

Explicit Bounds for the Distribution Function of the Sum of Dependent Normally Distributed Random Variables

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Abstract

In this paper an analytic expression is given for the bounds of the distribution function of the sum of dependent normally distributed random variables. Using the theory of copulas and the important Fréchet bounds the dependence structure is not restricted to any specific type. Numerical illustrations are provided to assess the quality of the derived bounds.

1 Introduction

Many problems in mathematical probability theory involve the computation of the distribution function for the sum of random variables (RVs). In case of independent RVs X and Y the distribution F_Z of the sum $Z = X + Y$ is given by the integral

$$F_Z(t) = P(X + Y \leq t) = \int_{-\infty}^t \left(\int f(z - y)g(y)dy \right) dz \quad (1)$$

with f and g as the corresponding density functions of X and Y respectively.

If the RVs however are stochastically dependent, which is a common situation in practice, the convolution of the marginal densities f and g in the integrand of (1) is no longer valid. Under the assumption that the joint density function f_{XY} of X and Y is specified, the distribution F_Z can be calculated by

$$F_Z(t) = P(X + Y \leq t) = \int_{-\infty}^t \left(\int f_{XY}(z - y, y)dy \right) dz \quad (2)$$

An analytical exact solution of the integral both in (1) and (2) is feasible for certain marginals f, g or joint densities f_{XY} (e.g. the normal one). In situations where f, g or f_{XY} is too cumbersome to work with, one could use Monte-Carlo simulation for numerical solution of the integrals.

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In many circumstances, however, the joint density of the RVs X and Y is not known, despite of given marginal distributions. This leads to the question, whether it is possible to provide bounds for the distribution of the sum, which are valid for all possible joint distributions. The original problem was formulated by A. N. Kolmogorov: Let X and Y be RVs with given distributions F_X and F_Y . Find bounds G^\wedge (upper bound) and G^\vee (lower bound) for the distribution G of the sum $Z = X + Y$, such that

$$G^\vee(z) = \inf P(X + Y < z) \quad (3)$$

$$G^\wedge(z) = \sup P(X + Y < z) \quad (4)$$

where the infimum and supremum are taken over all possible joint distributions having the marginal distributions F_X, F_Y . In this situation, it is said, that the joint distribution has fixed margins.

By now a rich literature is available on this subject. G.D. Makarov [1] solved Kolmogorov's problem via a cumbersome, ad hoc argument. Other authors, e.g. Frank et al. [2] have applied theory originally studied by Fréchet which leads to copulas naturally. The authors in [5], [6] provide examples of distributions for which the bounds G^\vee and G^\wedge can be explicitly computed. These distributions include the uniform, the Cauchy and the exponential families. For further reading the interested reader is referred to [7] - [10].

In this contribution we compute explicit bounds for normally distributed RVs by means of the copula based theory. Compared to the work of Frank et al. [2] who addressed this problem 1987 by using the method of Lagrange multipliers we will use a simple variable substitution.

The rest of the paper is organized as follows. In Section 2 we give a brief outline of the copula theory and present the general theorem for bounding the distribution of the sum of RVs with unknown dependence. Then in Section 3 lower and upper bounds for the distribution of the sum of normally distributed RVs are presented. Finally we illustrate these bounds by numerical examples in Section 4.

2 Preliminaries

2.1 Copulas

Copulas are used in probability theory and statistics for modeling the dependence between RVs X_1, X_2, \dots, X_d at a deeper level allowing to understand dependence measures different from a simple correlation coefficient approach. For any natural d , a (d -dimensional) copula is a distribution function on $[0, 1]^d$ with standard uniform margins.

For the formal definition of a copula as well as a general introduction into the wide fields of copulas the reader is referred to [3], [4]. In this paper the following theorems are of particular importance.

Theorem 1 (Sklar's Theorem) *Let H be a joint distribution function for the RVs X and Y with marginals F and G . Then there exists a copula C such that for all $x, y \in \mathbb{R}$:*

$$H(x, y) = C(F(x), G(y)) \quad (5)$$

Theorem 2 (Fréchet Bounds) *Let $(u, v) \in I^2 = ([0, 1], [0, 1]) \subset \mathbb{R}^2$, then for every copula C and every $(u, v) \in I^2$ the Fréchet-bounds inequality is valid:*

$$W(u, v) \leq C(u, v) \leq M(u, v) \quad (6)$$

with $M(u, v) = \min(u, v)$ and $W(u, v) = \max(u + v - 1, 0)$ as the Fréchet bounds.

