Nonlinear arctan-electrodynamics and charged black holes

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Abstract

We investigate a model of nonlinear electrodynamics with the Lagrangian density $\mathcal{L} = -(1/\beta)\arctan(\beta F_{\mu\nu}F^{\mu\nu}/4)$. The phenomenon of vacuum birefringence is studied. The model of electromagnetic fields coupled with the gravitation field is considered. The black hole solution is obtained possessing the asymptotic Reissner-Nordström solution. The corrections to Reissner-Nordström solution are found.

1 Introduction

Some models of nonlinear electrodynamics (NLE) possess finite self-energy of charged particles and do not have singularity of the electric field at the classical level [1], [2], [3], [4], [5]. Models of NLE can be considered as effective models taking into account quantum corrections. For example, nonlinear Heisenberg-Euler Lagrangian [6] takes into consideration one-loop quantum corrections to Maxwell's electrodynamics. The Born-Infeld model [1] represents NLE that does not have singularity of the electric field and can be used for strong electromagnetic fields. NLE was introduced to eliminate infinite electric fields and to generalize classical electrodynamics. Maxwell's theory may be considered as an approximation to NLE for weak fields. If the electromagnetic field strength is huge the self-interaction of photons should be taken into account and classical electrodynamics has to be modified [7].

The problems of the initial Big Bang singularity and early time inflation in cosmological models can be solved with the help of NLE. Cosmological models explore the classical Einstein equation and the classical NLE can be considered in the theory of gravity. The electromagnetic and gravitational fields in the early epoch of the universe are very strong, and therefore effects of NLE are important. Thus, high energies in early universe may produce

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nonlinear electromagnetic effects. Models of NLE are of interest in general relativity (GR) because they take into account processes of vacuum polarizations and can give an impact on the evolution of the early universe near the Planck era. In some inflationary models nonlinear electromagnetic fields can also mimic the dark energy. In early universe the magnetic fields can be greater than 10^{15} G and, therefore, nonlinear electromagnetic fields influence on spacetime. NLE can be considered as an effective electrodynamics in late epochs and such phenomenological approach [8] can mimic a material medium where electric permittivity and magnetic permeability depend on the field strength [9]. Thus, NLE can be used to create inflation in the early universe [10], [11]. To have accelerated expansion of the universe some models of NLE were considered [12]-[14]. The effects of coupling NLE to gravity can give negative pressures that result in the accelerated expansion of the universe [13]-[15]. In the Λ-Cold Dark Matter (ΛCDM) model the cosmological constant Λ drives the present cosmic acceleration and there is a similarity between the trace anomaly of NLE models and the cosmological constant [16]. The Einstein-Born-Infeld equations take into account nonlinear effects in strong electromagnetic and gravitational fields and were studied in [17]. Here we propose the NLE model coupled to gravitational field, and it depends on a dimensional constant β. We investigate Einstein-NLE model that influences on the universe evolution and does not have singularities. The energy-momentum tensor trace $\mathcal{T}$ may contribute to the cosmological constant [18] and, as a result, in curved spacetime nonperturbative effects of self-interacting quantum fields can mimic the cosmological constant. Classical gravity theory may be considered as an effective gravity theory at low energy and the Einstein-Hilbert classical action of GR have to possess the energy-momentum tensor trace anomaly [19]. In NLE the violation of the scale invariance, due to a dimensional constant β, can result in the negative pressure.

The charged black hole can be described by Reissner-Nordström (RN) solution and it may be the final state of charged stars. We investigate new NLE model and study the Einstein-NLE solution that gives some corrections to the RN black hole solution. The static and spherically symmetric charged black holes with a source of NLE are considered. The solutions obtained give some modification to the RN geometry. In the weak-field limit the black hole geometry, within NLE, is converted into Einstein-Maxwell geometry. The static and spherically symmetric spacetime of black hole with the Heisenberg-Euler effective Lagrangian of QED as a source was studied in [20].

