words, $A \subseteq I \times \Omega_{\mathrm{det}}$ mod μ_*. Since A is invariant, the orbit of a.e. point of A is contained in $I \times \Omega_{\mathrm{det}}$, which contradicts Lemma 3.1, for $\mu_*(A) > 0$.

So $f^+ > 0$ almost everywhere. On the other hand, from earlier considerations, it is easy to verify that, for $n \geq 1$,

$$m\left(I_{\mathbf{i}_n(s,\omega)}(\omega)\right) = \prod_{k=0}^{n-1} q_{i(s_k,\omega_k)}(\omega_k) = \exp\left(-\sum_{k=0}^{n-1} f(s_k,\omega_k)\right). \tag{3.4}$$

Due to the almost sure positivity of (3.3), the exponent in the rightmost term above is asymptotically linear in n, for a.a. (s,ω), which yields the assertion. Q.E.D.

Proposition 3.3 *As dynamical systems on probability spaces, $(\mathcal{M},\mu,T)$ is isomorphic to $((\mathbb{Z}^d)^{\mathbb{N}} \times \Omega, \mathbb{P}, \mathcal{F})$, and $(\mathcal{M},\mu_*,T)$ is isomorphic to $((\mathbb{Z}^d)^{\mathbb{N}} \times \Omega, \mathbb{P}_*, \mathcal{F})$.*

Proof. We hope the reader was already convinced in Section 2 that the PVP dynamical system describes exactly the annealed process with the PVP dynamics. On the other hand, Lemma 3.2 provides the ingredients for a formal proof, which we just sketch here.

For both pairs of systems, a natural isomorphism $\Phi : \mathcal{M} \longrightarrow (\mathbb{Z}^d)^{\mathbb{N}} \times \Omega$ is given by

$$\Phi(s,\omega) := \left((X_n(s,\omega))_{n\in\mathbb{N}}, \omega\right), \tag{3.5}$$

cf. (2.6). One sees that Φ is almost-everywhere bijective because of the following: By Lemma 3.2, a.e. $(s,\omega) \in \mathcal{M}$ is the unique intersection point of the nested sequence of right-open intervals $(I_{\omega,\mathbf{i}_n(s,\omega)})_{n\in\mathbb{N}}$, i.e., is uniquely determined by the sequence $(\mathbf{i}_n(s,\omega))$, equivalently, by the realization $(X_n(s,\omega))$ of the RW. Viceversa, for an environment ω, every realization of the walk determines a nested sequence of intervals which, except for a null set of realizations, gives a point $(s,\omega) \in I_\omega$. (Since the intervals are right-open, this correspondence is ill-defined at the endpoints of all such intervals. But this amounts to a null set of points in I_ω and a null set of realizations in $(\mathbb{Z}^d)^{\mathbb{N}}$.)

Finally, it is clear by the considerations of Section 2 that $\mu = \mathbb{P} \circ \Phi$, $\mu_* = \mathbb{P}_* \circ \Phi$ and $T = \Phi^{-1} \circ \mathcal{F} \circ \Phi$. Q.E.D.

Lemma 3.4 *The ergodic components of $(\mathcal{M}, \mu_*, T)$ contain whole horizontal fibers, that is, every invariant set is of the form $I \times B$, mod μ_* (equivalently, mod μ), where B is a measurable subset of Ω.*

Proof. Suppose the assertion is false. There exists an invariant set A whose intersection with many horizontal fibers is neither the full fiber nor empty, mod m_ω. That is, for some $\varepsilon > 0$, the Π_*-measure of

$$B_\varepsilon := \{\omega \in \Omega \mid m_\omega(A \cap I_\omega) \in [\varepsilon, 1-\varepsilon]\} \tag{3.6}$$

is positive. By the Poincaré Recurrence Theorem and the Lebesgue Density Theorem it is possible to pick $(s, \omega) \in A \cap (I \times B_\varepsilon)$ that is recurrent to $I \times B_\varepsilon$ and such that (s, ω) is a density point of $A \cap I_\omega$, within I_ω, relative to m_ω. We claim that there exist a sufficiently large n and a multi-index $\mathbf{i} \in \mathbb{Z}_+^n$ for which

$$m_\omega(A \cap I_{\omega,\mathbf{i}}) > (1-\varepsilon)\, m_\omega(I_{\omega,\mathbf{i}}) \tag{3.7}$$

and

$$T^n I_{\omega,\mathbf{i}} = I_{\omega_n} \subset I \times B_\varepsilon. \tag{3.8}$$

