The arrangement field theory (AFT)

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Abstract

We introduce the concept of "non-ordered space-time" and formulate a quaternionic field theory over such generalized non-ordered space. The imposition of an order over a non-ordered space appears to spontaneously generate gravity, which is revealed as a fictitious force. The same process gives rise to gauge fields that are compatible with those of Standard Model. We suggest a common origin for gravity and gauge fields from a unique entity called "arrangement matrix" (M) and propose to quantize all fields by quantizing M. Finally we give a proposal for the explanation of black hole entropy and area law inside this paradigm.

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1 Introduction to formalism

The scalar field paradigm describes the universe be means of a graph (ie an ensemble of vertices and edges). However there is a considerable difference between this framework and the usual modeling with spin-foams or spin-networks. The existence of an edge which connects two vertices is in fact probabilistic. In this way we consider the vertices as fundamental physical quantities, while the edges become dynamic fields.

In section 2.1 we introduce the concept of non-ordered space-time, ie an ensemble of vertices without any information on their mutual positions. In section 2.2 we define the "arrangement matrix" (M), which is a matricial field whose entries define the probability amplitudes for the existence of edges. The arrangement matrix regulates the order of vertices in the space-time, determining the topology of space-time itself. In the same section we extend the concept of derivative on such non-ordered space-time.

In section 3 we define a simple "toy-action" for a quaternionic field in a non-ordered space-time. We show how the imposition of an arrangement in such space-time generates automatically a metric h which is strictly determined by M.

In section 4 we discover a low energy limit under which the "toy-action" becomes a local action after the arrangement imposition.

In section 5 we show that a new interpretation of spin nature arises spontaneously from our framework. In the same section, the role of "arrangement matrix" is compared to the role of an external observer.

In section 6 we anticipate some unpublished results regarding the availment of our framework to describe all standard model interactions.

In section 7 we apply a second quantization to the "arrangement matrix", turning it in an operator which creates or annihilates edges. We show how this process can give a new interpretation to black hole entropy and area law. We infer that quantization of M automatically quantizes h, apparently without renormalization problems.

2 A non-ordered universe

2.1 Reciprocal relationship between space-time points

Every euclidean 4-dimensional space can be approximated by a graph Λ^4, that is a collection of vertices connected by edges of length Δ. We recover the continuous space in the limit $\Delta \to 0$. Moreover we can pass from the euclidean space to the lorenzian space-time by extending holomorphically any function in the fourth coordinate $x_4 \to ix_4$ [6].

In non commutative geometry, one can assume that a first vertex is connected to a second, without the second is connected to the first. This means that connections between vertices are made by two oppositely oriented edges, which we can represent by a couple of arrows.

We assume the vertices as fundamental quantities. Then we can select what couples of vertices are connected by edges; different choices of couple generated different graphs, which in the limit $\Delta \to 0$ correspond to different spaces.

Our fundamental assumption is that the existence of an edge follows a probabilistic law, like any other quantity in QM. We draw any pair of vertices, denoted by v_1 and v_2, and we connect each other by a couple of arrows oriented in opposite directions.

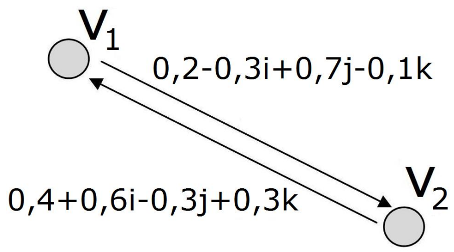
Before proceeding, we extend the common definition of amplitude probability. Usually this is a complex number, whose square module represents a probability and so is minor or equal to one.

We define instead the amplitude probability as an element in the division ring of quaternionic numbers, commonly indicated with $\mathbf{H}$. Its square module represents yet a probability and so is minor or equal to one. A quaternion q have the form $q = a + ib + jc + kd$ with $a, b, c, d \in \mathbf{R}$, $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$,

$jk = -kj = i$, $ki = -ik = j$.

We write a quaternionic number near the arrow which moves from v_1 to v_2. It corresponds to the probability amplitude for the existence of an edge which connects v_1 with v_2. We do the same thing for the other arrow, writing the probability amplitude for the existence of an edge which connects v_2 with v_1 .

A non-drawn arrow corresponds to an arrow with number 0. In principle, for every pair of vertices exists a couple of arrows which connect each other, eventually with label 0.



We can describe our universe by means of vertices connected by couple of arrows, with a quaternionic number next to each arrow, as shown in figure 1, below.

What we are building is another variation of the Penrose's spin-network model [15] or the Spin-Foam models [16], [17] in Loop Quantum Gravity [18], which generalize Feynman diagrams.

2.2 The Matrix relating couples of points

Given a spin-network, like the one in figure 1, we can move from picture to the "Arrangement Matrix" M, which is a simple table constructed as follows. We enumerate all the vertices in the graph at our will, provided we enumerate all of them. Typically we think of indexing the vertices by the usual sequence of integers $1, 2, 3, 4, 5, \ldots$.

Thus we create such matrix, whose rows and columns are enumerated in the same way as the vertices in the graph. Then we look at the vertices v_i and v_j: in

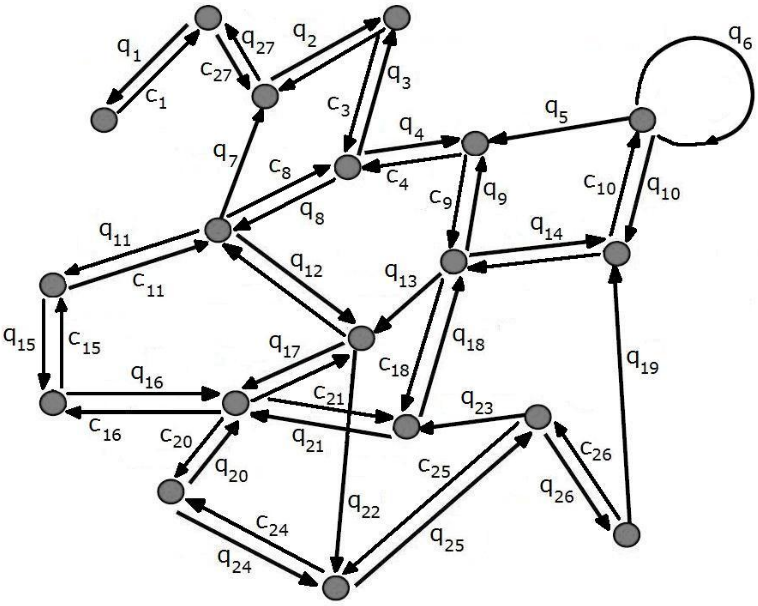
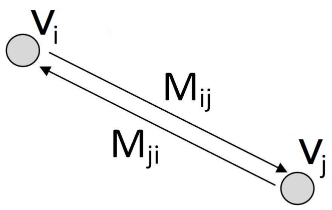


Figure 1: We can describe our universe by means of vertices connected by couple of arrows, with a quaternionic number next to each arrow.

the entry (i, j) we report the number situated near the arrow which moves from v_i to v_j. Similarly, in the entry (j, i) we report the number written near the opposite arrow. Remember that an absent arrow is an arrow with number 0 and consider for the moment $|M^{ij}| \leq 1$ for every ij.



In principle, we can image an entry $M_{ij} \neq M_{ji}$, even with $|M_{ij}|^2 \neq |M_{ji}|^2$. This means that v_i may be connected to v_j even if v_j is not connected to v_i. In that case, a non-commutative geometry is involved. The probability amplitude that v_i and v_j are mutually connected (we could talk about "classical" connection), is:

$$Cl.ampl. \propto M_{ij}M_{ji}$$

The probability amplitude for the vertex v_i to be classically connected with any other vertex (hence it will be not isolated) is:

$$Cl.ampl. \propto \sum_j M_{ij} M_{ji} = (M \cdot M)_{ii}$$

We can imagine our table with elements M_{ij} as a machine which "creates" jointures between vertices, by connecting each other or closing a single vertex onto itself through a loop. The loops are obviously represented by diagonal elements of matrix, with the form (i, i).

Now let's ask ourselves: is it necessary to know where the vertices are located? Let's look at the Standard Model action: it is given by a sum (or more properly, an integral), over *all* the points of the universe, of locally defined terms. Any term is defined on a single point. Since the terms are separated - a term for each point - and we integrate all of them, we do not need to know where the points physically are.

However, there are terms which are not strictly local, ie those containing the derivative operator ∂. The operator ∂, acting on a field φ in the point v_j, calculates the difference between the value of φ in a point immediately "after" v_j, and the value of φ immediately " before" v_j.