The paper is organized as follows. In Sec. 2 we formulate a new model of NLE with the dimensional parameter β. The energy-momentum tensor and its non-zero trace were obtained. We found the electric permittivity, ε, and the magnetic permittivity, μ, depending on the electromagnetic fields. It was demonstrated that the scale invariance and dual invariance are broken in the model proposed. The effect of vacuum birefringence was studied in Sec. 3. It was demonstrated that the phenomenon of the vacuum birefringence takes place in the order of $\beta^2 B_0^4$. In Sec. 4 the model of NLE coupled with the gravitation field was studied. We found the black hole solution possessing the asymptotic Reissner-Nordström solution. The corrections to Reissner-Nordström solution are obtained. Section 5 is devoted to the conclusion.

We use the units with $c = \hbar = 1$.

2 Nonlinear arctan-electrodynamics

Let us consider NLE with the Lagrangian density

$$\mathcal{L} = -\frac{1}{\beta}\arctan(\beta\mathcal{F}), \tag{1}$$

where β is dimensional parameter with the dimension of (length)^4 and $\beta\mathcal{F}$ is dimensionless, $F_{\mu\nu}$ is the field strength and $\mathcal{F} = (1/4)F_{\mu\nu}F^{\mu\nu}$. The fundamental length of the model is $L = \beta^{1/4}$ and it goes probably from quantum gravity and it is connected with the maximum of the electric field strength. If $L \to 0$ ($\beta \to 0$) the Lagrangian density (1) converts into Maxwell's Lagrangian density $\mathcal{L} \to -\mathcal{F}$. We obtain the symmetric energy-momentum tensor by varying the action (1) with respect to the metric [21]

$$T^{\mu\nu} = H^{\mu\lambda}F^{\nu}{}_{\lambda} - g^{\mu\nu}\mathcal{L}, \tag{2}$$

where

$$H^{\mu\lambda} = \frac{\partial\mathcal{L}}{\partial F_{\mu\lambda}} = \frac{\partial\mathcal{L}}{\partial\mathcal{F}}F^{\mu\lambda} = -\frac{F^{\mu\lambda}}{1+(\beta\mathcal{F})^2}. \tag{3}$$

From Eqs. (2),(3) one finds the symmetric energy-momentum tensor

$$T^{\mu\nu} = -\frac{F^{\mu\lambda}F^{\nu}{}_{\lambda}}{1+(\beta\mathcal{F})^2} - g^{\mu\nu}\mathcal{L}. \tag{4}$$

The symmetric energy-momentum tensor (4) possesses nonzero trace

$$\mathcal{T} \equiv T_{\mu}{}^{\mu} = \frac{4}{\beta}\arctan(\beta\mathcal{F}) - \frac{4\mathcal{F}}{1+(\beta\mathcal{F})^2}. \tag{5}$$

When $\beta \to 0$ one comes to classical electrodynamics, and trace (5) approaches to zero, $\mathcal{T} \to 0$. As the energy-momentum tensor trace is not zero, due to the dimensional parameter β, $\mathcal{T} \neq 0$, the scale invariance is violated. One can obtain the dilatation current $D_\mu = x_\nu T_{\mu}{}^{\nu}$ with the divergence $\partial_\mu D^\mu = \mathcal{T}$. The electric displacement field is defined by the expression $\mathbf{D} = \partial\mathcal{L}/\partial\mathbf{E}$. From Eq. (1) we find the electric displacement field

$$\mathbf{D} = \frac{\mathbf{E}}{1+(\beta\mathcal{F})^2}. \tag{6}$$

Using the definition $\mathbf{D} = \varepsilon\mathbf{E}$, one obtains the electric permittivity

$$\varepsilon = \frac{1}{1+(\beta\mathcal{F})^2}. \tag{7}$$

From the relation $\mathbf{H} = -\partial\mathcal{L}/\partial\mathbf{B}$ we find the magnetic field

