Non-Abelian Vortices in Supersymmetric Gauge Field Theory via Direct Methods

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Abstract

Vortices in supersymmetric gauge field theory are important constructs in a basic conceptual phenomenon commonly referred to as the dual Meissner effect which is responsible for color confinement. Based on a direct minimization approach, we present a series of sharp existence and uniqueness theorems for the solutions of some non-Abelian vortex equations governing color-charged multiply distributed flux tubes, which provide an essential mechanism for linear confinement. Over a doubly periodic domain, existence results are obtained under explicitly stated necessary and sufficient conditions that relate the size of the domain, the vortex numbers, and the underlying physical coupling parameters of the models. Over the full plane, existence results are valid for arbitrary vortex numbers and coupling parameters. In all cases, solutions are unique.

1 Introduction

A fundamental puzzle in physics, known as the quark confinement, is that quarks, which make up elementary particles such as mesons and baryons, cannot be observed in isolation. A well accepted confinement mechanism, known as the linear confinement model, states that, when one tries to separate a pair of quarks, such as a quark and

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an anti-quark constituting a meson, the energy consumed would grow linearly with respect to the the separation distance between the quarks so that it would require an infinite amount of energy in order to split the pair. The quark and anti-quark may be regarded as a pair of source and sink of color-charged force fields. The source and sink interact through color-charged fluxes which are screened in the bulk of space but form thin tubes in the form of color-charged vortex-lines so that the strength of the force remains constant over arbitrary distance, resulting in a linear dependence relation for the potential energy with regard to the separation distance. Such a situation is similar to that of a magnetic monopole and anti-monopole pair immersed in a type-II superconductor. The magnetic fluxes mediating the interacting monopoles are not governed by the Maxwell equations, which would otherwise give rise to an inverse-square-power law type of decay of the forces and lead to non-confinement, but rather by the Ginzburg–Landau equations, which produce thin vortex-lines, known as the Abrikosov vortices or the Nielsen–Olesen strings. The repulsion of the magnetic field in the bulk region of the superconductor is due to the Meissner effect and the partial magnetic penetration of the superconductor in the form of vortex-lines is a consequence of type-II superconductivity. Thus, one may conclude that a monopole and anti-monopole immersed in a type-II superconductor would be linearly confined.

Inspired by the above-described monopole confinement in a type-II superconductor, Mandelstam [36, 37], Nambu [39], and 't Hooft [54, 56] proposed in the 1970s that the ground state of quantum-chromodynamics (QCD) is a condensate of chromomagnetic (color-charged) monopoles, causing the chromoelectric fluxes between quarks to be squeezed into narrowly formed tubes or vortex-lines, similar to the electron condensation in the bulk of a superconductor, in the form of the Cooper pairs, resulting in the formation of magnetic flux-tubes or vortex-lines which mediate the interaction between monopoles, following a non-Abelian version of the Meissner effect, called the 'dual Meissner effect' [9, 20, 33, 52], which is responsible for the screening of chromoelectric fluxes [49, 50].

Interestingly, although finite-mass monopoles of the 't Hooft [53] and Polyakov [43] type in non-Abelian gauge theory have long been demonstrated to exist [2, 18], the magnetic fluxes are Coulomb-like which spread out radially and will not give rise to confinement. In 1994, Seiberg and Witten [45] came up with an $\mathcal{N} = 2$ supersymmetric gauge-field-theoretical formalism of non-Abelian monopole condensation and studied its implications to color confinement. Later, in 1997, Hanany, Strassler, and Zaffaroni [25] showed that the flux tubes or strings produced in the Seiberg–Witten formalism are of the Abrikosov–Nielsen–Olesen (Abelian) type [1, 23, 40, 57, 58] which are not exactly what anticipated in QCD [25, 50, 49]. In 2002, Marshakov and Yung [38] constructed Abelian vortices in a softly broken $\mathcal{N} = 2$ supersymmetric QCD (SQCD) model and showed that, although confinement is due to Abelian flux tubes, the multiplicity of the meson spectrum is the same as expected in a theory with non-Abelian confinement. In 2003, Hanany and Tong [26] derived a broad class of non-Abelian vortex equations and computed the dimensions of the associated moduli spaces, and Auzzi, Bolognesi, Evslin, Konishi, and Yung [5] analyzed non-Abelian vortices and confinement in $\mathcal{N} = 2$ SQCD in the context of non-Abelian superconductors. Since then, the subject of non-Abelian vortices, monopole condensation, and confinement has been extensively developed [12, 13, 15, 22, 27, 47, 48]. See [14, 21, 32, 46, 49, 50] for surveys and further literature. Mathematically, these studies unveil a broad spectrum of systems of elliptic equations with exponential nonlinearities and rich properties and structures, which present new challenges.

Recently, C. S. Lin and one of the authors (Y. Y.) carried out a systematic study [34, 35] of the multiple vortex equations obtained in [13, 15, 14, 22, 47, 48, 49, 50]. A series of sharp existence and uniqueness theorems were established. The methods used include monotone iterations, *a priori* estimates and degree-theory argument, and constrained minimization. In the present paper, we do two things. One is to develop and prove another series of sharp existence and uniqueness theorems for the multiple vortex equations derived in [5, 6], which are not covered in [34, 35]. The second thing is to develop a methodology that has not normally been used for these kinds of problems, over doubly periodic domains, which are often more difficult to approach due to the appearance of some integral constraints naturally associated with the equations. This is the highly efficient direct minimization approach which enables us to identify the key analytic ingredients and pursue a complete understanding of the problems almost immediately. As a by-product, such an approach also provides a constructive method for solutions. It is hopeful that our method may be explored further to study various multiple vortex equations, arising in non-Abelian gauge field theory, of more difficult structures. It should be noted that, after solving the problems here by the direct method, we gained true insight to solve them by the usual constrained minimization method. But we were not successful in all cases. Thus, the direct method, besides being simpler, is sometimes the only constructively workable one.

The content of the rest of the paper is outlined as follows.

In Section 2, we recall the SQCD multiple vortex equations of Auzzi, Bolognesi, Evslin, Konishi, and Yung [5] obtained in 2003 in which vortices are induced from three complex scalar fields. We then state our existence and uniqueness theorem, In the next two sections, we give the proofs for the various parts of the theorem. In Section 5, we turn our attention to a study of the multiple vortex equations derived by Auzzi and Kumar [6] in a supersymmetric Chern–Simons–Higgs theory formulated by Aharony, Bergman, Jafferis and Maldacena [3], known as the ABJM model. In this problem, vortices are generated from m complex scalar fields and the governing elliptic system consists of m equations. Again, we are able to obtain a sharp existence and uniqueness theorem for solutions over a doubly periodic domain and the full plane. Proofs of results are sketched in the subsequent two sections, as earlier.

In Section 8, we further illustrate how our direct methods may be used in tackling other problems of similar structures. Specifically, in §8.1, we revisit the $SO(2N)$ BPS vortex equations of Gudnason, Jiang, and Konishi [22], studied in [35] where solutions are constructed by a constrained minimization method when the total vortex number n does not exceed 3 and existence of solutions is established for arbitrary n using a degree-theory argument.

The main difficulty encountered in [35] is an extra term in one of the equations that makes it hard to resolve the constraints explicitly, which may be flown away by a homotopy flow. Here is an example where the direct method seems to be the only (constructively) successful one. With it we are able to obtain an existence proof for an arbitrary n. In fact, we will carry out our study in the most general situation where vortices are induced from the sets of zeros of the two complex scalar fields of the model.

In §8.2, we present a sharp existence and uniqueness theorem for the multiple vortex equations obtained by Marshakov and Yung [38] in 2002, which may be regarded as the earliest non-Abelian SQCD vortex equations for which the vortex-lines are taking values in the Cartan subalgebra of $SU(3)$ and, also, the starting point of the later development of the subject of non-Abelian vortices and monopoles in SQCD and their applications to color confinement. The method is again centered around direct minimization.

2 Vortices in Yang–Mills–Higgs theory

Following Auzzi, Bolognesi, Evslin, Konishi, and Yung [5], the Yang–Mills–Higgs action hosting the gauge field theory undergoing the spontaneous symmetry breaking

$$SU(N) \to SU(N-1) \times U(1), \tag{2.1}$$

within the context of the critical BPS coupling, assumes the form

$$\begin{aligned} S \;=\; \int \Bigg\{ & -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + (\nabla_\mu q_A)^\dagger \nabla^\mu q^A \\ & -\frac{g^2}{2} (q^\dagger_A t^a q^A)^2 - \frac{e^2}{4K(K+1)} (q^\dagger_A q^A - K\xi)^2 \Bigg\} \, \mathrm{d}x, \end{aligned} \tag{2.2}$$

where $K = N - 1$, the index $a = 1, 2, \cdots, K^2 - 1$, labels the group generators $\{t^a\}$ of $SU(K)$, $g, e > 0$ are the $SU(K)$ and $U(1)$ gauge-field coupling constants, respectively, $\xi > 0$ determines the vacuum expectation value of the quark fields q^A lying in the fundamental representation of $SU(K) \times U(1)$, $A = 1, 2, \cdots, N_{\text{flavor}}$ runs over the quark flavors,

$$\nabla_\mu = \partial_\mu - \mathrm{i} A^a_\mu t^a - \mathrm{i} A_\mu t^0, \quad t^0 = \frac{1}{\sqrt{2K(K+1)}} \begin{pmatrix} \mathbf{1}_K & 0 \\ 0 & -K \end{pmatrix}, \tag{2.3}$$

denotes the gauge-covariant derivative, with $\mathbf{1}_m$ the $m \times m$ identity matrix, and the Minkowski spacetime is of the signature $(+ - --)$. As a consequence of the Bogomol'nyi reduction [8, 31] for static vortex solutions, the BPS vortex equations [5] are of the form

$$\frac{1}{g^2} F^a_{12} + (q^\dagger_A t^a q^A) \;=\; 0, \quad a = 1, 2, \cdots, K^2 - 1, \tag{2.4}$$

$$
\begin{aligned}
\frac{1}{e^2}F_{12} + (q_A^\dagger q^A - K\xi) &= 0, && (2.5)\\
\nabla_1 q^A + \mathrm{i}\nabla_2 q^A &= 0, \quad A = 1, 2, \cdots, N_{\text{flavor}}. && (2.6)
\end{aligned}
$$

In the specific situation of $N = 4$ so that the unbroken symmetry is given by the group $SU(3) \times U(1)$, the non-Abelian vortex solutions may be described by gauge fields solely given in the $a = 3, 8$ (these are the Cartan subalgebra indices in the Gell-Mann matrix representation) and the $U(1)$ sectors, and the quark fields are represented by the complex matrix

$$
(q^{kA}) = \begin{pmatrix} \phi & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \chi \end{pmatrix}, \tag{2.7}
$$

where $k = 1, 2, 3$, is the color index which runs vertically, $A = 1, 2, 3$, is the flavor index which runs horizontally, and the winding numbers of ϕ, ψ, χ, away from a local region where ϕ, ψ, χ may vanish, say n_1, n_2, n_3, characterize the quark fields, which will be identified as the vortex charges or vortex numbers.

Now set

$$
A_j^3 = a_j, \quad A_j^8 = \frac{1}{\sqrt{3}} b_j, \quad A_j = \frac{1}{3} c_j, \quad j = 1, 2. \tag{2.8}
$$

where a_j, b_j, c_j ($j = 1, 2$) are real-valued vector fields. Then, in terms of the fields given in (2.7) and (2.8), the non-Abelian BPS multiple vortex equations (2.4)–(2.6) are found [5] to be

$$
\begin{aligned}
(\partial_1 + \mathrm{i}\partial_2)\phi &= \mathrm{i}\left(\frac{1}{2}(a_1 + \mathrm{i}a_2) + \frac{1}{6}(b_1 + \mathrm{i}b_2) + \frac{1}{3}(c_1 + \mathrm{i}c_2)\right)\phi, && (2.9)\\
(\partial_1 + \mathrm{i}\partial_2)\psi &= \mathrm{i}\left(-\frac{1}{2}(a_1 + \mathrm{i}a_2) + \frac{1}{6}(b_1 + \mathrm{i}b_2) + \frac{1}{3}(c_1 + \mathrm{i}c_2)\right)\psi, && (2.10)\\
(\partial_1 + \mathrm{i}\partial_2)\chi &= \mathrm{i}\left(-\frac{1}{3}(b_1 + \mathrm{i}b_2) + \frac{1}{3}(c_1 + \mathrm{i}c_2)\right)\chi, && (2.11)\\
a_{12} &= -\frac{\alpha}{2}(|\phi|^2 - |\psi|^2), && (2.12)\\
b_{12} &= -\frac{\alpha}{2}(|\phi|^2 + |\psi|^2 - 2|\chi|^2), && (2.13)\\
c_{12} &= -\beta(|\phi|^2 + |\psi|^2 + |\chi|^2 - 3\xi), && (2.14)
\end{aligned}
$$

where

$$
a_{12} = \partial_1 a_2 - \partial_2 a_1, \quad \text{etc}, \tag{2.15}
$$

are the reduced field curvatures, and $\alpha = g^2, \beta = 3e^2$. For convenience, we now use the complexified variables

$$
z = x^1 + \mathrm{i}x^2, \quad a = a_1 + \mathrm{i}a_2, \quad b = b_1 + \mathrm{i}b_2, \quad c = c_1 + \mathrm{i}c_2, \tag{2.16}
$$

and the complex derivatives

$$\partial = \partial_z = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2), \quad \overline{\partial} = \partial_{\overline{z}} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2) \tag{2.17}$$

to convert (2.9)–(2.14) into the system

$$\begin{aligned}
\overline{\partial} \ln \phi &= \frac{\mathrm{i}}{2}\left(\frac{a}{2} + \frac{b}{6} + \frac{c}{3}\right), &(2.18)\\
\overline{\partial} \ln \psi &= \frac{\mathrm{i}}{2}\left(-\frac{a}{2} + \frac{b}{6} + \frac{c}{3}\right), &(2.19)\\
\overline{\partial} \ln \chi &= \frac{\mathrm{i}}{2}\left(-\frac{b}{3} + \frac{c}{3}\right), &(2.20)\\
\mathrm{i}(\partial a - \overline{\partial}\overline{a}) &= \frac{\alpha}{2}(|\phi|^2 - |\psi|^2), &(2.21)\\
\mathrm{i}(\partial b - \overline{\partial}\overline{b}) &= \frac{\alpha}{2}(|\phi|^2 + |\psi|^2 - 2|\chi|^2), &(2.22)\\
\mathrm{i}(\partial c - \overline{\partial}\overline{c}) &= \beta(|\phi|^2 + |\psi|^2 + |\chi|^2 - 3\xi), &(2.23)
\end{aligned}$$

away from the zeros of ϕ, ψ, χ. From (2.18)–(2.20), we have

$$\begin{aligned}
a &= -2\mathrm{i}\overline{\partial}(\ln \phi - \ln \psi), &(2.24)\\
b &= -2\mathrm{i}\overline{\partial}(\ln \phi + \ln \psi - 2\ln \chi), &(2.25)\\
c &= -2\mathrm{i}\overline{\partial}(\ln \phi + \ln \psi + \ln \chi). &(2.26)
\end{aligned}$$