In the discretized theory, the integral over points becomes a sum over vertices of the graph. Similarly, the derivative becomes a finite difference. Hence, for terms containing ∂, we need a clear definition of "before" and "after", that is an arrangement of the vertices, as defined by the matrix M.

We consider a scalar field but don't represent it with the usual function (or distribution) $\varphi(x)$. Instead we denote it with a column of elements (an array) where each element is the value of the field in a specific vertex of the graph. For example (with only 7 vertices):

$$\varphi = \begin{pmatrix} \varphi(p_0) \\ \varphi(p_1) \\ \varphi(p_2) \\ \varphi(p_3) \\ \varphi(p_4) \\ \varphi(p_5) \\ \varphi(p_6) \end{pmatrix} \tag{1}$$

For simplicity, we start with a one-dimensional graph: it's easy to see how the derivative operator is proportional to an antisymmetric matrix $\tilde{M}$ whose elements are different from zero only immediately above the diagonal (where they count +1), and immediately below (where they count -1). We can see this, for example, in a "toy-graph" formed by only 12 separated vertices (figure 2). The argument remains true while increasing the number of vertices.

$$\partial\varphi = \frac{1}{2\Delta} \begin{pmatrix} 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \\ \varphi(5) \\ \varphi(6) \\ \varphi(7) \\ \varphi(8) \\ \varphi(9) \\ \varphi(10) \\ \varphi(11) \end{pmatrix} \tag{2}$$

$$= \begin{pmatrix} \varphi(1) - \varphi(11) \\ \varphi(2) - \varphi(0) \\ \varphi(3) - \varphi(1) \\ \varphi(4) - \varphi(2) \\ \varphi(5) - \varphi(3) \\ \varphi(6) - \varphi(6) \\ \varphi(7) - \varphi(5) \\ \varphi(8) - \varphi(6) \\ \varphi(9) - \varphi(7) \\ \varphi(10) - \varphi(8) \\ \varphi(11) - \varphi(9) \\ \varphi(0) - \varphi(10) \end{pmatrix} \tag{3}$$

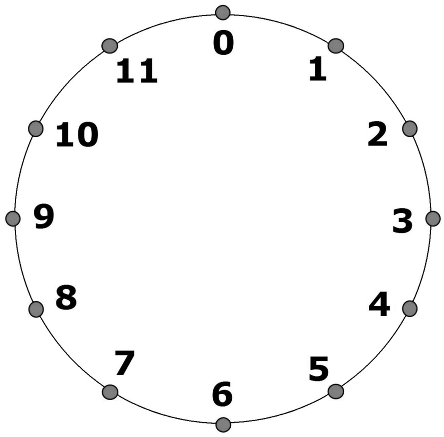


Figure 2:

Δ is the length of graph edges. In the continuous limit, $\Delta \to 0$ (that occurs in Hausdorff spaces, where matricial product turns into a convolution), we obtain

$$\begin{aligned}\partial\varphi(x) &= \lim_{\Delta\to 0}\frac{1}{2\Delta}\int \tilde{M}(x,y)\varphi(y)dy \\ \partial\varphi(x) &= \lim_{\Delta\to 0}\frac{1}{2\Delta}\int \left[\delta(y-(x+\Delta))-\delta(y-(x-\Delta))\right]\varphi(y)dy \\ \partial\varphi(x) &= \lim_{\Delta\to 0}\frac{\varphi(x+\Delta)-\varphi(x-\Delta)}{2\Delta} = \partial\varphi(x) \qquad (4)\end{aligned}$$

In this way our definition is consistent with the usual definition of derivative.

While increasing the number of points, a (-1) still remains in the up right corner of the matrix, and a $(+1)$ in the down left corner as well. To remove those two non-null terms, it is sufficient to make them unnecessary, by imposing boundary conditions that make the field null in the first and in the last point.

In fact we can describe an open universe (a straight line in one dimension), starting from a closed universe (a circle) and making the radius to tend to infinity. Hence we see that the conditions of null field in the first and in the last point become the traditional boundary conditions for the Standard Model fields.

Remark 1 *Note that in spaces with more than one dimension, a derivative matrix $\tilde{M}_\mu$ assumes the form (2) only if we number the vertices progressively along the coordinate μ. However, two different numberings can be always related by a vertices permutation.*

3 A quaternionic field action in a non-ordered space-time

Definition 2 *For any graph Λ^4 we define its* associated non-ordered space $\mathbf{S}_\Lambda$ *as the ensemble of all its vertices.*

The graph includes vertices plus edges (ordered connections between vertices), while the *associated non-ordered space* contains only vertices. In some sense, $\mathbf{S}_\Lambda$

doesn't know where any vertex is.

Consider a *numbering function* π, that is whatever bijection from $X \subset \mathbf{N}$ to the non-ordered space.

$$\begin{aligned} \pi \quad : \quad & X \subset \mathbf{N} \longrightarrow \mathbf{S}_\Lambda \\ & i \longrightarrow v_i = \pi(i) \end{aligned}$$

In this way, every vertex v_i in $\mathbf{S}_\Lambda$ is one to one with an integer $i \in X \subset \mathbf{N}$. This means that the ensemble of vertices has to be at most numerable.

We consider a generic invertible matrix M and interpret any entry M^{ij} of M as the probability amplitude for the existence in Λ^4 of an edge which connects $\pi(i)$ with $\pi(j)$. Remember that a couple of vertices can be connected by at most two oriented edges with different orientations. M^{ij} defines the probability amplitude for the edge which moves from $\pi(i)$ to $\pi(j)$, while M^{ji} defines the probability amplitude for the edge which moves from $\pi(j)$ to $\pi(i)$.

Take care that in four dimensions we have to number the vertices by elements (i, j, k, l) in $\mathbf{N}^4$ before taking the limit $\Delta \to 0$. In this way $\sum_{(i,j,k,l)} \Delta^4$ becomes $\int dx^0 dx^1 dx^2 dx^3$. If, as we have suggested, the vertices have been already numbered with elements of $\mathbf{N}$, we can change the numbering by using the natural bijection ϑ between $\mathbf{N}$ and $\mathbf{N}^4$, with $(i, j, k, l) = \vartheta(a)$, $(i, j, k, l) \in \mathbf{N}^4$ and $a \in \mathbf{N}$.

Definition 3 (covariant derivative) *Given any skew hermitian matrix A_μ, with entries in $\mathbf{H}$, and a skew hermitian matrix $\tilde{M}_\mu$, which assumes the form (2) when the vertices are numbered along the coordinate μ, their associated covariant derivative is*

$$\nabla_\mu = \tilde{M}_\mu + A_\mu. \tag{5}$$

Definition 4 (arrangement) *We indicate with n the number of elements inside $X \subset \mathbf{N}$. Given a normal matrix $\hat{M}$ and four covariant derivatives ∇_μ ($\mu =$*

$0,1,2,3$*) with dimensions* $n \times n$*, an* arrangement *for* $\hat{M}$ *is a quadruplet of couples* $(\hat{D}^\mu, \hat{U})$*, with* $\hat{D}^\mu$ *diagonal and* $\hat{U}$ *hyperunitary, such that*

$$\hat{M} = \sum_\mu \hat{U}\hat{D}^\mu \nabla_\mu \hat{U}^\dagger. \tag{6}$$

We require that covariant derivative will be form-invariant under the action of a transformation $V \in U(n, \mathbf{H})$ which acts both on $\tilde{M}_\mu$ and A_μ. We explicit $V\nabla_\mu V^\dagger$:

$$\begin{aligned} V\nabla_\mu V^\dagger &= V\left(\tilde{M}_\mu + A_\mu\right)V^\dagger \\ &= \underbrace{VV^\dagger}_{=1}\tilde{M}_\mu + V\left[\tilde{M}_\mu, V^\dagger\right] + VA_\mu V^\dagger. \end{aligned} \tag{7}$$

Setting

$$A'_\mu = V\left[\tilde{M}_\mu, V^\dagger\right] + VA_\mu V^\dagger, \tag{8}$$

we obtain

$$V\nabla_\mu V^\dagger = \tilde{M}_\mu + A'_\mu \stackrel{def}{=} \nabla'_\mu \tag{9}$$

that means

$$V\nabla_\mu[A]V^\dagger = \nabla_\mu[A'].$$

Hence the transformation law for the matrix A_μ is like we expect:

$$A_\mu \to A'_\mu = V\left[\tilde{M}_\mu, V^\dagger\right] + VA_\mu V^\dagger. \tag{10}$$

We observe that (10) preserves the hermiticity of A_μ. In fact

$$
\begin{aligned}
A'^{\dagger}_{\mu} &= (V[\tilde{M}_\mu, V^\dagger] + VA_\mu V^\dagger)^\dagger \\
&= (V\tilde{M}_\mu V^\dagger - \tilde{M}_\mu + VA_\mu V^\dagger)^\dagger \\
&= V\tilde{M}^\dagger_\mu V^\dagger - \tilde{M}^\dagger_\mu + VA^\dagger_\mu V^\dagger \\
&= -V\tilde{M}_\mu V^\dagger + \tilde{M}_\mu - VA_\mu V^\dagger \\
&= -(V[\tilde{M}_\mu, V^\dagger] + VA_\mu V^\dagger) = -A'_\mu \qquad (11)
\end{aligned}
$$

It's easy to see that (10) reduces to the usual transformation for a gauge field $A'_\mu = V\partial_\mu V^\dagger + VA_\mu V^\dagger$ in the limit $\Delta \to 0$.