$$\mathbf{H} = \frac{\mathbf{B}}{1+(\beta\mathcal{F})^2}, \tag{8}$$

and from the definition $\mathbf{B} = \mu\mathbf{H}$ the magnetic permeability is given by $\mu = 1/\varepsilon$. It follows from Eqs. (6),(8) that $\mathbf{D}\cdot\mathbf{H} = \varepsilon^2\mathbf{E}\cdot\mathbf{B}$ and $\mathbf{D}\cdot\mathbf{H} \neq \mathbf{E}\cdot\mathbf{B}$. Therefore, according to the criterion [22], the dual symmetry is broken in the model proposed. From the Lagrangian density (1), with the help of Eqs. (6),(8), the field equations can be represented in the form of the Maxwell equations

$$\nabla\cdot\mathbf{D} = 0, \qquad \frac{\partial\mathbf{D}}{\partial t} - \nabla\times\mathbf{H} = 0. \tag{9}$$

Using the Bianchi identity $\partial_\mu \widetilde{F}_{\mu\nu} = 0$, where $\widetilde{F}_{\mu\nu}$ is a dual tensor, we obtain the second pair of Maxwell's equations

$$\nabla\cdot\mathbf{B} = 0, \qquad \frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0. \tag{10}$$

The electric permittivity ε and the magnetic permittivity μ depend on the electromagnetic fields $\mathbf{E}$, $\mathbf{B}$, and therefore, Eqs. (6), (8), (9), (10) are the nonlinear Maxwell equations.

3 Vacuum birefringence

There is the effect of vacuum birefringence in QED due to one-loop quantum corrections described by the Heisenberg-Euler effective Lagrangian [23], [24]. In classical Maxwell's electrodynamics and BI electrodynamics the phenomenon of vacuum birefringence is absent but in generalized BI electrodynamics with two parameters [25] the effect of birefringence takes place. Now we investigate the possible effect of vacuum birefringence in the NLE described by Lagrangian density (1). Let us consider the superposition of the external constant and uniform magnetic induction field $\mathbf{B}_0 = B_0(1,0,0)$ and the plane electromagnetic wave $(\mathbf{e},\mathbf{b})$,

$$\mathbf{e} = \mathbf{e}_0 \exp\left[-i\left(\omega t - kz\right)\right], \quad \mathbf{b} = \mathbf{b}_0 \exp\left[-i\left(\omega t - kz\right)\right] \tag{11}$$

which propagates in the z-direction. Let the magnetic induction field $\mathbf{B}$ is strong and the total electromagnetic fields are $\mathbf{E} = \mathbf{e}$, $\mathbf{B} = \mathbf{b} + \mathbf{B}_0$. The electromagnetic wave fields are weak compared to the external magnetic induction field, $e_0, b_0 \ll B_0$. Then the Lagrangian density (1) is given by

$$\mathcal{L} = -\frac{1}{\beta}\arctan\left[\frac{\beta}{2}\left(\mathbf{B}_0 + \mathbf{b}\right)^2 - \frac{\beta}{2}\mathbf{e}^2\right]. \tag{12}$$

Defining the electric displacement field [26] $d_i = \partial\mathcal{L}/\partial e_i$ and the magnetic field $h_i = -\partial\mathcal{L}/\partial b_i$, one can linearize equations with respect to the wave fields $\mathbf{e}$ and $\mathbf{b}$. We suppose that $\beta^2 B_0^4 \ll 1$ and the electric permittivity tensor ε_{ij} and the inverse magnetic permeability tensor $(\mu^{-1})_{ij}$, up to $\mathcal{O}(e_0^2)$, $\mathcal{O}(b_0^2)$, become

$$\varepsilon_{ij} = \frac{\delta_{ij}}{1+(\beta B_0^2/2)^2}, \qquad (\mu^{-1})_{ij} = \frac{\delta_{ij}}{1+(\beta B_0^2/2)^2} - \frac{\beta^2 B_0^2 B_{0i} B_{0j}}{\left[1+(\beta B_0^2/2)^2\right]^2}, \tag{13}$$

and we have the relations $d_i = \varepsilon_{ij} e_j$, $b_i = \mu_{ij} h_j$. From Eq. (13) one obtains the elements of the electric permittivity and magnetic permeability tensors

$$\varepsilon_{11} = \varepsilon_{22} = \frac{1}{1+(\beta B_0^2/2)^2},$$

$$\mu_{11} = \frac{\left[1+(\beta B_0^2/2)^2\right]^2}{1-3(\beta B_0^2/2)^2}, \qquad \mu_{22} = 1+(\beta B_0^2/2)^2. \tag{14}$$