Inserting (2.24)–(2.26) into (2.21)–(2.23) and using the result $\Delta = 4\partial\overline{\partial} = 4\overline{\partial}\partial$, we obtain

$$\begin{aligned}
\Delta(\ln |\phi|^2 - \ln |\psi|^2) &= \alpha(|\phi|^2 - |\psi|^2), &(2.27)\\
\Delta(\ln |\phi|^2 + \ln |\psi|^2 - 2\ln |\chi|^2) &= \alpha(|\phi|^2 + |\psi|^2 - 2|\chi|^2), &(2.28)\\
\Delta(\ln |\phi|^2 + \ln |\psi|^2 + \ln |\chi|^2) &= 2\beta(|\phi|^2 + |\psi|^2 + |\chi|^2 - 3\xi), &(2.29)
\end{aligned}$$

again away from the zeros of ϕ, ψ, χ. Following [31], we know that the equations (2.9)–(2.11) or (2.18)–(2.20) imply that the zeros of ϕ, ψ, χ are discrete and of integer multiplicities. We use Z_ϕ, Z_ψ, Z_χ to denote the sets of zeros of ϕ, ψ, χ,

$$Z_\phi = \{p_{1,1}, \cdots, p_{1,n_1}\}, \quad Z_\psi = \{p_{2,1}, \cdots, p_{2,n_2}\}, \quad Z_\chi = \{p_{3,1}, \cdots, p_{3,n_3}\}, \tag{2.30}$$

so that the repetitions among the points $p_{\ell,s}$, $\ell = 1, 2, 3, s = 1, \cdots, n_\ell$, take account of the multiplicities of these zeros. Then the substitutions

$$u_1 = \ln |\phi|^2, \quad u_2 = \ln |\psi|^2, \quad u_3 = \ln |\chi|^2, \tag{2.31}$$

enable us to recast the equations (2.27)–(2.29) into the following elliptic system

$$\Delta(u_1 - u_2) = \alpha(\mathrm{e}^{u_1} - \mathrm{e}^{u_2}) + 4\pi\left(\sum_{s=1}^{n_1} \delta_{p_{1,s}}(x) - \sum_{s=1}^{n_2} \delta_{p_{2,s}}(x)\right), \tag{2.32}$$

$$\Delta(u_1+u_2-2u_3) = \alpha(\mathrm{e}^{u_1}+\mathrm{e}^{u_2}-2\mathrm{e}^{u_3}) +4\pi\left(\sum_{s=1}^{n_1}\delta_{p_{1,s}}(x)+\sum_{s=1}^{n_2}\delta_{p_{2,s}}(x)-2\sum_{s=1}^{n_3}\delta_{p_{3,s}}(x)\right), \quad (2.33)$$

$$\Delta(u_1+u_2+u_3) = 2\beta(\mathrm{e}^{u_1}+\mathrm{e}^{u_2}+\mathrm{e}^{u_3}-3\xi) +4\pi\left(\sum_{s=1}^{n_1}\delta_{p_{1,s}}(x)+\sum_{s=1}^{n_2}\delta_{p_{2,s}}(x)+\sum_{s=1}^{n_3}\delta_{p_{3,s}}(x)\right), \quad (2.34)$$

now defined over the entire domain.

Two situations are of interest, namely, the situation where the equations are considered over a doubly periodic domain, Ω, governing multiple vortices hosted in Ω so that the field configurations are subject to the 't Hooft periodic boundary condition [55, 60, 61] under which periodicity is achieved modulo gauge transformations, and the situation where the equations are considered over the full plane $\mathbb{R}^2$ and the solutions satisfy the boundary condition

$$u_\ell(x)\to\ln\xi\quad\text{as } |x|\to\infty,\quad \ell=1,2,3. \quad (2.35)$$

Concerning these situations, our main existence and uniqueness theorem for solutions of (2.9)–(2.14) or (2.32)–(2.34) may be stated as follows.

Theorem 2.1. *Consider the BPS system of multiple vortex equations (2.9)–(2.14) for $(\phi,\psi,\chi,a_j,b_j,c_j)$ with the prescribed sets of zeros given in (2.30) so that ϕ,ψ,χ have n_1,n_2,n_3 arbitrarily distributed zeros, respectively.*

(i) For this problem over a doubly periodic domain Ω, a solution exists if and only if the following three conditions

$$\frac{1}{3\alpha}(n_1+n_2-2n_3)+\frac{1}{\alpha}(n_1-n_2)+\frac{1}{3\beta}(n_1+n_2+n_3) < \frac{\xi|\Omega|}{2\pi}, \quad (2.36)$$

$$\frac{1}{3\alpha}(n_1+n_2-2n_3)-\frac{1}{\alpha}(n_1-n_2)+\frac{1}{3\beta}(n_1+n_2+n_3) < \frac{\xi|\Omega|}{2\pi}, \quad (2.37)$$

$$-\frac{2}{3\alpha}(n_1+n_2-2n_3)+\frac{1}{3\beta}(n_1+n_2+n_3) < \frac{\xi|\Omega|}{2\pi}, \quad (2.38)$$

hold simultaneously. Moreover, whenever a solution exists, it is unique.

(ii) For this problem over the full plane $\mathbb{R}^2$ subject to the boundary condition

$$|\phi|^2,|\psi|^2,|\chi|^2\to\xi\quad\text{as } |x|\to\infty, \quad (2.39)$$

there exists a unique solution up to gauge transformations so that the boundary behavior stated above is realized exponentially rapidly.

In either case, the excited total vortex fluxes are quantized quantities given explicitly by the formulas

$$\Phi_a=\int a_{12}\,\mathrm{d}x = 2\pi(n_1-n_2), \quad (2.40)$$

$$\Phi_b = \int b_{12}\,\mathrm{d}x \;=\; 2\pi(n_1+n_2-2n_3), \tag{2.41}$$

$$\Phi_c = \int c_{12}\,\mathrm{d}x \;=\; 2\pi(n_1+n_2+n_3), \tag{2.42}$$

respectively

This theorem will be established in the following two sections.

3 Proof of existence for doubly periodic case

In this section, we consider the equations (2.32)–(2.34) defined over a doubly periodic domain, Ω. Let u_ℓ^0 be a solution of the equation

$$\Delta u_\ell^0 = -\frac{4\pi n_\ell}{|\Omega|} + 4\pi \sum_{s=1}^{n_\ell} \delta_{p_{\ell,s}}(x), \quad x\in\Omega, \quad \ell=1,2,3. \tag{3.1}$$

Then the substitutions

$$u_\ell = u_\ell^0 + v_\ell, \quad \ell = 1,2,3,$$

recast the equations (2.32)–(2.34) into

$$\Delta(v_1-v_2) \;=\; \alpha(\mathrm{e}^{u_1^0+v_1}-\mathrm{e}^{u_2^0+v_2}) + \frac{4\pi}{|\Omega|}(n_1-n_2), \tag{3.2}$$

$$\Delta(v_1+v_2-2v_3) \;=\; \alpha(\mathrm{e}^{u_1^0+v_1}+\mathrm{e}^{u_2^0+v_2}-2\mathrm{e}^{u_3^0+v_3}) + \frac{4\pi}{|\Omega|}(n_1+n_2-2n_3), \tag{3.3}$$

$$\Delta(v_1+v_2+v_3) \;=\; 2\beta(\mathrm{e}^{u_1^0+v_1}+\mathrm{e}^{u_2^0+v_2}+\mathrm{e}^{u_3^0+v_3}-3\xi) + \frac{4\pi}{|\Omega|}(n_1+n_2+n_3). \tag{3.4}$$

Naturally, we should use the transformation

$$\left\{\begin{array}{rcl} w_1 &=& v_1-v_2,\\ w_2 &=& v_1+v_2-2v_3,\\ w_3 &=& v_1+v_2+v_3, \end{array}\right. \qquad \left\{\begin{array}{rcl} v_1 &=& \frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3,\\ v_2 &=& -\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3,\\ v_3 &=& -\frac{1}{3}w_2+\frac{1}{3}w_3, \end{array}\right. \tag{3.5}$$

to change (3.2)–(3.4) into the equations

$$\Delta w_1 \;=\; \alpha(\mathrm{e}^{u_1^0+\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3}-\mathrm{e}^{u_2^0-\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3}) + \frac{4\pi}{|\Omega|}(n_1-n_2), \tag{3.6}$$

$$\begin{aligned}\Delta w_2 \;=\;& \alpha(\mathrm{e}^{u_1^0+\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3}+\mathrm{e}^{u_2^0-\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3}-2\mathrm{e}^{u_3^0-\frac{1}{3}w_2+\frac{1}{3}w_3})\\ &+\frac{4\pi}{|\Omega|}(n_1+n_2-2n_3),\end{aligned} \tag{3.7}$$

$$\begin{aligned}\Delta w_3 \;=\;& 2\beta(\mathrm{e}^{u_1^0+\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3}+\mathrm{e}^{u_2^0-\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3}+\mathrm{e}^{u_3^0-\frac{1}{3}w_2+\frac{1}{3}w_3}-3\xi)\\ &+\frac{4\pi}{|\Omega|}(n_1+n_2+n_3),\end{aligned} \tag{3.8}$$

which are easily seen to be the Euler–Lagrange equations of the functional

$$\begin{aligned} I(w_1,w_2,w_3) \;=\; & \int_\Omega \left\{ \frac{1}{4\alpha}|\nabla w_1|^2 + \frac{1}{12\alpha}|\nabla w_2|^2 + \frac{1}{12\beta}|\nabla w_3|^2 + \mathrm{e}^{u_1^0+\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3} \right. \\ & + \mathrm{e}^{u_2^0-\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3} + \mathrm{e}^{u_3^0-\frac{1}{3}w_2+\frac{1}{3}w_3} + \frac{2\pi}{\alpha|\Omega|}(n_1-n_2)w_1 \\ & \left. + \frac{2\pi}{3\alpha|\Omega|}(n_1+n_2-2n_3)w_2 + \left(\frac{2\pi}{3\beta|\Omega|}(n_1+n_2+n_3)-\xi\right) w_3 \right\} \mathrm{d}x. \end{aligned} \tag{3.9}$$

This functional is not bounded from below when

$$2\pi(n_1+n_2+n_3) > 3\beta\xi|\Omega|. \tag{3.10}$$

In fact, we can show that (3.10) will never happen for (3.6)–(3.8). Indeed, integrating (3.6)–(3.8), we obtain the conditions

$$\begin{aligned} & \int_\Omega \mathrm{e}^{u_1^0+\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3}\,\mathrm{d}x \\ & = \xi|\Omega| - 2\pi\left(\frac{1}{3\alpha}(n_1+n_2-2n_3) + \frac{1}{\alpha}(n_1-n_2) + \frac{1}{3\beta}(n_1+n_2+n_3)\right) \\ & \equiv \eta_1 > 0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} & \int_\Omega \mathrm{e}^{u_2^0-\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3}\,\mathrm{d}x \\ & = \xi|\Omega| - 2\pi\left(\frac{1}{3\alpha}(n_1+n_2-2n_3) - \frac{1}{\alpha}(n_1-n_2) + \frac{1}{3\beta}(n_1+n_2+n_3)\right) \\ & \equiv \eta_2 > 0, \end{aligned} \tag{3.12}$$

$$\begin{aligned} & \int_\Omega \mathrm{e}^{u_3^0-\frac{1}{3}w_2+\frac{1}{3}w_3}\,\mathrm{d}x \\ & = \xi|\Omega| - 2\pi\left(-\frac{2}{3\alpha}(n_1+n_2-2n_3) + \frac{1}{3\beta}(n_1+n_2+n_3)\right) \\ & \equiv \eta_3 > 0, \end{aligned} \tag{3.13}$$

which are exactly the conditions (2.36)–(2.38). In particular, we have

$$\eta_1+\eta_2+\eta_3 > 0, \tag{3.14}$$

which rules out (3.10) immediately.

Below, we shall show that, under the conditions (3.11)–(3.13), the equations (3.6)–(3.8) have a solution. We will use both a direct minimization method and a constrained minimization method to approach the problem. These methods may be of independent practical value for computational purposes.

3.1 Direct minimization

We use $W^{1,2}(\Omega)$ to denote the usual Sobolev space of scalar-valued or vector-valued Ω-periodic L^2-functions whose derivatives are also in $L^2(\Omega)$. In the scalar case, we may decompose $W^{1,2}(\Omega)$ into $W^{1,2}(\Omega)=\mathbb{R}\oplus\dot{W}^{1,2}(\Omega)$ so that any $f\in W^{1,2}(\Omega)$ can be expressed as

$$f=\underline{f}+\dot{f},\quad \underline{f}\in\mathbb{R},\quad \dot{f}\in\dot{W}^{1,2}(\Omega),\quad \int_\Omega \dot{f}\,\mathrm{d}x=0. \tag{3.15}$$

It is useful to recall the Moser–Trudinger inequality [4, 16]

$$\int_\Omega \mathrm{e}^u\mathrm{d}x\leq C\exp\left(\frac{1}{16\pi}\int_\Omega|\nabla u|^2\,\mathrm{d}x\right),\quad u\in\dot{W}^{1,2}(\Omega). \tag{3.16}$$

With (3.16), it is clear that the functional I defined by (3.9) is a C^1-functional with respect to its argument $(w_1,w_2,w_3)\in W^{1,2}(\Omega)$ which is strictly convex and lower semi-continuous in terms of the weak topology of $W^{1,2}(\Omega)$.