Theorem 5 *For every invertible normal matrix $\hat{M}$ and every covariant derivative $\nabla[A]_\mu$ which is invertible (in the matricial sense), there exist*

1. *A new quadruplet of covariant derivatives $\nabla'_\mu = \nabla[A']_\mu$ such that $D^\mu \nabla'_\mu = 1$ for some diagonal matrix D^μ, where A'_μ is the gauge transformed of A_μ for some unitary transformation U;*
2. *An arrangement $(\hat{D}^\mu, \hat{U})$ between $\hat{M}$ and ∇'_μ.*

Proof. According to spectral theorem, $\forall \hat{M} \in \mathbb{M}^{(N)}\ \exists \hat{U}$ hyperunitary such that $\hat{U}\hat{M}\hat{U}^\dagger = K$ with K diagonal. $\hat{M}$ is invertible, so the same is true for K. Setting $\hat{D} = K^{-1}$:

$$
\begin{aligned}
\hat{U}\hat{M}\hat{U}^\dagger\hat{D} = K\hat{D} = KK^{-1} = 1 \qquad (12) \\
\hat{D}\hat{U}\hat{M}\hat{U}^\dagger = \hat{D}K = K^{-1}K = 1.
\end{aligned}
$$

At this point we choice a covariant derivative ∇_μ (which is also a normal matrix) and we reason as we did above for $\hat{M}$, putting

$$1 = D^\mu U \nabla_\mu U^\dagger = U \nabla_\mu U^\dagger D^\mu \tag{13}$$

for some D^μ diagonal and U unitary. No sum over repeated indices is implied.

A well known theorem states that U can be chosen in such a way that D^μ takes values in $\mathbf{C}$. Moreover we can always find a quaternion s with $|s| = 1$ such that, if D^μ takes values in $\mathbf{C} = \mathbf{R} \oplus i\mathbf{R}$, then $s^* D^\mu s$ will take values in $\mathbf{C} = \mathbf{R} \oplus (ri + tj + pk)\mathbf{R}$, with fixed $r, t, p \in \mathbf{R}$ and $r^2 + t^2 + p^2 = 1$. Every s with $|s| = 1$ describes in fact a rotation in the 3 dimensional space with base elements i, j, k.

Introducing such s, the equation (13) becomes

$$s1s = s^* D^\mu s s^* U \nabla_\mu U^\dagger s. \tag{14}$$

Now we note that $s^* U$ is another hyperunitary transformation. Redefining $s^* D_\mu s \to D_\mu$, $s^* U \to U$ we obtain newly

$$1 = D^\mu U \nabla_\mu U^\dagger. \tag{15}$$

In this way we can always choose in what complex plane is D_μ. In the following we call this propriety "s-invariance". Using (9) into (13):

$$1 = D^\mu \nabla'_\mu = \nabla'_\mu D^\mu \Longrightarrow \left[\nabla'_\mu, D^\mu\right] = 0. \tag{16}$$

Taking into account (12):

$$\begin{aligned} \hat{D}\hat{U}\hat{M}\hat{U}^\dagger &= D^\mu \nabla'_\mu \\ \hat{U}\hat{M}\hat{U}^\dagger \hat{D} &= \nabla'_\mu D^\mu. \end{aligned} \tag{17}$$

Summing on μ we obtain:

$$4\hat{D}\hat{U}\hat{M}\hat{U}^{\dagger} = \sum_{\mu} D^{\mu}\nabla'_{\mu} \tag{18}$$
$$4\hat{U}\hat{M}\hat{U}^{\dagger}\hat{D} = \sum_{\mu} \nabla'_{\mu}D^{\mu}.$$

Solving for $\hat{M}$:

$$\hat{M} = \frac{1}{4}\sum_{\mu} \hat{U}^{\dagger}\hat{D}^{-1}D^{\mu}\nabla'_{\mu}\hat{U} = \frac{1}{4}\sum_{\mu} \hat{U}^{\dagger}\nabla'_{\mu}D^{\mu}\hat{D}^{-1}\hat{U}. \tag{19}$$

Defining $\hat{D}^{\mu}$ as $\frac{1}{4}\hat{D}^{-1}D^{\mu}$

$$\hat{M} = \sum_{\mu} \hat{U}^{\dagger}\hat{D}^{\mu}\nabla'_{\mu}\hat{U} \tag{20}$$

QED ■

Note that in general $\hat{M} \neq \sum_{\mu} \hat{U}^{\dagger}\nabla'_{\mu}\hat{D}^{\mu}\hat{U}$ because $\hat{D}^{-1}D^{\mu} \neq D^{\mu}\hat{D}^{-1}$ for the non commutativity of quaternions.

Theorem 6 *For every invertible matrix M with entries in $\mathbf{H}$, a normal matrix $\hat{M} = U_M M$ exists, where U_M is unitary and $\hat{M}$ is neither hermitian nor skew hermitian.*

Proof. Given an invertible matrix M, a unique choice of matrices U and P always exists, with U unitary and P hermitian positive, such that $UM = P$. Moreover, a well known theorem states that, for every hermitian matrix P with entries in $\mathbf{H}$, there exist I, J, K skew hermitian unitary matrices which commute with P. Moreover I, J, K achieve the same algebra of quaternionic imaginary unities i, j, k.

Consider then the unitary matrix $p = exp((bI + cJ + dK)P)$, with $b, c, d \in \mathbf{R}$. It's easy to see that $[p, P] = 0$. Moreover the matrix $\hat{M} = pP$ is normal and it is neither hermitian or skew hermitian. In fact

$$(pP)^\dagger = p^\dagger P = p^{-1} P = \neq \pm pP$$

$$(pP)(pP)^\dagger = (Pp)(Pp)^\dagger = Ppp^\dagger P^\dagger = PP = Pp^\dagger pP = P^\dagger p^\dagger pP = (pP)^\dagger (pP)$$

Moreover

$$\hat{M} = pUM = U_M M \qquad U_M = pU \ \ unitary.$$

■

Definition 7 (associated normal matrix) *For every invertible matrix M, we define an* associated normal matrix *as a normal matrix obtained trough the construction above. We indicate it with $\hat{M}$ and use the notation U_M for the unitary transformation which transforms M in $\hat{M} = U_M M$.*

Theorem 8 *For every $n \times n$ invertible matrix M with entries in $\mathbf{H}$ and every quadruplet of covariant derivatives $\nabla[A]_\mu$ which are invertible (in the matricial sense), there exist*

1. *An associated normal matrix $\hat{M} = U_M M$ with U_M unitary;*
2. *A new quadruplet of covariant derivatives ${\nabla'}_\mu = \nabla[A']_\mu$ such that $D^\mu {\nabla'}_\mu = 1$ for some diagonal matrix D^μ, where A'_μ is the gauge transformed of A_μ for some unitary transformation U;*
3. *An arrangement $(\hat{D}_\mu, \hat{U})$ between $\hat{M}$ and ∇'_μ such that*

$$S = (M\phi)^\dagger \cdot (M\phi) = \sum_{i=1}^{n} \sum_{\mu,\nu} \sqrt{|h|} h^{\mu\nu}(x_i)({\nabla'}_\mu \phi'(x_i))^* ({\nabla'}_\nu \phi'(x_i)). \tag{21}$$