If the polarization of the electromagnetic wave is parallel to the external magnetic induction field, $\mathbf{e} = e_0(1, 0, 0)$, one obtains from the Maxwell equations the relation $\mu_{22}\varepsilon_{11}\omega^2 = k^2$. Therefore the index of refraction is given by

$$n_\| = \sqrt{\mu_{22}\varepsilon_{11}} = 1. \tag{15}$$

But when the polarization of the electromagnetic wave is perpendicular to the external induction magnetic field, $\mathbf{e} = e_0(0, 1, 0)$, we have the equality $\mu_{11}\varepsilon_{22}\omega^2 = k^2$, and the index of refraction becomes

$$n_\perp = \sqrt{\mu_{11}\varepsilon_{22}} = \sqrt{\frac{1 + (\beta B_0^2/2)^2}{1 - 3(\beta B_0^2/2)^2}} \approx 1 + \frac{\beta^2 B_0^4}{2}. \tag{16}$$

As a result, the phase velocities depend on the polarization of the electromagnetic wave, and the effect of vacuum birefringence takes place. If the polarization of the electromagnetic wave is parallel to the external magnetic field, $\mathbf{e}_0 \| \mathbf{B}_0$, the speed of electromagnetic wave is $v_\| = 1/n_\| = c = 1$. But when the polarization of the electromagnetic wave is perpendicular to the external magnetic field, $\mathbf{e} \perp \mathbf{B}_0$, the speed of the electromagnetic wave is given by $v_\perp = 1/n_\perp < c$.

In accordance with the Cotton-Mouton (CM) effect [27] the difference in the indices of refraction is given by

$$\triangle n_{CM} = n_\| - n_\perp = k_{CM} B_0^2, \tag{17}$$

and it is proportional to B_0^2. Because $\beta^2 B_0^4$ is much less than $\beta B_0^2 \ll 1$ the effect of vacuum birefringence is very weak, $n_\perp \approx 1$, and we can neglect the vacuum birefringence. The Cotton-Mouton coefficient k_{CM} obtained in the BMV [28], and PVLAS [29] experiments are bounded by

$$k_{CM} = (5.1 \pm 6.2) \times 10^{-21}\mathrm{T}^{-2} \qquad \text{(BMV)},$$

$$k_{CM} = (4 \pm 20) \times 10^{-23}\mathrm{T}^{-2} \qquad \text{(PVLAS)}. \tag{18}$$

The value of k_{CM} calculated within QED and taking into account loop corrections is smaller than the experimental data (18) [28], $k_{CM}^{QED} \approx 4.0 \times 10^{-24}\mathrm{T}^{-2}$.

4 Nonlinear electromagnetic fields and black holes

We consider the GR action coupled with the nonlinear electromagnetic field described by the Lagrangian density (1)

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R+\mathcal{L}\right], \tag{19}$$

where R is the Ricci scalar and $\kappa^{-1}=M_{Pl}$, M_{Pl} is the reduced Planck mass. The Einstein and electromagnetic equations, followed from Eq. (19), are

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa^2T_{\mu\nu}, \tag{20}$$

$$\partial_\mu\left(\frac{\sqrt{-g}F^{\mu\nu}}{1+(\beta\mathcal{F})^2}\right)=0. \tag{21}$$

Our goal is to obtain the static charged black hole solutions to Eqs. (20),(21). The spherically symmetric line element in $(3+1)$-dimensional spacetime is given by

$$ds^2=-f(r)dt^2+\frac{1}{f(r)}dr^2+r^2(d\vartheta^2+\sin^2\vartheta d\phi^2). \tag{22}$$

