A study of quantum pseudodot system with a two-dimensional pseudoharmonic potential using Nikiforov-Uvarov method

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Abstract

We use the Nikiforov-Uvarov method to calculate the bound states (energy spectra and wave functions) of a two-dimensional (2D) electron gas interacted with an exactly solvable pseudoharmonic confinement potential in a strong uniform magentic field inside dot and Aharonov-Bohm flux field inside a pseudodot. We give a unified treatment for both Schrödinger and spin-0 Klein-Gordon energy spectrum and wave functions as functions of chemical potential parameter, magnetic field strength, AB flux field and magnetic quantum number. We obtain analytic expression for the light interband absorption coefficient and threshold frequency of absorption as functions of applied magnetic field and geometrical size of quantum pseudodot. The temperature dependence energy levels for GaAs are also calculated.

Keywords: Pseudoharmonic potential, Quantum dot, Quantum antidot, Bound states, Magnetic field, Flux field, Light interband transition, Threshold frequency of absorption, Temperature dependence effective mass, Nikiforov-Uvarov method.

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I. INTRODUCTION

Over a long time, a considerable interest has been paid for studying size effects in orbital magnetism [1,2] and the magnetic properties of low-dimensional metallic and semiconducting structures with restricted geometries [3] on nanostructures such as dots, wires, wells, antidots, well wires and antiwells [4,5,6]. Such structures can confine charge carriers in one, two and three dimensions. Experimental research is currently made to study the optical and quantum properties of low-dimensional semiconducting structures for the fabrication purposes and subsequent working of electronic and optical devices. More studies analyzing these structures have been focused on the interband light absorption coefficient in the spherical [7,8,9], parabolic, cylindrical and rectangular [10] quantum dots in the presence and absence of magnetic field [11]. More other works on optical properties in nanostructures [12,13], band structure calculations, transport properties of Aharonov-Bohm (AB) type oscillations [14] and Altshuler-Aharonov-Spivak (AAS) type oscillation [15].

The quantum antidot structure has been modeled in the presence and absence of repulsive antidot potential, harmonic confining oscillator potential, the presence and absence of magnetic and Aharonov-Bohm (AB) flux fields in cylindrical coordinates [16]. This allows one to obtain an exact bound state solutions for the Schrödinger equation. The influence of dots and antidots on thermodynamic properties (e.g., magnetization) of the system, the magnetotransport properties and also the magneto-optical (MO) spectroscopic characteristics of a two-dimensional (2D) electron gas in a magnetic field are studied in [16]. The nature of MO transitions in this system demonstrate the appearance of rich spectrum of nonequidistant frequencies are different from the MO spectrum for a dot modeled by a harmonic confining potential. The quantum antidot is modeled as an electron moving outside a cylinder of radius a in the presence of magnetic and AB flux fields to find analytic expressions for energy and wave function [17]. The numerical and analytical solutions obtained for the dynamics of two classical electrons interacting via a Coulomb field in a 2D antidot superlatice potential in the presence of crossed electric and magnetic fields are quite different than the noninteracting electrons [18]. Some authors have studied a 2D theoretical model for the quantum dot in which electrons were confined by a nonhomogenous magnetic field (the so-called magnetic antidot) [19]. The pseudoharmonic (PH) potential [20,21] is used in modeling the quantum dots (QDs) and quantum antidots (QADs) in nanostructures

[22]. The spectral properties in a 2D electron confined by a pseudoharmonic quantum dot (PHQD) potential in the presence of external strong uniform magnetic field $\overrightarrow{B}$ along the z direction in the presence of AB flux field created by a selenoid inserted inside the pseudodot have been studied. The Schrödinger and spinless Klein-Gordon equation are solved exactly for their bound states (energy spectrum and wave function) [22]. The advantage of the Klein-Gordon solution is that it provides us relativistic corrections to the commonly known nonrelativistic solution.

It is well-known that factors such as impurity, electric and magnetic fields, pressure, and temperature play important roles in the electronic, optical and transport properties of low-dimensional semiconductor nanostructures [4,23-28]. In this regard, we carry out detailed exact analytic analysis of one-particle energetic spectrum and wave functions of both Schrödinger and Klein-Gordon equations with a pseudoharmonic potential in the presence of magnetic field and Aharonov-Bohm flux field by using the Nikiforov-Uvarov method [29,30]. The resulting energy spectrum serves as a base for calculating the corresponding interband light (optical) absorption coefficient and the threshold frequency value of absorption for the given model. In addition, the effect of the temperature on the effective mass is also calculated.

The structure of the paper is as follows. In Sec. 2, the basic formulas of the Nikiforov-Uvarov (NU) method are outlined in short. In Sec. 3, we studied the nonrelativistic quantum dot and antidot with the pseudoharmonic potential in the presence of magnetic and Aharonov-Bohm flux fields. The exact analytic expressions for the energy spectra and wave functions are calculated. In Sec. 4, the analytic expressions for the bound states of the KG electron interacted via the pseudoharmonic potential in the presence of magnetic field and AB flux field are calculated. These basic formulas are also reduced to Schrödinger solutions for the pseudoharmonic potential model and free-field interactions under the non-relativistic limits. Results and discussions are performed in Sec. 5. The conclusions and outlook are presented in Sec. 6.

II. NIKIFOROV-UVAROV METHOD

This method is usually used in solving a second-order hypergeometric-type differential equations satisfying special orthogonal functions [29]. In spherical or cylindrical coordinates,

the resulting Schrödinger-like equation with a given potential is reduced to a hypergeometric type equation through making a convenient change of variables, say, $r \to s$ and then solved systematically for its exact or approximate eigensolutions (energy levels and wave functions). The most convenient equation, we consider here, takes the standard form [30]

$$f''(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)} f'(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)} f(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials at most of second order, and $\widetilde{\tau}(s)$ is a first-degree polynomial and $f(s)$ is a hypergeometric type polynomial.

Next, we try to reduce Eq. (1) to a more comprehensible form by taking $f(s) = \phi(s)y(s)$ and choosing an appropriate function $\phi(s)$:

$$y''(s) + \left(2\frac{\phi'(s)}{\phi(s)} + \frac{\widetilde{\tau}(s)}{\sigma(s)}\right) y'(s) + \left(\frac{\phi''(s)}{\phi(s)} + \frac{\phi'(s)}{\phi(s)}\frac{\widetilde{\tau}(s)}{\sigma(s)} + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\right) y(s) = 0. \tag{2}$$

which appears to be more complicated than the standard form given in (1). To simplify (2), at first, we take the coefficient of $y'(s)$,

$$2\frac{\phi'(s)}{\phi(s)} + \frac{\widetilde{\tau}(s)}{\sigma(s)} = \frac{\tau(s)}{\sigma(s)}, \tag{3}$$

and set

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}, \tag{4}$$

to obtain

$$\pi(s) = \frac{1}{2}[\tau(s) - \widetilde{\tau}(s)], \tag{5}$$

where $\pi(s)$ is a polynomial of degree at most one. Overmore, the above equation can be rewritten in the form:

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s), \tag{6}$$

in which $\tau(s)$ is a polynomial of order one. On the other hand, we can express the term $\phi''(s)/\phi(s)$ appearing as one of the coefficients of Eq. (2) as

$$\frac{\phi''(s)}{\phi(s)} = \left(\frac{\phi'(s)}{\phi(s)}\right)' + \left(\frac{\phi'(s)}{\phi(s)}\right)^2 = \left(\frac{\pi(s)}{\sigma(s)}\right)' + \left(\frac{\pi(s)}{\sigma(s)}\right)^2. \tag{7}$$

In this case, the coefficient of $y(s)$ can be simply recasted in the form:

$$\frac{\phi''(s)}{\phi(s)} + \frac{\phi'(s)}{\phi(s)}\frac{\widetilde{\tau}(s)}{\sigma(s)} + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)} = \frac{\bar{\sigma}(s)}{\sigma^2(s)} \tag{8}$$

where

$$\bar{\sigma}(s) = \widetilde{\sigma}(s) + \pi^2(s) + \pi(s)[\widetilde{\tau}(s) - \sigma'(s)] + \pi'(s)\sigma(s). \tag{9}$$

Substituting the right-hand sides of Eq. (3) and Eq. (8) into Eq. (2), we finally obtain

$$y''(s) + \frac{\tau(s)}{\sigma(s)}y'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}y(s) = 0. \tag{10}$$

The above transformation allows one to set the hypergeometric function $f(s) = \phi(s)y(s)$, where $\phi(s)$ needs to satisfy the relation (4) with an arbitrary linear polynomial $\pi(s)$. Thus, making the substitution:

$$\bar{\sigma}(s) = \lambda\!\!\!^{-}\sigma(s),$$

where $\lambda\!\!\!^{-}$ is a constant. Hence, Eq. (10) turns into the so-called hypergeometric type equation:

$$\sigma(s)y'' + \tau(s)y' + \lambda\!\!\!^{-}y = 0, \tag{11}$$

whose solution is already been given in [31]. Now, comparing Eq. (9) with Eq. (11) leads to the following quadratic equation:

$$\pi^2(s) + [\widetilde{\tau}(s) - \sigma'(s)]\pi(s) + \widetilde{\sigma}(s) - k\sigma(s) = 0, \tag{12}$$

where

$$k = \lambda\!\!\!^{-} - \pi'(s). \tag{13}$$

Thus, the solution of quadratic equation (12) is given by

$$\pi(s) = \frac{\sigma'(s) - \widetilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \widetilde{\tau}(s)}{2}\right)^2 - \widetilde{\sigma}(s) + k\sigma(s)}, \tag{14}$$

where the parameter k inside the square root sign must be found explicitly to enable one to find the physical solutions of Eq. (14) for the plus and minus signs. Therefore, the expression under the square root sign has to be the square of a polynomial, since $\pi(s)$ is a polynomial of degree at most one which provides an equation of the quadratic form available for the constant k. Having set the discriminant of this quadratic equal to zero, the constant k is determined clearly. Once the constant k is found, the task of the determination of the polynomial $\pi(s)$ from (14) becomes simple and straightforward. Further, $\tau(s)$ and $\lambda\!\!\!^{-}$ can also be found from Eq. (6) and Eq. (13), respectively.