With the notation (3.15), we may apply the transformation (3.5) to arrive at

$$\begin{aligned} & I(w_1,w_2,w_3)-\int_\Omega\left\{\frac{1}{4\alpha}|\nabla\dot{w}_1|^2+\frac{1}{12\alpha}|\nabla\dot{w}_2|^2+\frac{1}{12\beta}|\nabla\dot{w}_3|^2\right\}\mathrm{d}x\\ =\ & \int_\Omega\left\{\mathrm{e}^{u_1^0+\underline{v}_1+\dot{v}_1}+\mathrm{e}^{u_2^0+\underline{v}_2+\dot{v}_2}+\mathrm{e}^{u_3^0+\underline{v}_3+\dot{v}_3}\right\}\mathrm{d}x-\eta_1\underline{v}_1-\eta_2\underline{v}_2-\eta_3\underline{v}_3\\ \geq\ & \sum_{\ell=1}^3\left(\sigma_\ell\mathrm{e}^{\underline{v}_\ell}-\eta_\ell\underline{v}_\ell\right)\geq\sum_{\ell=1}^3\eta_\ell\left(1+\ln\left[\frac{\sigma_\ell}{\eta_\ell}\right]\right), \end{aligned} \tag{3.17}$$

where we have used the Jensen inequality to obtain the lower bounds

$$\begin{aligned}\int_\Omega \mathrm{e}^{u_\ell^0+\underline{v}_\ell+\dot{v}_\ell}\,\mathrm{d}x &\geq |\Omega|\exp\left(\frac{1}{|\Omega|}\int_\Omega(u_\ell^0+\underline{v}_\ell+\dot{v}_\ell)\,\mathrm{d}x\right)\\ &= \left(|\Omega|\exp\left[\frac{1}{|\Omega|}\int_\Omega u_\ell^0\,\mathrm{d}x\right]\right)\mathrm{e}^{\underline{v}_\ell}\equiv\sigma_\ell\mathrm{e}^{\underline{v}_\ell},\quad \ell=1,2,3,\end{aligned} \tag{3.18}$$

in (3.17). Thus, in particular, we see that I is bounded from below and we may consider the following direct minimization problem

$$\eta_0\equiv\inf\left\{I(w_1,w_2,w_3)\,\Big|\,w_1,w_2,w_3\in W^{1,2}(\Omega)\right\}. \tag{3.19}$$

Let $\{(w_1^{(n)},w_2^{(n)},w_3^{(n)})\}$ be a minimizing sequence of (3.19). Since the function

$$F(u)=\sigma\mathrm{e}^u-\eta u, \tag{3.20}$$

where $\sigma,\eta>0$ are constants, enjoys the property that $F(u)\to\infty$ as $u\to\pm\infty$, we see from (3.17) that the sequences $\{\underline{v}_\ell^{(n)}\}$ ($\ell=1,2,3$), hence $\{\underline{w}_\ell^{(n)}\}$ ($\ell=1,2,3$), are all bounded. Without loss of generality, we may assume

$$\underline{w}_\ell^{(n)}\to\text{ some point }\underline{w}_\ell^{(\infty)}\in\mathbb{R}\text{ as }n\to\infty,\quad \ell=1,2,3. \tag{3.21}$$

On the other hand, in view of (3.17) and the Poincaré inequality, we see that all the sequences $\{\dot{w}_\ell^{(n)}\}$ are bounded in $\dot{W}^{1,2}(\Omega)$, $\ell = 1,2,3$. Without loss of generality, we may assume

$$\dot{w}_\ell^{(n)} \to \text{ some element } \dot{w}_\ell^{(\infty)} \in W^{1,2}(\Omega) \text{ weakly as } n \to \infty, \quad \ell = 1,2,3. \tag{3.22}$$

Of course, $\dot{w}^{(\infty)} \in \dot{W}^{1,2}(\Omega)$ ($\ell = 1,2,3$). Set $w_\ell^{(\infty)} = \underline{w}_\ell^{(\infty)} + \dot{w}_\ell^{(\infty)}$ ($\ell = 1,2,3$). Then (3.21) and (3.22) lead us to $w^{(n)} \to w^{(\infty)}$ weakly in $W^{1,2}(\Omega)$ as $n \to \infty$ ($\ell = 1,2,3$). The weakly lower semi-continuity of I enables us to conclude that $(w_1^{(\infty)}, w_2^{(\infty)}, w_3^{(\infty)})$ solves (3.19), which is a critical point of I. As a critical point of I, it satisfies the equations (3.6)–(3.8). Since I is strictly convex, it can have at most one critical point. Thus, the uniqueness of the solution of (3.6)–(3.8) follows immediately.

3.2 Constrained minimization

For convenience, we rewrite the constraints (3.11)–(3.13) collectively as

$$J_\ell(w_1, w_2, w_3) \equiv \int_\Omega \mathrm{e}^{u_\ell^0 + v_\ell}\,\mathrm{d}x = \eta_\ell, \quad \ell = 1,2,3, \tag{3.23}$$

and consider the constrained minimization problem

$$\eta_0 \equiv \inf\left\{ I(w_1, w_2, w_3) \,\middle|\, (w_1, w_2, w_3) \in W^{1,2}(\Omega) \text{ and satisfies (3.23)} \right\}. \tag{3.24}$$

Suppose that (3.24) allows a solution, say $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$. Then there are numbers (the Lagrange multipliers) in $\mathbb{R}$, say $\lambda_1, \lambda_2, \lambda_3$, such that

$$\begin{aligned} &D(I + \lambda_1 J_1 + \lambda_2 J_2 + \lambda_3 J_3)(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)(w_1, w_2, w_3) = 0, \\ &\quad \forall (w_1, w_2, w_3) \in W^{1,2}(\Omega). \end{aligned} \tag{3.25}$$

Now take the trial configurations, $(w_1, w_2, w_3) = (1,0,0), (0,1,0), (0,0,1)$ consecutively, in (3.25). Since $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ satisfies the constraints (3.23), as a result, we have

$$\frac{1}{2}\lambda_1\eta_1 - \frac{1}{2}\lambda_2\eta_2 = 0, \quad \frac{1}{6}\lambda_1\eta_1 + \frac{1}{6}\lambda_2\eta_2 - \frac{1}{3}\lambda_3\eta_3 = 0, \quad \frac{1}{3}\lambda_1\eta_1 + \frac{1}{3}\lambda_2\eta_2 + \frac{1}{3}\lambda_3\eta_3 = 0. \tag{3.26}$$

Consequently, $\lambda_1 = \lambda_2 = \lambda_3 = 0$. In other words, the Lagrange multipliers disappear automatically and, thus, the search for a critical point of the functional I is converted to obtaining a solution of the constrained minimization problem (3.24).

In order to approach (3.24), we resolve (3.23) to write down

$$\underline{v}_\ell = \ln \eta_\ell - \ln\left(\int_\Omega \mathrm{e}^{u_\ell^0 + \dot{v}_\ell}\,\mathrm{d}x \right), \quad \ell = 1,2,3. \tag{3.27}$$

Hence, in view of the left-hand side of (3.17), we get

$$\begin{aligned} & I(w_1,w_2,w_3) - \int_\Omega \left\{ \frac{1}{4\alpha}|\nabla \dot{w}_1|^2 + \frac{1}{12\alpha}|\nabla \dot{w}_2|^2 + \frac{1}{12\beta}|\nabla \dot{w}_3|^2 \right\} \mathrm{d}x \\ \geq & -\sum_{\ell=1}^3 \eta_\ell \ln \eta_\ell + \sum_{\ell=1}^3 \eta_\ell \ln \left(\int_\Omega \mathrm{e}^{u_\ell^0 + \dot{v}_\ell} \, \mathrm{d}x \right) \geq \sum_{\ell=1}^3 \eta_\ell \ln \left(\frac{\sigma_\ell}{\eta_\ell} \right), \end{aligned} \tag{3.28}$$

where we have used the Jensen inequality and the definition of the quantities σ_ℓ ($\ell = 1,2,3$) given in (3.18). Thus the problem (3.24) is well defined.

Let $\{(w_1^{(n)}, w_2^{(n)}, w_3^{(n)})\}$ be a minimizing sequence of (3.24). Then (3.28) says that the sequence $\{(\dot{w}_1^{(n)}, \dot{w}_2^{(n)}, \dot{w}_3^{(n)})\}$ is bounded in $W^{1,2}(\Omega)$. Hence we may assume that $\{(\dot{w}_1^{(n)}, \dot{w}_2^{(n)}, \dot{w}_3^{(n)})\}$ is weakly convergent in $W^{1,2}(\Omega)$. The inequality (3.16) and the expressions (3.27) indicate that $\{(\underline{w}_1^{(n)}, \underline{w}_2^{(n)}, \underline{w}_3^{(n)})\}$ is a convergent sequence in $\mathbb{R}^3$. Thus $\{(w_1^{(n)}, w_2^{(n)}, w_3^{(n)})\}$ is weakly convergent in $W^{1,2}(\Omega)$. In view of the weak continuity of the constraint functionals J_ℓ defined in (3.23) and the weak lower semi-continuity of the functional I defined in (4.11), we see that the weak limit of $\{(w_1^{(n)}, w_2^{(n)}, w_3^{(n)})\}$ in $W^{1,2}(\Omega)$ is a solution of (3.24). As a critical of I, it is also unique. Therefore, a constrained minimization proof for the existence of a unique solution of the equations (3.6)–(3.8) is obtained.

4 Proof of existence for planar case

With the correspondence relations stated in (3.5), we have

$$3w_1^2 + w_2^2 + 2w_3^2 = 6(v_1^2 + v_2^2 + v_3^2), \tag{4.1}$$

$$3|\nabla w_1|^2 + |\nabla w_2|^2 + 2|\nabla w_3|^2 = 6(|\nabla v_1|^2 + |\nabla v_2|^2 + |\nabla v_3|^2), \tag{4.2}$$

which will be useful for our analysis to follow.

To proceed further, here and elsewhere in the paper when we deal with the planar cases, we use the method developed in [31] and introduce the background functions [31]

$$u_\ell^0(x) = -\sum_{s=1}^{n_\ell} \ln(1 + \mu|x - p_{\ell,s}|^{-2}), \quad \mu > 0, \quad \ell = 1,2,3. \tag{4.3}$$

(Here and in the sequel, the parameter $\mu > 0$ should not be confused with the spacetime index μ used in the context of various field equations.)

Then we have

$$\Delta u_\ell^0 = -h_\ell + 4\pi \sum_{s=1}^{n_\ell} \delta_{p_{\ell,s}}(x), \quad h_\ell(x) = 4\sum_{s=1}^{n_\ell} \frac{\mu}{(\mu + |x - p_{\ell,s}|^2)^2}, \quad \ell = 1,2,3. \tag{4.4}$$

Now use the substitutions

$$u_\ell = \ln \xi + u_\ell^0 + v_\ell, \quad \ell = 1,2,3, \tag{4.5}$$

$$\alpha\xi \mapsto \alpha, \quad \beta\xi \mapsto \beta, \tag{4.6}$$

and (3.5) in (2.32)–(2.34), we obtain the governing equations

$$\Delta w_1 = \alpha\left(\mathrm{e}^{u_1^0+\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3} - \mathrm{e}^{u_2^0-\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3}\right) + (h_1 - h_2), \tag{4.7}$$

$$\Delta w_2 = \alpha\left(\mathrm{e}^{u_1^0+\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3} + \mathrm{e}^{u_2^0-\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3} - 2\mathrm{e}^{u_3^0-\frac{1}{3}w_2+\frac{1}{3}w_3}\right) +(h_1+h_2-2h_3), \tag{4.8}$$

$$\Delta w_3 = 2\beta\left(\mathrm{e}^{u_1^0+\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3} + \mathrm{e}^{u_2^0-\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3} + \mathrm{e}^{u_3^0-\frac{1}{3}w_2+\frac{1}{3}w_3} - 3\right) +(h_1+h_2+h_3), \tag{4.9}$$

over the full plane $\mathbb{R}^2$. The boundary condition for w_1, w_2, w_3 reads

$$w_\ell(x) \to 0 \quad \text{as } |x| \to \infty, \quad \ell = 1,2,3. \tag{4.10}$$

In order to obtain a solution of (4.7)–(4.9) subject to the boundary condition (4.10), we look for a critical point of the action functional

$$\begin{aligned} I(w_1,w_2,w_3) = \int_{\mathbb{R}^2} \Bigg\{ & \frac{1}{4\alpha}|\nabla w_1|^2 + \frac{1}{12\alpha}|\nabla w_2|^2 + \frac{1}{12\beta}|\nabla w_3|^2 \\ & + \left(\mathrm{e}^{u_1^0+\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3} - \mathrm{e}^{u_1^0} - \left[\frac{1}{2}w_1 + \frac{1}{6}w_2 + \frac{1}{3}w_3\right]\right) \\ & + \left(\mathrm{e}^{u_2^0-\frac{1}{2}w_1+\frac{1}{6}w_2+\frac{1}{3}w_3} - \mathrm{e}^{u_2^0} - \left[-\frac{1}{2}w_1 + \frac{1}{6}w_2 + \frac{1}{3}w_3\right]\right) \\ & + \left(\mathrm{e}^{u_3^0-\frac{1}{3}w_2+\frac{1}{3}w_3} - \mathrm{e}^{u_3^0} - \left[-\frac{1}{3}w_2 + \frac{1}{3}w_3\right]\right) \\ & + \frac{1}{2\alpha}(h_1-h_2)w_1 + \frac{1}{6\alpha}(h_1+h_2-2h_3)w_2 + \frac{1}{6\beta}(h_1+h_2+h_3)w_3 \Bigg\}\,\mathrm{d}x, \end{aligned} \tag{4.11}$$

which is C^1 and strictly convex over $W^{1,2}(\mathbb{R}^2)$. After some algebra, it can be seen that the Fréchet derivative of I enjoys the following property,

$$\begin{aligned} &(DI(w_1,w_2,w_3))(w_1,w_2,w_3) - \int_{\mathbb{R}^2}\left\{\frac{1}{2\alpha}|\nabla w_1|^2 + \frac{1}{6\alpha}|\nabla w_2|^2 + \frac{1}{6\beta}|\nabla w_3|^2\right\}\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\left\{\sum_{\ell=1}^3 \mathrm{e}^{u_\ell^0}(\mathrm{e}^{v_\ell}-1)v_\ell + \sum_{\ell=1}^3(\mathrm{e}^{u_\ell^0}-1)v_\ell + \sum_{\ell=1}^3 g_\ell v_\ell\right\}\mathrm{d}x, \end{aligned} \tag{4.12}$$

where g_ℓ's are some linear combinations of h_ℓ's. Now setting $\gamma = \max\{\alpha, 2\beta\}$ and applying (4.2), we derive from (4.12) the lower bound

$$\begin{aligned} &(DI(w_1,w_2,w_3))(w_1,w_2,w_3) \\ &\geq \int_{\mathbb{R}^2}\left\{\frac{1}{\gamma}\sum_{\ell=1}^3|\nabla v_\ell|^2 + \sum_{\ell=1}^3\left(\mathrm{e}^{u_\ell^0}[\mathrm{e}^{v_\ell}-1]v_\ell + [\mathrm{e}^{u_\ell^0}-1]v_\ell + g_\ell v_\ell\right)\right\}\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\left\{\frac{1}{\gamma}\sum_{\ell=1}^3|\nabla v_\ell|^2 + \sum_{\ell=1}^3 v_\ell\left(\mathrm{e}^{u_\ell^0+v_\ell} - 1 + g_\ell\right)\right\}\mathrm{d}x. \end{aligned} \tag{4.13}$$

We can now follow the analysis in [31]. To simplify the notation, we suppress the subscript ℓ and rewrite a typical part on the right-hand side of (4.13) as

$$M(v)=\int_{\mathbb{R}^2} v\left(\mathrm{e}^{u^0+v}-1+g\right)\,\mathrm{d}x. \tag{4.14}$$

Here g should not be confused with the coupling constant used before in the field-theoretical context. Thus $\mathrm{e}^t-1\geq t$ $(t\in\mathbb{R})$ gives us

$$\mathrm{e}^{u^0+v}-1+g\geq u^0+v+g,$$

which leads to

$$\begin{aligned} M(v_+) &\geq \int_{\mathbb{R}^2} v_+^2\,\mathrm{d}x+\int_{\mathbb{R}^2} v_+(u^0+g)\,\mathrm{d}x \\ &\geq \frac{1}{2}\int_{\mathbb{R}^2} v_+^2\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^2}(u^0+g)^2\,\mathrm{d}x. \end{aligned} \tag{4.15}$$

