Quantum fields with integer spins

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Abstract

The analogy with the spin 1, 2 and 3 situation enables us to introduce action for the arbitrary integer spin situation. It is not excluded that the theory has possible application in quantum cosmology with spin stars.

1 Introduction

Spin was originally introduced as the rotation with an angular momentum of a particle around some axis. On the other hand, spin has some peculiar properties that distinguish it from orbital angular momenta: a) Spin quantum numbers may take half-integer values. b) Although the direction of its spin can be changed, an elementary particle cannot be made to spin faster or slower. c) The spin of a charged particle is associated with a magnetic dipole moment with a g-factor differing from 1. This could only occur classically if the internal charge of the particle were distributed differently from its mass.

The conventional definition of the spin quantum number, s, is s = n/2, where n can be any non-negative integer. So, the allowed values of s are 0, 1/2, 1, 3/2, 2, etc. The value of s for an elementary particle depends only on the type of particle, and cannot be altered in any known way (in contrast to the spin direction).

We approach here the way from the spin 1 fields to the spin 2 fields and spin 3 fields for massive and massless particles and we derive the corresponding action for spin gravity from this base. Then, we approach the integer spin.

2 The spin 1 field equations

We show that the natural construction of the field of the particles with spin 1 is presented in source theory method. The relation

$$|\langle 0_+|0_-\rangle|^2 = \exp\{-2\mathrm{ImW}\} \le 1 \tag{1}$$

is postulated to be valid for all spin fields. Let us show here the construction of action and field equations concerning spin one.

If spin zero particles and fields are described by the scalar source, then a vector source denoted here as $J^\mu(x)$ can be considered as a candidate for the description of the spin 1 fields and particles. However, there exist some obstacles because source $J^\mu(x)$ has four components and spin one particles have only three spin possibilities. Nevertheless first, let us investigate by analogy with the spin zero fields the following form of the action for the unit spin fields:

$$W(J) = \frac{1}{2}\int (dx)(dx')J^\mu(x)\Delta_+(x-x')J_\mu(x'). \tag{2}$$

Then,

$$|\langle 0_+|0_-\rangle|^2 = e^{iW}e^{iW^*} = \exp\left\{-\int d\omega_p J^{*\mu}(p)J_\mu(p)\right\}. \tag{3}$$

However,

$$J^{*\mu}(p)J_\mu(p) = |\mathbf{J}(p)|^2 - |J^0(p)|^2 \le 0, \quad \text{or}, > 0 \tag{4}$$

and it means that the quantity defined by eq. (3) cannot be considered as the probability of the persistence of vacuum.

The difficulty can be overcome by replacing the original form $J^{*\mu}(x)J_\mu(x)$ by the following invariant structure:

$$J^{*\mu}(p)\left[g_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu\right]J^\nu(p), \tag{5}$$

which can be with regard to its invariance, determined in the rest frame of the time-like vector p^μ, where $p^\mu = (m,0,0,0)$ in the rest frame. Then, with $g_{\alpha\alpha} = (-1,1,1,1)$ we have

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu = \begin{cases} \delta_{kl}; & \mu = k; \quad \nu = l \\ 0; & \mu = 0; \quad \nu = 0 \\ 0; & \mu = k; \quad \nu = 0 \end{cases} \tag{6}$$

and

$$J^{*\mu}(p)\bar{g}_{\mu\nu}J^\nu(p) \equiv |\mathbf{J}|^2, \tag{7}$$

and now the quantity $|\langle 0_+|0_-\rangle|^2$ can be interpreted as the vacuum persistence probability.

At the same time $|\mathbf{J}|^2$ contains three independent source components, transforming among themselves under spatial rotation, as it is appropriate to unit spin.

After using eq. (5) it may be easy to get $W(J)$ in the space-time representation by the Fourier transformation, as it follows

$$W(J) = \frac{1}{2}\int (dx)(dx') \quad \times$$
$$\left\{J_\mu(x)\Delta_+(x-x')J^\mu(x') + \frac{1}{m^2}\partial_\mu J^\mu\Delta_+(x-x')\partial'_\nu J^\nu(x')\right\}. \tag{8}$$

The field of spin one particles can be defined using the definition of the test source $\delta J^\mu(x)$ by the relation

$$\delta W(J) = \int (dx)\delta J^\mu(x)\varphi_\mu(x), \tag{9}$$

where φ_μ is the field of particles with spin 1. After performing variation of the formula (8) and comparison with eq. (9) we get the equation for field of spin 1 in the following form:

$$\varphi_\mu(x) = \int (dx')\Delta_+(x-x')J_\mu(x') - \frac{1}{m^2}\partial_\mu\int (dx')\Delta_+(x-x')\partial'_\nu J^\nu(x'). \tag{10}$$

The divergence of the vector field $\varphi_\mu(x)$ is given by the relation

$$\partial_\mu \varphi^\mu(x) = \int (dx')\Delta_+(x-x')\partial'_\mu J^\mu(x') \quad -$$

$$\frac{1}{m^2}\partial^2\Delta_+(x-x')\partial'_\nu J^\nu(x') = \frac{1}{m^2}\partial_\mu J^\nu(x), \tag{11}$$

where we used relation $-\partial^2\Delta_+ = \delta(x-x') - m^2\Delta_+$.