To make the solutions of Eq. (11) more general, we try to show that all the derivatives of hypergeometric type function are also of hypergeometric type. This can be easily acheived by differentiating Eq. (11) and letting $v_1(s) = y'(s)$

$$\sigma(s)v_1''(s) + \tau_1(s)v_1'(s) + \mu_1 v_1(s) = 0, \tag{15}$$

where $\tau_1(s) = \tau(s) + \sigma'(s)$ and $\mu_1 = \lambda + \tau'(s)$. $\tau_1(s)$ is a polynomial of degree at most one and μ_1 is independent of the variable s. Equation (15) is obviously a hypergeometric type equation again. Further, taking $v_2(s) = y''(s)$ as a new representation and making the differentiation for the second time, we obtain

$$\sigma(s)v_2''(s) + \tau_2(s)v_2'(s) + \mu_2 v_2(s) = 0, \tag{16}$$

where

$$\tau_2(s) = \tau_1(s) + \sigma'(s) = \tau(s) + 2\sigma'(s), \tag{17}$$

$$\mu_2 = \mu_1 + \tau_1'(s) = \lambda + 2\tau'(s) + \sigma''(s). \tag{18}$$

Repeating this process, a general equation of hypergeometric type for $v_n(s) = y^{(n)}(s)$ is constructed as a family of particular solutions corresponding to a given λ;

$$\sigma(s)v_n''(s) + \tau_n(s)v_n'(s) + \mu_n v_n(s) = 0, \tag{19}$$

and hence the general recurrence relations for $\tau_n(s)$ and μ_n can be found as

$$\tau_n(s) = \tau(s) + n\sigma'(s), \tag{20}$$

$$\mu_n = \lambda + n\tau'(s) + \frac{n(n-1)}{2}\sigma''(s), \tag{21}$$

respectively. When we set $\mu_n = 0$, then Eq. (21) becomes

$$\lambda = \lambda_{n_r} = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad n = 0, 1, 2, \ldots \tag{22}$$

and hence Eq. (19) has a particular solution

$$y(s) = y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{dr^n}\left[\sigma^n(s)\rho(s)\right], \tag{23}$$

which is known as the Rodrigues relation of degree n and $\rho(s)$ is the weight function satisfying

$$\left[\sigma(r)\rho(r)\right]' = \tau(r)\rho(r). \tag{24}$$

Finally, to obtain an eigenvalue solution through the NU method, the relationship between λ and λ_{n_r} must be set up by means of Eq. (13) and Eq. (22).

III. NONRELATIVISTIC QDS AND QADS INFLUENCED BY MAGNETIC AND AB FLUX FIELDS

A. Exactly solvable bound states

Consider a two-dimensional ($2D$) single charged electron, e with an electronic effective mass, μ interacting via a radially symmetrical dot (electron) and antidot (hole). We will study the spectral properties of such dot and an antidot in a uniform magnetic field, $\overrightarrow{B} = B\widehat{z}$ and an AB flux field, applied simultanously. The Schrödinger equation is given by [32]

$$\left[\frac{1}{2\mu}\left(\overrightarrow{p} + \frac{e}{c}\overrightarrow{A}\right)^2 + V_{\text{conf}}(\vec{r})\right]\psi(\vec{r},\phi) = E\psi(\vec{r},\phi), \tag{25}$$

where $\overrightarrow{A}$ is the vector potential and the repulsive pseudoharmonic confinement quantum dot (PHQD) potential, $V_{\text{conf}}(\vec{r})$, describing the harmonic quantum dot and antidot structures, $V_D(r) = V_0 r^2/r_0^2$ and $V_{AD}(r) = V_0 r_0^2/r^2$, respectively, is taken as [20,21]

$$V_{\text{conf}}(\vec{r}) = V_0\left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2, \tag{26}$$

where r_0 and V_0 are the zero point (effective radius) and the chemical potential. The vector potential $\overrightarrow{A}$ may be represented as a sum of two terms, $\overrightarrow{A} = \overrightarrow{A}_1 + \overrightarrow{A}_2$ such that $\overrightarrow{\nabla} \times \overrightarrow{A}_1 = \overrightarrow{B}$ and $\overrightarrow{\nabla} \times \overrightarrow{A}_2 = 0$, where $\overrightarrow{B} = B\widehat{z}$ is the applied magnetic field, and $\overrightarrow{A}_2$ describes the additional magnetic flux Φ_{AB} created by a selenoid inserted inside the antidot (pseudodot). Hence, the vector potentials have azimuthal components given by [22]

$$\overrightarrow{A}_1 = \frac{Br}{2}\widehat{\phi},\ \overrightarrow{A}_2 = \frac{\Phi_{AB}}{2\pi r}\widehat{\phi},\ \overrightarrow{A} = \left(\frac{Br}{2} + \frac{\Phi_{AB}}{2\pi r}\right)\widehat{\phi}. \tag{27}$$

Let us consider the 2D cylindrical form of the wave functions:

$$\psi(\vec{r},\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}g(r),\ m = 0, \pm1, \pm2, \ldots, \tag{28}$$

where m is the magnetic quantum number. Now, inserting the wave functions (28) into the Schrödinger equation (25), we obtain the following equation for the radial wave function $g(r)$:

$$g''(r) + \frac{1}{r}g'(r) + \left(\nu^2 - \frac{\beta^2}{r^2} - \gamma^2 r^2\right)g(r) = 0, \tag{29}$$

where we have defined the parameters:

$$\nu^2 = \frac{2\mu}{\hbar^2}\left(E + 2V_0\right) - \frac{\mu\omega_c}{\hbar}\left(m + \xi\right), \tag{30a}$$

$$\beta^2 = (m+\xi)^2 + a^2, \tag{30b}$$

$$\gamma^2 = \frac{2\mu}{\hbar^2}\frac{V_0}{r_0^2} + \left(\frac{\mu\omega_c}{2\hbar}\right)^2, \tag{30c}$$

where $\xi = \Phi_{AB}/\Phi_0$ with the flux quantum $\Phi_0 = hc/e$, $\omega_c = eB/\mu c$ is the cyclotron frequency and $a = k_F r_0$ with $k_F = \sqrt{2\mu V_0/\hbar^2}$ is the fermi wave vector of the electron. The magnetic quantum number m relates to the quantum number β [Eq. (30b)].[1] Consequently, the radial wave function $g(r)$ is required to satisfy the boundary conditions, *i.e.*, $g(0) = 0$ and $g(\ r \to \infty) = 0$. In order to solve Eq. (29) by NU method, it is necessary to introduce the following variable $s = r^2$, $r \in (0,\infty)$ $\to$s$\in$ $(0,\infty)$ which recasts Eq. (29) in the form of hypergeometric type differential equation (1) as

$$g''(s) + \frac{2}{(2s)}g'(s) + \frac{1}{(2s)^2}\left(-\gamma^2 s^2 + \nu^2 s - \beta^2\right) g(s) = 0, \tag{31}$$

where we set $g(r) \equiv g(s)$. Applying the basic ideas of Ref. [30], by comparing Eq. (31) with Eq. (1) gives us the following polynomials:

$$\widetilde{\tau}(s) = 2, \quad \sigma(s) = 2s, \quad \widetilde{\sigma}(s) = -\gamma^2 s^2 + \nu^2 s - \beta^2. \tag{32}$$

In the present case, if we substitute the polynomials given by Eq. (32) into Eq. (14), the following equality for the polynomial $\pi(s)$ can be obtained

$$\pi(s) = \pm\frac{1}{2}\sqrt{\gamma^2 s^2 + (2k - \nu^2)s + \beta^2}. \tag{33}$$

The expression under the square root of the above equation must be the square of a polynomial of first degree. This is possible only if its discriminant is zero and the constant parameter k can be determined from the condition that the expression under the square root has a double zero. Hence, k is obtained as $k_{+,-} = \nu^2/2 \pm \beta\gamma$. In that case, it can be written in the four possible forms of $\pi(s)$;

$$\pi(s) = \begin{cases} +\left(\gamma s \pm \beta\right), \text{ for } k_+ = \frac{1}{2}\nu^2 + \beta\gamma, \\ -\left(\gamma s \pm \beta\right), \text{ for } k_- = \frac{1}{2}\nu^2 - \beta\gamma. \end{cases} \tag{34}$$

One of the four possible forms of $\pi(s)$ must be chosen to obtain an energy spectrum formula. Therefore, the most suitable form can be established by the choice:

$$\pi(s) = \beta - \gamma s,$$

[1] For this system, only two independent integer quantum numbers are required.

for k_-. The trick in this selection is to find the negative derivative of $\tau(s)$ given in Eq. (6). Hence, $\tau(s)$ and $\tau'(s)$ are obtained as

$$\tau(s) = 2\,(1+\beta) - 2\gamma s,\ \tau'(s) = -2\gamma < 0\ . \tag{35}$$

In this case, a new eigenvalue equation becomes

$$\lambda_n = 2\gamma n,\ n = 0, 1, 2, \ldots \tag{36}$$

where it is beneficial to invite the quantity $\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s)$ in Eq. (22) with n is the radial quantum number. Another eigenvalue equation is obtained from the equality $\lambda = k_- + \pi'$ in Eq. (13),

$$\lambda = \frac{\nu^2}{2} - \gamma\,(\beta+1)\,. \tag{37}$$

In order to find an eigenvalue equation, the right-hand sides of Eq. (36) and Eq. (37) must be compared with each other, i.e., $\lambda_n = \lambda$. In this case the result obtained will depend on $E_{n,m}$ in the closed form:

$$\nu^2 = 2\,(2n+1+\beta)\,\gamma. \tag{38}$$

Upon the substitution of the terms of right-hand sides of Eqs. (30a)-(30c) into Eq. (38), we can immediately obtain the following expression for the energy spectrum formula in the presence of PH potential :

$$E_{n,m}(\xi,\beta) = \hbar\Omega\left(n + \frac{|\beta|+1}{2}\right) + \frac{1}{2}\hbar\omega_c\,(m+\xi) - 2V_0,\ \Omega = \sqrt{\omega_c^2 + 4\omega_D^2}, \tag{39}$$

where $|\beta| = \sqrt{(m+\xi)^2 + a^2} > 0$ is an integer and $\omega_D = \sqrt{2V_0/\mu r_0^2}$. We have two sets of quantum numbers (n, m, β) and (n', m', β') for dot (electron) and antidot (hole), respectively. Therefore, expression (39) for the energy levels of the electron (hole) may be readily used for a study of the thermodynamic properties of quantum structures with dot and antidot in the presence and absence of magnetic field.

If we ignore the last $-2V_0$ term, the above formula becomes the Bogachek-Landman [16] energy levels, $E_{n,m}(\xi,\beta) = \hbar\Omega\left(n + \frac{|\beta|+1}{2}\right) + \frac{1}{2}\hbar\omega_c\,(m+\xi)$, in the presence of dot and antidot potential. In the absence of pseudoharmonic quantum dot (PHQD), i.e., $V_0 = 0$, $\Omega \to \omega_c$, then $E_{n,m}(\xi) = \hbar\omega_c\left[n + \frac{1}{2}(|m+\xi|+1)\right] + \frac{1}{2}\hbar\omega_c\,(m+\xi)$ which is the formula in the presence of magnetic and AB flux fields [16]. If we put $\xi = 0$, i.e., in the absence of AB flux field, we find $E_{n,m} = \hbar\omega_c\left[n + \frac{1}{2}(|m|+m+1)\right]$ which is the Landau energy

levels [33]. In the absence of magnetic field ($\omega_c = 0$) and an AB flux field ($\xi = 0$), we find $E_{n,m} = (4\hbar V_0/\mu r_0^2)\left[n + \left(\sqrt{m^2 + 2\mu V_0 r_0^2/\hbar^2} + 1\right)/2\right] - 2V_0$. When $m = 0$, we have $E_n = (4\hbar V_0/\mu r_0^2)(n + 1/2)$ for harmonic oscillator energy spectrum.