The Fréchet bounds inequality together with Sklar's Theorem leads to the Fréchet bounds for the joint distribution function H ,

$$\max(F(x) + G(y) - 1, 0) \leq H(x, y) \leq \min(F(x), G(y)) \quad (7)$$

so the joint distribution H is bounded in terms of its own marginals. The proofs of these theorems as well as (7) can be found in [3].

2.2 Bounding the sum of RVs with unknown dependence

The problem of calculating the distribution G of the sum $Z = X + Y$, where the dependence¹ between X and Y is unknown, is called the Kolmogorov problem. Using the theory of copulas the following theorem and its proof are given in [3].

Theorem 3 *Let X and Y be RVs with distribution functions F_X and F_Y . Let G denote the distribution function of $X + Y$. Then*

$$G^\vee(z) \leq G(z) \leq G^\wedge(z) \quad (8)$$

where

$$G^\vee(z) = \sup_{x+y=z} \{W(F_X(x), F_Y(y))\} \quad (9)$$

$$G^\wedge(z) = \inf_{x+y=z} \{\tilde{W}(F_X(x), F_Y(y))\} \quad (10)$$

with $\tilde{W}(u, v) = u + v - W(u, v) = \min(u + v, 1)$ and $W(u, v)$ as in Theorem 2.

In the following G^\wedge is named as upper bound and G^\vee as lower bound for the distribution G .

¹A note on terminology: The term "dependence" will be used for a common measure in the study of the dependence between RVs and therefore is not restricted to a measure of the linear dependence between RVs.

3 Bounds for the sum of normally distributed RVs

Let X and Y be normally distributed with means μ_X, μ_Y and standard deviations σ_X, σ_Y , denoted by $X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2)$. Their distribution functions F_X and F_Y are given by

$$F_X(x) = \Phi\left(\frac{x - \mu_X}{\sigma_X}\right) \quad \text{and} \quad F_Y(y) = \Phi\left(\frac{y - \mu_Y}{\sigma_Y}\right)$$

where Φ denotes the standard normal distribution

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{s^2}{2}} ds$$

In case of $\sigma_X \neq \sigma_Y$ the lower and upper bound for the distribution G of the sum $Z = X + Y$ are given by the following proposition.

Proposition 1 *The lower bound G^\vee and the upper bound G^\wedge for G are calculated by*

$$G^\vee(z) = \begin{cases} \max(\Psi(x_1) - 1, 0) : & \Psi(x_1) > \Psi(x_2) \\ \max(\Psi(x_2) - 1, 0) : & \Psi(x_1) < \Psi(x_2) \end{cases} \quad (11)$$

and

$$G^\wedge(z) = \begin{cases} \min(\Psi(x_1), 1) : & \Psi(x_1) < \Psi(x_2) \\ \min(\Psi(x_2), 1) : & \Psi(x_1) > \Psi(x_2) \end{cases} \quad (12)$$

with the function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$:

$$\Psi(x) = \Phi\left(\frac{x - \mu_X}{\sigma_X}\right) + \Phi\left(\frac{z - x - \mu_Y}{\sigma_Y}\right) \quad (13)$$

and x_1, x_2 as the local extremas of Ψ .

Proof. Using Theorem 3 together with the Fréchet bounds and (7) introduced in section 2, G^\vee and G^\wedge are given by

$$G^\vee(z) = \sup_{x+y=z} \max\left\{\Phi\left(\frac{x - \mu_X}{\sigma_X}\right) + \Phi\left(\frac{y - \mu_Y}{\sigma_Y}\right) - 1, 0\right\} \quad (14)$$

$$G^\wedge(z) = \inf_{x+y=z} \min\left\{\Phi\left(\frac{x - \mu_X}{\sigma_X}\right) + \Phi\left(\frac{y - \mu_Y}{\sigma_Y}\right), 1\right\} \quad (15)$$

For the sum of the distributions F_X and F_Y in (14), (15) we introduce the function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$:

$$\Psi(x) = \Phi\left(\frac{x - \mu_X}{\sigma_X}\right) + \Phi\left(\frac{z - x - \mu_Y}{\sigma_Y}\right) \quad (16)$$

where the variable y has been substituted by $z - x$.