Further, we have after applying operator $(-\partial^2 + m^2)$ on the equation (10) the following equation:

$$\left(-\partial^2 + m^2\right)\varphi_\mu(x) = J_\mu(x) - \frac{1}{m^2}\partial_\mu\partial_\nu J^\nu(x) \tag{12}$$

$$\left(-\partial^2 + m^2\right)\varphi_\mu(x) + \partial_\mu\partial_\nu\varphi^\nu(x) = J_\mu(x) \tag{13}$$

as a consequence of eq. (11).

It may be easy to cast the last equation into the following form

$$\partial^\nu G_{\mu\nu} + m^2\varphi_\mu = J_\mu, \tag{14}$$

where

$$G_{\mu\nu}(x) = -G_{\nu\mu}(x) = \partial_\mu\varphi_\nu - \partial_\nu\varphi_\mu. \tag{15}$$

Identifying $G_{\mu\nu}$ with $F_{\mu\nu}$ of the electromagnetic field we get instead of eq. (14) so called the Proca equation for the electromagnetic field with the massive photon.

It is evident that the zero mass limit does not exist for $\partial_\mu J^\mu(x) \neq 0$. In such a way we are forced to redefine action $W(J)$. One of the possibilities is to put

$$\partial_\mu J^\mu(x) = mK(x) \tag{16}$$

and identify $K(x)$ in the limit $m \to 0$ with the source of massless spin zero particles.

Since the zero mass particles with zero spin are experimentally unknown in any event, we take $K(x) = 0$ and we write

$$W_{[m=0]}(J) = \frac{1}{2}\int (dx)(dx')J_\mu(x)D_+(x-x')J^\mu(x'), \tag{17}$$

where

$$\partial_\mu J^\mu(x) = 0 \tag{18}$$

and

$$D_+(x-x') = \Delta_+(x-x'; m=0). \tag{19}$$

The detail discussion concerning helicity, angular momentum, etc. of this new particle (photon) can be found in the Schwinger book (Schwinger, 1970; 2018).

3 Spin 2 fields

Exploiting the experience with spin 1 fields we form the combinations

$$T_{\mu\nu}(x), \tag{20}$$

$$\partial_\mu T^{\mu\nu}(x), \tag{21}$$

$$\partial_\mu \partial_\nu T^{\mu\nu}(x) \tag{22}$$

and $T(x)$ with appropriate coefficients in order to get the plausible form of action $W(T)$ for particles with spin 2.

While the spin one particles are described by the four-vector fields and sources the possible mathematical object describing particles with spin 2 should be the tensor field $\varphi_{\mu\nu}$ and the tensor source $T_{\mu\nu}$. Let us suppose that the tensor source is symmetrical, or, $T_{\mu\nu} = T_{\nu\mu}$.

Then, it has ten independent components. The vector source

$$\partial_\mu T^{\mu\nu}(x) \tag{23}$$

has 3 + 1 components and the scalar source

$$T(x) = g_{\mu\nu} T^{\mu\nu}(x) \tag{24}$$

is the one-component object. If we eliminate them, then the multiplicity of the system will be equal to five and this situation corresponds to the particle with spin 2.

Now, the question arises, what is the mathematical structure of the action $W(T)$ for particles with spin 2. Exploiting the experience with spin 1 fields we observe that $W(J)$ as the scalar quantity is formed by suitable combinations of sources and their derivatives. Similarly, in case with spin 2 particles we use the combinations of $T_{\mu\nu}(x), \partial_\mu T^{\mu\nu}(x), \partial_\mu\partial_\nu T^{\mu\nu}(x)$ and $T(x)$ with appropriate coefficients. The plausible form of $W(T)$ for particles with spin 2 is as follows:

$$\begin{gathered} W(T) = \frac{1}{2}\int (dx)(dx')\left\{T^{\mu\nu}(x)\Delta_+(x-x')T_{\mu\nu}(x) \quad + \right. \\ \frac{2}{m^2}\partial_\nu T^{\mu\nu}(x)\Delta_+(x-x')\partial'^\lambda T_{\mu\lambda}(x') \quad + \\ \frac{1}{m^4}\partial_\mu\partial_\nu T^{\mu\nu}(x)\Delta_+(x-x')\partial'_\alpha\partial'_\beta T^{\alpha\beta}(x') \quad - \\ \left. \frac{1}{3}\left(T(x) - \frac{1}{m^2}\partial_\mu\partial_\nu T^{\mu\nu}(x)\right)\Delta_+(x-x')\left(T(x') - \frac{1}{m^2}\partial'_\alpha\partial'_\beta T^{\alpha\beta}(x')\right)\right\}, \end{gathered} \tag{25}$$

where the coefficients follow (as we will see in the next text) from the probability condition $|\langle 0_+|0_-\rangle|^2 \leq 1$.

The probability of the vacuum persistence generated by action (25) calculated in the momentum space is of the form:

$$|\langle 0_+|0_-\rangle|^2 = \exp\left\{-\int d\omega_p T^{*\mu\nu}(p)\Pi_{\mu\nu,\alpha\beta}(p)T^{\alpha\beta}(p)\right\}, \tag{26}$$

where (Schwinger, 1970; 2018):

$$\Pi_{\mu\nu,\alpha\beta}(p) = \frac{1}{2}\left[\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} + \bar{g}_{\nu\alpha}\bar{g}_{\mu\beta}\right] - \frac{1}{3}\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} \tag{27}$$

with

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu \tag{28}$$

being the invariant tensor.