Here ϕ is a one-component quaternionic field, while

$$x_i \equiv \pi(i)$$
$$\phi'(x_i) = {\phi'}^i(x) = \sum_j \hat{U}^{ij}\phi^j(x) = \sum_j \hat{U}^{ij}\phi(x_j)$$
$$\sqrt{h}h^{\mu\nu}(x_i) = \frac{1}{2}d^\mu d^{*\nu}(x_i) + c.c. \qquad \hat{D}^{ij}_\mu = d^\mu(x_i)\delta_{ij}. \tag{22}$$

Proof. The existence of $\nabla'_\mu = \nabla[A']_\mu$ follows from the proof of theorem 5, while the existence of an associated normal matrix $\hat{M} = U_M M$ descends from theorem 6. Hence we see that the first action in (21) is invariant for transformations (U_1, U_2) in $U(n, \mathbf{H}) \otimes U(n, \mathbf{H})$ which send M in $U_2 M U_1^\dagger$ and ϕ in $U_1\phi$. In fact

$$\begin{aligned} S[\phi] &= \phi^\dagger M^\dagger M \phi \\ &\rightarrow \phi^\dagger U_1^\dagger (U_2 M U_1^\dagger)^\dagger (U_2 M U_1^\dagger) U_1 \phi \\ &= \phi^\dagger U_1^\dagger U_1 M^\dagger U_2^\dagger U_2 M U_1^\dagger U_1 \phi \\ &= \phi^\dagger M^\dagger M \phi = S[\phi] \end{aligned} \tag{23}$$

If we set $U_1 = 1$ and $U_2 = U_M$ we have

$$S[\phi] \rightarrow \phi^\dagger M^\dagger U_M^\dagger U_M M \phi = \begin{cases} = \phi^\dagger M^\dagger M \phi = S[\phi] \\ = \phi^\dagger \hat{M}^\dagger \hat{M} \phi \qquad . \end{cases} \tag{24}$$

We substitute (20) in (24) with $\hat{M}$ in place of M.

$$
\begin{aligned}
S[\phi] &= \sum_{\mu,\nu} \left(\hat{U}^\dagger \hat{D}^\mu \nabla'_\mu \hat{U}\phi\right)^\dagger \left(\hat{U}^\dagger \hat{D}^\nu \nabla'_\nu \hat{U}\phi\right) \\
&= \sum_{\mu,\nu} \left(\phi^\dagger \hat{U}^\dagger \nabla'^\dagger_\mu \hat{D}^{\mu\dagger} \hat{U} \hat{U}^\dagger \hat{D}^\nu \nabla'_\nu \hat{U}\phi\right) \\
&= \sum_{\mu,\nu} \left(\phi^\dagger \hat{U}^\dagger \nabla'^\dagger_\mu \hat{D}^{\mu\dagger} \underbrace{\hat{U} \hat{U}^\dagger}_{=1} \hat{D}^\nu \nabla'_\nu \hat{U}\phi\right) \\
&= \sum_{\mu,\nu} \left(\phi^\dagger \hat{U}^\dagger \nabla'^\dagger_\mu \hat{D}^{\mu\dagger} \hat{D}^\nu \nabla'_\nu \hat{U}\phi\right) \\
&= \sum_{\mu,\nu} \left(\phi'^\dagger \nabla'^\dagger_\mu \hat{D}^{\mu\dagger} \hat{D}^\nu \nabla'_\nu \phi'\right).
\end{aligned}
$$

In the last step we have taken in account the definition (22). Finally

$$
S = \frac{1}{2} \sum_{\mu,\nu} \phi'^\dagger \nabla'^\dagger_\nu \left(\hat{D}^{\mu\dagger} \hat{D}^\nu + c.c.\right) \nabla'_\mu \phi'. \tag{25}
$$

It is remarkable that $\hat{D}_\mu$ is diagonal:

$$
\hat{D}^\mu_{ij} = d^\mu (x_i) \delta_{ij}. \tag{26}
$$

We can set

$$
\sqrt{|h|} h^{\mu\nu} (x_i) = \frac{1}{2} d^{\mu *} d^\nu (x_i) + c.c. \tag{27}
$$

and then

$$
S = \sum_{i,\mu,\nu} \sqrt{|h|} h^{\mu\nu}(x_i) \left(\nabla'_\mu \phi'\right)^{*i} \left(\nabla'_\nu \phi'\right)^i. \tag{28}
$$

■ QED.

The action of a transformation (U_1, U_2) on ∇' follows from its action on M. We can always use the invariance under $U(n, \mathbf{H}) \otimes U(n, \mathbf{H})$ to put M in the form

$M = \sum_\mu \hat{D}^\mu \nabla'_\mu$. Starting from this we have

$$U_2 M U_1^\dagger = \sum_\mu U_2 \hat{D}^\mu \nabla'_\mu U_1^\dagger = \sum_\mu U_2 \hat{D}^\mu U_1^\dagger U_1 \nabla'_\mu U_1^\dagger. \tag{29}$$

We define $\nabla''_\mu = U_1 \nabla'_\mu U_1^\dagger$ the transformed of ∇' under (U_1, U_2) and $\hat{D}'^\mu = U_2 \hat{D}^\mu U_1^\dagger$ the transformed of $\hat{D}^\mu$. We assume that A'_μ inside ∇'_μ transforms correctly as a gauge field, so that

$$\nabla'[A']_\mu \phi' = \nabla'[A']_\mu U_1^\dagger \phi'' = U_1^\dagger \nabla''[A']_\mu \phi'' = U_1^\dagger \nabla'[A'_{U1}]_\mu \phi''$$

$$\phi'' = U_1 \phi'.$$

We want $\hat{D}'^\mu$ remain diagonal and $h' = h[\hat{D}'] = h[\hat{D}]$. In this case there are two relevant possibilities:

1. $\hat{D}$ is a matrix made by blocks $m \times m$ with m integer divisor of n and every block proportional to identity. In this case the residual symmetry is $U(1, \mathbf{H})^n \times U(m, \mathbf{H})^{n/m}$ with elements (sV, V), s both diagonal and unitary, $V \in U(m, \mathbf{H})^{n/m}$;

2. h is any diagonal matrix. The symmetry reduces to $U(1, \mathbf{H})^n \otimes U(1, \mathbf{H})^n$ which is local $U(1, \mathbf{H}) \otimes U(1, \mathbf{H}) \sim SU(2) \otimes SU(2) \sim SO(4)$.

In this way, if we keep fixed the metric h and keep diagonal $\hat{D}$, the new action will be invariant at least under $U(1, \mathbf{H})^n \otimes U(1, \mathbf{H})^n$ which doesn't modify h.

Note however that the action (28) is highly non local, because the fields $A_\mu(x^a, x^b)$ with $a \neq b$ can relate couples of vertices very far each other. In fact the transformations in $U(n, \mathbf{H})$ mix all the vertices in the universe independently from their position. In the next section we'll discover in what limit (besides $\Delta \to 0$) the (28) becomes a local action. Let us now pause on the metric $h^{\mu\nu}$.

Remark 9 *We observe how the metric h has appeared from nowhere. We get the "impression" that the metric does not exist "a priori", but is generated by the* *matrices $\hat{D}$. In other words: the metric is simply the result of our desire to see an ordered universe at any cost.*

Remark 10 *Note that we have chosen the matrix ∇ between skew hermitian matrices, so that the gauge fields AR_i have real eigenvalues, corresponding to effectively measurable quantities*[1] *Conversely, $\hat{M}$ must remain generically normal. In fact, if $\hat{M}$ was (skew) hermitian, the fields d would become (imaginary) real, and there would not be enough degrees of freedom to construct the metric h.*

We focus on the relationship:

$$\sqrt{|h|}h^{\mu\nu}(x_i) = \frac{1}{2}d^{\mu*}d^{\nu}(x_i) + c.c. \tag{30}$$

We set:

$$d = \begin{pmatrix} a_0 + ib_0 + jc_0 + kd_0 \\ a_1 + ib_1 + jc_1 + kd_1 \\ a_2 + ib_2 + jc_2 + kd_2 \\ a_3 + ib_3 + jc_3 + kd_3 \end{pmatrix} \tag{31}$$

It's easy to see how s-invariance permits us to choose the D_μ in such a way that the real vectors

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}, \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix} \tag{32}$$

will be linearly independent.