Now, we can proceed with the computation of the local extremas of Ψ as function of one variable. Therefore the first derivative $\Psi'(x)$ is built:

$$\Psi'(x) = \frac{1}{\sigma_X \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} - \frac{1}{\sigma_Y \sqrt{2\pi}} \cdot e^{-\frac{(x-(z-\mu_Y))^2}{2\sigma_Y^2}} \quad (17)$$

Finding the zeros of $\Psi'(x)$ leads to

$$\frac{1}{\sigma_X \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} = \frac{1}{\sigma_Y \sqrt{2\pi}} \cdot e^{-\frac{(x-(z-\mu_Y))^2}{2\sigma_Y^2}} \quad (18)$$

which - after some technical calculation - is equivalent to the quadratic equation

$$\alpha x^2 + \beta x + \gamma = 0 \quad (19)$$

with the variables α, β, γ as follows:

$$\alpha = \frac{1}{2\sigma_X^2} - \frac{1}{2\sigma_Y^2} \quad (20)$$

$$\beta = \frac{z - \mu_Y}{\sigma_Y^2} - \frac{\mu_X}{\sigma_X^2} \quad (21)$$

$$\gamma = \frac{-(z - \mu_Y)^2}{2\sigma_Y^2} + \frac{\mu_X^2}{2\sigma_X^2} - \ln \frac{\sigma_Y}{\sigma_X} \quad (22)$$

The solutions of (19) are given by $x_{1,2}$

$$x_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \quad (23)$$

which are the candidates for the local extremas of the function $\Psi(x)$.

As $\sigma_X \neq \sigma_Y$ ($\alpha \neq 0$) it is guaranteed that the division in (23) is defined. Moreover, any possible values for $\mu_X, \mu_Y, \sigma_X, \sigma_Y$ will lead to two solutions $x_1 \neq x_2$ of the quadratic equation (19). In order to show this, we compute the limits of Ψ for $x \rightarrow \pm\infty$.

$$\lim_{x \rightarrow -\infty} \Psi(x) = \lim_{x \rightarrow -\infty} \left(\underbrace{\Phi\left(\frac{x - \mu_X}{\sigma_X}\right)}_{\rightarrow 0} + \underbrace{\Phi\left(\frac{z - x - \mu_Y}{\sigma_Y}\right)}_{\rightarrow 1} \right) = 1 \quad (24)$$

$$\lim_{x \rightarrow \infty} \Psi(x) = \lim_{x \rightarrow \infty} \left(\underbrace{\Phi\left(\frac{x - \mu_X}{\sigma_X}\right)}_{\rightarrow 1} + \underbrace{\Phi\left(\frac{z - x - \mu_Y}{\sigma_Y}\right)}_{\rightarrow 0} \right) = 1 \quad (25)$$

The same limits of Ψ for $x \rightarrow \pm\infty$ together with the fact that Ψ is continuous but not constant on \mathbb{R} implies that Ψ has one extremum at least. If Ψ had only one extremum there would be exactly one solution $x_1 = x_2$ (zero of order 2) of

(19). A zero of order 2 for $\Psi'(x) = 0$ however cannot be an extreme value. So, in any case there exist exactly two extreme values.

Now we consider the set $U := \{\Psi(x_1) - 1, \Psi(x_2) - 1, 0\}$, which contains both a minimum and maximum value. If $\Psi(x_1) > \Psi(x_2)$ then $\max(U) = \max(\Psi(x_1), 0)$, otherwise $\max(U) = \max(\Psi(x_2), 0)$. As for each set M having a maximum value there is $\sup M = \max M$, equation (11) of Proposition 1 is shown. Similarly equation (12) of Proposition 1 can be proven using the set $V := \{\Psi(x_1), \Psi(x_2), 1\}$. If $\Psi(x_1) < \Psi(x_2)$ then $\min(U) = \min(\Psi(x_1), 1)$, otherwise $\min(U) = \min(\Psi(x_2), 1)$. \square

In the special case, that the RVs have the same standard deviation ($\sigma_X = \sigma_Y$) the bounds G^\vee and G^\wedge can be expressed in closed form by the following corollary.