It may be easy to calculate $T^*\Pi T$ in the rest frame of p_μ where

$$\bar{g}_{\mu\nu} \to \delta_{kl}. \tag{29}$$

Under this condition it is

$$T^*\Pi T \to \bar{T}^{*kl}\bar{T}^{kl} \geq 0; \quad k, l \neq 0, \tag{30}$$

where

$$\bar{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{3}g^{\mu\nu}\bar{g}_{\varrho\sigma}T^{\varrho\sigma} \tag{31}$$

and the following relations have been used:

$$\bar{g}_{\mu\nu}\bar{T}^{\mu\nu} = 0 \tag{32}$$

$$p^\mu \bar{g}_{\mu\nu} = 0 \tag{33}$$

$$\bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = 0. \tag{34}$$

By analogy with spin 1 fields we define the spin 2 field $\varphi_{\mu\nu}(x)$ by relation

$$\delta W(T) = \int (dx)\delta T^{\mu\nu}(x)\varphi_{\mu\nu}(x), \tag{35}$$

which can be easily transformed into momentum space as

$$\delta W(T) = \int \frac{(dp)}{(2\pi)^4}\delta T^{\mu\nu}(-p)\varphi_{\mu\nu}(p). \tag{36}$$

The symmetrical field $\varphi_{\mu\nu}(x)$ following from eq. (36) has the following form (Schwinger, 1970; 2018)

$$\begin{aligned}\varphi_{\mu\nu}(x) = \int (dx')\Delta_+(x-x')T_{\mu\nu}(x') \quad - \\ \frac{1}{m^2}\partial_\mu \int (dx')\Delta_+(x-x')\partial'^\lambda T_{\lambda\nu}(x') \quad - \\ \frac{1}{m^2}\partial_\nu \int (dx')\Delta_+(x-x')\partial'^\lambda T_{\mu\lambda}(x') \quad + \\ \frac{1}{m^4}\partial_\mu\partial_\nu \int (dx')\Delta_+(x-x')\partial'_\kappa\partial'_\lambda T^{\kappa\lambda}(x') \quad - \\ \frac{1}{3}(g_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu)\int (dx')\Delta_+(x-x') \quad \times \\ (T(x') - \frac{1}{m^2}\partial'_\kappa\partial'_\lambda T^{\kappa\lambda}(x')).\end{aligned} \tag{37}$$

From this equation follows immediately the divergence of $\varphi_{\mu\nu}(x)$:

$$\partial^\mu\varphi_{\mu\nu}(x) = \frac{1}{m^2}\partial^\mu T_{\mu\nu}(x) - \frac{1}{3m^2}\partial_\nu[T(x) - \frac{2}{m^2}\partial_\kappa\partial_\lambda T^{\kappa\lambda}(x)] \tag{38}$$

and

$$\varphi = g_{\mu\nu}\varphi^{\mu\nu} = -\frac{1}{3m^2}[T(x) + \frac{2}{m^2}\partial_\kappa\partial_\lambda T^{\kappa\lambda}(x)]. \tag{39}$$

The combination of eq. (38) and (39) forms

$$\partial^\mu\varphi_{\mu\nu}(x) - \partial_\nu\varphi = \frac{1}{m^2}\partial^\mu T_{\mu\nu}(x). \tag{40}$$

The differential equation following from eq. (37) is of the form:

$$(-\partial^2 + m^2)\varphi_{\mu\nu}(x) = T_{\mu\nu}(x) \quad -$$

$$\frac{1}{m^2}\left[\partial_\mu\partial^\lambda T_{\lambda\nu}(x) + \partial_\nu\partial^\lambda T_{\mu\lambda}(x)\right] + g_{\mu\nu}\frac{1}{m^2}\partial_\kappa\partial_\lambda T^{\kappa\lambda}(x) \quad -$$

$$\frac{1}{3}(g_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu)\left[T(x) + \frac{2}{m^2}\partial_\kappa\partial_\lambda T^{\kappa\lambda}(x)\right]. \tag{41}$$

If we replace the scalar and vector combinations of sources in eq. (41) by the field objects from eqs. (38)–(40), we get the following differential equation for $\varphi_{\mu\nu}$:

$$(-\partial^2 + m^2)\varphi_{\mu\nu} + \partial_\mu\partial^\lambda\varphi_{\lambda\nu}(x) + \partial_\nu\partial^\lambda\varphi_{\mu\lambda}(x) - \partial_\mu\partial_\nu\varphi(x) \quad -$$

$$g_{\mu\nu}\left[(-\partial^2 + m^2)\varphi(x) + \partial_\kappa\partial_\lambda\varphi^{\kappa\lambda}(x)\right] = T_{\mu\nu}(x), \tag{42}$$

which also follows from $\delta W(T) = 0$, where

$$W(T) = \int (dx)\left[T^{\mu\nu}(x)\varphi_{\mu\nu}(x) + \pounds\right], \tag{43}$$

where £ is the Lagrange function and it has the following mathematical structure:

$$\pounds = -\frac{1}{2}\left[\partial^\alpha\varphi^{\mu\nu}(x)\partial_\alpha\varphi_{\mu\nu}(x) + m^2\varphi^{\mu\nu}(x)\varphi_{\mu\nu}(x) \quad -\right.$$

$$\left.\partial^\alpha\varphi(x)\partial_\alpha\varphi(x) - m^2\varphi^2(x)\right] \quad -$$

$$\partial_\mu\varphi^{\mu\nu}(x)\partial_\nu\varphi(x) + \partial_\mu\varphi^{\mu\nu}(x) + \partial_\mu\varphi^{\mu\nu}(x)\partial^\alpha\varphi_{\alpha\nu}(x). \tag{44}$$