Next, we calculate the corresponding wave functions for the present PH potential model. We find the first part of the wave function through Eq. (4), i.e.,

$$\phi_m(s) = \exp\left(\int \frac{\pi(s)}{\sigma(s)} ds\right) = s^{|\beta|/2} e^{-\gamma s/2}. \tag{40}$$

Then, the weight function defined by Eq. (24) as

$$\rho(s) = \frac{1}{\sigma(s)} \exp\left(\int \frac{\tau(s)}{\sigma(s)} ds\right) = s^{|\beta|} e^{-\gamma s}, \tag{41}$$

which gives the second part of the wave function (Rodrigues formula); namely, Eq.(23):

$$y_{n,m}(s) \sim s^{-|\beta|} e^{\gamma s} \frac{d^{n_r}}{ds^{n_r}} \left(s^{n+|\beta|} e^{-\gamma s}\right) \sim L_n^{(|\beta|)}(\gamma s), \tag{42}$$

where $L_a^{(b)}(x) = \frac{(a+b)!}{a!b!} F(a, b+1; x)$ is the associated Laguarre polynomial and $Fa, b; x)$ is the confluent hypergeometric function. Using $g(s) = \phi_m(s) y_{n,m}(s)$, in this way we may write the radial wave function in the following fashion

$$g(r) = C_{n,m} r^{|\beta|} e^{-\gamma r^2/2} F\left(-n, |\beta| + 1; \gamma r^2\right), \tag{43}$$

and finally the total wave function (28) becomes

$$\psi_{n,m}(\vec{r}, \phi) = \sqrt{\frac{\gamma^{|\beta|+1} n!}{\pi (n + |\beta|)!}} r^{|\beta|} e^{-\gamma r^2/2} L_n^{(|\beta|)}\left(\gamma r^2\right) e^{im\phi}$$

$$= \frac{1}{|\beta|!} \sqrt{\frac{\gamma^{|\beta|+1} (n + |\beta|)!}{\pi n!}} r^{|\beta|} e^{-\gamma r^2/2} F\left(-n, |\beta| + 1; \gamma r^2\right) e^{im\phi}. \tag{44}$$

The energy levels in Eq. (39) differ from the usual Landau levels in cylindrical coordinate system [34] to which it transforms when $\xi = 0$ (i.e., $\Phi_{AB} = 0$), and $a \to 0$ (i.e., when the chemical potential of dot and antidot vanishes, i.e., $V_0 \to 0$). Nevertheless, the Landau levels are nearly continuous discrete spectrum for a particle confined to a large box with $B = 0$ to equally spaced levels corresponding to $B > 0$. Each increment of energy, $\hbar\omega_c$, corresponding to free particle states, which is the degeneracy of each Landau level leading to a larger spacing as magnetic field B tends to become stronger [33]. The present model removes this degeneracy with energy levels spectrum becomes

$$E_{n,m} = \hbar\omega_c \left[n + \frac{1}{2}(|m| + m + 1)\right], \tag{45}$$

and the wave function reads as

$$\psi_{n,m}(\vec{r},\phi)=\frac{1}{m!}\sqrt{\frac{\gamma^{m+1}\,(n+m)!}{\pi n!}}r^{m}e^{-\gamma r^{2}/2}F\left(-n,m+1;\gamma r^{2}\right)e^{im\phi}, \tag{46}$$

where $\gamma=(\mu\omega_c)/2\hbar$. In the limit when $\omega_c \ll g=\sqrt{\frac{8V_0}{\mu}}\frac{c}{r_0}$,then we have

$$E_{nm}=\varepsilon_0+\varepsilon_1\omega_c+\varepsilon_2\omega_c^2-\varepsilon_4\omega_c^4+..., \tag{47}$$

where

$$\varepsilon_0=-2V_0+N_{nm}g,\ \varepsilon_1=\frac{\hbar m}{2},\ \varepsilon_2=\frac{N_{nm}}{2g},\ \varepsilon_4=\frac{N_{nm}}{8g^3},\ N_{nm}=\hbar\left(n+\frac{m+1}{2}\right),\ g=\frac{1}{r_0}\sqrt{\frac{8V_0}{\mu}}. \tag{48}$$

B. Interband light absorption coefficient

Expressions (39) and (44), obtained above for charge carriers (electron or hole) energy spectrum and the corresponding wave function in quantum pseudodot under the influence of external magnetic field and AB flux field, allow to calculate the direct interband light absorption coefficient $K(\overline{\omega})$ in such system and the threshold frequency of absorption. The light absorption coefficient can be expressed as [11,12,35]:

$$K(\overline{\omega})=N\sum_{n,m,\beta}\sum_{n',m',\beta'}\left|\int\psi^{e}_{n,m,\beta}(\vec{r},\phi)\psi^{h}_{n',m',\beta'}(\vec{r},\phi)rdrd\phi\right|^{2}\delta\left(\Delta-E^{e}_{n,m,\beta}-E^{h}_{n',m',\beta'}\right),$$

$$=N\sum_{n,m,\beta}\sum_{n',m',\beta'}\frac{\gamma^{|\beta|+|\beta'|+2}\,(n+|\beta|)!\,(n'+|\beta'|)!}{\pi^{2}n!n'!\left(|\beta|!\right)^{2}\left(|\beta'|!\right)^{2}}\left|\int_{0}^{2\pi}e^{i(m+m')\phi}d\phi\int_{0}^{\infty}rdre^{-(\gamma+\gamma')r^{2}/2}r^{|\beta|+|\beta'|}\right.$$
$$\left.\times\ F\left(-n,|\beta|+1;\gamma r^{2}\right)F\left(-n',|\beta'|+1;\gamma' r^{2}\right)\right|^{2}\delta\left(\Delta-E^{e}_{n,m,\beta}-E^{h}_{n',m',\beta'}\right), \tag{49}$$

where $\Delta=\hbar\overline{\omega}-\varepsilon_g$, ε_g is the width of forbidden energy gap, $\overline{\omega}$ is the frequency of incident light, N is a quantity proportional to the square of dipole moment matrix element modulus, $\psi^{e(h)}$ is the wave function of the electron (hole) and $E^{e(h)}$ is the corresponding energy of the electron (hole).

Now, we use the integrals [33]

$$\int_{0}^{2\pi}e^{i(m+m')\phi}d\phi=\begin{cases}2\pi & \text{if } m=-m',\\ 0 & \text{if } m\neq -m',\end{cases} \tag{50}$$

and

$$\int_0^\infty e^{-\kappa x} x^{\lambda-1} F\left(-n,\lambda;qx\right) F\left(-n',\lambda;q'x\right) dx = \Gamma(\lambda)\kappa^{n+n'-\lambda}\left(\kappa-q\right)^{-n}\left(\kappa-q'\right)^{-n'}$$

$$\times\ {}_2F_1\left(n,n',\lambda;\frac{qq'}{\left(\kappa-q\right)\left(\kappa-q'\right)}\right), \tag{51}$$

where $\Gamma(x)$ is the Euler-Gamma function and ${}_2F_1\ (a,b,c;z)$ is the hypergeometric function, to calculate the light absorption coefficient:

$$K(\overline{\omega}) = N\sum_{n,m,\beta}\sum_{n',m',\beta'} P^{\beta}_{n,n'} Q^{\beta}_{n,n'} \delta\left(\Delta - E^{e}_{n,m,\beta} - E^{h}_{n',m',\beta'}\right), \tag{52}$$

where

$$P^{\beta}_{n,n'} = \frac{1}{\left(|\beta|!\right)^4}\left(\gamma\gamma'\right)^{|\beta|+1}\left(\frac{\gamma+\gamma'}{\gamma-\gamma'}\right)^{2(n+n')}\frac{(n+|\beta|)!\,(n'+|\beta|)!}{n!n'!}, \tag{53}$$

and

$$Q^{\beta}_{n,n'} = \left[|\beta|!\left(\frac{2}{\gamma+\gamma'}\right)^{|\beta|+1}\ {}_2F_1\ \left(n,n',|\beta|+1;-\frac{4\gamma\gamma'}{\left(\gamma-\gamma'\right)^2}\right)\right]^2. \tag{54}$$

Using Eqs. (39) and (49), we find the threshold frequency value of absorption as

$$\hbar\overline{\omega} = \varepsilon_g + \hbar\left(n+\frac{\sqrt{\left(m+\Phi_{AB}/\Phi_0\right)^2+2\mu V_0 r_0^2/\hbar^2}+1}{2}\right)\sqrt{\left(\frac{qB}{\mu c}\right)^2+\frac{8V_0}{\mu r_0^2}}+\frac{q\hbar B}{2\mu c}\left(m+\frac{\Phi_{AB}}{\Phi_0}\right)$$

$$+\hbar\left(n'+\frac{\sqrt{\left(m'+\Phi_{AB}/\Phi_0\right)^2+2\mu' V_0 r_0^2/\hbar^2}+1}{2}\right)\sqrt{\left(\frac{qB}{\mu' c}\right)^2+\frac{8V_0}{\mu' r_0^2}}+\frac{q\hbar B}{2\mu' c}\left(m'+\frac{\Phi_{AB}}{\Phi_0}\right)-4V_0. \tag{55}$$

When $n=m=0$, then

$$\hbar\overline{\omega}_{00} = \varepsilon_g + \frac{\hbar}{2}\left(\sqrt{\left(\Phi_{AB}/\Phi_0\right)^2+2\mu V_0 r_0^2/\hbar^2}+1\right)\sqrt{\left(\frac{qB}{\mu c}\right)^2+\frac{8V_0}{\mu r_0^2}}+\frac{q\hbar B}{2\mu c}\frac{\Phi_{AB}}{\Phi_0}$$

$$+\frac{\hbar}{2}\left(\sqrt{\left(m'+\Phi_{AB}/\Phi_0\right)^2+2\mu' V_0 r_0^2/\hbar^2}+1\right)\sqrt{\left(\frac{qB}{\mu' c}\right)^2+\frac{8V_0}{\mu' r_0^2}}+\frac{q\hbar B}{2\mu' c}\frac{\Phi_{AB}}{\Phi_0}-4V_0. \tag{56}$$

C. Temperature dependence of the effective mass

The variation of the effective mass with temperature is determined according to the expression [28,36,37]

$$\frac{\mu_e}{\mu(T)} = \frac{1}{f(T)} = 1 + E_p^{\Gamma}\left[\frac{2}{E_g^{\Gamma}(T)}+\frac{1}{E_g^{\Gamma}(T)+\Delta_0}\right], \tag{57}$$

where μ_e is the electronic mass, $E_p^{\Gamma} = 7.51\ eV$ is the energy related to the momentum matrix element, $\Delta_0 = 0.341\ eV$ is the spin-orbit splitting and $E_g^{\Gamma}(T)$ is the temperature-dependence of the energy gap (in eV units) at the Γ point which is given by [12,36,38,39]

$$E_g^{\Gamma}(T) = 1.519 - \frac{(5.405 \times 10^{-4})\, T^2}{T + 204}\ (eV). \tag{58}$$

Table 1 lists the temperature-dependent effective mass to the effective mass of donor electron, i.e., $\mu(T)/\mu_e$ for different values of temperatures. It is seen from Table 1 that raising the temperature will decrease the value of $f(T) = \mu(T)/\mu_e$. As a matter of fact, the decrease in this value means that kinetic energy of the donor electron decrease and consequently lowering the binding energy. The results are similar to Ref. [28]. Hence the temperature dependence energy spectrum formula can be expressed as

$$E_{n,m}(B,T) = \frac{\hbar\omega_c}{f(T)}\left[\sqrt{1 + 4\frac{\omega_D^2}{\omega_c^2}f(T)}\left(n + \frac{\sqrt{(m+\xi)^2 + a^2 f(T)} + 1}{2}\right) + \frac{m+\xi}{2}\right] - 2V_0,$$

which for GaAs turns to be

$$E_{n,m}(B,T) = 14.9254\hbar\omega_c\left[\sqrt{1 + 0.268\frac{\omega_D^2}{\omega_c^2}}\left(n + \frac{\sqrt{(m+\xi)^2 + 0.067a^2} + 1}{2}\right) + \frac{m+\xi}{2}\right] - 2V_0,$$

where we have used $\mu = 0.067\mu_e$.