Corollary 1 *Let $\sigma_X = \sigma_Y = \sigma$ the bounds G^\vee and G^\wedge for the distribution G of the sum of two normally distributed random variables are given by*

$$G^\vee(z) = \begin{cases} 0, & z \leq \mu_X + \mu_Y \\ 2\Phi\left(\frac{z - \mu_X - \mu_Y}{2\sigma}\right) - 1, & z \geq \mu_X + \mu_Y \end{cases} \quad (26)$$

and

$$G^\wedge(z) = \begin{cases} 2\Phi\left(\frac{z - \mu_X - \mu_Y}{2\sigma}\right), & z \leq \mu_X + \mu_Y \\ 1, & z \geq \mu_X + \mu_Y \end{cases} \quad (27)$$

Proof. By using the same arguments as in the proof of Proposition 1, we get $\alpha = 0$ from (20). This means we have to solve a linear equation $\beta x + \gamma = 0$. From (21) and (22) we get

$$\beta = \frac{z - \mu_Y}{\sigma^2} - \frac{\mu_X}{\sigma^2} = \frac{z - \mu_Y - \mu_X}{\sigma^2} \quad (28)$$

$$\gamma = \frac{-(z - \mu_Y)^2}{2\sigma^2} + \frac{\mu_X^2}{2\sigma^2} - \ln 1 = \frac{-z^2 + 2z\mu_Y - \mu_Y^2 + \mu_X^2}{2\sigma^2} \quad (29)$$

Then the solution x_0 for $\beta x + \gamma = 0$ is given by

$$x_0 = -\frac{\gamma}{\beta} = \frac{z^2 - 2z\mu_Y + \mu_Y^2 - \mu_X^2}{2(z - \mu_Y - \mu_X)} = \frac{z - \mu_Y + \mu_X}{2} \quad (30)$$

As x_0 is a zero of order 1 of $\Psi'(x) = 0$ it is guaranteed, that x_0 is indeed an extreme value. Inserting x_0 into $\Psi(x)$ leads to

$$\begin{aligned} \Psi(x_0) &= \Phi\left(\frac{\frac{z - \mu_Y + \mu_X}{2} - \mu_X}{\sigma}\right) + \Phi\left(\frac{z - \frac{z - \mu_Y + \mu_X}{2} - \mu_Y}{\sigma}\right) \\ &= \Phi\left(\frac{z - \mu_Y - \mu_X}{2\sigma}\right) + \Phi\left(\frac{z - \mu_X - \mu_Y}{2\sigma}\right) \\ &= 2\Phi\left(\frac{z - \mu_Y - \mu_X}{2\sigma}\right) \end{aligned} \quad (31)$$

If $z > \mu_X + \mu_Y$ then the argument of the distribution function Φ in (31) is positive. As $\Phi(t) > 0.5$ for any positive value $t \in \mathbb{R}$ we have $\Psi(x_0) > 1$. Conversely, if $z < \mu_X + \mu_Y$ then the argument of the distribution function Φ in (31) is negative and as a result $\Psi(x_0) < 1$. Together with the equations for G^\vee and G^\wedge in (14), (15) the corollary is proven. \square

Remark 1 *The result of corollary 1 is in accordance with the result of M. J. Frank [2]. However the bounds from Frank in the common case for different standard deviations ($\sigma_X \neq \sigma_Y$) cannot be reproduced. Simple numerical examples have revealed, that Frank's bounds have also negative values, which is in contradiction to a distribution function.*

4 Illustrative Examples

In this section we want to illustrate the Proposition 1 in section 3 by a simple example with the following prerequisites:

- The RVs X and Y are normally distributed, where the parameters μ and σ are chosen as follows: $X \sim N(1, 0.1)$, $Y \sim N(1.5, 0.15)$.
- For modeling the dependence amongst X and Y two different strategies are used: 1) X, Y are bivariate normal distributed with a given correlation coefficient ρ . 2) The dependence of X and Y is modeled using either the Clayton (C_θ^{Cl}) or the Gumbel (C_θ^{Gu}) Copula [4].

$$\begin{aligned} C_\theta^{Cl}(u_1, u_2) &= (u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta}} \\ C_\theta^{Gu}(u_1, u_2) &= \exp\left(-\left((-\ln(u_1))^\theta + (-\ln(u_2))^\theta\right)^{\frac{1}{\theta}}\right) \end{aligned}$$

These copulas have been chosen, because the Gumbel (Clayton) copula turns out to have upper (lower) tail dependence. A method for generating realizations from a particular copula within Matlab is described in [11]. The value θ has been set to 2.5 for both copulas.

- The bound computation G^\vee and G^\wedge as well as generating dependent random variables using the concept of copulas is done via Matlab.

In figure 1 the bounds G^\vee , G^\wedge are compared to the sum distribution F assuming X, Y are bivariate normally distributed. The corner cases $\rho = 0$ and $\rho = 1$ have been used for the bivariate normal distribution.

In figure 2 the bounds G^\vee , G^\wedge are compared to the sum distribution F assuming that X, Y are dependent either by Clayton or Gumbel copula.