On the other hand, in view of the inequality $1-\mathrm{e}^{-t}\geq t/(1+t)$ $(t\geq 0)$, we have

$$\begin{aligned} v_-\left(1-g-\mathrm{e}^{u^0-v_-}\right) &= v_-\left(1-g+\mathrm{e}^{u^0}[1-\mathrm{e}^{-v_-}]-\mathrm{e}^{u^0}\right) \\ &\geq v_-\left(1-g+\mathrm{e}^{u^0}\frac{v_-}{1+v_-}-\mathrm{e}^{u^0}\right) \\ &= \frac{v_-^2}{1+v_-}(1-g)+\frac{v_-}{1+v_-}\left(1-\mathrm{e}^{u^0}-g\right). \end{aligned} \tag{4.16}$$

Of course, we may choose $\mu>0$ in (4.3) large enough so that $g<1/2$ (say). Furthermore, since $1-\mathrm{e}^{u^0}$ and g are in $L^2(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2}\frac{v_-}{1+v_-}\left|1-\mathrm{e}^{u^0}-g\right|\,\mathrm{d}x\leq\frac{1}{4}\int_{\mathbb{R}^2}\frac{v_-^2}{1+v_-}\,\mathrm{d}x+\int_{\mathbb{R}^2}\left(1-\mathrm{e}^{u^0}-g\right)^2\,\mathrm{d}x. \tag{4.17}$$

Combining (4.16) and (4.17), we obtain

$$M(-v_-)=\int_{\mathbb{R}^2} v_-\left(1-g-\mathrm{e}^{u^0-v_-}\right)\,\mathrm{d}x\geq\frac{1}{4}\int_{\mathbb{R}^2}\frac{v_-^2}{1+v_-}\,\mathrm{d}x-C, \tag{4.18}$$

where and in the sequel $C>0$ denotes an irrelevant constant. Summarizing (4.15) and (4.18), we arrive at

$$M(v)\geq\frac{1}{4}\int_{\mathbb{R}^2}\frac{v^2}{1+|v|}\,\mathrm{d}x-C. \tag{4.19}$$

We now recall the standard Sobolev inequality

$$\int_{\mathbb{R}^2} v^4\,\mathrm{d}x\leq 2\int_{\mathbb{R}^2} v^2\,\mathrm{d}x\int_{\mathbb{R}^2}|\nabla v|^2\,\mathrm{d}x,\quad v\in W^{1,2}(\mathbb{R}^2). \tag{4.20}$$

Consequently, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^2} v^2\,\mathrm{d}x\right)^2 &= \left(\int_{\mathbb{R}^2} \frac{|v|}{1+|v|}(1+|v|)|v|\,\mathrm{d}x\right)^2\\ &\leq 2\int_{\mathbb{R}^2}\frac{v^2}{(1+|v|)^2}\,\mathrm{d}x\int_{\mathbb{R}^2}(v^2+v^4)\,\mathrm{d}x\\ &\leq 4\int_{\mathbb{R}^2}\frac{v^2}{(1+|v|)^2}\,\mathrm{d}x\int_{\mathbb{R}^2}v^2\,\mathrm{d}x\left(1+\int_{\mathbb{R}^2}|\nabla v|^2\,\mathrm{d}x\right)\\ &\leq \frac{1}{2}\left(\int_{\mathbb{R}^2}v^2\,\mathrm{d}x\right)^2 + C\left(1+\left[\int_{\mathbb{R}^2}\frac{v^2}{(1+|v|)^2}\,\mathrm{d}x\right]^4+\left[\int_{\mathbb{R}^2}|\nabla v|^2\,\mathrm{d}x\right]^4\right).\end{aligned} \tag{4.21}$$

As a result of (4.21), we have

$$\left(\int_{\mathbb{R}^2} v^2\,\mathrm{d}x\right)^{\frac{1}{2}} \leq C\left(1+\int_{\mathbb{R}^2}|\nabla v|^2\,\mathrm{d}x+\int_{\mathbb{R}^2}\frac{v^2}{(1+|v|)^2}\,\mathrm{d}x\right). \tag{4.22}$$

Now set $C_0=\min\{1/\gamma,1/4\}$. In view of (4.13) and (4.19), we have

$$(DI(w_1,w_2,w_3))(w_1,w_2,w_3)\geq C_0\sum_{\ell=1}^{3}\int_{\mathbb{R}^2}\left(|\nabla v_\ell|^2+\frac{v_\ell^2}{1+|v_\ell|}\right)\,\mathrm{d}x - C. \tag{4.23}$$

As a consequence of (4.22), (4.23), (4.1), (4.2), we may conclude with the coercive lower bound

$$(DI(w_1,w_2,w_3))(w_1,w_2,w_3)\geq C_1\sum_{\ell=1}^{3}\|w_\ell\|_{W^{1,2}(\mathbb{R}^2)} - C_2, \tag{4.24}$$

where $C_1,C_2>0$ are constants. In view of the estimate (4.24), the existence of a critical point of the functional I in the space $W^{1,2}(\mathbb{R}^2)$ follows. In fact, from (4.24), we may choose $R>0$ large enough so that

$$\inf\left\{(DI(w_1,w_2,w_3))(w_1,w_2,w_3)\,\Big|\,\sum_{\ell=1}^{3}\|w_\ell\|_{W^{1,2}(\mathbb{R}^2)}=R\right\}\geq 1 \tag{4.25}$$

(say). Consider the minimization problem

$$\eta_0\equiv\inf\left\{I(w_1,w_2,w_3)\,\Big|\,\sum_{\ell=1}^{3}\|w_\ell\|_{W^{1,2}(\mathbb{R}^2)}\leq R\right\}. \tag{4.26}$$

This problem obviously has a solution due to the fact that the functional (4.11) is weakly lower semi-continuous. Let $(\tilde{w}_1,\tilde{w}_2,\tilde{w}_3)$ be a solution of (4.26). We show that it must be an interior point. Otherwise, if

$$\sum_{\ell=1}^{3}\|\tilde{w}_\ell\|_{W^{1,2}(\mathbb{R}^2)}=R, \tag{4.27}$$

then, with the vector notation $\mathbf{w} = (w_1, w_2, w_3)$, the result (4.25) gives us

$$\lim_{t\to 0} \frac{I([1-t]\tilde{\mathbf{w}}) - I(\tilde{\mathbf{w}})}{t} = \frac{\mathrm{d}}{\mathrm{d}t} I([1-t]\tilde{\mathbf{w}})\Big|_{t=0} = -(DI(\tilde{\mathbf{w}}))(\tilde{\mathbf{w}}) \leq -1. \tag{4.28}$$

Thus, when $t > 0$ is sufficiently small, with $\mathbf{w}^t = (1-t)\tilde{\mathbf{w}}$, we have

$$I(w_1^t, w_2^t, w_3^t) < I(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) = \eta_0, \quad \sum_{\ell=1}^{3} \|w_\ell^t\|_{W^{1,2}(\mathbb{R}^2)} = (1-t)R < R, \tag{4.29}$$

which contradicts the definition of η_0 made in (4.26). Thus $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ must an interior point for the problem (4.26). Consequently, it is a critical point of the functional (4.11). The strict convexity of the functional implies that such a critical point must be unique. In the following, we rewrite $\tilde{w}_\ell$ as w_ℓ.

Besides, using the standard embedding inequality

$$\|f\|_{L^p(\mathbb{R}^2)} \leq \left(\pi \left[\frac{p}{2} - 1\right]\right)^{\frac{p-2}{2p}} \|f\|_{W^{1,2}(\mathbb{R}^2)}, \quad p > 2, \tag{4.30}$$

and the MacLaurin series

$$(\mathrm{e}^f - 1)^2 = f^2 + \sum_{s=3}^{\infty} \frac{2^s - 2}{s!} f^s, \tag{4.31}$$

it is seen that $\mathrm{e}^f - 1 \in L^2(\mathbb{R}^2)$ when $f \in W^{1,2}(\mathbb{R}^2)$. Applying this in (4.7)–(4.9) and using elliptic estimates, we have $w_\ell \in W^{2,2}(\mathbb{R}^2)$ ($\ell = 1, 2, 3$). In particular, $w_\ell(x) \to 0$ as $|x| \to \infty$, $\ell = 1, 2, 3$. In view of this property and (4.7)–(4.9), we see that the right-hand sides of (4.7)–(4.9) all lie in $L^p(\mathbb{R}^2)$ for any $p > 2$, which establishes $w_\ell \in W^{2,p}(\mathbb{R}^2)$ ($\ell = 1, 2, 3$) by elliptic L^p-estimates. Consequently, $|\nabla w_\ell|(x) \to 0$ as $|x| \to \infty$ ($\ell = 1, 2, 3$). Linearizing (4.7)–(4.9), we see that w_ℓ vanishes exponentially fast and ∇w_ℓ vanishes like $\mathrm{O}(|x|^{-3})$ at infinity, $\ell = 1, 2, 3$. Thus, we have

$$\int_{\mathbb{R}^2} \Delta w_\ell \, \mathrm{d}x = 0, \quad \ell = 1, 2, 3. \tag{4.32}$$

Integrating (4.7)–(4.9) over $\mathbb{R}^2$ and inserting (4.32) and the definitions of h_ℓ ($\ell = 1, 2, 3$), we have

$$\begin{aligned} &\alpha \int_{\mathbb{R}^2} \left(\mathrm{e}^{u_1^0 + \frac{1}{2}w_1 + \frac{1}{6}w_2 + \frac{1}{3}w_3} - \mathrm{e}^{u_2^0 - \frac{1}{2}w_1 + \frac{1}{6}w_2 + \frac{1}{3}w_3} \right) \mathrm{d}x \\ &= -\int_{\mathbb{R}^2} (h_1 - h_2) \, \mathrm{d}x = -4\pi(n_1 - n_2), \end{aligned} \tag{4.33}$$

$$\begin{aligned} &\alpha \int_{\mathbb{R}^2} \left(\mathrm{e}^{u_1^0 + \frac{1}{2}w_1 + \frac{1}{6}w_2 + \frac{1}{3}w_3} + \mathrm{e}^{u_2^0 - \frac{1}{2}w_1 + \frac{1}{6}w_2 + \frac{1}{3}w_3} - 2\mathrm{e}^{u_3^0 - \frac{1}{3}w_2 + \frac{1}{3}w_3} \right) \mathrm{d}x \\ &= -\int_{\mathbb{R}^2} (h_1 + h_2 - 2h_3) \, \mathrm{d}x = -4\pi(n_1 + n_2 - 2n_3), \end{aligned} \tag{4.34}$$

$$\begin{aligned} &2\beta \int_{\mathbb{R}^2} \left(\mathrm{e}^{u_1^0 + \frac{1}{2}w_1 + \frac{1}{6}w_2 + \frac{1}{3}w_3} + \mathrm{e}^{u_2^0 - \frac{1}{2}w_1 + \frac{1}{6}w_2 + \frac{1}{3}w_3} + \mathrm{e}^{u_3^0 - \frac{1}{3}w_2 + \frac{1}{3}w_3} - 3 \right) \mathrm{d}x \\ &= -\int_{\mathbb{R}^2} (h_1 + h_2 + h_3) \, \mathrm{d}x = -4\pi(n_1 + n_2 + n_3), \end{aligned} \tag{4.35}$$

as stated in the theorem.

5 Vortices in Chern–Simons–Higgs theory

In the context of supersymmetric Chern–Simons–Higgs theory in the standard (2+1)-dimensional Minkowski spacetime, recently developed by Aharony, Bergman, Jafferis, and Maldacena [3], known also as the ABJM model [6, 7, 19, 24, 51, 59], which is a Chern–Simons theory within which the matter fields are four complex scalars,

$$C^I = (Q^1, Q^2, R^1, R^2), \quad I = 1, 2, 3, 4, \tag{5.1}$$

in the bi-fundamental $(\mathbf{N}, \overline{\mathbf{N}})$ representation of the gauge group $U(N) \times U(N)$, which are all $N \times N$ complex matrices, of the gauge fields A_μ and B_μ, and the associated Chern–Simons terms for A_μ and B_μ are set at the levels κ and $-\kappa$ so that they give rise to the Lagrangian density

$$\mathcal{L}_{\rm CS} = \frac{\kappa}{4\pi}\epsilon^{\mu\nu\gamma}\mathrm{Tr}\left(A_\mu\partial_\nu A_\gamma + \frac{2\mathrm{i}}{3}A_\mu A_\nu A_\gamma - B_\mu\partial_\nu B_\gamma - \frac{2\mathrm{i}}{3}B_\mu B_\nu B_\gamma\right), \tag{5.2}$$

and the gauge-covariant derivative

$$D_\mu C^I = \partial_\mu C^I + \mathrm{i}A_\mu C^I - \mathrm{i}C^I B_\mu, \quad I = 1, 2, 3, 4. \tag{5.3}$$

The scalar potential density is of the mass-deformed form [19]

$$V = \mathrm{Tr}(M^{\alpha\dagger}M^\alpha + N^{\alpha\dagger}N^\alpha), \tag{5.4}$$

where

$$\begin{aligned} M^\alpha &= \sigma Q^\alpha + \frac{2\pi}{\kappa}(2Q^{[\alpha}Q^\dagger_\beta Q^{\beta]} + R^\beta R^\dagger_\beta Q^\alpha - Q^\alpha R^\dagger_\beta R^\beta \\ &\qquad + 2Q^\beta R^\dagger_\beta R^\alpha - 2R^\alpha R^\dagger_\beta Q^\beta), \end{aligned} \tag{5.5}$$

$$\begin{aligned} N^\alpha &= -\sigma R^\alpha + \frac{2\pi}{\kappa}(2R^{[\alpha}R^\dagger_\beta R^{\beta]} + Q^\beta Q^\dagger_\beta R^\alpha - R^\alpha Q^\dagger_\beta Q^\beta \\ &\qquad + 2R^\beta Q^\dagger_\beta Q^\alpha - 2Q^\alpha Q^\dagger_\beta R^\beta), \end{aligned} \tag{5.6}$$

the Kronecker symbol $\epsilon^{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ is used to lower or raise indices, and $\sigma > 0$ a massive parameter. Thus, when the spacetime metric is of the signature $(+--)$, the total (bosonic) Lagrangian density of the ABJM model can be written as

$$\mathcal{L} = -\mathcal{L}_{\rm CS} + \mathrm{Tr}([D_\mu C^I]^\dagger [D^\mu C^I]) - V, \tag{5.7}$$

which is of a pure Chern–Simons type for the gauge field sector. As in [6], we focus on a reduced situation where (say) $R^\alpha = 0$. Then, by virtue of (5.5) and (5.6), the scalar potential density (5.4) takes the form

$$V = \mathrm{Tr}(M^{\alpha\dagger}M^\alpha), \quad M^\alpha = \sigma Q^\alpha + \frac{4\pi}{\kappa}(Q^\alpha Q^\dagger_\beta Q^\beta - Q^\beta Q^\dagger_\beta Q^\alpha). \tag{5.8}$$