[1] The operator R_i acts on any array ψ as $R_i\psi = \psi i$.

$$\sqrt{|h|}h^{-1} = \begin{pmatrix} a_0^2+b_0^2+c_0^2+d_0^2 & a_0a_1+b_0b_1+c_0c_1+d_0d_1 & a_0a_2+b_0b_2+c_0c_2+d_0d_2 & a_0a_3+b_0b_3+c_0c_3+d_0d_3 \\ a_1a_0+b_1b_0+c_1c_0+d_1d_0 & a_1^2+b_1^2+c_1^2+d_1^2 & a_1a_2+b_1b_2+c_1c_2+d_1d_2 & a_1a_3+b_1b_3+c_1c_3+d_1d_3 \\ a_2a_0+b_2b_0+c_2c_0+d_2d_0 & a_2a_1+b_2b_1+c_2c_1+d_2d_1 & a_2^2+b_2^2+c_2^2+d_2^2 & a_2a_3+b_2b_3+c_2c_3+d_2d_3 \\ a_3a_0+b_3b_0+c_3c_0+d_3d_0 & a_3a_1+b_3b_1+c_3c_1+d_3d_1 & a_3a_2+b_3b_2+c_3c_2+d_3d_2 & a_3^2+b_3^2+c_3^2+d_3^2 \end{pmatrix} \quad (33)$$

Note that we have 10 independent metric components as it should be. What would have happened if the entries of M were been simply complex numbers?

In that case we could always take a one-form X_ν such that $X_\nu(Im\, d^\nu) = X_\nu(Re\, d^\nu) = 0$. The contraction of X_ν with the metric would be

$$\sqrt{h}h^{\mu\nu}X_\nu = d^{*\mu}(d^\nu X_\nu) + d^\mu(d^{*\nu}X_\nu) = 0.$$

Hence the metric would be degenerate. For $d^\mu \in \mathbf{H}$ this can't happen, because no one-form can be orthogonal to 4 vectors linearly independent in a 4-dimensional space. Moreover a such one-form exists in spaces with dimension > 4. For this reason our theory hasn't meaning in presence of extra dimensions.

4 A local action from the quaternionic field action

Here we expose how to get a local action from the quaternionic field action in the limit of low energy. We can add to action quadratic $\sim M^2$ and quartic $\sim M^4$ terms, provided they are gauge invariant. In general we obtain a non-trivial potential of form $\alpha M^4 - \beta M^2$. We suppose that a minimum for such potential breaks the symmetry $U(n,\mathbf{H})\otimes U(n,\mathbf{H})$ and provides a mass to gauge fields A_μ. To view it is sufficient to rewrite M as a function of A_μ and consider a quartic term:

$$h^{\mu\alpha}A_\mu A_\alpha h^{\nu\beta}A_\nu A_\beta. \tag{34}$$

For a minimum of M there is a minimum of A which gives sense to the expansion:

$$A_\mu = A_\mu^{\min} + \delta A_\mu. \tag{35}$$

Therefore the (34) generates a factor:

$$m\left(x\right)^2 h^{\nu\beta}A_\nu A_\beta \tag{36}$$

$$m\left(x\right)^2 = h^{\mu\alpha}A_\mu^{\min}A_\alpha^{\min} \tag{37}$$

Hence the gauge fields acquire a mass, varying from point to point in the universe and essentially dependent on the metric.

Theorem 11 *Given a potential for M, which is both hermitian and invariant for $U(n,\mathbf{H})\otimes U(n,\mathbf{H})$, his minimum configurations are always invariant at least for $U(1,\mathbf{H})^n\otimes U(1,\mathbf{H})^n$, that is a local $U(1,\mathbf{H})\otimes U(1,\mathbf{H})$.*

Proof. A such potential contains only terms of type $tr((MM^\dagger)^j)$, $j\in\mathbf{N}$. All we can measure are eigenvalues of hermitian operators, and a hermitian operator has only real eigenvalues q which are invariant under $U(1,\mathbf{H})^n$, ie $sqs^*=qss^*=q$ for $|s|=1$. The simpler hermitian operators made by M are $MM^\dagger$ and $M^\dagger M$, whose eigenvalues are invariant under

$$M\to s_1Ms_2^* \qquad (s_1,s_2)\in U(1,\mathbf{H})\otimes U(1,\mathbf{H})$$

$$MM^\dagger\to s_2Ms_1^*s_1M^\dagger s_2^* = s_2MM^\dagger s_2^*$$

$$M^\dagger M\to s_1M^\dagger s_2^*s_2Ms_1^* = s_1M^\dagger Ms_1^*$$

■

In this manner we have always $m = 0$ for diagonal fields $A_\mu(x^a, x^a) \stackrel{!}{=} A_\mu(x^a)$.

A transformation $(s_1, s_2) \in U(1, \mathbf{H}) \otimes U(1, \mathbf{H})$ acts inside action in the expected way (see formula (29))

$$\phi' \to s_1 \phi' \stackrel{!}{=} \phi''$$

$$\nabla'[A']_\mu \to s_1 \nabla'[A']_\mu s_1^* = \nabla'[A'_{s1}]_\mu$$

$$d^\mu \to s_2 d^\mu s_1^*$$

$$S[\phi', A'] = S'[\phi'', A'_{s1}] = \sum_{\mu\nu} (s_2 d^\mu s_1^* \nabla'[A'_{s1}]_\mu \phi'')^\dagger (s_2 d^\nu s_1^* \nabla'[A'_{s1}]_\nu \phi'') \qquad (38)$$

We use the natural correspondence

$$(1, i, j, k) \longleftrightarrow i(\sigma^0, \sigma^1, \sigma^2, \sigma^3), \qquad \sigma^0 = -i\mathbf{1}, \qquad (39)$$

and define the complex field $\hat{\phi}$ as a complex 2×2 matrix,

$$\hat{\phi}^a = \begin{pmatrix} \phi_1^a + i\phi_2^a & \phi_3^a + i\phi_4^a \\ -\phi_3^a + i\phi_4^a & \phi_1^a - i\phi_2^a \end{pmatrix}$$

with $\phi'' = \phi_1 + i\phi_2 + j\phi_3 + k\phi_4$ and $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathbf{R}$. Every term between parenthesis becomes

$$W_2 (i\sigma^k) d_k^\mu W_1^\dagger \nabla'[A'_{s1}]_\mu \hat{\phi} \qquad (40)$$

where σ are Pauli matrices and $(W_1, W_2) \in SU(2) \otimes SU(2)$.

Theorem 12 *For every $SO(4)$ transformation Λ, a transformation $(W_1, W_2) \in SU(2) \otimes SU(2)$ exists, such that for every vector $d_j \in R^4$ we find*

$$\Lambda_i{}^j d_j \sigma^i = d_i W_2 \sigma^i W_1^\dagger.$$

Proof. We write $W_1 = U_1'^\dagger U_1$ and $W_2 = U_1' U_1$. In this manner we decompose $SU(2) \otimes SU(2)$ in $SU(2)_{rot} \otimes SU(2)_{boosts}$. $SU(2)_{rot}$ is generated by the couples (U_1, U_1), while $SU(2)_{boosts}$ by the couples $(U_1'^\dagger, U_1')$. After a wick rotation, the first one describes rotation in R^3, while the second one describes boosts.

A generic vector $d = \begin{pmatrix} d_0 & d_1 & d_2 & d_3 \end{pmatrix}$ gives

$$d_i(i\sigma^i) = \begin{pmatrix} d_0 + id_3 & id_1 + d_2 \\ id_1 - d_2 & d_0 - id_3 \end{pmatrix}$$

with $|d|^2 = det\, d_i(i\sigma^i)$. A transformation in $SO(4)$ doesn't change the norm $|d|$. Moreover, for every d exists a transformation in $SO(4)$ which put it in the normal form

$$d = \begin{pmatrix} |d| & 0 & 0 & 0 \end{pmatrix}.$$

The same properties have to be true for $SU(2) \otimes SU(2)$. The first one is banally verified because $det\, W_1 = det\, W_2 = 1$ and then $det\, d_i(i\sigma^i) = det\, d_i(W_2 i\sigma^i W_1^\dagger)$. Being $d_i(i\sigma^i)$ normal, we can use a transformation in $SU(2)_{rot}$ to put it in a diagonal form

$$U_1 d_i(i\sigma^i) U_1^\dagger = \begin{pmatrix} d_0 + id_3 & 0 \\ 0 & d_0 - id_3 \end{pmatrix}.$$

Define now the matrix U_1' as

$$U_1' = \frac{1}{\sqrt{|d|}} \begin{pmatrix} \sqrt{d_0 + id_3} & 0 \\ 0 & \sqrt{d_0 - id_3} \end{pmatrix}.$$