IV. THE SPINLESS KLEIN-GORDON PARTICLE IN MAGNETIC AND AB FLUX FIELDS

The Klein-Gordon (KG) equation is wave equation mostly used in describing particle dynamics in relativistic quantum mechanics. Nonetheless, physically this equation describes a scalar particle (spin 0). Moreover, this wave equation, for free particles, is constructed using two objects: the four-vector linear momentum operator $P_\mu = i\hbar\partial_\mu$ and the scalar rest mass M, allows one to introduce naturally two types of potential coupling. One is the gauge-invariant coupling to the four-vector potential $\{A_\mu(\overrightarrow{r})\}_{\mu=0}^{3}$ which is introduced via the minimal substitution $P_\mu \to P_\mu - gA_\mu$, where g is a real coupling parameter. The other, is an additional coupling to the space-time scalar potential $S_{\rm conf}(\overrightarrow{r})$ which is introduced by the substitution $M \to M + S_{\rm conf}(\overrightarrow{r})$. The term "four-vector" and "scalar" refers to the corresponding unitary irreducible representation of the Poincaré space-time symmetry

group (the group of rotations and translations in (3+1)-dimensional Minkowski space-time). Gauge invariance of the vector coupling allows for the freedom to fix the gauge (eliminating the non physical gauge modes) without altering the physical content of the problem. Many choose to simplify the solution of the problem by taking the space component of the vector potential to vanish (i.e., $\overrightarrow{A}$). One may write the time-component of the four-vector potential as $gA_0 = V_{\rm conf}(\vec{r})$, then it ends up with two independent potential functions in the KG equation. These are the "vector" potential $V_{\rm conf}(\overrightarrow{r})$ and the "scalar" potential $S_{\rm conf}(\overrightarrow{r})$ [40,41].

The free KG equation is written as

$$(\partial^{\mu}\partial_{\mu} + M^2)\psi_{KG}(t, \overrightarrow{r}) = 0. \tag{59}$$

Moreover, the vector and scalar couplings mentioned above introduce potential interactions by mapping the free KG equation as

$$\left\{c^2\left(\overrightarrow{p} + \frac{e}{c}\overrightarrow{A}\right)^2 + [M + S_{\rm conf}(\vec{r})]^2\right\}\psi(\overrightarrow{r}, \phi) = [E - V_{\rm conf}(\vec{r})]^2\,\psi(\overrightarrow{r}, \phi), \tag{60}$$

where $\psi(\overrightarrow{r}, \phi)$ is $2D$ cylindrical wave function defined as in (28). This type of coupling attracted a lot of attention in the literature due to the resulting simplification in the solution of the relativistic problem. The scalar-like potential coupling is added to the scalar mass so that in case when $S_{\rm conf}(\vec{r}) = \pm V_{\rm conf}(\overrightarrow{r})$, the KG equation could always be reduced to a Schrödinger-type second order differential equation as follows

$$\left[c^2\left(\overrightarrow{p} + \frac{e}{c}\overrightarrow{A}\right)^2 + 2\left(E \pm Mc^2\right)V_{\rm conf}(\overrightarrow{r}) + M^2c^4 - E^2\right]\psi(\overrightarrow{r}, \phi) = 0. \tag{61}$$

Hence, the bound state solutions of the above two cases are to be treated separetely as follows.

A. The $S_{\rm conf}(\overrightarrow{r}) = +V_{\rm conf}(\overrightarrow{r})$ case

The positive energy states (corresponding to $S_{\rm conf}(\overrightarrow{r}) = +V_{\rm conf}(\overrightarrow{r})$ in the nonrelativistic limit (taking $E - Mc^2 \cong E$ and $E + Mc^2 \cong 2\mu c^2$, where $|E| \ll Mc^2$) are solutions of

$$\left[\frac{1}{2\mu}\left(\overrightarrow{p} + \frac{e}{c}\overrightarrow{A}\right)^2 + 2V_{\rm conf}(\overrightarrow{r}) - E\right]\psi(\overrightarrow{r}, \phi) = 0. \tag{62}$$

where $\psi(\overrightarrow{r}, \phi)$ stands for either $\psi^{(+)}(\overrightarrow{r}, \phi)$ or $\psi^{(\rm KG)}(\overrightarrow{r}, \phi)$. This is the Schrödinger equation for the potential $2V_{\rm conf}(\overrightarrow{r})$. Thus, the choice $S_{\rm conf}(\overrightarrow{r}) = +V_{\rm conf}(\overrightarrow{r})$ produces a nontrivial

nonrelativistic limit with a potential function $2V_{\rm conf}(\overrightarrow{r})$, and not $V_{\rm conf}(\overrightarrow{r})$. Accordingly, it would be natural to scale the potential term in Eq. (61) and Eq. (62) so that in the nonrelativistic limit the interaction potential becomes $V_{\rm conf}$, not $2V_{\rm conf}$. thus, we need to recast Eq. (61) and Eq. (62) as [41]

$$\left[c^2\left(\overrightarrow{p}+\frac{e}{c}\overrightarrow{A}\right)^2+\left(E+Mc^2\right)V_{\rm conf}(\overrightarrow{r})+M^2c^4-E^2\right]\psi(\overrightarrow{r},\phi)=0. \tag{63a}$$

$$\left[\frac{1}{2\mu}\left(\overrightarrow{p}+\frac{e}{c}\overrightarrow{A}\right)^2+V_{\rm conf}(\overrightarrow{r})-E\right]\psi_{nm}(\overrightarrow{r},\phi)=0, \tag{63b}$$

with $V_{\rm conf}(\overrightarrow{r})$ and $\overrightarrow{A}$ are given in Eq. (26) and Eq. (27), respectively. To avoid repeatition in solving Eq. (63a), we follow the same steps of solution explained before by taking $\psi_{nm}(\overrightarrow{r},\phi)=g(r)e^{im\phi}/\sqrt{2\pi}$ to obtain an equation satisfying the radial part: of the wave function:

$$g''(s)+\frac{2}{(2s)}g'(s)+\left(-b_1^2s^2+\lambda_1^2s-a_1^2\right)g(s)=0, \tag{64}$$

where we have used

$$\lambda_1^2=\frac{1}{\hbar^2c^2}\left[E^2+2\left(E+Mc^2\right)V_0-M^2c^4\right]-\frac{M\omega_c}{\hbar}\left(m+\xi\right), \tag{65a}$$

$$a_1^2=\left(m+\xi\right)^2+\frac{r_0^2}{\hbar^2c^2}\left(E+Mc^2\right)V_0, \tag{65b}$$

$$b_1^2=\left(\frac{M\omega_c}{2\hbar}\right)^2+\frac{1}{\hbar^2c^2r_0^2}\left(E+Mc^2\right)V_0. \tag{65c}$$

The solution of Eq. (64) can be easily constructed on making the changes: $\nu\rightarrow\lambda_1$, $\beta\rightarrow a_1$, and $\gamma\rightarrow b_1$. Thus, the equation for the KG positive energy states can be easily found from Eq. (38) as

$$\lambda_1^2=2\left(2n+1+a_1\right)b_1, \tag{66}$$

and further inserting Eqs. (65a)-(65c), we finally obtain the transcendental energy formula

$$\hbar\left(1+2n+\sqrt{m'^2+\frac{a'^2}{2M}\gamma_1}\right)\sqrt{\omega_c^2+\frac{2\omega_D'^2}{M}\gamma_1}=\frac{1}{M}\left(\gamma_2+2V_0\right)\gamma_1-\hbar\omega_cm',$$

$$\gamma_1=\frac{\left(E+Mc^2\right)}{c^2},\ \gamma_2=E-Mc^2,\ \omega_D'=\sqrt{\frac{2V_0}{Mr_0^2}},\ a'^2=\frac{2MV_0r_0^2}{\hbar^2} \tag{67}$$

where $m'=m+\xi$ is a new quantum number. We may find solution to the above transcendental equation as $E=E_{KG}^{(+)}$. In the nonrelativistic limit ($\gamma_1\rightarrow 2M$ and $\gamma_2\rightarrow E$), the above equation can be easily reduced to the simple energy spectrum formula given in Eq. (39).

Overmore, under the above parameters mapping, the $2D$ KG wave function can be found directly from Eq. (44) as

$$\psi_{n,m}^{(+)}(\vec{r},\phi)=\sqrt{\frac{b_1^{|a_1|+1}n!}{\pi\left(n+|a_1|\right)!}}r^{|a_1|}e^{-b_1r^2/2}L_n^{(a_1)}(b_1r^2)e^{im\phi}. \tag{68}$$

B. The $S_{\mathbf{conf}}(\overrightarrow{r})=-V_{\mathbf{conf}}(\overrightarrow{r})$ case

In this case, we follow the same steps of solution in the previous subsection:

$$g''(s)+\frac{2}{(2s)}g'(s)+\left(-b_2^2s^2+\lambda_2^2s-a_2^2\right)g(s)=0, \tag{69}$$

where we have used

$$\lambda_1^2\rightarrow\lambda_2^2=\frac{1}{\hbar^2c^2}\left[E^2+2\left(E-Mc^2\right)V_0-M^2c^4\right]-\frac{M\omega_c}{\hbar}\left(m+\xi\right), \tag{70a}$$

$$a_1^2\rightarrow a_2^2=\left(m+\xi\right)^2+\frac{r_0^2}{\hbar^2c^2}\left(E-Mc^2\right)V_0, \tag{70b}$$

$$b_1^2\rightarrow b_2^2=\left(\frac{M\omega_c}{2\hbar}\right)^2+\frac{1}{\hbar^2c^2r_0^2}\left(E-Mc^2\right)V_0. \tag{70c}$$

Thus, the equation for the KG negative energy states can be readily found as

$$\lambda_2^2=2\left(2n+1+a_2\right)b_2, \tag{71}$$

which provides the transcendental energy spectrum formula

$$\left(\left(2n+1\right)\hbar c+\sqrt{\hbar^2c^2\left(m+\xi\right)^2+r_0^2V_0\gamma_2}\right)\sqrt{M^2\omega_c^2+\frac{4V_0}{r_0^2}\gamma_2}$$
$$=\left(c^2\gamma_1+2V_0\right)\gamma_2-\hbar cM\omega_c\left(m+\xi\right), \tag{72}$$

and the corresponding $2D$ KG wave function is found as

$$\psi_{n,m}^{(-)}(\vec{r},\phi)=\sqrt{\frac{b_2^{|a_2|+1}n!}{\pi\left(n+|a_2|\right)!}}r^{|a_2|}e^{-b_2r^2/2}L_n^{(a_2)}(b_2r^2)e^{im\phi}. \tag{73}$$