If we take the spur of eq. (42), we get

$$(-\partial^2 + m^2)\varphi(x) + \partial_\mu\partial_\nu\varphi^{\mu\nu}(x) = -\frac{1}{2}\left[T(x) + m^2\varphi(x)\right]. \tag{45}$$

After inserting eq. (45) into eq. (42), we get

$$(-\partial^2 + m^2)\varphi_{\mu\nu}(x) + \partial_\mu\partial^\lambda\varphi_{\lambda\nu}(x) + \partial_\nu\partial^\lambda\varphi_{\mu\lambda}(x) \quad -$$

$$\partial_\mu\partial_\nu\varphi(x) + g_{\mu\nu}\frac{m^2}{2}\varphi(x) = T_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}T(x). \tag{46}$$

Introducing

$$G_{\mu\lambda\nu}(x) = -G_{\nu\lambda\mu}(x) = \partial_\mu\varphi_{\lambda\nu}(x) - \partial_\nu\varphi_{\lambda\mu}(x) \tag{47}$$

we can write eq. (46) in the following form:

$$\partial^\lambda G_{\mu\nu\lambda}(x) - \partial_\nu G^\lambda_{\mu\lambda}(x) + m^2 \left[\varphi_{\mu\nu}(x) + \frac{1}{2} g_{\mu\nu}\varphi(x)\right] \quad =$$

$$T_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu} T(x). \tag{48}$$

Further relations concerning spin 2 fields can be found in monograph by Schwinger (1970; 2018).

4 The massless limit of the spin 2 theory

It is evident that in order action (25) continue to exist in the limit $m \to 0$ it is natural to put

$$\partial_\nu T^{\mu\nu}(x) = \frac{m}{\sqrt{2}} J^\mu(x) \tag{49}$$

and

$$\partial_\mu J^\mu(x) = m\left[\sqrt{3}K(x) - \frac{1}{\sqrt{2}} T(x)\right], \tag{50}$$

where $J^\mu(x)$ and $K(x)$ are independent sources. The independence of them is expressed by constants $\sqrt{3}$ and $1/\sqrt{2}$ which are chosen to eliminate any coupling between sources $K(x)$ and $T(x)$. After insertion of eqs. (49) and (50) into action (43), we get for $m \to 0$:

$$W(T) = \frac{1}{2} \int (dx)(dx') \Big\{ T^{\mu\nu}(x) D_+(x - x') T_{\mu\nu}(x) \quad -$$

$$\frac{1}{2} T(x) D_+(x - x') T(x') + J^\mu(x) D_+(x - x') J_\mu(x') \quad +$$

$$K(x) D_+(x - x') K(x') \Big\}, \tag{51}$$

where $D_+(x - x') = \Delta_+(x - x'; m = 0)$ and for $m = 0$

$$\partial_\mu T^{\mu\nu}(x) = 0 \tag{52}$$

$$\partial_\mu J^\mu(x) = 0. \tag{53}$$

The formula (51) represents the invariant decomposition by means of sources of massless particles with spin 2, 1 and 0. The massless particles of helicity ± 2 is called graviton and the action which corresponds to this particle follows from action (51) in the form

$$W(T) = \frac{1}{2} \int (dx)(dx') \left\{ T^{\mu\nu}(x) D_+(x - x') T_{\mu\nu}(x) \quad - \right.$$

$$\left. \frac{1}{2} T(x) D_+(x - x') T(x') \right\} \tag{54}$$

with

$$\partial_\mu T^{\mu\nu}(x) = 0. \tag{55}$$

The graviton is the particle which was not hitherto experimentally discovered. Nevertheless, we can suppose it is the mediate boson which initiates the gravitational phenomena, just as the photon initiates the electromagnetic ones.

The source restriction $\partial_\mu T^{\mu\nu}(x) = 0$ for the graviton source states the existence of a conservation law of the vector

$$p^\nu = \int d\sigma_\mu T^{\mu\nu}(x), \tag{56}$$

where $d\sigma_\mu$ is an invariant element of area, which can be identified with the energy-momentum vector. The connection between the mechanical tensor $T^{\mu\nu}_{mech}$ and the gravitational tensor $T^{\mu\nu}_{grav}$ is postulated by relation

$$T^{\mu\nu}_{grav} = \kappa^{1/2} T^{\mu\nu}_{mech}, \tag{57}$$

where κ is the gravitational constant of the magnitude

$$\kappa = 8\pi G, \tag{58}$$

and $= \kappa = 6.67430(15)10^{-11} m^3.kg^{-1}.s^{-2}$ (CODATA, 2018).