It's easy to verify that $U_1' U_1'^\dagger = 1$ and $det\, U_1' = 1$. Applying to $U_1 d_i(i\sigma^i) U_1^\dagger$ this transformation in $SU(2)_{boosts}$ we obtain

$$U_1'U_1 d_i(i\sigma^i)U_1^\dagger U_1' = \begin{pmatrix} |d| & 0 \\ 0 & |d| \end{pmatrix}.$$

So, for every d, a transformation in $SU(2)\otimes SU(2)$ exists, which puts it in the normal form. In this way, d transforms exactly as a vielbein field in the Palatini formulation of General Relativity, giving then the correspondence

$$\begin{aligned}
\Lambda_i{}^j d_j(i\sigma^i) &= U_1'U_1(i\sigma^j)U_1^\dagger U_1' d_j \\
\Lambda_i{}^j d_j\sigma^i &= W_2\sigma^j W_1^\dagger d_j \\
\frac{1}{2}tr(\Lambda_i{}^j d_j\sigma^i\sigma^k) &= \frac{1}{2}tr(W_2\sigma^j W_1^\dagger\sigma^k)d_j \\
\Lambda_k{}^j d_j &= \frac{1}{2}tr(W_2\sigma^j W_1^\dagger\sigma^k)d_j \\
\Lambda_k{}^j &= \frac{1}{2}tr(W_2\sigma^j W_1^\dagger\sigma^k).
\end{aligned} \qquad (41)$$

So, at every $\Lambda\in SO(4)$ corresponds a couple $(U_1,U_2)\in SU(2)\otimes SU(2)$. ■

Applying this to (40), it becomes

$$W_2(i\sigma^k)d_k^\mu W_1^\dagger\nabla'[A'_{s1}]_\mu\hat{\phi} = \Lambda_k{}^i d_i^\mu\sigma^k\nabla'[A'_{s1}]_\mu\hat{\phi}. \qquad (42)$$

Note that if we write $\hat{\phi} = (\hat{\phi}_1\ \ \hat{\phi}_2)$, with $\hat{\phi}_1,\hat{\phi}_2$ complex column arrays 1×2, then $\hat{\phi}_2 = i\sigma_2\hat{\phi}_1^*$. This implies that the column array 1×4 $\begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_2\end{pmatrix}$ transforms under $SO(4)$ as a Majorana spinor.

Applying newly the correspondence (39) to (42), we obtain

$$s_2 d^\mu s_1^*\nabla'[A'_{s1}]_\mu\phi'' = \Lambda d^\mu\nabla'[A'_{s1}]_\mu\phi''.$$

Inserting it in the action (38)

$$
\begin{aligned}
S'[A'_{s1},\phi''] &= \sum_{\mu\nu}(\nabla'[A'_{s1}]_\nu\phi'')^\dagger d^{*\nu}\Lambda^\dagger\Lambda\, d^\mu(\nabla'[A'_{s1}]_\mu\phi'') \\
&= \sum_{\mu\nu}(\nabla'[A'_{s1}]_\nu\phi'')^\dagger d^{*\nu} d^\mu(\nabla'[A'_{s1}]_\mu\phi'') \\
&= \sum_{\mu\nu}(d^\nu\nabla'[A'_{s1}]_\nu\phi'')^\dagger(d^\mu\nabla'[A'_{s1}]_\mu\phi'') \\
&= S[A'_{s1},\phi''].
\end{aligned}
\tag{43}
$$

The diagonal gauge field $A(x^a)$ compensates the action of $SU(2)\otimes SU(2)$ inside ∇'. Moreover we have just demonstrated that the field d^μ transforms under this group as a vielbein field in the Palatini formulation of General Relativity. This implies $A(x^a)$ is a gravitational spin-connection. Consequently, every purely imaginary quaternion defines a spin operator $\vec{S}$ via the correspondence $(i,j,k)\leftrightarrow 2i(S_1,S_2,S_3)$. In fact, each element in $U(1,\mathbf{H})$ is the exponential of a purely imaginary quaternion, in the same way as an element in $SU(2)$ is the exponential of $i\vec{\alpha}\cdot\vec{S}$ for some real vector $\vec{\alpha}$.

Note that a majorana spinor in an euclidean space can't distinguish if s_2 belongs to $SU(2)_{rot}$, $SU(2)_{boosts}$ or if it is a mixed combination. Only after the wick rotation it feels a difference, because the generator of $SU(2)_{boosts}$ moves from $i\sigma^i$ to σ^i, while $SU(2)_{rot}$ remains unchanged.

Someone can infer that, if ϕ transforms as a majorana spinor, our action has not the standard form. We don't care this now: what exposed is only a toy model. In another work (under review) we show explicitly how to get the correct Dirac action for these and all the others fields (both fermions and bosons).

To finish, we suppose that masses of other fields ($A(x^a,x^b)$ with $a\neq b$) are sufficiently large, so that the experimental physics of nowadays is unable to locate them. For the same reason, in the low energy approximation, they can be omitted from the action. Neglecting the "ultra-massive" fields, the scalar field action becomes a local action

$$S=\sum_{i=1}^{n}\sum_{\mu,\nu}\sqrt{|h|}h^{\mu\nu}\left(x_i\right)\left(\overset{G}{\nabla}_\mu\phi(x_i)\right)^*\left(\overset{G}{\nabla}_\nu\phi(x_i)\right) \tag{44}$$

where $\overset{G}{\nabla}$ are standard gravitational covariant derivatives.

5 The origin of spin

Consider the spin operator S_3

$$\hat{S}_3=\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{45}$$

and calculate the normalized eigenvectors and eigenstates.

$$|\uparrow\rangle=e^{i\phi}\begin{pmatrix}1\\ 0\end{pmatrix},\ \text{ with eigenvalue } \lambda_1=+\frac{1}{2}\ \text{ (in unit } \hbar=1) \tag{46}$$

$$|\downarrow\rangle=e^{i\phi}\begin{pmatrix}0\\ 1\end{pmatrix},\ \text{ with eigenvalue } \lambda_2=-\frac{1}{2} \tag{47}$$

where ϕ is an arbitrary phase. The eigenvectors completeness guarantees that the field $\hat{\phi}_1$, which appears in the precedent section, can be always decomposed in a sum of such eigenstates.

The projectors on a single eigenstate of S_3 are

$$\hat{\pi}^+=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \tag{48}$$

$$\hat{\pi}^-=\frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{49}$$

We see that $\hat{\pi}^{\pm}$ are idempotent, while $\hat{\pi}^{+}\hat{\pi}^{-} = 0$, as it should be. A rotation by an angle θ around the axe 1 is represented by the unitary matrix:

$$U_1(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{50}$$

where

$$U_1(\theta)\hat{\phi}_1 = (\widehat{s(U_1)\phi})_1 \qquad \hat{\phi}_1 = (\widehat{\phi})_1$$

for some quaternion $s(U)$ with $|s| = 1$. In the special case of a rotation by π:

$$U_1(\pi) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{51}$$

We suppose now that the system is in the eigenstate $|\uparrow\rangle$; following a rotation around the axis 1 the state will be:

$$|\uparrow\rangle_R = U_1(\theta)\,|\uparrow\rangle\,.$$

For $\theta = \pi$:

$$|\uparrow\rangle_R = U_1(\pi)\,|\uparrow\rangle = -i\,|\downarrow\rangle = e^{-i\pi/2}\,|\downarrow\rangle \to |\downarrow\rangle\,, \tag{52}$$

$$|\downarrow\rangle_R = U_1(\pi)\,|\downarrow\rangle = -i\,|\uparrow\rangle = e^{-i\pi/2}\,|\uparrow\rangle \to |\uparrow\rangle\,, \tag{53}$$

since the state is defined up to an inessential phase factor. We observe that a rotation by π around the axe 1 is equivalent to exchange $|\uparrow\rangle$ with $|\downarrow\rangle$, as we have just verified by (52) and (53).

Surely we can expand the matrix M as follows

$$M\left(x^a, x^b\right) = M'\left(x^a, x^b\right) + |s(x^a)|e^{r(x^a)}\delta^{ab} \tag{54}$$

with $M'\left(x^a, x^b\right) = 0$ for $a = b$.

The element $r(x^a) = arg[s(x^a)]$ is a purely imaginary quaternion: when it acts on ϕ, it determines uniquely the result of a spin measure, exchanging the states $|\uparrow\rangle$ - $|\downarrow\rangle$. This seems to suggest an identification between the arrangement field M and the observer who performs the measurement.