The equations of motion of the Lagrangian (5.7) are rather complicated. However, in the static limit, Auzzi and Kumar [6] showed that these equations may be reduced into the following first-order BPS system of equations

$$\begin{aligned} D_0 Q^1 - \mathrm{i} W^1 &= 0, \quad D_1 Q^2 + \mathrm{i} D_2 Q^2 = 0, & (5.9) \\ D_1 Q^1 &= 0, \quad D_2 Q^1 = 0, \quad D_0 Q^2 = 0, \quad W^2 = 0, & (5.10) \end{aligned}$$

coupled with the Gauss law constraints which are the temporal components of the Chern–Simons equations

$$\begin{aligned} \frac{\kappa}{4\pi}\epsilon^{\mu\nu\gamma} F^{(A)}_{\nu\gamma} &= \mathrm{i}(Q^\alpha [D^\mu Q^\alpha]^\dagger - [D^\mu Q^\alpha] Q^{\alpha\dagger}), & (5.11) \\ \frac{\kappa}{4\pi}\epsilon^{\mu\nu\gamma} F^{(B)}_{\nu\gamma} &= \mathrm{i}([D^\mu Q^\alpha]^\dagger Q^\alpha - Q^{\alpha\dagger}[D^\mu Q^\alpha]), & (5.12) \end{aligned}$$

where

$$\begin{aligned} F^{(A)}_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu], \\ F^{(B)}_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + \mathrm{i}[B_\mu, B_\nu], \\ W^1 &= \sigma Q^1 + \frac{2\pi}{\kappa}(Q^1 Q^{2\dagger} Q^2 - Q^2 Q^{2\dagger} Q^1), \\ W^2 &= \sigma Q^2 + \frac{2\pi}{\kappa}(Q^2 Q^{1\dagger} Q^1 - Q^1 Q^{1\dagger} Q^2), \end{aligned}$$

provided that [6] one takes the ansatz that Q^1 assumes its vacuum expectation value

$$Q^1 = \sqrt{\frac{\sigma\kappa}{2\pi}}\,\mathrm{diag}\left(0, 1, \cdots, \sqrt{N-2}, \sqrt{N-1}\right), \tag{5.13}$$

the non-trivial entries of Q^2 are given by $(N-1)$ complex scalar fields ψ and ϕ_ℓ $(\ell = 1, \cdots, N-2)$ according to

$$Q^2_{N,N-1} = \sqrt{\frac{\sigma\kappa}{2\pi}}\psi, \quad Q^2_{N-\ell,N-\ell-1} = \sqrt{\frac{\sigma\kappa(\ell+1)}{2\pi}}\phi_\ell, \quad \ell = 1, \cdots, N-2, \tag{5.14}$$

and the spatial components of the gauge fields A_j and B_j $(j = 1, 2)$ are expressed in terms of $(N-1)$ real-valued vector potentials $a = (a_j))$ and $b^\ell = (b^\ell_j)$ $(j = 1, 2; \ell = 1, \cdots, N-2)$ satisfying

$$A_j = B_j = \mathrm{diag}\left(0, b^{N-2}_j, \cdots, b^1_j, a_j\right), \quad j = 1, 2. \tag{5.15}$$

Within the above described formalism, the non-Abelian BPS vortex equations obtained by Auzzi and Kumar in [6], without restricting to the radially symmetric configurations, are of the form

$$\begin{aligned} (\partial_1 + \mathrm{i}\partial_2)\psi &= \mathrm{i}(a - b^1)\psi, & (5.16) \\ (\partial_1 + \mathrm{i}\partial_2)\phi_\ell &= \mathrm{i}(b^\ell - b^{\ell+1})\phi_\ell, \quad 1 \le \ell \le N-3, & (5.17) \end{aligned}$$

$$
\begin{aligned}
(\partial_1 + \mathrm{i}\partial_2)\phi_{N-2} &= \mathrm{i}b^{N-2}\phi_{N-2}, && (5.18)\\
a_{12} &= 2(N-1)\sigma^2(1-|\psi|^2), && (5.19)\\
b_{12}^1 &= 2(N-2)\sigma^2(1+|\psi|^2-2|\phi_1|^2), && (5.20)\\
b_{12}^\ell &= 2(N-1-\ell)\sigma^2(1+\ell|\phi_{\ell-1}|^2-(\ell+1)|\phi_\ell|^2), &&\\
& \quad 2 \leq \ell \leq N-2, && (5.21)
\end{aligned}
$$

where

$$
a = (a_1, a_2) = a_1 + \mathrm{i}a_2, \quad b^\ell = (b_1^\ell, b_2^\ell) = b_1^\ell + \mathrm{i}b_2^\ell, \quad \ell = 1, \cdots, N-2,
$$

are the conveniently complexified gauge vector fields and the summation convention is not applied to the repeated index ℓ. As before, the structure of the equations (5.16)–(5.18) implies that the zeros of the fields ψ, ϕ_ℓ ($\ell = 1, \cdots, N-2$) are discrete and of integer multiplicities which may collectively be expressed in the form of the respective finite sets

$$
Z_\psi = \{p_{1,1}, \cdots, p_{1,n_1}\}, \quad Z_{\phi_\ell} = \{p_{\ell+1,1}, \cdots, p_{\ell+1,n_{\ell+1}}\}, \quad \ell = 1, \cdots, N-2. \quad (5.22)
$$

For the prescribed sets of zeros given in (5.22), we are to construct a solution of (5.16)–(5.21) to realize these zeros. For this problem, here is our main theorem.

Theorem 5.1. *Consider the general BPS system of multiple vortex equations (5.16)–(5.21) for $(\psi, \phi_\ell, a, b^\ell)$ with the prescribed sets of zeros given in (5.22) so that ψ, ϕ_ℓ have $n_1, n_{\ell+1}$, $\ell = 1, \cdots, N-2$, arbitrarily distributed zeros, respectively.*

(i) For this problem over a doubly periodic domain Ω, a solution exists if and only if the following $(N-1)$ conditions

$$
4\pi \sum_{k=1}^{N-1} n_k < (N-1)\lambda|\Omega|, \quad (5.23)
$$

$$
4\pi \left(\frac{1}{N-1} \sum_{k=1}^{N-1} n_k + \frac{1}{N-2} \sum_{k=2}^{N-1} n_k \right) < 2\lambda|\Omega|, \quad (5.24)
$$

$$
4\pi \left(\frac{1}{N-1} \sum_{k=1}^{N-1} n_k + \frac{1}{N-2} \sum_{k=2}^{N-1} n_k + \cdots + \frac{1}{N-\ell} \sum_{k=\ell}^{N-1} n_k \right) < \ell\lambda|\Omega|, \quad \ell = 3, \cdots, N-1, \quad (5.25)
$$

are fulfilled simultaneously. Moreover, whenever a solution exists, it is unique.

(ii) For this problem over the full plane $\mathbb{R}^2$ subject to the boundary condition

$$
|\psi|^2, |\phi_\ell|^2 \to 1 \quad \text{as } |x| \to \infty, \quad \ell = 1, \cdots, N-2, \quad (5.26)
$$

there exists a unique solution up to gauge transformations so that the boundary behavior stated above is realized exponentially rapidly.

In either case, the excited total vortex fluxes are quantized quantities given explicitly by the formulas

$$\Phi_a = \int a_{12}\,\mathrm{d}x \quad = \quad 2\pi \sum_{k=1}^{N-1} n_k, \tag{5.27}$$

$$\Phi_{b^1} = \int b^1_{12}\,\mathrm{d}x \quad = \quad 2\pi \sum_{k=2}^{N-1} n_k, \tag{5.28}$$

$$\Phi_{b^\ell} = \int b^\ell_{12}\,\mathrm{d}x \quad = \quad 2\pi \sum_{k=\ell+1}^{N-1} n_k, \quad \ell = 2, \cdots, N-2, \tag{5.29}$$

respectively.

To approach the problem, we proceed as before. Firstly, note that, away from the zero sets given in (5.22), we may resolve the equations (5.16)–(5.18) to find

$$\begin{aligned} a &= -2\mathrm{i}\overline{\partial}\left(\ln\psi + \ln\phi_1 + \ln\phi_2 + \cdots + \ln\phi_{N-2}\right), && (5.30)\\ b^1 &= -2\mathrm{i}\overline{\partial}\left(\ln\phi_1 + \ln\phi_2 + \cdots + \ln\phi_{N-2}\right), && (5.31)\\ b^\ell &= -2\mathrm{i}\overline{\partial}\left(\ln\phi_\ell + \cdots + \ln\phi_{N-2}\right), \quad \ell = 2, \cdots, N-3, && (5.32)\\ b^{N-2} &= -2\mathrm{i}\overline{\partial}\ln\phi_{N-2}. && (5.33) \end{aligned}$$

Thus, following the same procedure as before to substitute (5.30)–(5.33) into (5.19)–(5.21), we arrive at

$$\begin{aligned} \Delta\left(\ln|\psi|^2 + \ln|\phi_1|^2 + \cdots + \ln|\phi_{N-2}|^2\right) &= \lambda(N-1)(|\psi|^2 - 1), && (5.34)\\ \Delta\left(\ln|\phi_1|^2 + \cdots + \ln|\phi_{N-2}|^2\right) &= \lambda(N-2)(2|\phi_1|^2 - |\psi|^2 - 1), && (5.35)\\ \Delta\left(\ln|\phi_\ell|^2 + \cdots + \ln|\phi_{N-2}|^2\right) &= \lambda(N-1-\ell)([\ell+1]|\phi_\ell|^2 - \ell|\phi_{\ell-1}|^2 - 1),\\ &\qquad \ell = 2, \cdots, N-3, && (5.36)\\ \Delta\ln|\phi_{N-2}|^2 &= \lambda([N-1]|\phi_{N-2}|^2 - [N-2]|\phi_{N-3}|^2 - 1), && (5.37) \end{aligned}$$

away from the zero sets (5.22), where $\lambda = 4\mu^2$. Next, set $m = N-1$ and

$$u_1 = \ln|\psi|^2, \quad u_\ell = \ln|\phi_{\ell-1}|^2, \quad \ell = 2, \cdots, N-1 = m. \tag{5.38}$$

Then, the equations (5.34)–(5.37) are converted into

$$\begin{aligned} \Delta(u_1 + u_2 + \cdots + u_m) &= \lambda m\left(\mathrm{e}^{u_1} - 1\right) + 4\pi \sum_{k=1}^{m}\sum_{s=1}^{n_k} \delta_{p_{k,s}}(x), && (5.39)\\ \Delta(u_2 + \cdots + u_m) &= \lambda(m-1)\left(2\mathrm{e}^{u_2} - \mathrm{e}^{u_1} - 1\right)\\ &\quad + 4\pi \sum_{k=2}^{m}\sum_{s=1}^{n_k} \delta_{p_{k,s}}(x), && (5.40) \end{aligned}$$

$$\Delta(u_\ell+\cdots+u_m) = \lambda(m-\ell+1)\left(\ell \mathrm{e}^{u_\ell}-[\ell-1]\mathrm{e}^{u_{\ell-1}}-1\right) +4\pi\sum_{k=\ell}^{m}\sum_{s=1}^{n_k}\delta_{p_{k,s}}(x), \quad \ell=3,\cdots,m-1, \quad (5.41)$$

$$\Delta u_m = \lambda\left(m\mathrm{e}^{u_m}-[m-1]\mathrm{e}^{u_{m-1}}-1\right) +4\pi\sum_{s=1}^{n_m}\delta_{p_{m,s}}(x). \quad (5.42)$$

The above formalism may be viewed as an SQCD extension of those of Hong, Kim, and Pac [28], Jackiw and Weinberg [30], and Dunne [10, 11], of the Abelian and non-Abelian Chern–Simons–Higgs theory. See also the survey [29].

In the following two sections, we first consider the equations over a doubly periodic domain. Then we consider the equations over the full plane.

6 Proof of existence for doubly periodic case

Now consider the equations (5.39)–(5.42) defined over a doubly periodic domain, say Ω. We use the direct method to solve them. Let u_ℓ^0 be some doubly periodic source functions over Ω satisfying

$$\Delta u_\ell^0 = -\frac{4\pi n_\ell}{|\Omega|}+4\pi\sum_{s=1}^{n_\ell}\delta_{p_{\ell,s}}(x), \quad \ell=1,2,\cdots,m. \quad (6.1)$$

Then the substitutions $u_\ell=u_\ell^0+v_\ell$, $\ell=1,2,\cdots,m$, give us the regularized equations

$$\Delta(v_1+v_2+\cdots+v_m) = \lambda m\left(\mathrm{e}^{u_1^0+v_1}-1\right)+\frac{4\pi}{|\Omega|}\sum_{k=1}^{m}n_k, \quad (6.2)$$

$$\Delta(v_2+\cdots+v_m) = \lambda(m-1)\left(2\mathrm{e}^{u_2^0+v_2}-\mathrm{e}^{u_1^0+v_1}-1\right)+\frac{4\pi}{|\Omega|}\sum_{k=2}^{m}n_k, \quad (6.3)$$

$$\Delta(v_\ell+\cdots+v_m) = \lambda(m-\ell+1)\left(\ell\mathrm{e}^{u_\ell^0+v_\ell}-[\ell-1]\mathrm{e}^{u_{\ell-1}^0+v_{\ell-1}}-1\right)+\frac{4\pi}{|\Omega|}\sum_{k=\ell}^{m}n_k,$$
$$j=3,\cdots,m-1, \quad (6.4)$$

$$\Delta v_m = \lambda\left(m\mathrm{e}^{u_m^0+v_m}-[m-1]\mathrm{e}^{u_{m-1}^0+v_{m-1}}-1\right)+\frac{4\pi}{|\Omega|}n_m. \quad (6.5)$$

Integrating the above equations, we obtain the constraints

$$\int_\Omega \mathrm{e}^{u_1^0+v_1}\,\mathrm{d}x = |\Omega|-\frac{4\pi}{\lambda}\left(\frac{1}{m}\sum_{k=1}^{m}n_k\right)\equiv\eta_1>0, \quad (6.6)$$

$$\int_\Omega \mathrm{e}^{u_2^0+v_2}\,\mathrm{d}x = |\Omega|-\frac{4\pi}{2\lambda}\left(\frac{1}{m}\sum_{k=1}^{m}n_k+\frac{1}{m-1}\sum_{k=2}^{m}n_k\right)\equiv\eta_2>0, \quad (6.7)$$

$$\int_\Omega \mathrm{e}^{u_\ell^0+v_\ell}\,\mathrm{d}x \;=\; |\Omega|-\frac{4\pi}{\ell\lambda}\left(\frac{1}{m}\sum_{k=1}^m n_k+\frac{1}{m-1}\sum_{k=2}^m n_k+\cdots+\frac{1}{m-\ell+1}\sum_{k=\ell}^m n_k\right)$$
$$\equiv\; \eta_\ell>0,\quad \ell=3,\cdots,m, \tag{6.8}$$

which are the conditions stated in (5.23)–(5.25). Moreover, it will be convenient to introduce the transformation

$$\left\{\begin{array}{lcl} w_1 &=& v_1+v_2+\cdots+v_m,\\ w_2 &=& v_2+\cdots+v_m,\\ \cdots &=& \cdots\cdots\cdots\cdots\\ w_\ell &=& v_\ell+\cdots+v_m,\\ \cdots &=& \cdots\cdots\\ w_m &=& v_m, \end{array}\right. \qquad \left\{\begin{array}{lcl} v_1 &=& w_1-w_2,\\ v_2 &=& w_2-w_3,\\ \cdots &=& \cdots\cdots\\ v_\ell &=& w_\ell-w_{\ell+1},\\ \cdots &=& \cdots\cdots\\ v_m &=& w_m. \end{array}\right. \tag{6.9}$$