The action corresponding to the gravitational field initiated by the mechanical tensor T^{μ}_{mech} is now

$$W(T) = \frac{\kappa}{2}\int (dx)(dx')\left\{T^{\mu\nu}(x)D_+(x-x')T^{\mu\nu}(x) \quad - \right.$$

$$\left. \frac{1}{2}T(x)D_+(x-x')T(x')\right\} \tag{59}$$

with

$$\partial_\mu T^{\mu\nu}(x) = 0. \tag{60}$$

The corresponding gravitational field definition is

$$\delta W(T) = \int (dx)\delta T^{\mu\nu} h_{\mu\nu}, \tag{61}$$

which implies the field equation for $h_{\mu\nu}(x)$ in the following form:

$$-\partial^2 h_{\mu\nu}(x) + \partial_\mu\partial^\alpha h_{\alpha\nu}(x) + \partial_\nu\partial^\alpha h_{\mu\alpha}(x) - \partial_\mu\partial_\nu h(x) \quad =$$

$$\kappa(T_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}T(x)) \tag{62}$$

with

$$T(x) = g_{\mu\nu}T^{\mu\nu}(x) \tag{63}$$

$$h(x) = g_{\mu\nu}h^{\mu\nu}(x). \tag{64}$$

Let us remark that action (59) was used by author to determination of the spectral form of the emitted gravitons by the binary system and by the related systems (Pardy, 1983; 1994a; 1994b; 1994c; 1994d; 2011; 2018; 2019). Action (59) enables also the derivation of the Newton gravity potential and Einstein gravity field equations (Schwinger, 1970, 2018; Pardy, 1984). After the verbal investigation of the Schwinger book involving theory of gravity (Schwinger, 1970; 2018), we can honestly say that the words principle of equivalence are not involved in his book.

5 Spin 3 fields

The spin 3 fields is discussed as the analogue of the spin 2 situation. The spin 3 action is derived together with the spin 3 field equation. The massles situation is also discussed.

It may be easy to show that in case of the spin 2 fields the action can be expressed in the following form:

$$W(T) = \frac{1}{2}\int \frac{(dp)}{(2\pi)^4} T^{\mu\nu}(-p)\frac{\Pi_{\mu\nu,\alpha\beta}}{p^2+m^2-i\varepsilon}T^{\alpha\beta}(p), \tag{65}$$

where $\Pi_{\mu\nu,\alpha\beta}$ is given by eq. (27) of the spin 2 discussion. By analogy with spin 2 situation we can postulate the form of the action for fields with spin 3 as follows:

$$W(S) = \frac{1}{2}\int \frac{(dp)}{(2\pi)^4} S^{\lambda\mu\nu}(-p)\frac{\Pi_{\lambda\mu\nu,\alpha\beta\gamma}}{p^2+m^2-i\varepsilon}S^{\alpha\beta\gamma}(p), \tag{66}$$

where function $\Pi_{\lambda\mu\nu,\alpha\beta\gamma}$ it is easy to determine. While in case of spin 0, 1 and 2 fields the determination of Π is not very difficult, the situation changes with higher spins. Nevertheless, there is the general method how to determine function Π for all integer spins as it will be shown in the next text and we here write down the derived result for spin 3 situation (Schwinger, 1970; 2018).

$$\Pi_{\alpha\beta\gamma,\alpha'\beta'\gamma'} = \bar{g}_{\alpha\alpha'}\bar{g}_{\beta\beta'}\bar{g}_{\gamma\gamma'} - \frac{1}{5}\left[\bar{g}_{\alpha\beta}\bar{g}_{\gamma\gamma'}\bar{g}_{\alpha'\beta'} + \bar{g}_{\beta\gamma}\bar{g}_{\alpha\alpha'}\bar{g}_{\beta'\gamma'} + \bar{g}_{\gamma\alpha}\bar{g}_{\beta\beta'}\bar{g}_{\alpha'\gamma'}\right], \tag{67}$$

where

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu. \tag{68}$$

Let us write down some properties of tensor $\Pi_{\alpha\beta\gamma,\alpha'\beta'\gamma'}$.

$$p^\alpha \Pi_{\alpha\beta\gamma,\alpha'\beta'\gamma'} = \frac{p^2+m^2}{m^2} \quad \times$$

$$\left[p_{\alpha'}\bar{g}_{\beta\beta'}\bar{g}_{\gamma\gamma} - \frac{1}{5}\left(p_\beta\bar{g}_{\gamma\gamma'}\bar{g}_{\alpha'\beta'} + p_{\alpha'}\bar{g}_{\beta\gamma}\bar{g}_{\beta'\gamma'} + p_\gamma\bar{g}_{\alpha\alpha'}\bar{g}_{\alpha'\gamma'}\right)\right] \tag{69}$$

and

$$g^{\beta\gamma}\Pi_{\alpha\beta\gamma,\alpha'\beta'\gamma'} = \frac{p^2+m^2}{m^2} \quad \times$$

$$\left[\frac{p_{\beta'}p_{\gamma'}}{m^2}\bar{g}_{\alpha\alpha'} - \frac{1}{5}\left(\frac{p_\alpha p_{\gamma'}}{m^2}\bar{g}_{\alpha'\beta'} + \bar{g}_{\alpha\alpha'}\bar{g}_{\beta'\gamma'} + \frac{p_\alpha p_{\beta'}}{m^2}\bar{g}_{\gamma'\alpha'}\right)\right]. \tag{70}$$

The field of spin 3, $\varphi_{\alpha\beta\gamma}(p)$, in the momentum representation is defined in analogy with spin 0, 1 and 2 fields, or

$$\delta W(S) = \int \frac{(dp)}{(2\pi)^4}\delta S^{\alpha\beta\gamma}(-p)\varphi_{\alpha\beta\gamma}(p) \tag{71}$$

and it is obtained as

$$\varphi_{\alpha\beta\gamma}(p) = \frac{1}{p^2+m^2-i\varepsilon}\Pi_{\alpha\beta\gamma,\alpha'\beta'\gamma'}(p)S^{\alpha'\beta'\gamma'}(p). \tag{72}$$