In fact, among the infinitely many n that verify $T^n(s,\omega) \in I \times B_\varepsilon$, we can choose, by Lemma 3.2, one for which $I_{\omega,\mathbf{i}_n(s,\omega)}$ is so small that (3.7) is verified for $\mathbf{i} = \mathbf{i}_n(s,\omega)$. The equality in (3.8) is true by the Markov property of $\phi(\cdot, \omega_{n-1}) \circ \cdots \circ \phi(\cdot, \omega)$ (recall the notation (2.14)).

It is no loss of generality to assume that the n^{th} iterate of m_ω-a.e. point of $A \cap I_{\omega,\mathbf{i}}$ remains in A (in the choice of (s, ω), use the invariance of A mod μ_* and Fubini's Theorem). Since the restriction of T^n to $I_{\omega,\mathbf{i}}$ is linear, we deduce from (3.7)-(3.8) that $m_{\omega_n}(A \cap I_{\omega_n}) > 1 - \varepsilon$, which contradicts (3.6), because $\omega_n \in B_\varepsilon$ by (3.8).

Therefore, an invariant set mod μ_* can only occur in the form $I \times B$, with B measurable (by the same argument is in the proof of Lemma 3.1). Q.E.D.

Remark 3.5 The techniques of Lemma 3.4 (based on the fact that $\phi(\cdot, \omega)$ is a piecewise-linear Markov map of the interval) easily imply that any T-invariant measure that is smooth along the horizontal fibers must be uniform on them, i.e., must be of the type $m \otimes \Pi'$.

Theorem 3.6 $(\mathcal{M}, \mu_*, T)$ *is ergodic.*

Proof. Suppose the system is not ergodic. By Lemma 3.4, we have an invariant set $I \times B$, with $\Pi(B) \in (0,1)$. Therefore the probability $\mu' := \mu_*(\,\cdot\,| I \times B)$ is T-invariant and decomposes as $\mu' = m \otimes \Pi'$, where $\Pi' := \Pi_*(\,\cdot\,|B)$. So we can apply Proposition

2.3 with Π' in the role of Π_*. The result contradicts the hypothesis $\Pi(B) \in (0,1)$. Q.E.D.

We can now easily prove the main result of the paper.

PROOF OF THEOREM 1.3. Assertion *(a)* is Proposition 2.3. Assertion *(b)* follows from Proposition 2.1, Theorem 3.6 and Proposition 3.3. Q.E.D.

PROOF OF COROLLARY 1.4. A bounded measurable function $\phi : \Omega^{\mathbb{N}} \longrightarrow \mathbb{R}$ induces a bounded measurable function $f : \mathcal{M} \longrightarrow \mathbb{R}$ via $f(s,\omega) := \phi((\tau_{X_n(s,\omega)}\omega)_{n\in\mathbb{N}}) = \phi((\omega_n)_{n\in\mathbb{N}})$. By Theorem 3.6, the asymptotic Birkhoff average of f is constant μ_*-almost everywhere, i.e., from Proposition 3.3, for P_ω-a.e. realization $(x_{n,\omega})$ of the RW, in Π_*-a.e. environment ω. By the definition of the EVP process (compare (1.7) with (1.3)), this means that the asymptotic Birkhoff average of ϕ is constant for $\mathcal{P}_{\Pi_*}$-a.e. realization (ω_n) of the process. By density, the result extends to all $\phi \in L^1(\mathcal{P}_{\Pi_*})$. Q.E.D.