Consequently, the governing equations become

$$\Delta w_1 \;=\; m\lambda\left(\mathrm{e}^{u_1^0+w_1-w_2}-1\right)+\frac{4\pi}{|\Omega|}\sum_{k=1}^m n_k, \tag{6.10}$$

$$\Delta w_2 \;=\; (m-1)\lambda\left(2\mathrm{e}^{u_2^0+w_2-w_3}-\mathrm{e}^{u_1^0+w_1-w_2}-1\right)+\frac{4\pi}{|\Omega|}\sum_{k=2}^m n_k, \tag{6.11}$$

$$\Delta w_\ell \;=\; (m-\ell+1)\lambda\left(\ell\mathrm{e}^{u_\ell^0+w_\ell-w_{\ell+1}}-[\ell-1]\mathrm{e}^{u_{\ell-1}^0+w_{\ell-1}-w_\ell}-1\right)$$
$$+\frac{4\pi}{|\Omega|}\sum_{k=\ell}^m n_k,\quad \ell=3,\cdots,m-1, \tag{6.12}$$

$$\Delta w_m \;=\; \lambda\left(m\mathrm{e}^{u_m^0+w_m}-[m-1]\mathrm{e}^{u_{m-1}^0+w_{m-1}-w_m}-1\right)+\frac{4\pi}{|\Omega|}n_m, \tag{6.13}$$

whose variational functional is seen to be

$$I(w_1,\cdots,w_m)$$
$$=\int_\Omega\left\{\frac{1}{2m\lambda}|\nabla w_1|^2+\cdots+\frac{1}{2(m-\ell+1)\lambda}|\nabla w_\ell|^2+\cdots+\frac{1}{2\lambda}|\nabla w_m|^2\right\}\mathrm{d}x$$
$$+J(w_1,\cdots,w_m), \tag{6.14}$$

where

$$J(w_1,\cdots,w_m) \;=\; \int_\Omega\left\{\left(\mathrm{e}^{u_1^0+w_1-w_2}-\left[1-\frac{4\pi}{m\lambda|\Omega|}\sum_{k=1}^m n_k\right]w_1\right)\right.$$
$$+\cdots+\left(\ell\mathrm{e}^{u_\ell^0+w_\ell-w_{\ell+1}}-\left[1-\frac{4\pi}{(m-\ell+1)\lambda|\Omega|}\sum_{k=\ell}^m n_k\right]w_\ell\right)$$
$$\left.+\cdots+\left(m\mathrm{e}^{u_m^0+w_m}-\left[1-\frac{4\pi}{\lambda|\Omega|}n_m\right]w_m\right)\right\}\mathrm{d}x. \tag{6.15}$$

On the other hand, in view of (6.9), we obtain after some algebra the representation

$$J(w_1,\cdots,w_m)=\sum_{\ell=1}^m \ell\left(\int_\Omega \mathrm{e}^{u_\ell^0+v_\ell}\,\mathrm{d}x-\eta_\ell\underline{v}_\ell\right). \tag{6.16}$$

Thus, we may use the same direct minimization method as before in a verbatim way to establish the existence and uniqueness of a critical point of the functional (6.14).

For completeness, we now sketch how to establish the existence of a critical point of (6.14) by a constrained minimization approach. For this purpose, we rewrite (6.6)–(6.8) as

$$J_\ell(w_1,\cdots,w_m)\equiv\int_\Omega \mathrm{e}^{u_\ell^0+v_\ell}\,\mathrm{d}x=\eta_\ell,\quad \ell=1,\cdots,m, \tag{6.17}$$

and consider the problem

$$\min\left\{I(w_1,\cdots,w_m)\,|\,(w_1,\cdots,w_m)\text{ satisfies (6.17) and lies in }W^{1,2}(\Omega)\right\}. \tag{6.18}$$

We use the notation $\mathbf{w}=(w_1,\cdots,w_m)$. If $\tilde{\mathbf{w}}$ is a solution to (6.18), then there are some numbers (the Lagrange multipliers) $\lambda_1,\cdots,\lambda_m\in\mathbb{R}$ so that

$$\left(DI(\tilde{\mathbf{w}})+\lambda_1 DJ_1(\tilde{\mathbf{w}})+\cdots+\lambda_m DJ_m(\tilde{\mathbf{w}})\right)(\mathbf{w})=0,\quad \forall\mathbf{w}. \tag{6.19}$$

Now insert in (6.19) the test configurations $\mathbf{w}=\mathbf{w}_\ell=(\delta_{1\ell},\cdots,\delta_{m\ell})$, $\ell=1,\cdots,m$. We have, after applying (6.17), the relations

$$\lambda_1\eta_1=0;\quad -\lambda_{\ell-1}\eta_{\ell-1}+\lambda_\ell\eta_\ell=0,\quad \ell=2,\cdots,m, \tag{6.20}$$

which lead us to $\lambda_1=\cdots=\lambda_m=0$. In other words, the constraints do not give rise to the undesired Lagrange multiplier problem so that a solution of the constrained minimization problem (6.18) is a critical point of the functional (6.14) itself.

Moreover, from the constraints (6.17), we have

$$\underline{v}_\ell = \underline{w}_\ell-\underline{w}_{\ell+1}=\ln\eta_\ell-\ln\left(\int_\Omega \mathrm{e}^{u_\ell^0+\dot{w}_\ell-\dot{w}_{\ell+1}}\,\mathrm{d}x\right),\quad \ell=1,\cdots,m-1, \tag{6.21}$$

$$\underline{v}_m = \underline{w}_m=\ln\eta_m-\ln\left(\int_\Omega \mathrm{e}^{u_m^0+\dot{w}_m}\,\mathrm{d}x\right). \tag{6.22}$$

Inserting these into (6.16) and applying the condition $\eta_\ell>0$ ($\ell=1,\cdots,m$) and the Jensen inequality, we again arrive at the coerciveness for the functional (6.14),

$$I(w_1,\cdots,w_m)\geq C_1\sum_{\ell=1}^m\int_\Omega|\nabla\dot{w}_\ell|^2\,\mathrm{d}x-C_2, \tag{6.23}$$

where $C_1,C_2>0$ are some irrelevant constants. Consequently, the existence of a solution to the problem (6.18) follows as before.

7 Proof of existence for planar case

To proceed, we define u_ℓ^0 and h_ℓ as in (4.3) and (4.4) so that ℓ runs from 1 through m. Thus, in view of the translations $u_\ell = u_\ell^0 + v_\ell$ ($\ell = 1, \cdots, m$) and the transformation (6.9), the governing equations (5.39)–(5.42) become

$$\Delta w_1 = m\lambda\left(\mathrm{e}^{u_1^0+w_1-w_2} - 1\right) + \sum_{k=1}^m h_k, \tag{7.1}$$

$$\Delta w_\ell = (m-\ell+1)\lambda\left(\ell\mathrm{e}^{u_\ell^0+w_\ell-w_{\ell+1}} - [\ell-1]\mathrm{e}^{u_{\ell-1}^0+w_{\ell-1}-w_\ell} - 1\right) + \sum_{k=\ell}^m h_k,$$
$$\ell = 2, \cdots, m-1, \tag{7.2}$$

$$\Delta w_m = \lambda\left(m\mathrm{e}^{u_m^0+w_m} - [m-1]\mathrm{e}^{u_{m-1}^0+w_{m-1}-w_m} - 1\right) + h_m, \tag{7.3}$$

which are the Euler–Lagrange equations of the functional

$$I(w_1, \cdots, w_m) = \int_{\mathbb{R}^2}\left\{\frac{1}{2\lambda}\sum_{\ell=1}^m \frac{1}{(m-\ell+1)}|\nabla w_\ell|^2\right\}\mathrm{d}x + J(w_1, \cdots, w_m), \tag{7.4}$$

where

$$J(w_1, \cdots, w_m) =$$
$$\int_{\mathbb{R}^2}\left\{\sum_{\ell=1}^{m-1}\left(\ell\mathrm{e}^{u_\ell^0+w_\ell-w_{\ell+1}} - \ell\mathrm{e}^{u_\ell^0} - w_\ell\right) + \left(m\mathrm{e}^{u_m^0+w_m} - m\mathrm{e}^{u_m^0} - w_m\right)\right\}\mathrm{d}x$$
$$+\int_{\mathbb{R}^2}\left\{\frac{1}{\lambda}\sum_{\ell=1}^{m-1}\frac{1}{(m-\ell+1)}\left(\sum_{k=\ell}^m h_k\right)w_\ell + \frac{1}{\lambda}h_m w_m\right\}\mathrm{d}x. \tag{7.5}$$

It is important to note that, using the relation

$$\sum_{\ell=1}^m w_\ell = \sum_{\ell=1}^{m-1}\ell(w_\ell - w_{\ell+1}) + m w_m, \tag{7.6}$$

we can rewrite $J(w_1, \cdots, w_m)$ defined in (7.5) as

$$J(w_1, \cdots, w_m) =$$
$$\int_{\mathbb{R}^2}\left\{\sum_{\ell=1}^{m-1}\ell\left(\mathrm{e}^{u_\ell^0+w_\ell-w_{\ell+1}} - \mathrm{e}^{u_\ell^0} - [w_\ell - w_{\ell+1}]\right) + m\left(\mathrm{e}^{u_m^0+w_m} - \mathrm{e}^{u_m^0} - w_m\right)\right\}\mathrm{d}x$$
$$+\int_{\mathbb{R}^2}\left\{\frac{1}{\lambda}\sum_{\ell=1}^{m-1}\frac{1}{(m-\ell+1)}\left(\sum_{k=\ell}^m h_k\right)w_\ell + \frac{1}{\lambda}h_m w_m\right\}\mathrm{d}x. \tag{7.7}$$

Consequently, after some algebraic manipulation, we obtain

$$
\begin{aligned}
(DI(w_1,\cdots,w_m))(w_1,\cdots,w_m) &= \frac{1}{\lambda}\int_{\mathbb{R}^2}\left\{\sum_{\ell=1}^{m}\frac{1}{(m-\ell+1)}|\nabla w_\ell|^2\right\}\,\mathrm{d}x \\
&+\int_{\mathbb{R}^2}\left\{\sum_{\ell=1}^{m-1}\left(\ell \mathrm{e}^{u_\ell^0}\left[\mathrm{e}^{w_\ell-w_{\ell+1}}-1\right][w_\ell-w_{\ell+1}]+\ell\left[\mathrm{e}^{u_\ell^0}-1\right][w_\ell-w_{\ell+1}]\right)\right. \\
&+m\mathrm{e}^{u_m^0}\left[\mathrm{e}^{w_m}-1\right]w_m+m\left[\mathrm{e}^{u_m^0}-1\right]w_m \\
&\left.+\frac{1}{\lambda}\sum_{\ell=1}^{m-1}\frac{1}{(m-\ell+1)}\left(\sum_{k=\ell}^{m}h_k\right)w_\ell+\frac{1}{\lambda}h_m w_m\right\}\,\mathrm{d}x. \qquad (7.8)
\end{aligned}
$$

On the other hand, in view of the transformation (6.9), we have

$$
c_1\sum_{\ell=1}^{m}v_\ell^2 \;\leq\; \sum_{\ell=1}^{m}w_\ell^2\leq c_2\sum_{\ell=1}^{m}v_\ell^2, \qquad (7.9)
$$

$$
c_1\sum_{\ell=1}^{m}|\nabla v_\ell|^2 \;\leq\; \sum_{\ell=1}^{m}|\nabla w_\ell|^2\leq c_2\sum_{\ell=1}^{m}|\nabla v_\ell|^2, \qquad (7.10)
$$

where $c_1, c_2 > 0$ are some constants. Thus (7.8) and (7.10) enable us to arrive at

$$
\begin{aligned}
&(DI(w_1,\cdots,w_m))(w_1,\cdots,w_m)\geq \\
&c_0\sum_{\ell=1}^{m}\int_{\mathbb{R}^2}|\nabla v_\ell|^2+\sum_{\ell=1}^{m}\int_{\mathbb{R}^2}\left\{\ell\left(\mathrm{e}^{u_\ell^0}\left[\mathrm{e}^{v_\ell}-1\right]v_\ell+\left[\mathrm{e}^{u_\ell^0}-1\right]v_\ell\right)+g_\ell v_\ell\right\}\,\mathrm{d}x, \quad (7.11)
\end{aligned}
$$

where $c_0 > 0$ is a suitable constant and the functions g_ℓ's are some linear combinations of the functions h_ℓ's. It has been seen that the structure of the right-hand side of (7.11) indicates that there are constants $C_1, C_2 > 0$ such that

$$
(DI(w_1,\cdots,w_m))(w_1,\cdots,w_m)\geq C_1\sum_{\ell=1}^{m}\int_{\mathbb{R}^2}\left(|\nabla v_\ell|^2+\frac{v_\ell^2}{1+|v_\ell|}\right)\,\mathrm{d}x-C_2. \quad (7.12)
$$

Therefore, applying (7.9), (7.10), and (7.12), we can again conclude with the coercive lower bound

$$
(DI(w_1,\cdots,w_m))(w_1,\cdots,w_m)\geq C_3\sum_{\ell=1}^{m}\|w_\ell\|_{W^{1,2}(\mathbb{R}^2)}-C_4, \qquad (7.13)
$$

for some constants $C_3, C_4 > 0$. Hence, the existence of a critical point of the functional (7.4) in the space $W^{1,2}(\mathbb{R}^2)$ follows. Since (7.4) is strictly convex in $(w_1,\cdots,w_m)\in W^{1,2}(\mathbb{R}^2)$ and C^1, it may have at most one critical point in $W^{1,2}(\mathbb{R}^2)$.

The rest of the analysis regarding asymptotic estimates and computation of fluxes is similar to that of Section 4 and is thus omitted.

8 Further applications of direct methods

In this section, we show that our direct minimization methods may be used to study other non-Abelian BPS vortex equations of similar structures arising in SQCD. We will present two examples as further illustrations.

8.1 Vortices in an $SO(2N)$ theory

In this subsection, we use the direct method developed earlier to strengthen the existence results obtained in [35] for an $SO(2N)$ BPS vortex problem formulated in [22].