Indeed the operator M can simulate a measurement operation when it presents the form $M^{ab} = u^a w^b$:

$$\begin{aligned} M^{ab} &= u^a w^b \overset{continuous}{\longrightarrow} M(x,y) = \psi(x)\psi^*(y) \\ M^{ab}\varphi_b &= u^a(w^b\varphi_b) \overset{continuous}{\longrightarrow} \int dy\, M(x,y)\varphi(y) \\ &= \psi(x)\int dy\, \psi^*(y)\varphi(y) = \psi(x)(\psi,\varphi) \end{aligned}$$

$\psi(x)$ is any eigenstate, while (ψ, φ) denotes the scalar product between ψ and φ. We see that M projects φ along the eigenstate ψ, and in quantum mechanics a measurement is just a projection.

The latter argument gives also an indication about the spin nature. Consider the entries of M closest to the diagonal: they are the M^{ij+1} and M^{ij-1} which compose $\tilde{M}$. Moreover, they represent the probability amplitudes for the existence of connections between (numerically) consecutive vertices. In the limit $\Delta \to 0$, $\tilde{M}$ becomes ∂, which is proportional to $i\partial$, an operator which acts on a wave function $\psi(x)$ and returns the momentum p of the corresponding particle:

$$i\partial\psi(x) = p\psi(x).$$

In this way, the entries of $\tilde{M}$ represent both a momentum and a probability amplitude for connections between (numerically) consecutive vertices. In a certain sense, $\tilde{M}$ draws continuous paths and measures the momentums along these paths (figure 3).

If we describe a particle with a wave function ϕ, its spin is determined by diagonal components of M: in fact, $exp(r)$ acts on ϕ as a rotation in the tangent space. Consequentely, if r is applied to ϕ, it returns the spin of the associated particle.

The diagonal components of M represent also the probability amplitudes for a connection between a vertex and itself. Reasoning in analogy with the components of $\tilde{M}$, we associate at every such "pointwise" loop a circumference S^1: we interpret the spin as the rotational momentum due to the motion along these circumferences (figure 4).

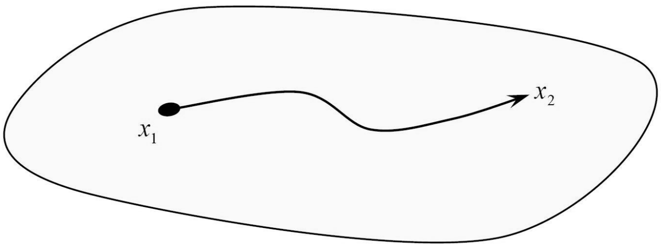


Figure 3: $\tilde{M}$ behaves as a derivative, that is proportional to a momentum operator. The non-empty entries of $\tilde{M}$ represent both a momentum and a probability amplitude for connections between (numerically) consecutive vertices. In a certain sense, $\tilde{M}$ draws continuous paths and measures the momentums along them.

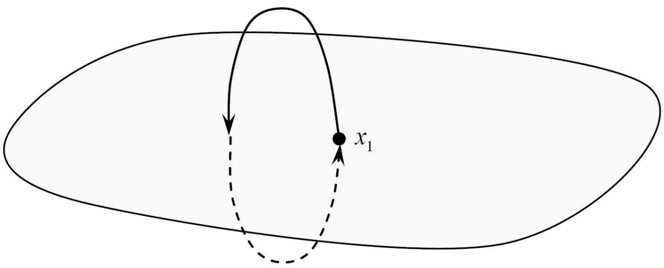


Figure 4: Each diagonal component of M represents the probability amplitude for a connection between a vertex and itself. The spin is a momentum along such pointwise loops.

It is remarkable that there exist two types of pointwise loops: the one in figure 4, where a particle assumes the same aspect after a complete rotation, and the one in figure 5, where a particle assumes the same aspect after two complete rotations.

The first case suggests a relationship with gauge fields of spin 1, the second with fermionic fields of spin 1/2.

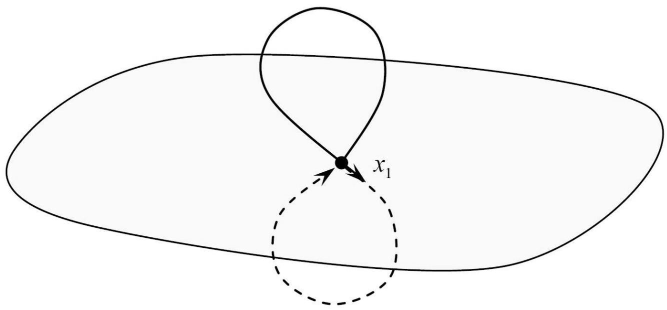


Figure 5: Pointwise loop associable with fermionic field.

6 Symmetry breaking

We imagine that the symmetry breaking of $U(n, \mathbf{H}) \otimes U(n, \mathbf{H})$ is not complete, but a residual symmetry remains for transformations in $U(1, \mathbf{H})^n \times U(m, \mathbf{H})^{n/m}$. Here m is an integer divisor of n.

In this case, it is possible to regroup the n points into n/m ensembles $\mathcal{U}^a$, with $a = 1, 2, \ldots, n/m$.

$$\mathcal{U}^a = \mathcal{U}^a(x_1^a, x_2^a, \ldots, x_m^a)$$

$$\varphi = (\varphi(x_i^a)) = \begin{pmatrix} \varphi(x_1^1) & \varphi(x_2^1) & \varphi(x_3^1) & \ldots & \varphi(x_m^1) \\ \varphi(x_1^2) & \varphi(x_2^2) & \varphi(x_3^2) & \ldots & \varphi(x_m^2) \\ \varphi(x_1^3) & \varphi(x_2^3) & \varphi(x_3^3) & \ldots & \varphi(x_m^3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi(x_1^{n/m}) & \varphi(x_2^{n/m}) & \varphi(x_3^{n/m}) & \ldots & \varphi(x_4^{n/m}) \end{pmatrix} \tag{55}$$

$$A = (A_{ij}^{ab}) = (A(x_i^a, x_j^b)).$$

Now the indices a, b of A act on the columns of φ, while the indices i, j act on the rows. The fields A_{ij}^{ab} with $a = b$ maintain null masses and so they continue to behave as gauge fields for $U(m, \mathbf{H})^{n/m}$. Every $U(m, \mathbf{H})$ term in $U(m, \mathbf{H})^{n/m}$ acts independently inside a single $\mathcal{U}^a$. So, if we consider the ensembles $\mathcal{U}^a$ as the real physical points, we can interpret $U(m, \mathbf{H})^{n/m}$ as a local $U(m, \mathbf{H})$. It's simple to verify:

$$\begin{aligned} h^{\mu\nu}(x_i^a) &= h^{\mu\nu}(x_j^a) \qquad \forall x_i, x_j \in \mathcal{U}^a \\ h(x^a) &\overset{!}{=} h(\mathcal{U}^a) = h^{\mu\nu}(x_i^a) \qquad \forall x_i^a \in \mathcal{U}^a \\ A(x_i^a, x_j^a) &= Tr\,\left[A(x^a)T^{(ij)}\right], \quad where \\ A(x^a) &= \sum_{ij} A(x_i^a, x_j^a)T^{(ij)}, \end{aligned} \tag{56}$$

with $T^{(ij)}$ generator of $U(m, \mathbf{H})$. Using these relations, in the next work we'll show how the terms $tr\,(MM^\dagger)$ and $tr\,(MM^\dagger MM^\dagger)$ generate respectively the Ricci scalar and the kinetic term for gauge fields. Extending M to grassmanian elements we have (up to a generalized $U(n, \mathbf{H})$ transformation)

$$M = \theta(\partial^\dagger + \psi^\dagger) + d^\mu(\partial_\mu + A_\mu)$$

$$M^\dagger = \theta^\dagger(\partial + \psi) + d^{*\mu}(\partial_\mu^\dagger + A_\mu^\dagger).$$

$\theta, \theta^\dagger$ are at the same time grassmanian coordinates and grassmanian equivalents of d, d^*. $\partial, \partial^\dagger$ are grassmanian derivatives and $\psi, \psi^\dagger$ grassmanian fields (ie fermions).

Our final action will be

$$S = tr\left(\frac{MM^\dagger}{16\pi G} - \frac{1}{4}MM^\dagger MM^\dagger\right)$$

This action resembles the action of a $\lambda\phi^4$ theory. Some preliminary results suggest that we can treat it through feynmann graphs by expanding $M \to \partial + \delta M$, apparently without renormalization problems.

We will see how the quartic term includes automatically the kinetic terms for gauge fields of $SO(4) \otimes SU(3) \otimes SU(2) \otimes U(1)$ and the dirac action for exactly three fermionic families.

