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$\mathbf{R}^3$ can be composed out of disjoint circles

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Abstract

In the paper we have succeeded to demonstrate that Euclidean $\mathbf{R}^3$ is composed out of disjoint circles. Moreover a distance topology is derived from the intuitive, perhaps some would say naive, geometrical argumentation.

Keywords: Intuitive Geometry, Computational Geometry

1 Introduction

Intuitive Geometry has the charme of simple elements and the ability to ask deep questions. Concerning the question whether or not the $\mathbf{R}^3$ is composed out of disjoint circles there has been previous proof [1] and [2].

The method presented below differs from the previous two because firstly a single finite object is created that consists entirely of disjoint circles. Because of its sharp boundary plus the fact that the center of the object is lying on a disjoint circle as the origin of an arbitrary local XYZ frame and a 'halo reduction' construction method it is, secondly, possible to populate $\mathbf{R}^3$ with disjoint objects, hence disjoint circles.

Let us start the construction by introducing some of the simple geometrical elements. In the first place let us select an arbitrary orthogonal system of axes: XYZ and an origin $O = (0,0,0)$. Secondly, let us define a cricle in the YOZ plane

$$\partial C_R = \left\{(x,y,z) \in \mathbf{R}^3 : x = 0,\ y^2 + z^2 = R^2\right\} \tag{1}$$

The interior of this circle ∂C_R is defined by

$$C_R = \left\{(x,y,z) \in \mathbf{R}^3 : x = 0,\ y^2 + z^2 < R^2\right\} \tag{2}$$

We also may find occasion to write $C_R \backslash \{O\} = Int(\partial C_R)$ and note that all the interior points lying on circles is intended. Hence, $O = (0,0,0) \notin Int(\partial C_R)$ because it is not on any circle in C_R. The exterior is defined by

$$\overline{C}_R = \left\{(x,y,z) \in \mathbf{R}^3 : x = 0,\ y^2 + z^2 > R^2\right\} \tag{3}$$

Concerning the exterior half-plane $\overline{C}_R$ let us define $\overline{C}_R^{\oplus}$ when $y > 0$ and $\overline{C}_R^{\ominus}$ when $y < 0$.

Another simple element is a circle in a plane // the XOY plane denoted by

$$\alpha_r(p,q,u) = \left\{(x,y,z) \in \mathbf{R}^3 : z = u,\ |x-p|^2 + |y-q|^2 = r^2,\ r > 0\right\} \tag{4}$$

Note that the $r > 0$ condition is there to ensure that $\alpha_r(p,q,u)$ does not collapse to a point. A point is not a genuine circle (i.e. a cricle with radius zero is not a circle). Likewise, a point is also not a line segment with length zero etc.

From the circle, with origin (p,q,u) one can construe the whole of the plane $\alpha(p,q,u)$, minus one point (p,q,u), as a union of *disjoint* circles:

$$\alpha(p,q,u) = \lim_{s\to\infty} \cup_{r=\frac{1}{s}}^{s} \alpha_r(p,q,u) \tag{5}$$

2 Construing a 3D disjoint circle object

2.1 Circle planes

In the first place let us try to construe planes // the XOY plane of our (local) coordinate frame intersecting with the circle $\partial C_R(O)$. Here O is the origin of the (local) coordinate frame. The first plane, called α_{No} intersects with $\partial C_R(O)$ at $(0,0,R)$. The second plane is α_{So} intersects with $\partial C_R(O)$ at $(0,0,-R)$. Note that $\alpha_{No} = \alpha(0,0,R)$ and $\alpha_{So} = \alpha(0,0,-R)$. Generally, when plane α with origin $(0, q_\alpha, u_\alpha)$ and α' with origin $(0, q_{\alpha'}, u_{\alpha'})$ are given a subsequent plane $\alpha''(0, q_{\alpha''}, u_{\alpha''})$ can be construed with

$$u_{\alpha''} = \frac{1}{2}\left(u_\alpha + u_{\alpha'}\right) \tag{6}$$

and

$$q_{\alpha''} = \sqrt{R^2 - \frac{1}{4}\left(u_\alpha + u_{\alpha'}\right)^2} \tag{7}$$

Thus construed α planes are gathered in a set A.

2.2 Line segments from circle planes

In the previous the $\partial C_R(O)$ circle in the YOZ plane is intersected with circle planes $\alpha \in A$ with origins $V_\alpha = (0, q_\alpha, u_\alpha)$ on the $y > 0$ side of $\partial C_R(O)$ lying // the XOY plane. The α_{No} and α_{So} are auxiliary planes that now can be removed. From this construction line segments β are subsequently obtained.

$$\beta(0, q, u) = \left(\overline{C}_R^{\oplus}(O) \cup \partial C_R(O) \cup C_R(O)\right) \cap \alpha(0, q, u) \tag{8}$$

The line segements β streches out to infinity on the $y > 0$ side of the YOZ plane.

Let us subsequently for brevity denote the line segment $\beta(V_\alpha)$ and use $\alpha \in A$ as a kind of index. The point W_α can be identified as the singleton set

$$\{W_\alpha\} = (\partial C_R(O) \cap \beta(V_\alpha)) \setminus \{V_\alpha\} \tag{9}$$

W_α lies on $\partial C_R(O)$ opposite the original α plane's origin V_α and represent the far $y < 0$ side of the line β cutting the circle ∂C_R.