It should be noted that the negative energy states (corresponding to $S_{\mathrm{conf}}(\overrightarrow{r})=-V_{\mathrm{conf}}(\overrightarrow{r})$) are free fields since under these conditions Eq. (61) reduces to

$$\left[-\frac{1}{2\mu}\left(\overrightarrow{p}+\frac{e}{c}\overrightarrow{A}\right)^2+E\right]\psi_{n,m}(\overrightarrow{r},\phi)=0. \tag{74}$$

which is a simple free-interaction mode. For the free fields, Eq. (74), the set of parameters in Eqs. (70a)-(70c) reads

$$\lambda_2 = \sqrt{\frac{2\mu E}{\hbar^2} - \frac{\mu\omega_c}{\hbar}(m+\xi)},\ a_2 = m+\xi,\ b_2 = \frac{\mu\omega_c}{2\hbar}, \tag{75}$$

which lead to the energy spectrum formula

$$E^{(-)} = \left(n + m + \xi + \frac{1}{2}\right)\hbar\omega_c, \tag{76}$$

and and wave function

$$\psi_{nm}^{(-)}(\vec{r},\phi) = \sqrt{\frac{\left(\frac{\mu\omega_c}{2\hbar}\right)^{m+\xi+1} n!}{\pi\,(n+m+\xi)!}} r^{m+\xi} e^{-\frac{\mu\omega_c}{4\hbar c}r^2} L_n^{(m+\xi)}\left(\frac{\mu\omega_c}{2\hbar}r^2\right) e^{im\phi}. \tag{77}$$

V. RESULTS AND DISCUSSION

We solved exactly the Schrödinger and Klein-Gordon equations for an electron under the pseudoharmonic interaction consisting of quantum dot potential and antidot potential in the presence of a uniform strong magnetic field $\overrightarrow{B}$ along the z axis and AB flux field created by an infinitely long selenoid inserted inside the pseudodot. We obtained bound state solutions including the energy spectrum formula (39) and wave function (44) for a Schrödinger electron. Overmore, for the Klein-Gordon electron, the positive energy equation (67) and wave function (68) is found for $S_{\rm conf}(\overrightarrow{r}) = +V_{\rm conf}(\overrightarrow{r})$ case. However, the negative energy equation (67) and wave function (68) are found for $S_{\rm conf}(\overrightarrow{r}) = +V_{\rm conf}(\overrightarrow{r})$ case. These two cases are reduced to the Schrödinger equation with a potential interaction $V_{\rm conf}(\overrightarrow{r})$ and free field interaction solutions, respectively. Now we study the effect of the pseudoharmonic potential, the presence and absence of magnetic field B, the presence and absence of AB flux density ξ and the antidot potential on the energy levels (39). To see the dependence of the energy spectrum on the magnetic quantum number, m, we take the following values: magnetic field $\overrightarrow{B} = (6\ T)\ \widehat{z}$, AB flux field $\xi = 8$, chemical potential $V_0 = 0.68346$ (meV) and $r_0 = 8.958 \times 10^{-6}\ cm$ [22]. Thus, we obtained $a = \sqrt{2\mu V_0 r_0^2/\hbar^2} = 11.997702$, $2\omega_D = \sqrt{8V_0/\mu r_0^2} = 0.3280381\ \omega_c$ and $\hbar\omega = 1.05243\hbar\omega_c$ [34], the dependence of the energy spectrum, (39) on the n and m is given by

$$\frac{E_{n,m}}{\hbar\omega_c} = 1.05243\left(n + \frac{\sqrt{(m+8)^2 + 12^2} + 1}{2}\right) + \frac{1}{2}(m+8) - 1.9678584,\ \text{for } B = 6\ T. \tag{78}$$

where $m = 0, \pm 1, \pm 2, \ldots$ and $n = 0, 1, 2, \ldots$. For the lowest ground state ($n = 0$): $E_{0,m}/\hbar\omega_c = 1.05243 \left(\sqrt{(m+8)^2 + 12^2} + 1 \right)/2 + (m+8)/2 - 1.9678584$, for $B = 6\ T$. Overmore, to show the effect of magnetic field B on the energy spectrum, we take values for parameters $\xi = 8$, $V_0 = 0.68459\ meV$ and $r_0 = 8.958 \times 10^{-6}\ cm$ [22], where $a = \sqrt{2\mu V_0 r_0^2/\hbar^2} = 12.007617$ and $4\omega_D^2 = 8V_0/\mu r_0^2 = 0.120039 \times 10^{24}\ (rad/s)^2$, the dependence of energy levels (39) on the magnetic field becomes

$$E_{n,m}\,(meV) = 0.1157705\sqrt{B^2 + 3.8803305}\left(n + \frac{\sqrt{(m+8)^2 + 12^2} + 1}{2} \right)$$

$$+ 0.1157705B\left(\frac{m+8}{2} \right) - 1.36918. \tag{79}$$

In Figure 1, we plot the pseudodot energy levels in the absence (presence) of pseudodot potential (i.e., $V_0 = 0 \rightarrow a = 0$ ($V_0 \neq 0 \rightarrow a = 12$)) and in the absence (presence) of AB flux field Φ_{AB} (i.e., $\xi = 0$ ($\xi = 8$)) as a function of magnetic quantum number m for $B = 6\ T$. As demonstrated in Figure 1, the Landau energy states [33] (i.e., $V_0 = 0 \rightarrow a = 0$, $\xi = 0$ and $\xi = 8$) are degenerate states (see, long dashed and dotted solid curves) for negative values of m, however, the pseudodot potential removes this degeneracy (case when $V_0 \neq 0$ $\rightarrow a = 12$), (see, solid and dotted dashed curves). In the absence of pseudodot potential ($a = 0$) and presence of AB flux field ($\xi = 8$), the degeneracy still exists (long dashed line). It is found that the energy levels of PHQD potential are approximately equal the Landau energy levels for large absolute m values. However, they are quite different for small absolute m values ($-12 \preceq m \preceq 13$ when $\xi = 0$ and $-20 \preceq m \preceq 5$ when $\xi = 8$). It is also noted that as the quantum number n increases ($n > 0$), the curves are quite similar to Figure 1 but the energy levels are pushed up toward the positive energy for all values of m. In Figure 2 to Figure 7, we plot the magnetic field dependence of the ground state energy $E_{0,m}(\xi, a)$ (in units of meV) in the presence and absence of pseudodot potential and AB flux field for several values of magnetic quantum numbers $m = 27, 35, 1, 0, -24$ and -16, respectively. It is shown in Figure 2 to Figure 7 that pseudodot energy increases with increasing magnetic field strength. Further, in the absence of pseudodot potential, magnetic field in the positive z direction removes the degeneracy for positive m values. In these Figures, the behavior of pseudodot energy as function of the magnetic field B is shown in the presence of pseudodot potential and AB flux field (solid curves), in the absence of pseudodot potential and presence

of AB flux field (dotted curves) and the absence of pseudodot potential and AB flux field (dashed curves).

To investigate the dependence of the energy levels on temperature, we take the values of parameters: $B = 6\ T$, $\xi = 8$, $V_0 = 0.68346\ (meV)$ and $r_0 = 8.958 \times 10^{-6}\ cm$ [22]. Hence, the temperature dependence of the energy levels (in the units of $\hbar\omega_c$) at the Γ point are given by

$$\frac{E_{n,m}(T)}{\hbar\omega_c} = \frac{1}{f(T)}\left[\sqrt{1+(0.32804)^2 f(T)}\left(n+\frac{\sqrt{(m+8)^2+144f(T)}+1}{2}\right)+\frac{m+8}{2}\right] - 1.9678584, \tag{80}$$

where $f(T)$ is calculated in Table 1 at any temperature value. In GaAs, we have $f(T) = 0.067$ [11]. Taking the special values of parameters $\xi = 8$, $V_0 = 0.68459\ meV$ and $r_0 = 8.958 \times 10^{-6}$ cm [22], two parameters (temperature and magnetic field) dependence of the energy levels (in units of meV) are calculated as

$$E_{n,m}(B,T) = \frac{1}{f(T)}\left[0.1157705\sqrt{B^2+3.8803305f(T)}\left(n+\frac{\sqrt{(m+8)^2+144f(T)}+1}{2}\right) + 0.1157705B\left(\frac{m+8}{2}\right)\right] - 1.36918\ (\text{units } meV). \tag{81}$$

which becomes

$$E_{n,m}(B) = 14.9254\left[0.1157705\sqrt{B^2+0.26}\left(n+\frac{\sqrt{(m+8)^2+9.648}+1}{2}\right) + 0.1157705B\left(\frac{m+8}{2}\right)\right] - 1.36918\ (\text{units } meV). \tag{82}$$

for GaAs. Figure 8 to Figure 12 show the variation of the pseudodot energy levels (in meV) as function of magnetic field B (in $Tesla$) (82) in the presence of pseudodot potential and AB flux field (solid curves), in the absence of pseudodot potential and presence of AB flux field (dotted curves) and the absence of pseudodot potential and AB flux field (dashed curves).for various values of radial quantum numbers n and magnetic quantum numbers m. For GaAs case, we consider the following cases (a) $n = m = 0$, (b) $n = 5$, $m = 0$, (c) $n = 0$, $m = 5$, (d) $n = 0$, $m = -5$ and (e) $n = 5$, $m = -5$ in Figures 8 to 12, respectively.

VI. CONCLUSIONS AND OUTLOOK

In this work, we have obtained bound state energies and wave functions of the KG particle in the field of pseudoharmonic quantum dot and antidot structure in the presence of a uniform magnetic and AB flux fields. The positive (negative) KG energy states corresponding to $S_{\rm conf}(\overrightarrow{r}) = +V_{\rm conf}(\overrightarrow{r})$ ($S_{\rm conf}(\overrightarrow{r}) = -V_{\rm conf}(\overrightarrow{r})$) are studied. Overmore, the Schrödinger bound state solutions are found. Under nonrelativistic limit, the KG equation with equal mixture of scalar and vector potentials $S_{\rm conf}(\overrightarrow{r}) = +V_{\rm conf}(\overrightarrow{r})$ and $S_{\rm conf}(\overrightarrow{r}) = -V_{\rm conf}(\overrightarrow{r})$ can be easily reduced into the well-known Schrödinger equation of a particle with an interaction potential field and a free field, respectively. Overmore, the nonrelativistic electron and hole energy spectra and the their corresponding wave functions are used to calculate the the interband light absorption coefficient and the the threshold frequency of absorption. Also, the energy spectrum of the electron (hole) may be used for a study of the thermodynamic properties of quantum structures with dot (antidot) in a magnetic field. The temperature dependence of the energy levels are calculated using the Table 1 at any temperature T (Kelvin).

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TABLE I: Calculated $f(T)$ with different values of temperature for GaAs.

$T\ (K)$	$\mu(T)/\mu_e$	$T\ (K)$	$\mu(T)/\mu_e$
0	0.0669984	170	0.0653679
10	0.0669886	180	0.0652177
20	0.0669608	190	0.0650643
30	0.0669174	200	0.0649080
40	0.0668603	210	0.0647490
50	0.0667911	220	0.0645874
60	0.0667112	230	0.0644235
70	0.0666217	240	0.0642573
80	0.0665236	250	0.0640891[a]
90	0.0664178	260	0.0639188
100	0.0663051	270	0.0637468
110	0.0661861	280	0.0635730
120	0.0660614	290	0.0633976
130	0.0659315	300	0.0632206[a]
140	0.0657968	350	0.0623154
150	0.0656577[a]	400	0.0613818[a]
160	0.0655147	500	0.0594513[a]

[a]See Ref. [28].