Recall that, in the work of Gudnason–Jiang–Konishi [22], the Lagrangian density of the non-Abelian Yang–Mills–Higgs model reads

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4e^2}F^0_{\mu\nu}F^{0\mu\nu} - \frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + (\mathcal{D}_\mu q_f)^\dagger \mathcal{D}^\mu q_f \\ &\quad -\frac{e^2}{2}\left|q_f^\dagger t^0 q_f - \frac{v_0^2}{\sqrt{4N}}\right|^2 - \frac{g^2}{2}\left|q_f^\dagger t^a q_f\right|^2, \end{aligned} \tag{8.1}$$

for which the gauge group G is of the general form $G = G' \times U(1)$ where G' is a compact simple Lie group which may typically be chosen to be $G' = SO(2N)$ or $G' = USp(2N)$ (the unitary symplectic group). Assume that $a = 1, \cdots, \dim(G')$ labels the generators of G', the index 0 indicates the $U(1)$ gauge field, $f = 1, \cdots, N_{\text{flavor}}$ labels the matter flavors or 'scalar quark' fields, q_f, all are assumed to lie in the fundamental representation of G'. The gauge fields, gauge-covariant derivatives, and field tensors are given by

$$A_\mu = A^0_\mu + A^a_\mu t^a, \quad \mathcal{D}_\mu q_f = \partial_\mu q_f + \mathrm{i} A_\mu q_f, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu], \tag{8.2}$$

respectively, where the generators of G' and $U(1)$, i.e., $\{t^a\}$ and t^0, are normalized to satisfy

$$\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}, \quad t^0 = \frac{1}{\sqrt{4N}}\mathbf{1}_{2N}, \tag{8.3}$$

with $\mathbf{1}_m$ denoting the $m \times m$ identity matrix. When the number of matter flavors is $N_{\text{flavor}} = 2N$, the scalar quark fields may be represented as a color-flavor mixed matrix q of size $2N \times 2N$. Restricting to static field configurations which are uniform with respect to the spatial coordinate x^3, a Bogomol'nyi completion [8] may be performed to yield the BPS [8, 44] vortex equations [5, 12, 22, 38, 47]

$$\mathcal{D}_1 q + \mathrm{i}\mathcal{D}_2 q = 0, \tag{8.4}$$

$$F^0_{12} - \frac{e^2}{\sqrt{4N}}(\mathrm{Tr}(qq^\dagger) - v_0^2) = 0, \tag{8.5}$$

$$F^a_{12}t^a - \frac{g^2}{4}(qq^\dagger - J^\dagger(qq^\dagger)^T J) = 0, \tag{8.6}$$

where J is the standard symplectic matrix

$$J = \begin{pmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & 0 \end{pmatrix}. \tag{8.7}$$

In its general form, the system of the non-Abelian BPS vortex equations (8.4)–(8.6) appears hard to approach and an ansatz-based reduction may be made as a tool for further simplification. In the case when $G' = SO(2N)$, the ansätze presented in [22] gives us the following matrix forms for the Higgs field,

$$q = \begin{pmatrix} \Phi \mathbf{1}_{2N-2} & 0 & 0 \\ 0 & \phi & 0 \\ 0 & 0 & \psi \end{pmatrix}, \tag{8.8}$$

where Φ, ϕ, ψ are three complex scalar fields, and for the gauge potential,

$$A_j = a_j \mathbf{1}_{2N} + b_j \text{diag}\{\mathbf{0}_{2N-2}, 1, -1\}, \quad j = 1, 2, \tag{8.9}$$

where a_j and b_j ($j = 1, 2$) are real-valued vector fields. Then, in terms of the complexified field a and b defined in (2.16),

the BPS system of vortex equations found in [22] assumes the form

$$\overline{\partial}\Phi = \mathrm{i}a\Phi, \tag{8.10}$$

$$\overline{\partial}\phi = \mathrm{i}(a+b)\phi, \tag{8.11}$$

$$\overline{\partial}\psi = \mathrm{i}(a-b)\psi, \tag{8.12}$$

$$a_{12} = \frac{e^2}{4N}(\xi^2 - 2(N-1)|\Phi|^2 - |\phi|^2 - |\psi|^2), \tag{8.13}$$

$$b_{12} = \frac{g^2}{4}(|\psi|^2 - |\phi|^2). \tag{8.14}$$

Thus, we can recast the system of equations (8.10)–(8.14) into

$$\Delta \ln |\Phi|^4 = \Delta \left(\ln|\phi|^2 + \ln|\psi|^2\right), \tag{8.15}$$

$$\Delta \left(\ln|\phi|^2 + \ln|\psi|^2\right) = \frac{2e^2}{N}\left(2(N-1)|\Phi|^2 + |\phi|^2 + |\psi|^2 - \xi^2\right), \tag{8.16}$$

$$\Delta \left(\ln|\phi|^2 - \ln|\psi|^2\right) = 2g^2(|\phi|^2 - |\psi|^2), \tag{8.17}$$

where we have stayed away from the possible zeros of the fields Φ, ϕ, ψ. We extend our study in [35] and consider a solution so that the zeros of Φ coincide with those of ϕ and ψ. As a consequence of the boundary condition, we see that (8.15) leads us to the simple relation

$$|\Phi|^4 = |\phi|^2|\psi|^2. \tag{8.18}$$

We are interested in constructing solutions over a doubly periodic domain, Ω. The multiple vortices are generated from the sets of zeros of ϕ and ψ, prescribed as

$$Z_\phi = \{p_1, \cdots, p_m\}, \quad Z_\psi = \{q_1, \cdots, q_n\}. \tag{8.19}$$

Therefore the vortex-governing equations are then given in terms of the new functions $u = \ln|\phi|^2$ and $v = \ln|\psi|^2$ as

$$\Delta(u+v) = \alpha\left(2(N-1)\mathrm{e}^{\frac{1}{2}(u+v)} + \mathrm{e}^u + \mathrm{e}^v - \gamma\right) + 4\pi\sum_{s=1}^m \delta_{p_s}(x) + 4\pi\sum_{s=1}^n \delta_{q_s}(x), \quad (8.20)$$

$$\Delta(u-v) = \beta\left(\mathrm{e}^u - \mathrm{e}^v\right) + 4\pi\sum_{s=1}^n \delta_{p_s}(x) - 4\pi\sum_{s=1}^n \delta_{q_s}(x), \quad (8.21)$$

where α, β, γ are positive constants given by

$$\alpha = \frac{2e^2}{N}, \quad \beta = 2g^2, \quad \gamma = \xi^2. \quad (8.22)$$

In [35], we proved an existence and uniqueness theorem for the solution of (8.20) and (8.21) when $Z_\psi = \emptyset$ (or $n = 0$) under the necessary and sufficient condition

$$4\pi m\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) < \gamma|\Omega|, \quad (8.23)$$

and we found suitable conditions under which the solution may be constructed by a constrained minimization method. The general existence proof in [35], however, is based on *a priori* estimates and a degree theory argument which is unfortunately non-constructive. Here we show that the solution can actually be obtained by the (constructive) direct minimization method used in the earlier sections of the present paper. For the broadest generality, we consider the presence of the zeros of ψ as well ($n \geq 0$). We are able to obtain the following sharp results.

Theorem 8.1. *For the non-Abelian vortex equations (8.20) and (8.21) defined over the doubly periodic domain Ω, a solution exists if and only if the condition*

$$4\pi\left(\frac{(m+n)}{\alpha} + \frac{|m-n|}{\beta}\right) < \gamma|\Omega|. \quad (8.24)$$

Furthermore, if a solution exists, it is unique and may be constructed by a direct minimization method.

To proceed, let u_0 and v_0 be solutions of the equations

$$\Delta u_0 = 4\pi\sum_{s=1}^m \delta_{p_s}(x) - \frac{4\pi m}{|\Omega|}, \quad \Delta v_0 = 4\pi\sum_{s=1}^n \delta_{q_s}(x) - \frac{4\pi n}{|\Omega|}. \quad (8.25)$$

Then the substitutions $u = u_0 + U$ and $v = v_0 + V$ change the equations (8.20) and (8.21) into

$$\begin{aligned}\Delta(U+V) &= \alpha\left(2(N-1)\mathrm{e}^{\frac{1}{2}(u_0+v_0)+\frac{1}{2}(U+V)} + \mathrm{e}^{u_0+U} + \mathrm{e}^{v_0+V} - \gamma\right) \\ &\quad + \frac{4\pi}{|\Omega|}(m+n), \quad (8.26)\\ \Delta(U-V) &= \beta\left(\mathrm{e}^{u_0+U} - \mathrm{e}^{v_0+V}\right) + \frac{4\pi}{|\Omega|}(m-n). \quad (8.27)\end{aligned}$$

Next, use the transformation

$$\left\{\begin{array}{ll} f &= \frac{1}{2}(U+V), \\ g &= \frac{1}{2}(U-V), \end{array}\right. \qquad \left\{\begin{array}{ll} U &= f+g, \\ V &= f-g. \end{array}\right. \tag{8.28}$$

We see that f, g satisfy

$$\Delta f = \alpha\left((N-1)\mathrm{e}^{\frac{1}{2}(u_0+v_0)+f} + \frac{1}{2}\mathrm{e}^{u_0+f+g} + \frac{1}{2}\mathrm{e}^{v_0+f-g} - \frac{\gamma}{2}\right) + \frac{2\pi}{|\Omega|}(m+n), \tag{8.29}$$

$$\Delta g = \frac{\beta}{2}\left(\mathrm{e}^{u_0+f+g} - \mathrm{e}^{v_0+f-g}\right) + \frac{2\pi}{|\Omega|}(m-n), \tag{8.30}$$

which are the Euler–Lagrange equations of the functional

$$\begin{aligned} I(f,g) = \int_\Omega \Bigg\{ & \frac{1}{2\alpha}|\nabla f|^2 + \frac{1}{2\beta}|\nabla g|^2 + (N-1)\mathrm{e}^{\frac{1}{2}(u_0+v_0)+f} \\ & + \frac{1}{2}\left(\mathrm{e}^{u_0+f+g} + \mathrm{e}^{v_0+f-g}\right) + \left(\frac{2\pi}{\alpha|\Omega|}[m+n] - \frac{\gamma}{2}\right) f + \frac{2\pi}{\beta|\Omega|}[m-n]g \Bigg\}\,\mathrm{d}x. \end{aligned} \tag{8.31}$$

On the other hand, integrating the equations (8.29) and (8.30), we obtain the natural constraints

$$\begin{aligned} \int_\Omega \left\{(N-1)\mathrm{e}^{\frac{1}{2}(u_0+v_0)+f} + \mathrm{e}^{u_0+f+g}\right\}\mathrm{d}x &= \frac{\gamma}{2}|\Omega| - \frac{2\pi}{\alpha}(m+n) - \frac{2\pi}{\beta}(m-n) \\ &\equiv \eta_1 > 0, \end{aligned} \tag{8.32}$$

$$\begin{aligned} \int_\Omega \left\{(N-1)\mathrm{e}^{\frac{1}{2}(u_0+v_0)+f} + \mathrm{e}^{v_0+f-g}\right\}\mathrm{d}x &= \frac{\gamma}{2}|\Omega| - \frac{2\pi}{\alpha}(m+n) + \frac{2\pi}{\beta}(m-n) \\ &\equiv \eta_2 > 0. \end{aligned} \tag{8.33}$$

In terms of the quantities η_1 and η_2 in (8.32) and (8.33), we may rewrite (8.31) as

$$\begin{aligned} I(f,g) &= \int_\Omega \left\{\frac{1}{2\alpha}|\nabla \dot{f}|^2 + \frac{1}{2\beta}|\nabla \dot{g}|^2 + (N-1)\mathrm{e}^{\frac{1}{2}(u_0+v_0)+f}\right\}\mathrm{d}x \\ &\quad + \frac{1}{2}\left(\int_\Omega \left\{\mathrm{e}^{u_0+\underline{U}+\dot{U}} + \mathrm{e}^{v_0+\underline{V}+\dot{V}}\right\}\mathrm{d}x - \eta_1\underline{U} - \eta_2\underline{V}\right). \end{aligned} \tag{8.34}$$

In view of the analysis presented earlier (e.g., §3.1 and (3.17)–(3.22) in particular), we see that the existence of a critical point of the functional (8.31) follows as a consequence of the condition $\eta_1, \eta_2 > 0$.

Finally, adding η_1 and η_2, we have

$$\eta \equiv \gamma|\Omega| - \frac{4\pi}{\alpha}(m+n) > 0. \tag{8.35}$$

Thus, in view of $\eta_1, \eta_2 > 0$ again, we have

$$-\eta < \frac{4\pi}{\beta}(m-n) < \eta, \tag{8.36}$$

which leads to the single condition (8.24). On the other hand, it is obvious that (8.24) also implies $\eta_1, \eta_2 > 0$. Thus the proof of the theorem follows.