References

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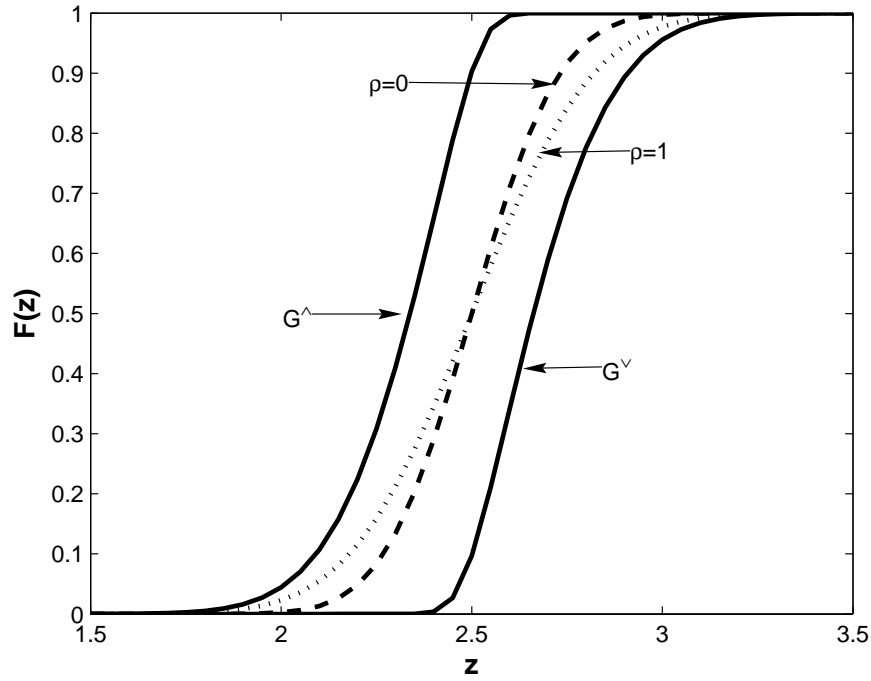


Figure 1: Computed bounds G^{\wedge} , G^{\vee} compared to the distribution F of the sum $X + Y$ assuming bivariate normally distributed random variables X, Y

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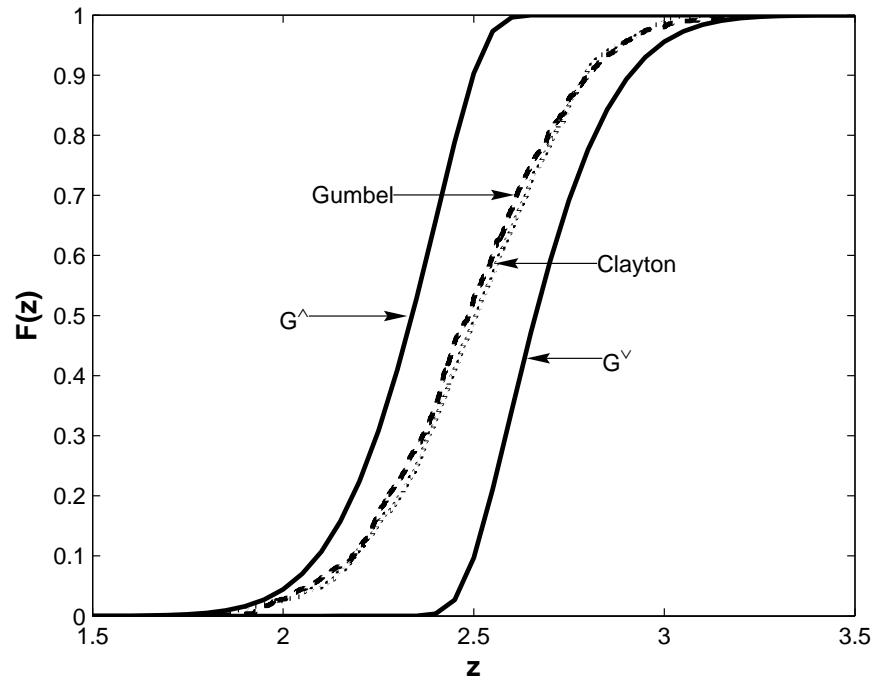


Figure 2: Computed bounds G^{\wedge} , G^{\vee} compared to the distribution F of the sum $X + Y$ where the dependence of X and Y is modeled by Clayton and Gumbel copula

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