7 Second quantization and black hole entropy

It is remarkable that in our model the gauge fields and the gravitational fields have different origins, although they are both born from M. The gravitational field in fact appears as a multiplicative factor for moving from M to the covariant derivative ∇'. The gauge fields are instead some additive elements in ∇'. This could be the reason for which the gravitational field seems non quantizable in the standard way. On the other side, quantizing the gauge fields is equivalent to quantize a partial piece of M in a flat space. But a similar equivalence does not exist for the gravitational field. In our framework this doesn't create problems, since we will quantize M directly, rather than gravitational and gauge fields.

What does it mean "to quantize" M? It's true that a matrice M is a quantum object from its birth, as they are quantum objects the wave functions which describe particles.

However, we will impose commutation relations on M, in the same way we impose commutation relations on the wave functions. This is the so called "second quantization".

The wave functions, which first had described the probability amplitude to find a particle, then have become operators which create or annihilate particle. Similarly, M describes first the probability amplitude for the existence of connections between vertices. After the second quantization it will become an operator which creates or annihilate connections. In particular, the operator $M(x^a, x^b)$ creates a connection between the vertices x^a and x^b.

M corresponds to $D^\mu \nabla'_\mu$ (by invariance respect $U(n, \mathbf{H})$): so it contains the various fields A_μ and $h^{\mu\nu}$. If we second quantize M, then, indirectly, we quantize the other fields, including the gravitational field.

To quantize M we put $[M^{ij}, M^{kl\dagger}] = \delta^{ik}\delta^{jl}$. Here the symbol $\dagger$ indicates the adjoint operator respect only the scalar product between states in the Fock space. The condition $[M^{ij}, M^{kl\dagger}] = \delta^{ik}\delta^{jl}$ means that every entry M^{ij} expands in a sum of 4 operators

$$M^{ij} = a + i(b_1 + b_2 + b_3) \qquad b_1^\dagger = b_1, \;\; b_2^\dagger = b_2, \;\; b_3^\dagger = b_3$$

The b's realize the $SU(2)$ algebra implicit in the imaginary part of quaternions.

$$[b_1, b_2] = b_3; \qquad [b_2, b_3] = b_1; \qquad [b_3, b_1] = b_2$$

The operators $a^\dagger$ and $b^\dagger = b_1 + ib_2$ create an edge which connects the vertex i with the vertex j. The number operator is

$$N^{ij} = M^{ij\dagger} M^{ij} = a^\dagger a + |\vec{b}|^2 \qquad \text{no sum on } \; ij$$

$a^\dagger a$ has eigenvalues $q \in \mathbf{N}$ with multiplicity 1. Moreover the eigenvalues of $|\vec{b}|^2$ are in the form $j(j+1)$ for $j \in \mathbf{N}/2$, with multiplicity $(2j+1)$. How about $N > 1$? We can consider a surface immersed into the graph. Its area is Δ^2 times the number of edges which pass through it. If we admit the possibility for the creation of many superimposed edges, we can interpret this superimposition as a "super-edge" which carries an area equal to $N\Delta^2$.

Remark 13 *Regarding diagonal components, we suggest a slightly different interpretation:* $a^\dagger$ *could create loops, while* $b^\dagger$ *could create perturbations which travel through the loops (ie particles with spin* j*). This suggest a duality between a loop on vertex* v_i *and a closed string (as intended in* String Theory*) situated approximately on the same vertex. Note that the two interpretations can be accommodated* *if we consider quanta of area as non-local perturbations.*

The only Black Hole information detectable from the exterior, is the information coded in the Horizon. So, the only distinguishable states of a Black Hole are distinguishable states of its horizon. For the Black Hole horizon we consider all the edges which pass through it, oriented only from the interior to the exterior.

If the horizon is crossed only by edges with $N = q + j(j+1)$ and $a^\dagger a = q$, the number of its distinguishable states is

$$num_S = (2j+1)^{A/(q+j(j+1))} .$$

We suppose now a generic partition with $A = \sum_{j,q} A_{j,q}$, where an area $A_{j,q}$ is crossed only by edges with $N = q + j(j+1)$ and $a^\dagger a = q$. The number of distinguishable states becomes

$$num_S = \sum_{\{A_{j,q}\}} \prod_{j,q} (2j+1)^{A_{j,q}/((q+j(j+1))\Delta^2)}$$

where the sum is over all the possible partitions of A. The "classical" contribution comes from $j = 0$ and gives $num_S = 1$ (We call it "classical" because it is the only one with $N = 1$). This implies no entropy and is related to the fact that $tr\, M_H^\dagger M_H \sim \int_H \sqrt{h_H} R(h_H) = 2\pi \chi_H$, where M_H is the restriction of M to the edges which cross the horizon, h_H is the induced metric on the horizon and χ is the Euler characteristic.

The dominant contribution comes from $q = 0$ and $j = 1/2$, which gives

$$num_S = 2^{4A_{1/2,0}/(3\Delta^2)}$$

So we can define entropy as

$$S = k_B \, log\, 2^{4A/3\Delta^2} = \frac{4\, log\, 2\, k_B A}{3\Delta^2}.$$

Our approach gives thus a proposal for the explanation of area law. Indeed our entropy formula corresponds to the one given by Bekenstein and Hawking if $3\Delta^2 = 16G\, log\, 2$.

What is our interpretation of black hole radiation? The proximity between vertices is probabilistic: we can have a high probability of receiving two vertices as "neighbors", but never a certainty. We look at a large number of vertices for a long time: some vertex, which first seems to be adjacent to some other, suddenly can appears far away. For this reason, some internal vertices in a Black Hole may happen to be found outside, so that the Black Hole slowly evaporates.

We can consider also the contribution from ($q = j = 0$). If it exists, clearly it is the dominant one. Indeed, an horizon means absence of connections between the exterior and the interior. For an external observer, the universe finishes with the horizon. In fact, respect the coordinate system of a statical observer infinitely distant from the horizon, every object, falling in the black hole, sits on the horizon for an infinite time. In relation to the proper time of the statical far away observer, the object never surpasses the horizon. If nothing surpasses the horizon, this means that the Hawking radiation comes from the deposit of all the objects fallen in the black hole, ie from the horizon. This resolves the information paradox proposed by Hawking.

Someone can infer that absence of connections is only illusory, because the horizon singularity is of the type called "apparent": it doesn't exist in several coordinate systems, as the system comoving with a free falling object.

We reply that it's true, because also the absence of connections depends strictly from the state on which the number operator acts. Every state can be associated to a particular coordinates system and, if we change coordinate system, we have to change the state. In this way, the connections can exist for an observer and not exist for some others.

It's the same which happens for the particles. The same particle can exist in a coordinate system and not exists in an another system (see Unruh effect). This is because the same number operator acts on different states.

Calculate now num_S for $q = 0, j \to 0$. It is

$$
\begin{aligned}
num_S &= \lim_{j \to 0} (2j+1)^{A/(j(j+1)\Delta^2)} \\
&= \lim_{j \to 0} (1+2j)^{A/(j\Delta^2)} \\
&= \lim_{j \to 0} (1+2j)^{2A/(2j\Delta^2)} \\
&= \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{2Ax/\Delta^2} \\
&= e^{2A/\Delta^2}
\end{aligned}
\tag{57}
$$

The entropy becomes

$$S = k_B \, log \, e^{2A/\Delta^2} = \frac{2k_B A}{\Delta^2}$$

This corresponds to the Bekenstein-Hawking result for $\Delta^2 = 8G$.

8 Conclusion

In this paper we have abandoned the preconceived existence of an order in the space-time structure, taking a probabilistic approach also to its topology and its homology.

This framework gives new suggestions about the origin of space-time metric and particles spin. At the same time it hints a possible emersion of all fields from an unique entity, ie the arrangement matrix, after the imposition of an order.

Unfortunately, there isn't space here to post an explicit calculation of terms $tr\,(M^\dagger M)$ and $tr\,(M^\dagger M M^\dagger M)$. We have already said that they generate the Ricci scalar, the kinetic terms for gauge fields and the Dirac actions for exactly three fermionic families.

In a next future we'll show how several phenomena can find a possible explanation inside this paradigm, as we have seen earlier for black hole entropy. These deal with the galaxy rotation curves, the inflation, the quantum entanglement, the values of matrices CKM and PMNS and the value of Newton constant G.

Here we have given a simple example by using a one-component field. Nevertheless, a potential for M causes a symmetry breaking which gives mass to gauge fields without need of Higgs mechanism. In the end, the one-component field action results unnecessary.

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