8.2 Vortices in a softly broken SQCD model

In this subsection, we construct multiple vortices in the SQCD model of Marshakov and Yung [38] in which the confinement is achieved through Abelian fluxes generated in the Cartan subalgebra sector of $SU(3)$ so that, in terms of the Gell-Mann matrices $\hat{\lambda}_3$ and $\hat{\lambda}_8$, the gauge field A_μ assumes the form

$$A_\mu = A_\mu^{(3)}\hat{\lambda}_3 + A_\mu^{(8)}\hat{\lambda}_8, \tag{8.37}$$

where $A_\mu^{(3)}$ and $A_\mu^{(8)}$ are two real-valued vector fields. As in [38], we use u and d to denote the up and down colors of quarks, which are represented by a pair of complex-valued Higgs scalar fields, say $\phi^{(\mathrm{u})}$ and $\phi^{(\mathrm{d})}$, respectively. The u- and d-fluxes will be the Cartan subalgebra valued which are induced from the gauge fields

$$A_\mu^{(\mathrm{u})} = \frac{\sqrt{3}}{2}A_\mu^{(3)} + \frac{1}{2}A_\mu^{(8)}, \quad A_\mu^{(\mathrm{d})} = -\frac{\sqrt{3}}{2}A_\mu^{(3)} + \frac{1}{2}A_\mu^{(8)}. \tag{8.38}$$

With the above notation, the effective action functional of the SQCD model of Marshakov and Yung [38] reads

$$\begin{aligned} S &= \int\Bigg\{\frac{1}{4g^2}F_{\mu\nu}^{(3)}F^{(3)\mu\nu} + \frac{1}{4g^2}F_{\mu\nu}^{(8)}F^{(8)\mu\nu} \\ &\quad + \left(\overline{\nabla_\mu^{(\mathrm{u})}\phi^{(\mathrm{u})}}\right)\nabla^{(\mathrm{u})\mu}\phi^{(\mathrm{u})} + \left(\overline{\nabla_\mu^{(\mathrm{d})}\phi^{(\mathrm{d})}}\right)\nabla^{(\mathrm{d})\mu}\phi^{(\mathrm{d})} \\ &\quad + \frac{g^2}{8}\left(|\phi^{(\mathrm{u})}|^2 - |\phi^{(\mathrm{d})}|^2 - \xi[1-\omega]\right)^2 + \frac{g^2}{24}\left(|\phi^{(\mathrm{u})}|^2 + |\phi^{(\mathrm{d})}|^2 - \xi[1+\omega]\right)^2\Bigg\}\,\mathrm{d}x, \end{aligned} \tag{8.39}$$

where $g, \xi, \omega > 0$ are all physical constants, and

$$\nabla_\mu^{(\mathrm{u},\mathrm{d})} = \partial_\mu - \frac{\mathrm{i}}{\sqrt{3}}A_\mu^{(\mathrm{u},\mathrm{d})}, \quad \mu = 0,1,2,3, \tag{8.40}$$

are gauge-covariant derivatives to be operated upon $\phi^{(\mathrm{u},\mathrm{d})}$, accordingly. Concentrating on static case for which the field configurations are uniform in a spatial direction, say x^3, we see that the method of Bogomol'nyi [8] may be used to show that the equations of motion may be reduced into the following BPS [8, 44] system [38]

$$\nabla_1^{(\mathrm{u},\mathrm{d})}\phi^{(\mathrm{u},\mathrm{d})} + \mathrm{i}\nabla_2^{(\mathrm{u},\mathrm{d})}\phi^{(\mathrm{u},\mathrm{d})} = 0, \tag{8.41}$$

$$F_{12}^{(3)} + \frac{g^2}{2}\left(|\phi^{(\mathrm{u})}|^2 - |\phi^{(\mathrm{d})}|^2 - \xi[1-\omega]\right) = 0, \tag{8.42}$$

$$F_{12}^{(8)} + \frac{g^2}{2\sqrt{3}}\left(|\phi^{(\mathrm{u})}|^2 + |\phi^{(\mathrm{d})}|^2 - \xi[1+\omega]\right) = 0, \tag{8.43}$$

subject to the boundary condition

$$|\phi^{(\mathrm{u})}|^2 \to \xi, \quad |\phi^{(\mathrm{d})}|^2 \to \xi\omega, \tag{8.44}$$

so that the (minimum) vortex-line energy or tension may be calculated via the flux formula

$$T=\frac{\xi}{\sqrt{3}}\int_{\mathbb{R}^2}\left(F_{12}^{(\mathrm{u})}+\omega F_{12}^{(\mathrm{d})}\right)\,\mathrm{d}x. \tag{8.45}$$

With the notation

$$\phi=\phi^{(\mathrm{u})},\quad \psi=\phi^{(\mathrm{d})},\quad a_\mu=\frac{1}{2}A_\mu^{(3)}+\frac{\sqrt{3}}{2}A_\mu^{(8)},\quad b_\mu=A_\mu^{(\mathrm{d})}, \tag{8.46}$$

and the relations

$$A_\mu^{(\mathrm{u})}=\frac{\sqrt{3}}{2}a_\mu-\frac{1}{2}b_\mu,\quad \nabla_\mu^{(\mathrm{u})}=\partial_\mu-\frac{\mathrm{i}}{2}\left(a_\mu-\frac{1}{\sqrt{3}}b_\mu\right),\quad \nabla_\mu^{(\mathrm{d})}=\partial_\mu-\frac{\mathrm{i}}{\sqrt{3}}b_\mu, \tag{8.47}$$

we may rewrite the equations (8.41)–(8.43) as

$$(\partial_1+\mathrm{i}\partial_2)\phi = \frac{\mathrm{i}}{2}\left([a_1+\mathrm{i}a_2]-\frac{1}{\sqrt{3}}[b_1+\mathrm{i}b_2]\right)\phi, \tag{8.48}$$

$$(\partial_1+\mathrm{i}\partial_2)\psi = \frac{\mathrm{i}}{\sqrt{3}}(b_1+\mathrm{i}b_2)\psi, \tag{8.49}$$

$$a_{12} = \frac{g^2}{2}(\xi-|\phi|^2), \tag{8.50}$$

$$b_{12} = \frac{g^2}{\sqrt{3}}\left(\xi\left[\omega-\frac{1}{2}\right]+\frac{1}{2}|\phi|^2-|\psi|^2\right). \tag{8.51}$$

Assume the sets of zeros of ϕ and ψ are as prescribed in (8.19). Then, as before, the substitution, $u=\ln|\phi|^2$ and $v=\ln|\psi|^2$, allows us to transform the equations (8.48)–(8.51) into

$$\Delta u+\frac{1}{2}\Delta v = \frac{g^2}{2}(\mathrm{e}^u-\xi)+4\pi\sum_{s=1}^{m}\delta_{p_s}(x)+2\pi\sum_{s=1}^{n}\delta_{q_s}(x), \tag{8.52}$$

$$\Delta v = \frac{g^2}{3}\left(-\mathrm{e}^u+2\mathrm{e}^v-\xi[2\omega-1]\right)+4\pi\sum_{s=1}^{n}\delta_{q_s}(x). \tag{8.53}$$

We consider a doubly periodic domain Ω first. Let u_0 and v_0 be given in (8.25). Then $u=u_0+U$ and $v=v_0+V$ recast (8.52)–(8.53) into

$$\Delta U+\frac{1}{2}\Delta V = \frac{g^2}{2}(\mathrm{e}^{u_0+U}-\xi)+\frac{4\pi m}{|\Omega|}+\frac{2\pi n}{|\Omega|}, \tag{8.54}$$

$$\Delta V = \frac{g^2}{3}\left(-\mathrm{e}^{u_0+U}+2\mathrm{e}^{v_0+V}-\xi[2\omega-1]\right)+\frac{4\pi n}{|\Omega|}. \tag{8.55}$$

Set $W=U+\frac{1}{2}V$. We arrive at

$$\Delta V = \frac{g^2}{3}\left(2\mathrm{e}^{v_0+V}-\mathrm{e}^{u_0-\frac{1}{2}V+W}-\xi[2\omega-1]\right)+\frac{4\pi n}{|\Omega|}, \tag{8.56}$$

$$\Delta W = \frac{g^2}{2}\left(\mathrm{e}^{u_0-\frac{1}{2}V+W}-\xi\right)+\frac{2\pi}{|\Omega|}(2m+n), \tag{8.57}$$

which are the Euler–Lagrange equations of the functional

$$I(V,W) = \int_\Omega \left\{ \frac{3}{4g^2}|\nabla V|^2 + \frac{1}{g^2}|\nabla W|^2 + \mathrm{e}^{u_0 - \frac{1}{2}V + W} + \mathrm{e}^{v_0+V} - \left(\xi\left[\omega - \frac{1}{2}\right] - \frac{6\pi n}{g^2|\Omega|}\right) V - \left(\xi - \frac{4\pi}{g^2|\Omega|}[2m+n]\right) W \right\} \mathrm{d}x. \tag{8.58}$$

To proceed further, we integrate (8.56) and (8.57) to obtain the constraints

$$\int_\Omega \mathrm{e}^{u_0 - \frac{1}{2}V+W}\,\mathrm{d}x = \xi|\Omega| - \frac{4\pi}{g^2}(2m+n) \equiv \eta_1 > 0, \tag{8.59}$$

$$\int_\Omega \mathrm{e}^{v_0+V}\,\mathrm{d}x = \xi\omega|\Omega| - \frac{4\pi}{g^2}(m+2n) \equiv \eta_2 > 0. \tag{8.60}$$

Therefore, we have

$$I(V,W) - \frac{1}{g^2}\int_\Omega \left\{\frac{3}{4}|\nabla \dot{V}|^2 + |\nabla \dot{W}|^2\right\}\mathrm{d}x = \left(\int_\Omega \mathrm{e}^{u_0 + \underline{U} + \dot{U}}\,\mathrm{d}x - \eta_1 \underline{U}\right) + \left(\int_\Omega \mathrm{e}^{v_0 + \underline{V} + \dot{V}}\,\mathrm{d}x - \eta_2 \underline{V}\right). \tag{8.61}$$

In view of (8.59)–(8.61), we see that the existence of a critical point, in fact, a global minimizer, of the functional (8.58) in $W^{1,2}(\Omega)$ follows as before.

We now consider (8.52) and (8.53) over the full plane. In (8.19), use the notation

$$p_s = p_{1,s}, \quad s = 1, \cdots, m \equiv n_1; \quad q_s = p_{2,s}, \quad s = 1, \cdots, n \equiv n_2. \tag{8.62}$$

Let u_ℓ^0 and h_ℓ be defined as in (4.3) and (4.4), respectively, $\ell = 1, 2$. Introduce the translations

$$u = \ln \xi + u_1^0 + v_1, \quad v = \ln(\xi\omega) + u_2^0 + v_2, \tag{8.63}$$

and the refined parameters

$$\alpha = \frac{1}{2}g^2\xi, \quad \beta = \frac{1}{3}g^2\xi, \quad \gamma = \frac{2}{3}g^2\xi\omega. \tag{8.64}$$

We do so since the ranges of these parameters will not be important for our existence theory over $\mathbb{R}^2$. The equations (8.52) and (8.53) now become

$$\Delta v_1 + \frac{1}{2}\Delta v_2 = \alpha\left(\mathrm{e}^{u_1^0+v_1} - 1\right) + h_1 + \frac{1}{2}h_2, \tag{8.65}$$

$$\Delta v_2 = -\beta\left(\mathrm{e}^{u_1^0+v_1} - 1\right) + \gamma\left(\mathrm{e}^{u_2^0+v_2} - 1\right) + h_2. \tag{8.66}$$

Set $v_1+\frac{1}{2}v_2=w_1, v_2=w_2$. Then we have the following modified system of equations

$$\begin{aligned}\Delta w_1 &= \alpha\left(\mathrm{e}^{u_1^0+w_1-\frac{1}{2}w_2}-1\right)+h_1+\frac{1}{2}h_2, &(8.67)\\ \Delta w_2 &= -\beta\left(\mathrm{e}^{u_1^0+w_1-\frac{1}{2}w_2}-1\right)+\gamma\left(\mathrm{e}^{u_2^0+w_2}-1\right)+h_2, &(8.68)\end{aligned}$$

which are the Euler–Lagrange equations of the functional

$$\begin{aligned}I(w_1,w_2) &= \int_{\mathbb{R}^2}\left\{\frac{1}{2\alpha}|\nabla w_1|^2+\frac{1}{4\beta}|\nabla w_2|^2+\mathrm{e}^{u_1^0+w_1-\frac{1}{2}w_2}-\mathrm{e}^{u_1^0}-\left[w_1-\frac{1}{2}w_2\right]\right.\\ &\quad\left.+\frac{\gamma}{2\beta}\left(\mathrm{e}^{u_2^0+w_2}-\mathrm{e}^{u_2^0}-w_2\right)+\frac{1}{\alpha}\left(h_1+\frac{1}{2}h_2\right)w_1+\frac{1}{2\beta}h_2w_2\right\}\mathrm{d}x. \qquad (8.69)\end{aligned}$$

It is clear that this functional is C^1 and strictly convex over $W^{1,2}(\mathbb{R}^2)$. Besides, we have

$$\begin{aligned}(DI(w_1,w_2))(w_1,w_2) &= \int_{\mathbb{R}^2}\left\{\frac{1}{\alpha}|\nabla w_1|^2+\frac{1}{2\beta}|\nabla w_2|^2\right\}\mathrm{d}x\\ &+\int_{\mathbb{R}^2}\left\{\left(\mathrm{e}^{u_1^0+w_1-\frac{1}{2}w_2}-1\right)\left(w_1-\frac{1}{2}w_2\right)+\frac{\gamma}{2\beta}\left(\mathrm{e}^{u_2^0+w_2}-1\right)w_2\right.\\ &\left.+\frac{1}{\alpha}\left(h_1+\frac{1}{2}h_2\right)w_1+\frac{1}{2\beta}h_2w_2\right\}\mathrm{d}x. \qquad (8.70)\end{aligned}$$

Thus, we can show that there are constants $C_1,C_2>0$ such that

$$(DI(w_1,w_2))(w_1,w_2)\geq C_1\left(\|w_1\|_{W^{1,2}(\mathbb{R}^2)}+\|w_2\|_{W^{1,2}(\mathbb{R}^2)}\right)-C_2,\quad\forall w_1,w_2. \qquad (8.71)$$

Consequently, the existence and uniqueness of a critical point of the functional (8.69) over $W^{1,2}(\mathbb{R}^2)$ is established.

In summary, we may state

Theorem 8.2. *Consider the BPS system of SQCD vortex equations (8.48)–(8.51) for (ϕ,ψ,a_j,b_j) with the prescribed sets of zeros given in (8.19) so that ϕ,ψ have m,n arbitrarily distributed zeros, respectively.*

(i) For this problem over a doubly periodic domain Ω, a solution exists if and only if the condition

$$\max\left\{\frac{1}{\omega}(m+2n),2m+n\right\}<\frac{g^2\xi|\Omega|}{4\pi} \qquad (8.72)$$

holds. Moreover, whenever a solution exists, it is unique.

(ii) For this problem over the full plane $\mathbb{R}^2$ subject to the finite-energy boundary condition

$$|\phi|^2\to\xi,\quad|\psi|^2\to\xi\omega,\quad\text{as }|x|\to\infty, \qquad (8.73)$$

there exists a unique solution up to gauge transformations so that the boundary behavior stated above is realized exponentially rapidly.

In either case, the excited total vortex fluxes are quantized quantities given explicitly by the formulas

$$\Phi_a = \int a_{12}\,\mathrm{d}x = 2\pi(2m+n), \quad \Phi_b = \int b_{12}\,\mathrm{d}x = 2\sqrt{3}\pi n, \tag{8.74}$$

respectively, and the solutions may be obtained by methods of direct minimization.

We note that the condition (8.72) is simply a suppressed restatement of the two simultaneous conditions (8.59) and (8.60).

We also note that, applying the relations between the gauge fields a_μ, b_μ and $A_\mu^{(3)}, A_\mu^{(8)}$, we easily obtain the Abelian (or the Cartan subalgebra valued) fluxes

$$\Phi^{(3)} = \int F_{12}^{(3)}\,\mathrm{d}x = 2\pi(m-n), \quad \Phi^{(8)} = \int F_{12}^{(8)}\,\mathrm{d}x = 2\sqrt{3}\pi(m+n). \tag{8.75}$$

Moreover, the u- and d- fluxes may be expressed by the formulas

$$\Phi^{(\mathrm{u})} = \int F_{12}^{(\mathrm{u})} = 2\sqrt{3}\pi m, \quad \Phi^{(\mathrm{d})} = \int F_{12}^{(\mathrm{d})} = 2\sqrt{3}\pi n, \tag{8.76}$$

which depend on the winding numbers of the u- and d-Higgs fields $\phi^{(\mathrm{u})}$ and $\phi^{(\mathrm{d})}$, respectively, and give rise to the tension or energy of the vortex-lines

$$T = 2\pi(m+\omega n)\xi, \tag{8.77}$$

according to (8.45).

We also note that our direct method works easily for the classical BPS Abelian Higgs vortex equations [31] defined over doubly periodic domains [60] or formulated over compact Riemann surfaces [41, 42].

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