We imply that the vector-potential possesses non-vanishing component $A_0=h(r)$ and $\mathcal{F}=-[h'(r)]^2/2$ (the prime means the derivative with respect to the argument). Then the electric field is given by $E=h'(r)$. As a result, Eq. (21) becomes

$$\partial_r\left(\frac{4r^2h'(r)}{4+\beta^2[h'(r)]^4}\right)=0. \tag{23}$$

Eq. (23) can be integrated and we find

$$4r^2h'(r)=Q\left(4+\beta^2[h'(r)]^4\right), \tag{24}$$

where Q is the constant of integration. It is convenient to introduce new dimensionless variables

$$y=\frac{4r^2}{Q\sqrt{\beta}},\qquad x=\sqrt{\beta}h'(r). \tag{25}$$

Then from Eq. (24) we obtain the algebraic equation

$$x^4-xy+4=0. \tag{26}$$

One can find with the help of Cardano's formulas the analytic solutions to Eq. (26), $x(y)$. The function $y(x)$ has a minimum at $x = (4/3)^{1/4}$ ($E = h'(r) = [4/(3\beta^2)]^{1/4}$) and $r_{min} = (4/3)^{3/8}\beta^{1/4}\sqrt{Q}$. Therefore, Eq. (26) possesses the real solutions if $r > r_{min}$. As a result, there is no singularity of the electric field E. It should be noted that there are two branches (solutions) of the function $E(r)$ with physical values that decrease with r and with nonphysical values which increase with r. Therefore, we imply that $E \leq [4/(3\beta^2)]^{1/4}$ ($x \leq (4/3)^{1/4} \approx 1.075$). As a result, the maximum electric field is $E_{max} = [4/(3\beta^2)]^{1/4}) \approx 1.075/\sqrt{\beta}$ and it decreases with r. The plot of the function $y(x)$ is presented in Fig. 1. Now we can find from Eqs. (25),(26) the integral

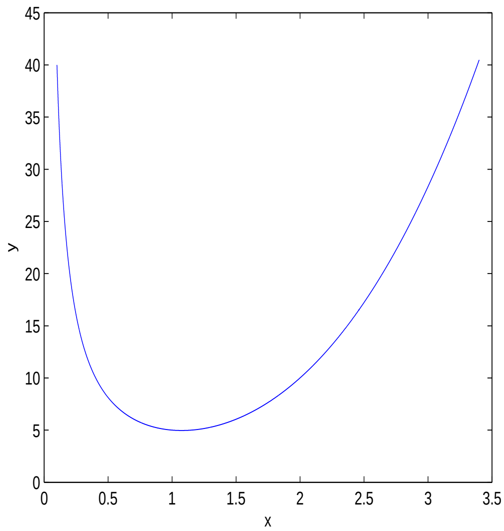

Figure 1: The function y versus x.

$$h(r) = \int h'(r)dr = \frac{\sqrt{Q}}{4\beta^{1/4}}\int \frac{(3x^4-4)dx}{\sqrt{x(x^4+4)}}, \tag{27}$$

where x is the solution to Eq. (26) and x, as a function of r, is given by Eqs. (25),(26). The integral (27) can be calculated but it is complicated. Instead of the exact value we use the simple approximate value. At $r \to \infty$ and physical value $x \to 0$, one obtains the Taylor series

$$\frac{(3x^4-4)}{\sqrt{x(x^4+4)}} = -\frac{2}{\sqrt{x}} + \frac{7x^{7/2}}{4} + \mathcal{O}(x^{11/2}). \tag{28}$$

As a result, according to Eq. (27), (28) at $r \to 0$, we have

$$A_0(r) = h(r) = \frac{\sqrt{Q}}{4\beta^{1/4}}\left(-4\sqrt{x} + \frac{7}{18}x^{9/2} + \mathcal{O}(x^{13/2})\right). \qquad (29)$$

From Eq. (24) for the leading term at $r \to \infty$, we obtain the expression for the electric field as in Maxwell's electrodynamics, $E = h'(r) \to Q/r^2$. Thus, the integration constant Q means the charge. From Eq. (29) taking into consideration only the first term, one finds, using $x = \sqrt{\beta}Q/r^2$, the ordinary potential $A_0 = -Q/r$. Next terms in Eq. (29) give some corrections to Coulomb's law.