FIG. 1: Pseudodot $n = 0$ energy levels (in $\hbar\omega_c$ unit) as a function of magnetic quantum number m in the presence and absence of PHQD potential ($a = 12$ and $a = 0$) and in the presence and absence of AB flux field ($\xi = 8$ and $\xi = 0$) for magnetic field $B = 6\ T$.

FIG. 2: Ground state pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for magnetic quantum number $m = 27$.

FIG. 3: Ground state pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for magnetic quantum number $m = 35$.

FIG. 4: Ground state pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for magnetic quantum number $m = 1$.

FIG. 5: Ground state pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for magnetic quantum number $m = 0$.

FIG. 6: Ground state pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for magnetic quantum number $m = -24$.

FIG. 7: Ground state pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for magnetic quantum number $m = -16$.

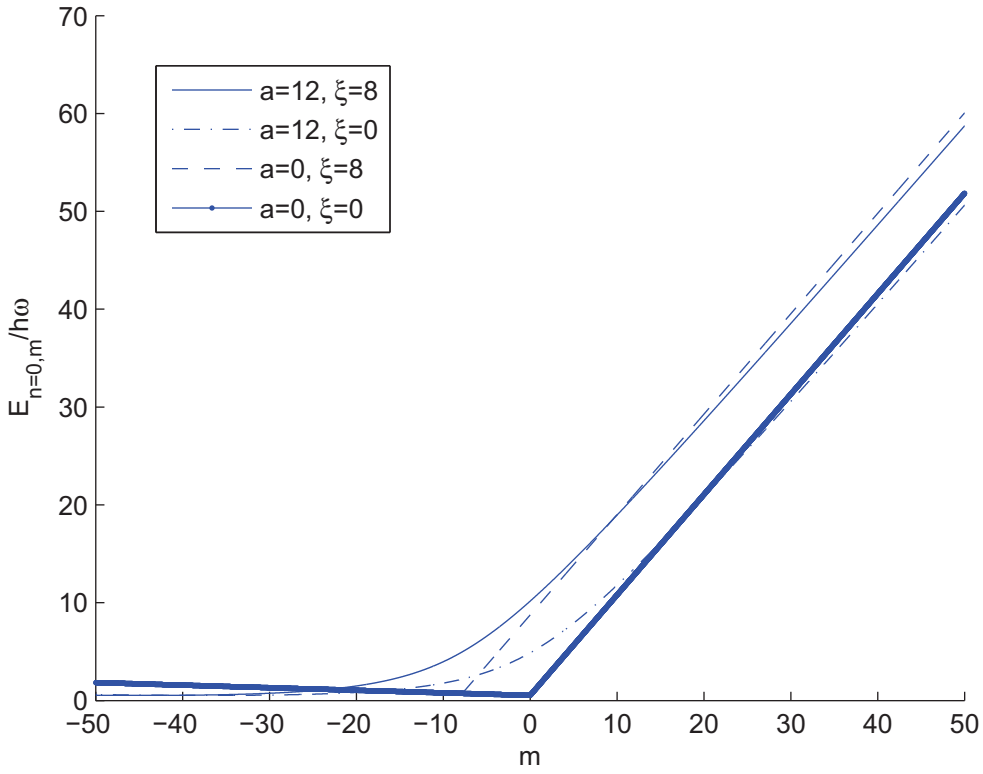
FIG. 8: Ground state GaAs pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for magnetic quantum number $m = 0$.

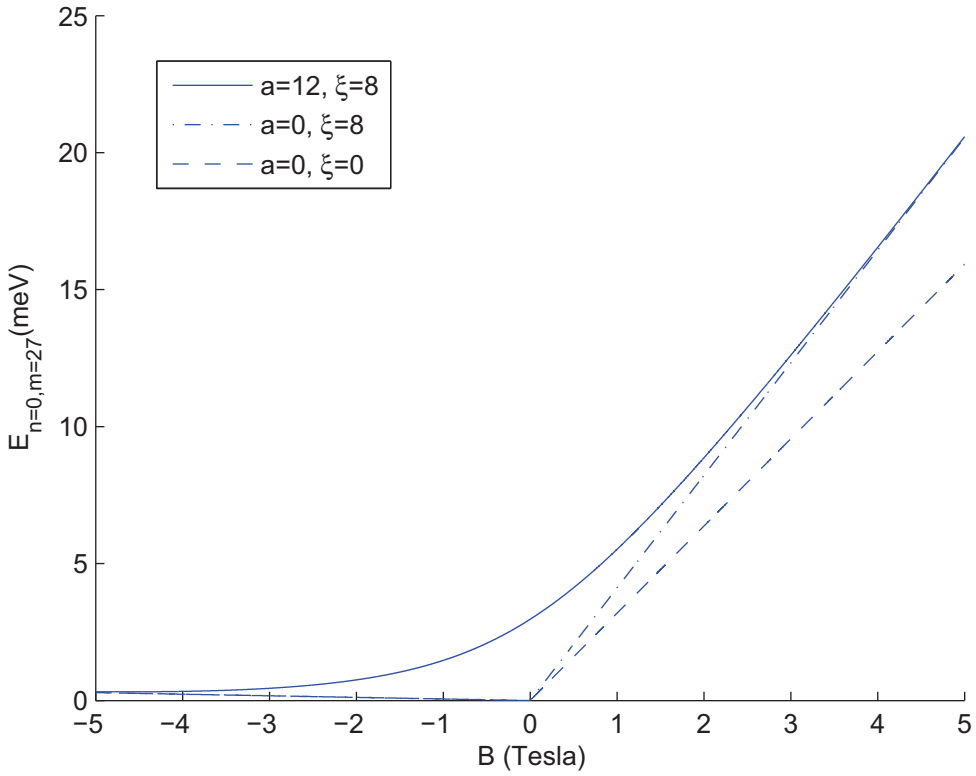
FIG. 9: GaAs pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for $n = 5$ and $m = 0$.

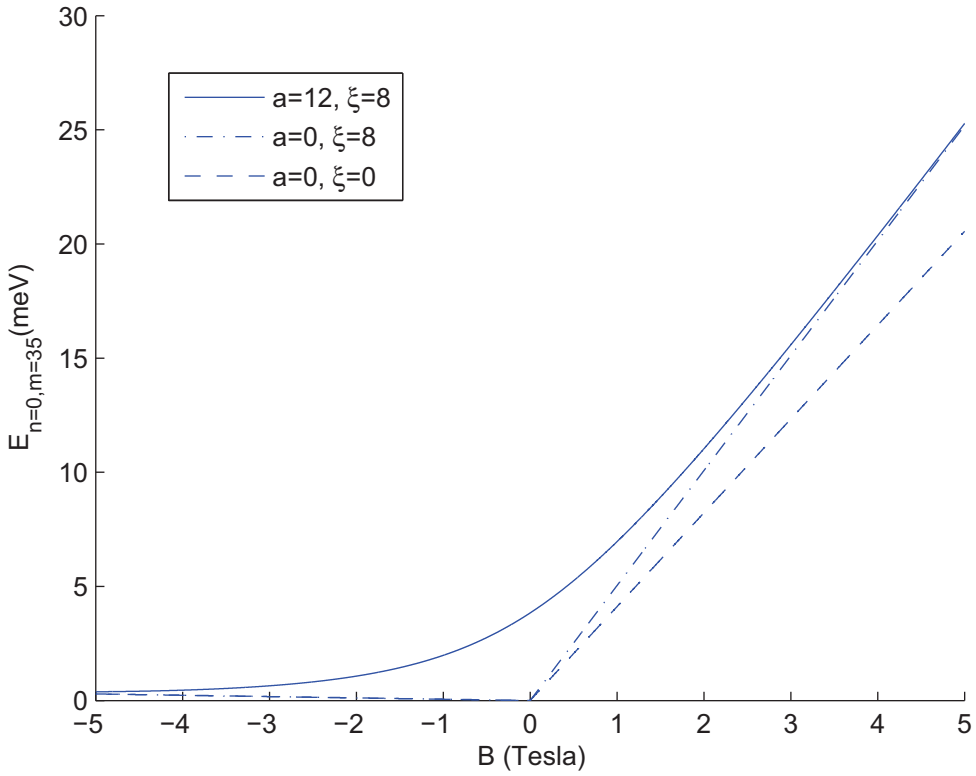
FIG. 10: Ground state GaAs pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for magnetic quantum number $m = 5$.

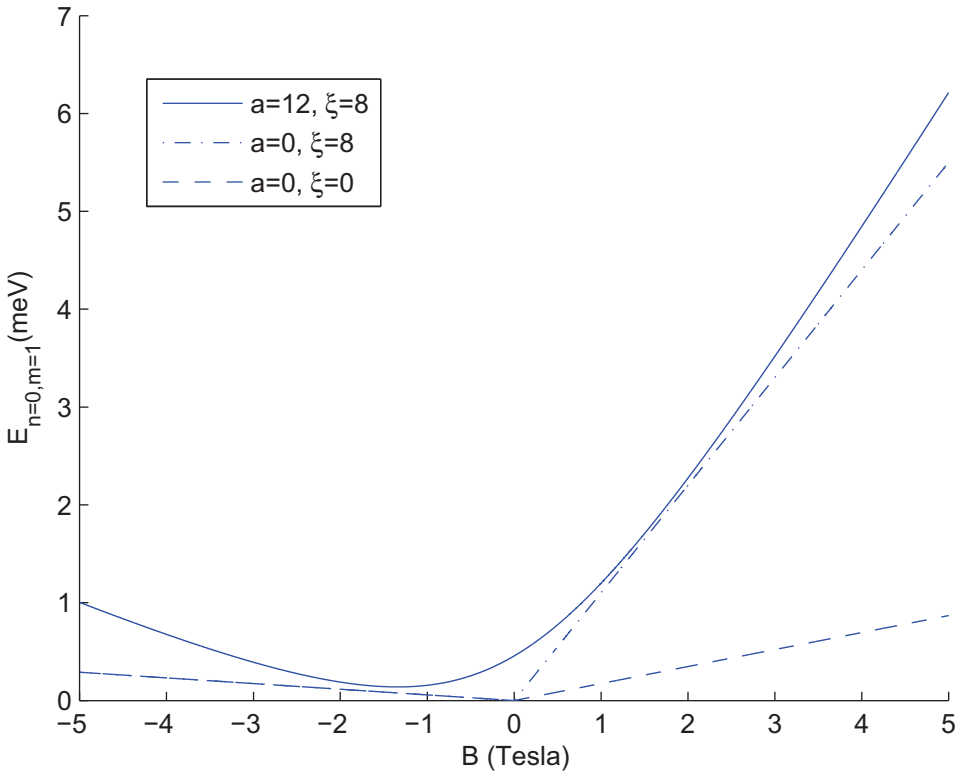
FIG. 11: Ground state GaAs pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for magnetic quantum number $m = -5$.