PROOF OF COROLLARY 1.5. Let us first notice that, by (1.8) and the definition of D from Section 2,

$$\mathcal{D}(\omega) = \sum_{i\in\mathbb{Z}^+} q_i(\omega)\, d_i = \int_I ds\, D(s,\omega). \tag{3.9}$$

We apply Theorem 3.6 to the displacement function D, cf. (2.6):

$$\begin{aligned}
\lim_{n\to\infty} \frac{X_n(s,\omega)}{n} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} D \circ T^k(s,\omega) \\
&= \int_{\mathcal{M}} \mu_*(ds'd\omega') D(s',\omega') \\
&= \int_\Omega \Pi_*(d\omega')\, \mathcal{D}(\omega'),
\end{aligned} \tag{3.10}$$

for μ_*- or μ-a.e. $(s,\omega) \in \mathcal{M}$, that is, P_ω-almost surely for Π-a.e. ω (Proposition 3.3). Q.E.D.

4 Bistochastic environments

In the remainder of this note we focus on the example of the bistochastic environments, as defined by (1.9). In the language of Section 2, (1.9) implies that

$$\sum_{i\in\mathbb{Z}_+} q_i(\tau_{-d_i}\,\omega) = \sum_{x\in\mathbb{Z}^d} p_{0x}(\tau_{-x}\,\omega) = \sum_{x\in\mathbb{Z}^d} p_{-x0}(\omega) = 1, \tag{4.1}$$

for Π-a.e. ω; cf. (2.1) and (1.1). Therefore, in view of the statement of Lemma 2.2,

$$\sum_{i\in\mathbb{Z}_+} \int_{\tau_{-d_i}(B)} \Pi(d\omega')\, q_i(\omega') = \sum_{i\in\mathbb{Z}_+} \int_B \Pi(d\omega)\, q_i(\tau_{-d_i}\omega) = \Pi(B), \tag{4.2}$$

having used the invariance of Π w.r.t. (τ_z) (in the first equality) and (4.1) (in the second equality). The above proves Lemma 2.2 for Π, and thus Proposition 2.1 for μ. Consequently, all the results of Sections 2 and 3 hold true if Π_* and μ_* are replaced, respectively, by Π and μ. In particular, this implies Corollary 1.6.

Let us now assume the hypotheses of Theorem 1.7. We point out that requiring $\gamma > 2$ in (A2) ensures that the variance of the distribution $p_x = (p_{xy})_{y \in \mathbb{Z}^d}$ is unifomly bounded in x and ω, almost everywhere. This is instrumental in proving that the process $(X_n)_{n \in \mathbb{N}}$, defined in (2.6), is a martigale with finite covariances, as we do in a moment.

Here we consider (X_n) mostly (but not exclusively) in its quenched interpretation, i.e., as a sequence of random variables on the probability space (I_ω, m_ω), for some $\omega \in \Omega$. At any rate, let $\mathbb{E}_\omega$ denote the expectation in (I_ω, m_ω), and $\mathbb{E}_\mu$ the expectation in $(\mathcal{M}, \mu)$. Also, Greek superscripts will indicate the components of a d-dimensional vector (for example, X_n^α is the α^{th} component of X_n).

As shown by (2.6), both the quenched and the annealed trajectories have increments given by the identity

$$X_{n+1} - X_n = D \circ T^n. \tag{4.3}$$

Lemma 4.1 *For a.e. ω, the quenched RW $(X_{n,\omega})$ is a martingale with asymptotic covariance matrix $C := (c_{\alpha\beta})_{\alpha,\beta=1}^d$, where*

$$c_{\alpha\beta} := \mathbb{E}_\mu(D^\alpha D^\beta) = \int_{\mathcal{M}} \mu(ds d\omega)\, D^\alpha(s,\omega) D^\beta(s,\omega) < \infty.$$

In particular, writing for simplicity $X_n = X_{n,\omega}$, one has:

(a) for $\mathscr{F}_n := \sigma(X_1, \ldots, X_n)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}_\omega\Big((X_{k+1}^\alpha - X_k^\alpha)(X_{k+1}^\beta - X_k^\beta) \,\Big|\, \mathscr{F}_k \Big) = c_{\alpha\beta},$$

in probability w.r.t. m_ω;

(b) the Lindeberg condition holds, i.e., $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}_\omega\big(|X_{k+1} - X_k|^2 \, 1_{\{|X_{k+1}-X_k| > \varepsilon\sqrt{n}\}} \big) = 0,$$

where $1_{\{\cdot\}}$ is the indicator function of a given event.