We know from the experience with spin 0, 1, and 2 particles that action W can be cast into various forms. If we take the obligate form, we write for the spin 3 action:

$$W(S) = \int (dx) \left[S^{\alpha\beta\gamma}(x)\varphi_{\alpha\beta\gamma}(x) + \mathcal{L}(x) \right], \tag{73}$$

where $\mathcal{L}\ (x)$ is the Lagrange function of the spin 3 fields and it can be shown that its mathematical structure is as follows (Schwinger, 1970; 2018):

$$\mathcal{L} = -\frac{1}{2}\left[\partial^{\kappa}\varphi^{\alpha\beta\gamma}\partial_{\kappa}\varphi_{\alpha\beta\gamma} + m^2\varphi^{\alpha\beta\gamma}\varphi_{\alpha\beta\gamma} - 3\partial_{\kappa}\varphi^{\kappa\beta\gamma}\partial^{\alpha}\varphi_{\alpha\beta\gamma} \quad + \right.$$

$$\left. 6\,\partial_{\beta}\varphi^{\alpha\beta\gamma}\partial_{\gamma}\varphi_{\alpha} - 3\partial^{\alpha}\varphi^{\gamma}\partial_{\alpha}\varphi_{\gamma} - 3m^2\varphi^{\kappa}\varphi_{\kappa} - \frac{3}{2}(\partial_{\kappa}\varphi^{\kappa})^2\right] \quad +$$

$$\frac{1}{2}m\left(\varphi^{\kappa}\partial_{\kappa}\Phi - \Phi\partial_{\kappa}\varphi^{\kappa}\right) + \partial^{\kappa}\Phi\partial_{\kappa}\Phi + 4m^2\Phi, \tag{74}$$

where the auxiliary function Φ is to receive independent variation in the stationary action principle and

$$\varphi_{\lambda}(x) = \varphi^{\kappa}_{\lambda\kappa}(x). \tag{75}$$

6 The massless fields with helicity 3

It is possible to show that from the massive field action for spin 3 generate the massless limit with the following action (Schwinger, 1970; 2018):

$$W(S, m=0) = \frac{1}{2}\int (dx)(dx')$$

$$\left[S^{\lambda\mu\nu}(x)D_{+}(x-x')S_{\lambda\mu\nu} - \frac{3}{4}S^{\lambda}(x)D_{+}(x-x')S_{\lambda}(x') \right], \tag{76}$$

where

$$S^{\lambda}(x) = g_{\mu\nu}S^{\lambda\mu\nu}(x) \tag{77}$$

and

$$\partial_{\lambda}S^{\lambda\mu\nu}(x) = 0. \tag{78}$$

The momentum space transformation of eqs. (72)–(74) is

$$W(S) = \frac{1}{2}\int \frac{(dp)}{(2\pi)^4}\frac{1}{p^2 - i\varepsilon} \quad \times$$

$$\left[S^{\lambda\mu\nu}(-p)S_{\lambda\mu\nu}(p) - \frac{3}{4}S^{\lambda}(-p)S_{\lambda}(p) \right] \tag{79}$$

with

$$S^{\lambda}(p) = S^{\lambda\nu}_{\nu}(p) \tag{80}$$

and

$$p_\lambda S^{\lambda\mu\nu}(p) = 0. \tag{81}$$

The field is defined as usually

$$\delta W(S) = \int \frac{(dp)}{(2\pi)^4} \delta S^{\lambda\mu\nu}(-p) S_{\lambda\mu\nu}(p) \tag{82}$$

and any additional term containing p_λ, p_μ or p_ν as factors will not contribute in eq. (82) because of the source restriction (81). Then, the general form of the field with mass $m = 0$ is

$$\varphi_{\lambda\mu\nu}(p) = \frac{1}{p^2 - i\varepsilon}\left[S_{\lambda\mu\nu}(p) - \frac{1}{4}(g_{\mu\nu}S_\lambda(p) + g_{\nu\lambda}S_\mu(p) + g_{\lambda\mu}S_\nu(p)\right] \quad +$$

$$p_\lambda\varphi_{\mu\nu}(p) + p_\mu\varphi_{\nu\lambda}(p) + p_\nu\varphi_{\lambda\mu}(p), \tag{83}$$

where the cyclically related terms are required by the total symmetry of the tensor $\varphi_{\lambda\mu\nu}$ and $\varphi_{\mu\nu}(p)$ is the new symmetrical tensor determined by the source restriction. In order to use the tensor $\varphi_{\mu\nu}(p)$, first, let us note that

$$\varphi_\lambda = \varphi^{\nu}_{\lambda\nu}(p) = \frac{1}{p^2 - i\varepsilon}\left(-\frac{1}{2}\right) S_\lambda(p) + p_\lambda\varphi(p) + 2p^\nu\varphi_{\lambda\nu}(p), \tag{84}$$

where

$$\varphi(p) = \varphi^{\nu}_{\nu}(p). \tag{85}$$

Multiplication of eq. (83) by p^λ then introduces just the combination equal to $\frac{1}{2}(\varphi_\lambda - p_\lambda\varphi)$ and we get:

$$p^2\varphi_{\mu\nu} = p^\lambda\varphi_{\lambda\mu\nu} - \frac{1}{2}(p_\mu\varphi_\nu + p_\nu\varphi_\mu) + p_\mu p_\nu\varphi. \tag{86}$$