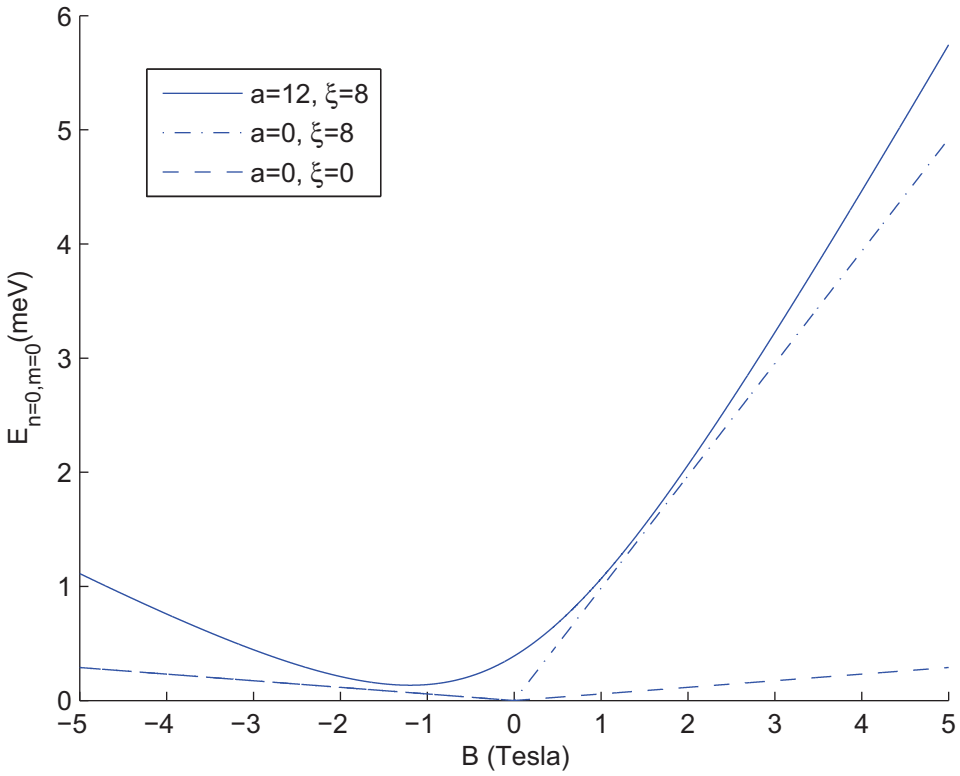
FIG. 12: GaAs pseudodot energy levels (in meV) as a function of magnetic field B (in $Tesla$). Solid, dotted and dashed curves represent the pseudodot energy levels in presence of AB flux field, Landau energy levels in presence of AB flux field and Landau levels in the absence of AB flux field, respectively for $n = 5$ and $m = -5$.

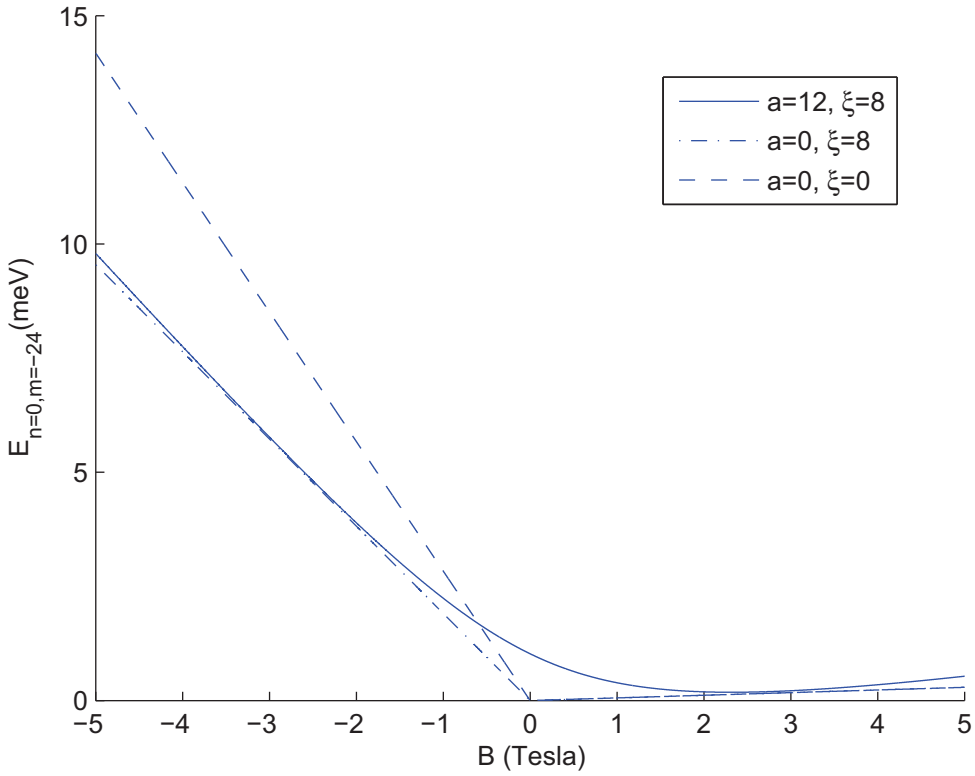

a=12, ξ=8
a=12, ξ=0
a=0, ξ=8
a=0, ξ=0
E_{n=0,m}/hω
m
70
60
50
40
30
20
10
0
−50
−40
−30
−20
−10
0
10
20
30
40
50

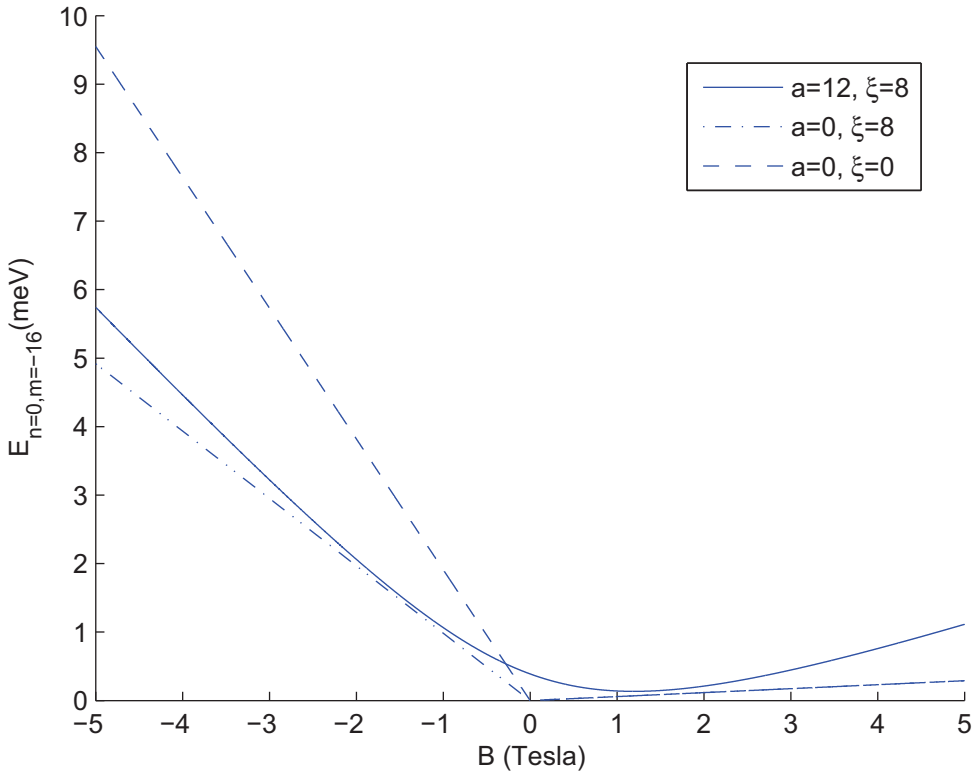

25
20
15
10
5
0
−5
−4
−3
−2
−1
0
1
2
3
4
5
B (Tesla)
E n=0,m=27 (meV)
a=12, ξ=8
a=0, ξ=8
a=0, ξ=0

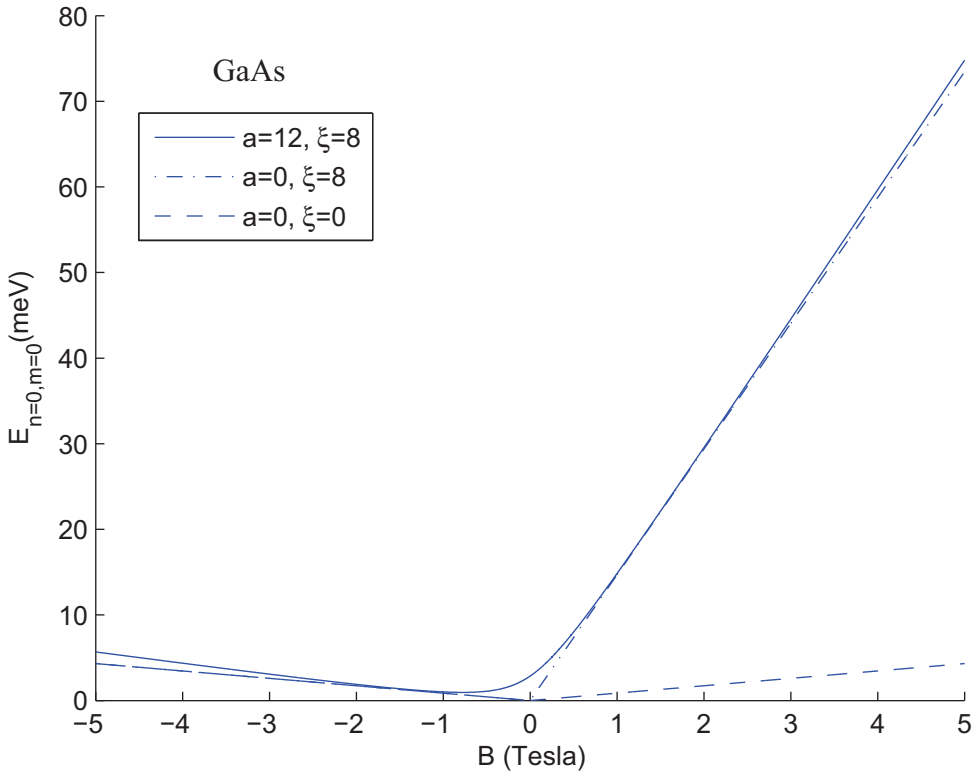

a=12, ξ=8
a=0, ξ=8
a=0, ξ=0
E_{n=0,m=35}(meV)
B (Tesla)
30
25
20
15
10
5
0
−5
−4
−3
−2
−1
0
1
2
3
4
5