Proof. The considerations outlined in the beginning of Section 3 show that the partition of I_ω corresponding to $\mathscr{F}_n$ is precisely $\{I_{\omega,\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^n}$, whence, by (4.3),

$$\mathbb{E}_\omega(D \circ T^n | \mathscr{F}_n) = \sum_{\mathbf{i}\in\mathbb{Z}_+^n} \left[\frac{1}{m(I_\mathbf{i})} \int_{I_\mathbf{i}} ds\, D \circ T^n(s,\omega) \right] 1_{I_{\omega,\mathbf{i}}}. \tag{4.4}$$

Let us show that all the integrals above vanish. For $\mathbf{i}$ fixed and s ranging in $I_{\mathbf{i}}$, the first n positions of the walk are determined by the values $x_k := X_k(s,\omega)$ ($k = 1, \ldots, n$). Thus, denoting $(\mathbf{i}, j) = (i_0, i_1, ..., i_{n-1}, j) \in \mathbb{Z}_+^{n+1}$,

$$\begin{aligned} \int_{I_{\mathbf{i}}} ds\, D \circ T^n(s,\omega) &= \sum_{j=1}^{\infty} m\big(I_{(\mathbf{i},j)}\big)\; d_j \\ &= p_{0x_1} \cdots p_{x_{n-1},x_n} \sum_{j=1}^{\infty} p_{x_n, x_n + d_j}\, d_j = 0, \end{aligned} \tag{4.5}$$

via (1.10). Therefore (X_n) is a martingale.

Now let us fix $\alpha, \beta \in \{1, \ldots, d\}$ and set

$$f(s,\omega) := f(\omega) := \mathbb{E}_\omega(D^\alpha D^\beta) = \int_I ds'\; D^\alpha(s',\omega) D^\beta(s',\omega). \tag{4.6}$$

Assumption (A2) with $\gamma > 2$ implies that f is bounded: in fact,

$$\begin{aligned} |f(\omega)| &\le \int_I ds'\; |D(s',\omega)|^2 = \sum_{i \in \mathbb{Z}_+} m(I_i)\, |d_i|^2 \\ &= \sum_{y \in \mathbb{Z}^d} p_{0y}\, |y|^2 \;\le\; K \sum_{y \in \mathbb{Z}^d} |y|^{-d-\gamma+2} \;=:\; K_1. \end{aligned} \tag{4.7}$$

Also, by the definition of $c_{\alpha\beta}$,

$$\int_{\mathcal{M}} \mu(ds d\omega)\, f(s,\omega) = \int_\Omega \Pi(d\omega)\, f(\omega) = c_{\alpha\beta}. \tag{4.8}$$

The above and (4.7) prove the finiteness of $c_{\alpha\beta}$.

Coming to assertion *(a)*, and considering (4.3), one easily checks that

$$\mathbb{E}_\omega\big((D^\alpha \circ T^k)(D^\beta \circ T^k) \,\big|\, \mathscr{F}_k \big) = \mathbb{E}_{\omega_k}(D^\alpha D^\beta) \circ T^k = f \circ T^k \tag{4.9}$$

(as usual, ω_k is given by (2.14)). In light of (4.8)-(4.9), *(a)* follows from Theorem 3.6 (with $\mu_* = \mu$): the Birkhoff average of f converges to $c_{\alpha\beta}$ for μ-a.e. $(s,\omega) \in \mathcal{M}$. That is, for Π-a.e. ω, we have convergence almost everywhere in I_ω, w.r.t. m_ω. Convergence in probability follows.