4.1 Asymptotic Reissner-Nordström black holes

The function $f(r)$ in Eq. (22) can be obtained by the relation [30]

$$f(r) = 1 + \frac{k_1}{r} + \frac{k_2}{r^2} + \frac{1}{r^2}\int dr\left[\int r^2 R(r)dr\right], \qquad (30)$$

where k_1, k_2 are constants of integration. One can find the Ricci scalar by the relation

$$R = -\kappa^2\mathcal{T}, \qquad \mathcal{T} = g^{\mu\nu}T_{\mu\nu}, \qquad (31)$$

where the trace of the energy-momentum tensor is given by Eq. (5). From Eqs. (5),(31) and $\mathcal{F} = -(1/2)(h'(r))^2$, one obtains the Ricci scalar

$$R = \kappa^2\left[\frac{4}{\beta}\arctan\left(\beta[h'(r)]^2/2\right) - \frac{2[h'(r)]^2}{1+\beta^2[h'(r)]^4/4}\right], \qquad (32)$$

where $h'(r)$ is the solution to Eq. (24). We find the approximate value for R with the help of Taylor series at $r \to \infty$ ($E = h'(r) << 1$).

$$R = \kappa^2\left(\frac{\beta^2[h'(r)]^6}{3} + \mathcal{O}([h'(r)]^{10})\right) \approx \kappa^2\left(\frac{\beta^2Q^6}{3r^{12}} + \mathcal{O}(r^{-20})\right). \qquad (33)$$

Replacing Eq. (33) into Eq. (30) one obtains after the integration the asymptotic value

$$f(r) = 1 + \frac{k_1}{r} + \frac{k_2}{r^2} + \frac{\kappa^2\beta^2Q^6}{216r^{10}} + \mathcal{O}(r^{-18}). \qquad (34)$$

The last terms in Eq. (34) give corrections to the Reissner-Nordström solution. We may identify the constant $k_2 = G^2Q^2$ and $k_1 = -2GM$, where M

is the mass of the black hole, and G is the gravitation constant. As a result, the metric function at $r \to \infty$ is given by

$$f(r) = 1 - \frac{2GM}{r} + \frac{G^2Q^2}{r^2} + \frac{\kappa^2\beta^2Q^6}{216r^{10}} + \mathcal{O}(r^{-18}). \tag{35}$$

It follows from Eq. (35) that asymptotically spacetime becomes flat (the Minkowski spacetime). If the parameter β is zero, $\beta = 0$, we come to linear Maxwell's electrodynamics and the solution (35) becomes the Reissner-Nordström solution. The asymptotic Reissner-Nordström black hole solution for another model of NLE was studied in [31]. It should be mentioned that corrections to the Reissner-Nordström solution change the expressions for the horizons, the event horizon r_+, and the Cauchy horizon r_-.

5 Conclusion

We have proposed a new model of NLE with a dimensional parameter β and investigated the effect of vacuum birefringence. In the leading order of βB_0^2 (B_0 is the external magnetic induction field) the phenomenon of birefringence disappears. The effect of vacuum birefringence is in the order of $\beta^2 B_0^4$. It is demonstrated that the scale invariance is violated in this model because the trace of the energy-momentum tensor is not zero. The dual symmetry of the electromagnetic fields in this model of NLE is also broken. We have studied NLE coupled with the gravitation field and the static spherically symmetric solutions were obtained which describe the charged black holes. It was shown that the electric field does not have singularity of the charged objects. We demonstrate that the black hole solution found has the asymptotic Reissner-Nordström solution. In the case of the Maxwell electrodynamics, ($\beta \to 0$), and we come to the Reissner-Nordström solution. The model of Maxwell's electromagnetic fields coupled with gravitation fields non-linearly was investigated in [32]. One can also generalize the study of NLE proposed introducing non-minimal coupling with the gravity. It is possible to investigate the instabilities of RN black holes and anti-evaporation effects in the current model similar to the study of these effects in the Maxwell-$F(R)$ theory [33].

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