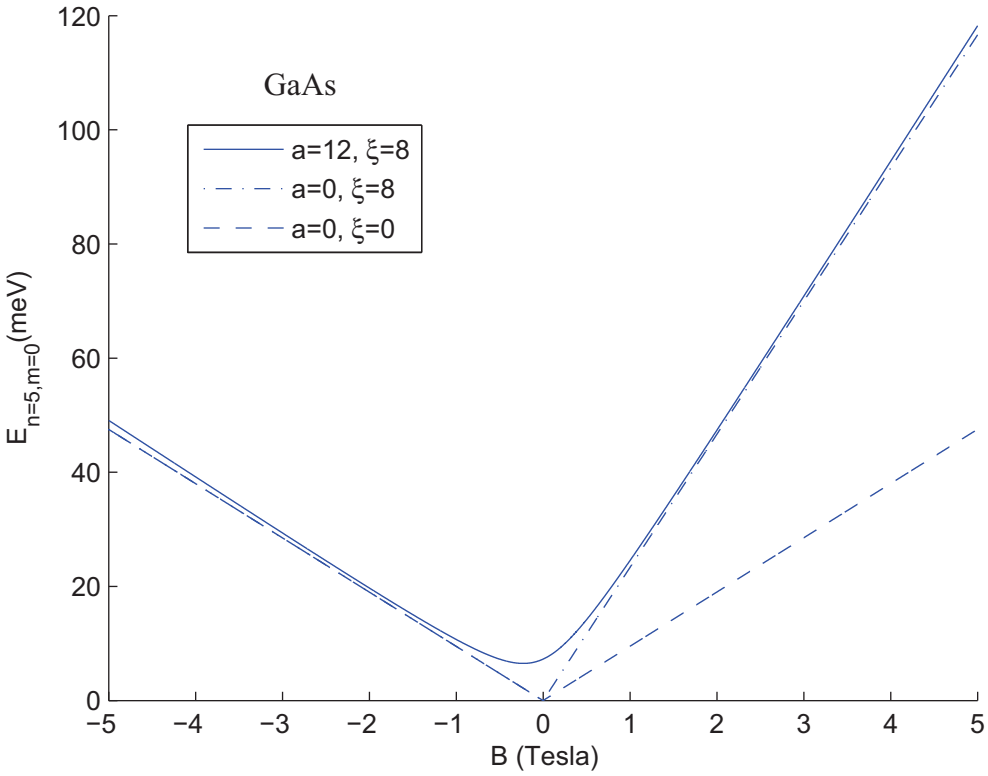

a=12, ξ=8
a=0, ξ=8
a=0, ξ=0
E_{n=0,m=1} (meV)
B (Tesla)
7
6
5
4
3
2
1
0
−5
−4
−3
−2
−1
0
1
2
3
4
5

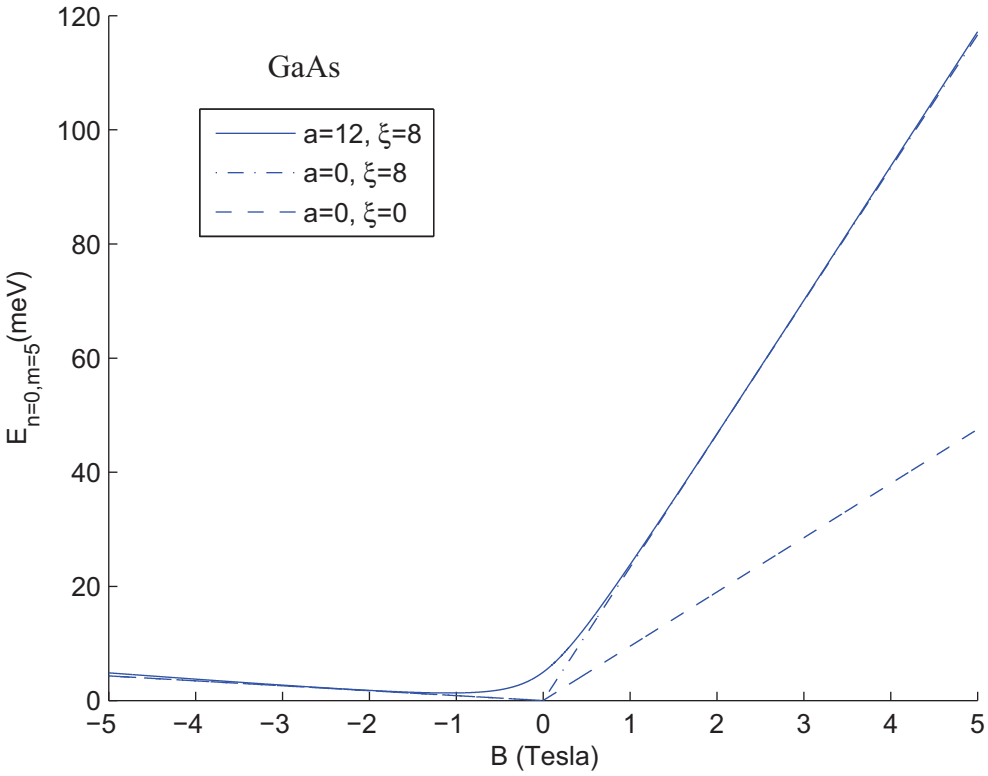

a=12, ξ=8
a=0, ξ=8
a=0, ξ=0
E_{n=0,m=0}(meV)
B (Tesla)
6
5
4
3
2
1
0
−5
−4
−3
−2
−1
0
1
2
3
4
5

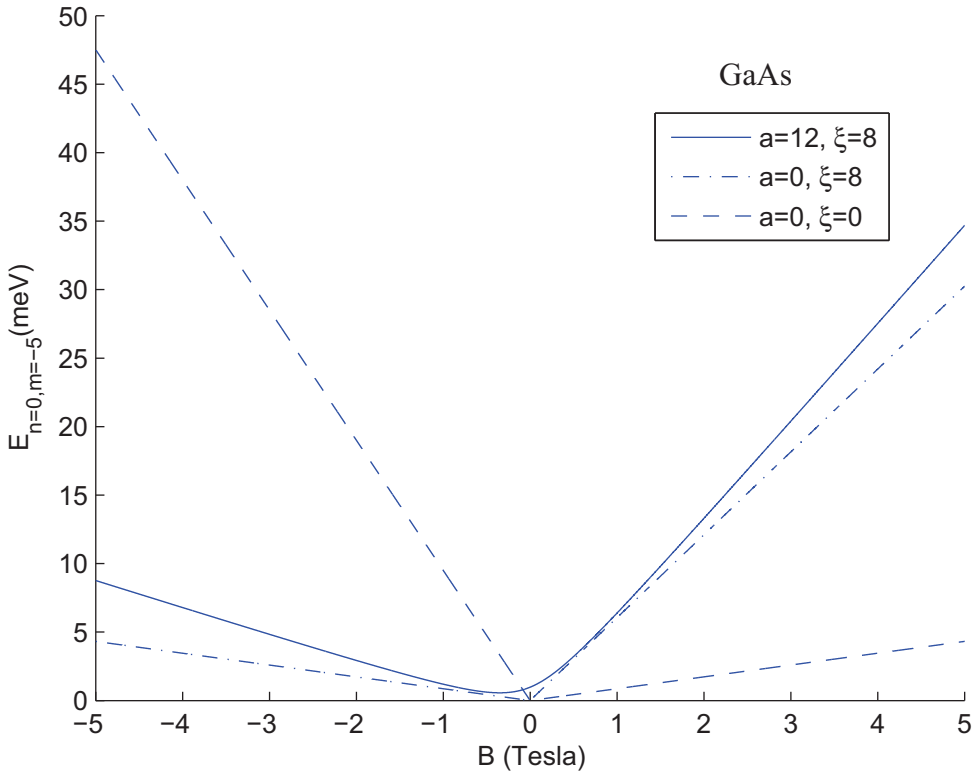

15
10
5
0
−5
−4
−3
−2
−1
0
1
2
3
4
5
B (Tesla)
E n=0,m=−24 (meV)
a=12, ξ=8
a=0, ξ=8
a=0, ξ=0

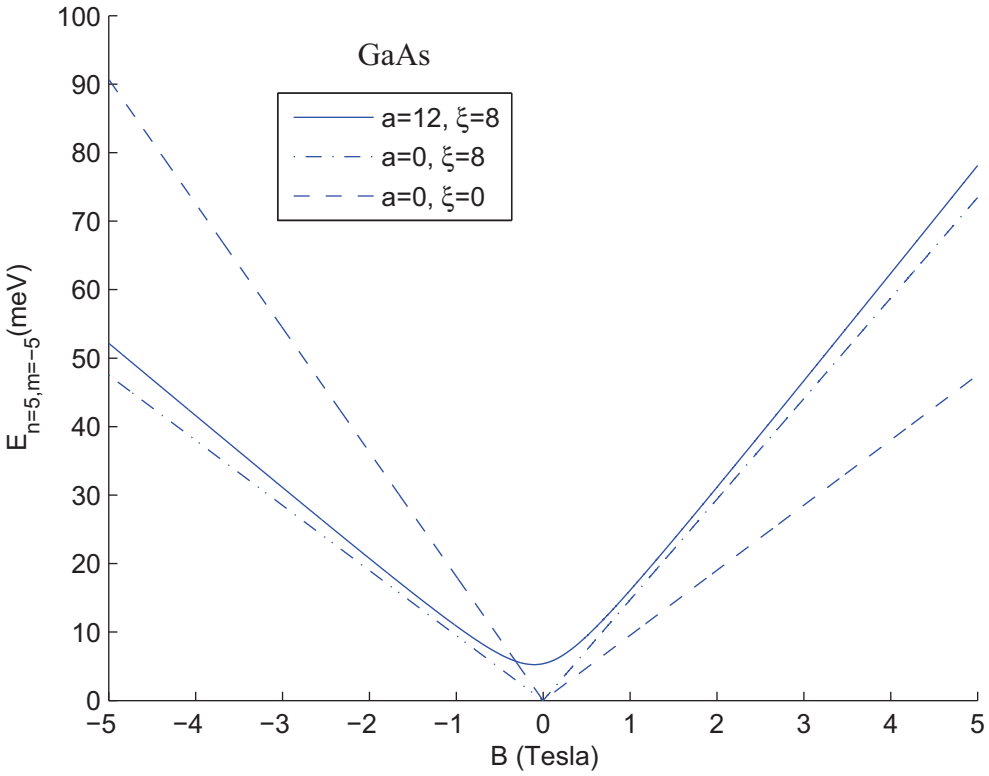

10
9
8
7
6
5
4
3
2
1
0
−5
−4
−3
−2
−1
0
1
2
3
4
5
B (Tesla)
E n=0,m=−16 (meV)
a=12, ξ=8
a=0, ξ=8
a=0, ξ=0


GaAs
a=12, ξ=8
a=0, ξ=8
a=0, ξ=0
$E_{n=0,m=0}$(meV)
B (Tesla)
80
70
60
50
40
30
20
10
0
−5
−4
−3
−2
−1
0
1
2
3
4
5


GaAs
a=12, ξ=8
a=0, ξ=8
a=0, ξ=0
E_{n=5,m=0}(meV)
B (Tesla)
120
100
80
60
40
20
0
−5
−4
−3
−2
−1
0
1
2
3
4
5


GaAs
a=12, ξ=8
a=0, ξ=8
a=0, ξ=0
E_{n=0,m=5}(meV)
B (Tesla)
120
100
80
60
40
20
0
−5
−4
−3
−2
−1
0
1
2
3
4
5


GaAs
a=12, ξ=8
a=0, ξ=8
a=0, ξ=0
$E_{n=0,m=-5}$(meV)
B (Tesla)
50
45
40
35
30
25
20
15
10
5
0
−5
−4
−3
−2
−1
0
1
2
3
4
5


GaAs
a=12, ξ=8
a=0, ξ=8
a=0, ξ=0
E_{n=5,m=−5}(meV)
B (Tesla)
100
90
80
70
60
50
40
30
20
10
0
−5
−4
−3
−2
−1
0
1
2
3
4
5