The proof of assertion *(b)* is achieved by a similar argument, except that we use the function

$$g_M(s,\omega) := g_M(\omega) := \mathbb{E}_\omega\big(|D|^2\, 1_{\{|D|>M\}}\big)\,, \tag{4.10}$$

which is bounded as in (4.7) (M is any positive number). Once again, the ergodicity of $(\mathcal{M}, \mu, T)$ entails that, for a.e. ω, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g_M \circ T^k(s,\omega) = \mathbb{E}_\mu(g_M) \tag{4.11}$$

holds for a.e. $s \in I_\omega$. Due to the boundedness of g_M, the above is true even if we integrate the summand in the l.h.s. on $s \in I_\omega$. Therefore, for every $M > 0$, there exists a full-measure set of ω for which

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}_\omega \left(\left| D \circ T^k \right|^2 1_{\{|D \circ T^k| > \varepsilon \sqrt{n}\}} \right) \\ \le \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}_\omega \left(g_M \circ T^k \right) \ = \ \mathbb{E}_\mu(g_M). \end{aligned} \tag{4.12}$$

It is easy to see (e.g., by letting M take values along a diverging sequence of positive numbers), that (4.12) holds for a.a. ω, *independently of M*. In view of (4.3), this yields *(b)* because $\inf_{M>0} \mathbb{E}_\mu(g_M) = 0$. Q.E.D.

In terms of the transition probabilities of Section 1, the entries of C are expressed by

$$c_{\alpha\beta} = \int_\Omega \Pi(d\omega) \sum_{y \in \mathbb{Z}^d} p_{0y}(\omega)\, y^\alpha y^\beta. \tag{4.13}$$

Understandably, C is the average (or annealed) covariance matrix of a jump, relative to steady state on the space of the environments.

The quenched IP is a corollary of Lemma 4.1.

Proof of Theorem 1.7. The result follows from the Lindeberg–Feller Theorem for martingales, whose hypotheses are precisely the assertions of Lemma 4.1. (See [D, Thm 7.7.4] for a convenient one-dimensional version of that theorem; cf. also [HH, Chap. 4]. The multidimensional version follows in a standard way, via the Cramér–Wold device [D, B1]: see, e.g., [BP, p. 1341]; cf. also [Z, Thm. 3.3.4] and [La].) Q.E.D.

As Corollary 1.8 is obvious, it remains to prove Corollary 1.9, i.e., the Π-a.s. recurrence of $(X_{n,\omega})$ in dimension 1 and 2. We do so via a general result of Schmidt [S], which we restate and adapt as follows.

Theorem 4.2 *Let $(\mathcal{M}, \mu, T)$ be an ergodic dynamical system, with $\mu(\mathcal{M}) = 1$, and f a measurable function $\mathcal{M} \longrightarrow \mathbb{Z}^d$. Call* cocycle of f *the family $(S_n)_{n\in\mathbb{N}}$ of random vectors on $(\mathcal{M}, \mu)$ thus defined: For $n = 0$, $S_0 := 0$; for $n \ge 1$,*

$$S_n := \sum_{k=0}^{n-1} f \circ T^k.$$

If there exist a positive-density sequence $(n_k)_{k\in\mathbb{N}}$ and two constants $\kappa, \rho_o > 0$ such that

$$\mu \left(\left| \frac{S_{n_k}}{{n_k}^{1/d}} \right| \le \rho \right) \ge \kappa \rho^d,$$

for all $k \in \mathbb{N}$ and $\rho \in (0, \rho_o)$, then the cocycle (S_n) is recurrent*, namely,*

$$\liminf_{n\to\infty} |S_n| = 0,$$

μ-almost surely.

Theorem 4.2 was proved by Schmidt in 1998 [S], with the extra assumption that T be invertible mod μ. The generalization to noninvertible measure-preserving maps is an easy exercise which can be found, e.g., in Appendix B of [L2].

PROOF OF COROLLARY 1.9. By (2.6) we see that the annealed trajectory (X_n) is the cocycle of D over the ergodic dynamical system $(\mathcal{M}, \mu, T)$. For $d \leq 2$, the hypotheses of Theorem 4.2 are evidently satisfied if (X_n) verifies the Central Limit Theorem (CLT) with zero mean, relative to μ. (In point of fact, much less is needed when $d = 1$.) But the CLT is a trivial consequence of Corollary 1.8.

Corollary 1.9 then follows from the isomorphism between $(\mathcal{M}, \mu, T)$ and the annealed process (Proposition 3.3). Q.E.D.

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