An equation for $\varphi(p)$ is produced by combination

$$p^\lambda p^\mu p^\nu\varphi_{\lambda\mu\nu} = 3p^2 p^\mu p^\nu\varphi_{\mu\nu} \tag{87}$$

with

$$p^2 p^\mu p^\nu\varphi_{\mu\nu} = p^\lambda p^\mu p^\nu\varphi_{\lambda\mu\nu} - p^2 p^\lambda\varphi_\lambda + (p)^2\varphi, \tag{88}$$

namely

$$(p^2)^2\varphi = p^2 p^\lambda\varphi_\lambda - \frac{2}{3}p^\lambda p^\mu p^\nu\varphi_{\lambda\mu\nu}. \tag{89}$$

The momentum space version of the field equation for $\varphi_{\lambda\mu\nu}(p)$ is now derived as

$$p^2\varphi_{\lambda\mu\nu} - p_\lambda p^{\lambda'}\varphi_{\lambda'\mu\nu} - p_\mu p^{\mu'}\varphi_{\lambda\mu'\nu} - p_\nu p^{\nu'}\varphi_{\lambda\mu\nu'} \quad +$$

$$p_\mu p_\nu\varphi_\lambda + p_\nu p_\lambda\varphi_\mu + p_\lambda p_\mu\varphi_\nu - 3p_\lambda p_\mu p_\nu\varphi \quad =$$

$$S_{\lambda\mu\nu} - \frac{1}{4}(g_{\mu\nu}S_\lambda + g_{\nu\lambda}S_\mu + g_{\lambda\mu}S_\nu). \tag{90}$$

From this equation we can derive eq. (89) by multiplication with $p^{\lambda}p^{\mu}p^{\nu}$. The field equation involves only the combination

$$\varphi_{\lambda\mu\nu}(p) - 3\frac{p_{\lambda}p_{\mu}p_{\nu}}{p^2 - i\varepsilon}\varphi(p), \tag{91}$$

since

$$p_{\lambda}p^{\lambda'}(p_{\lambda'}p_{\mu}p_{\nu}) - (p^2)p_{\mu}p_{\nu}p_{\lambda} = 0 \tag{92}$$

and it means that this combination is redefinition of $\varphi_{\lambda\mu\nu}$ which means that $\varphi(p)$ can be transformed away. Thus the final set of the field equations for spin 3 and $m = 0$ is with $\varphi = 0$:

$$p^2\varphi_{\lambda\mu\nu} - p_{\lambda}p^{\lambda'}\varphi_{\lambda'\mu\nu} - ... + p_m p_{\nu}\varphi_{\lambda} + ... \quad -$$

$$g_{\mu\nu}(p^2\varphi_{\lambda} + \frac{1}{2}p_{\lambda}p^{\lambda'}\varphi_{\lambda'} - p^{\mu'}p^{\nu'}\varphi_{\lambda\mu'\nu'}) - ... = S_{\lambda\mu\nu}, \tag{93}$$

where dots represents the terms that are generated by the cyclic permutation from the given ones.

The algebraic consequence of this equation is as follows:

$$p^{\lambda}S_{\lambda\mu\nu}(p) = g_{\mu\nu}\frac{1}{4}p^{\lambda}S_{\lambda}(p) \tag{94}$$

and it is consistent with the vanishing divergence of the source, but does not imply it.

We have seen that from the massive field action for spin 3, the massless limit is generated with the following action (Schwinger, 1970; 2018):

$$W(S, m = 0) = \frac{1}{2}\int (dx)(dx')$$

$$\left[S^{\lambda\mu\nu}(x)D_{+}(x - x')S_{\lambda\mu\nu} - \frac{3}{4}S^{\lambda}(x)D_{+}(x - x')S_{\lambda}(x')\right], \tag{95}$$

where

$$S^{\lambda}(x) = g_{\mu\nu}S^{\lambda\mu\nu}(x) \tag{96}$$

and

$$\partial_{\lambda}S^{\lambda\mu\nu}(x) = 0. \tag{97}$$

According to Schwinger (1970; 2018) - *Ordinary matter possesses no conserved physical properties that could be identified with the ones described by the local conservation law (97), or indeed for any* $n \geq 3$. *The inability to construct their sources strongly affirms the empirical absence of the particles. But perhaps one should not reject totally the possibility of eventually encountering such properties, and the associated particles, under circumstances that are presently unattainable.*

Here we think that matter with the spin 3 is the so called electromagnetic-gravity because it involves the electromagnetism (spin 1) and gravity (spin 2).

7 Field with arbitrary integer spin

The analogy with the spin 1, 2 and 3 situation enables to introduce action for the arbitrary integer spin situation.

The analogy with the situation of spins 1, 2 and 3 fields inspire us in postulating the action for fields with the arbitrary integer spin, in the following form:

$$W(S) = \frac{1}{2} \int \frac{(dp)}{(2\pi)^4} S^{\mu_1 \mu_2 \ldots \mu_n} \frac{\Pi_{\mu_1 \mu_2 \ldots \mu_n, \nu_1 \nu_2 \ldots \nu_n}}{p^2 + m^2 - i\varepsilon} S^{\nu_1 \nu_2 \ldots \nu_n}, \tag{98}$$

where n denotes integer spin and $S^{\mu_1 \mu_2 \cdots \mu_n}(p)$ is the n-component source. Our goal is the derivation of the $2n$-component mathematical object $\Pi_{\mu_1 \mu_2 \ldots \mu_n, \nu_1 \nu_2 \ldots \nu_n}$.

The tensor $S^{\mu_1 \mu_2 \cdots \mu_n}(p)$ of rang n describes a larger system than a particle with spin n. Part of necessary reduction to describe the spin n can be obtained by projection factor $\bar{g}_{\mu\nu}(p)$ in the following form of the combination:

$$S^{*\mu_1 \mu_2 \ldots \mu_n}(p) \bar{g}_{\mu_1 \nu_1} \bar{g}_{\mu_2 \nu_2} \ldots \bar{g}_{\mu_n \nu_n} S^{*\nu_1 \nu_2 \ldots \nu_n}(p) \quad \rightarrow \quad S^*_{k_1 k_2 \ldots k_n} S_{k_1 k_2 \ldots k_n}, \tag{99}$$

where the second form refers to the rest frame of the momentum p^μ.

The number of independent components of the symmetrical tensor $S_{k_1 k_2 \ldots k_n}$ is

$$\frac{1}{2}(n+1)(n+2) \tag{100}$$

and it corresponds to the number of states exhibited by symmetrical collections of n unit spins. The total spin quantum number ranges from $s = n$ through $s = n-2, n-4$ terminating at 1 or 0 as n is odd or even and

$$\sum_s (2s+1) = \frac{1}{2}(n+1)(n+2). \tag{101}$$

A combination of the two unit spins into a null resultant corresponds to expression such as $S_{k_1 k_2 \ldots k_n}$. To remove this possibility and thereby select only $s = n$, we must make $S_{k_1 k_2 \ldots k_n}$ traceless. The subtraction of the appropriate number of restrictions gives the number of independent components

$$\frac{1}{2}(n+1)(n+2) - \frac{1}{2}(n-1)n = 2n+1 \tag{102}$$

as expected for $s = n$.

The structure of the coupling between sources can be presented as

$$S^{*\mu_1 \mu_2 \ldots \mu_n} \Pi_{\mu_1 \mu_2 \ldots \mu_n, \nu_1 \nu_2 \ldots \nu_n} S^{\nu_1 \nu_2 \ldots \nu_n} \tag{103}$$

and Π is determined by equation (Schwinger, 1970; 2018):

$$\begin{aligned} x^{\mu_1} x^{\mu_2} \ldots x^{\mu_n} \Pi_{\mu_1 \mu_2 \ldots \mu_n, \nu_1 \nu_2 \ldots \nu_n} y^{\nu_1} y^{\nu_2} \ldots y^{\nu_3} \quad &= \\ (x \cdot y)^n + c_{n1} (x \cdot x)(x \cdot y)^{n-2} (y \cdot y) \quad &+ \ldots \quad \ldots \\ = \quad [(x \cdot x)(y \cdot y)]^{1/2} \; &\frac{2^n (n!)^2}{(2n)!} P_n(\mu), \end{aligned} \qquad (104$$

where

$$(x \cdot y) = x^{\mu} \bar{g}_{\mu\nu} y^{\nu} \tag{105}$$

$$\mu = \frac{(x \cdot y)}{[(x \cdot y)(y \cdot y)]^{1/2}} \tag{106}$$

and $P_n(\mu)$ is the Legendre polynomial.

The addition theorem of the spherical harmonics provides the factorization:

$$x^{\mu_1} x^{\mu_2} ... x^{\mu_n} \Pi_{\mu_1\mu_2...\mu_n, \nu_1\nu_2...\nu_n} y^{\nu_1} y^{\nu_2} ... y^{\nu_3} \quad =$$

$$\frac{2^n (n!)^2}{(2n)!} \frac{4\pi}{2n+1} \sum_{\lambda=-n}^{n} Y_{n\lambda}(x) Y^*_{n\lambda}(y). \tag{107}$$

The last equation enables to form the dyadic construction

$$\Pi^{\mu_1\mu_2...\mu_n, \nu_1\nu_2...\nu_n} \sum_{\lambda=-n}^{n} e^{\mu_1\mu_2...\mu_n}_{p\lambda} e^{*\nu_1\nu_2...\nu_n}_{p\lambda} \tag{108}$$

with

$$e^{\mu_1\mu_2...\mu_n}_{p\lambda} x_{\mu_1} x_{\mu_2} ... x_{\mu_n} = \left[\frac{2^n (n!)^2}{(2n)!} \frac{4\pi}{2n+1} \right]^{1/2} Y_{n\lambda}(x). \tag{109}$$

8 Discussion

Real particles have intrinsic, or, "inborn" angular momentum. This degree of freedom is called spin. This property of a particle does not change. It is fixed. This is qualitatively different than orbital angular momentum which does change. Classically, the angular momentum of a body orbiting another body, such as the earth orbiting the sun, would be described as orbital angular momentum, and the angular momentum of the earth revolving around its axis would be described as spin angular momentum. The quantum mechanical description is quite different. The quantum description of spin is also dramatically different than the classical description. Or, an electron is a point-like particle, and consequently it does not even have an axis about which it could spin.

So, the classical derivation of the particle spin is not possible and it means that the Schwinger formal approach to the integer spin is serious and meaningful.

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