From the line segments $\beta(V_\alpha)$ we subsequently construe new line segments

$$\gamma(V_\alpha) = \beta(V_\alpha) \setminus \{W_\alpha\} \tag{10}$$

2.3 Lines and circles

The line segments γ enable new circles that have origin V_α and $\alpha \in A$. The V_α are lying on circle $\partial C_R(O)$. The circles // the XOY plane with origin V_α for all $\alpha \in A$ are disjoint. If $||V_\alpha - W_\alpha||$ denotes the Euclidean distance between V_α and W_α then the interior of the circle $\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)$ in the XOY plane does belong to the construction. For completeness

$$\partial C_{||V_\alpha - W_\alpha||}(V_\alpha) = \left\{(x, y, z) \in \mathbf{R}^3 : z = u_\alpha,\ x^2 + |y - q_\alpha|^2 = ||V_\alpha - W_\alpha||^2\right\} \tag{11}$$

The $\partial C_{||V_\alpha - W_\alpha||}(V_\alpha) \setminus \{W_\alpha\}$ and the exterior of $\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)$ do not belong to the construction. Let us call our 3D circular entity $E_R(O)$ and write

$$E_R(O) = \partial C_R(O) \cup \left(\bigcup_{\alpha \in A} Int\left(\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)\right)\right) \tag{12}$$

Here Int denotes the interior in the sense of $C_R(O) \setminus \{O\} = Int\,(\partial C_R(O))$. Note however that $Int\left(\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)\right)$ lies in a plane // XOY. Note also that V_α is not an element of $Int\left(\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)\right)$ because it is not lying on any circle in $Int\left(\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)\right)$. The point V_α is lying on the circle $\partial C_R(O)$.

3 Inspection of the object $E_R(O)$

In the previous two sections we made sure that the object we have called $E_R(O)$ is built out of disjoint circles. Circle $\partial C_R(O)$ does not intersect the circles in $Int\left(\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)\right)$ but merely contains the origin V_α of the circles contained in $Int\left(\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)\right)$. In addition the circles in $Int\left(\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)\right)$ are disjoint concentric around the point V_α.

Let us, subsequently, introduce a distance to an object like $E_R(O)$.

$$d(\vec{x}, E_R(O)) = \inf\{||\vec{x} - \vec{y}|| \,:\, \vec{y} \in E_R(O)\} \tag{13}$$

A point $\vec{x} = (x, y, z) \in \mathbf{R}^3$.

Let us also define the epsilon sphere around $\vec{x} \in \mathbf{R}^3$.

$$B_\epsilon(\vec{y}) = \left\{\vec{x} \in \mathbf{R}^3 \,:\, ||\vec{x} - \vec{y}||^2 \leq \epsilon^2\right\} \tag{14}$$

We may subsequently ask if we only have one type of point $\vec{x} \notin E_R(O)$. That is: there are points $\vec{x} \notin E_R(O)$ with $\epsilon > 0$ and $B_\epsilon(\vec{x}) \cap E_R(O) = \emptyset$.

One could argue that the object $E_R(O)$ does not have a sharp boundary in planes // the XOY plane. Hence, I could take a point $\vec{z}$ on $\left(\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)\right) \backslash \{W_\alpha\}$ and argue that for every finite small $\epsilon > 0$ the set $B_\epsilon(\vec{z})$ would contain a (tiny) fraction of $E_R(O)$ because $E_R(O)$ has no sharply defined boundary in planes// XOY, despite $d(\vec{z}, E_R(O)) > 0$.

3.1 The reduction of $E_R(O)$ to $G_R(O)$

Suppose we reduce the $E_R(O)$ object such that in $Int\left(\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)\right)$ only circles with size $\leq R$ are allowed. This eliminiates the outer border $\partial C_{||V_\alpha - W_\alpha||}(V_\alpha)$ when the $y < 0$ points are eliminated too. Hence, the $y < 0$ side of $\partial C_R(O)$ is eliminated as well. The object thus obtained is called $F_R(O)$. If subsequently the $y < 0$ part of $\partial C_R(O)$ is 'glued'back on again to $F_R(O)$, object $G_R(O)$ is constructed.

All points in $G_R(O)$ are lying on disjoint circles and every boundary of $G_R(O)$ is included in the object and is 'sharp'. Hence, there is for $G_R(O)$ only one type of point $\vec{x} \notin G_R(O)$.

$$\forall_{\vec{x} \notin G_R(O)} \exists_{\epsilon > 0} \; [B_\epsilon(\vec{x}) \cap G_R(O) = \emptyset] \tag{15}$$

Or in terms of distances

$$\forall_{\vec{x} \in \mathbf{R}^3 : d(\vec{x}, G_R(O)) > 0} \exists_{\epsilon > 0} \; [B_\epsilon(\vec{x}) \cap G_R(O) = \emptyset] \tag{16}$$

3.2 Local frames

Let us denote the origin of a local XYZ coordinate frame with O_{loc}. Based upon equations (15) and/or (16) we can either have $\vec{x} \in G_R(O_{loc})$ or if $\vec{x} \notin G_R(O_{loc})$ then *either*

$$\exists_{R' \in \mathbf{R}}\ \vec{x} = O'_{loc} \tag{17}$$

such that $\vec{x} \notin G_R(O_{loc})$ has $\vec{x} \in G_{R'}(O'_{loc})$ and $G_R(O_{loc}) \cap G_{R'}(O'_{loc}) = \emptyset$, *or* we have, $\vec{x} \notin G_R(O_{loc})$ and $\exists_{R'' \in \mathbf{R}}\ \exists_{O''_{loc} \in \mathbf{R}^3}$ such that $\vec{x} \in G_{R''}(O''_{loc})$ but $\vec{x} \neq O''_{loc}$. However, in this latter case we also maintain $G_R(O_{loc}) \cap G_{R''}(O''_{loc}) = \emptyset$. For completeness it is noted that the orientation of the $G_{R''}(O''_{loc})$ object is arbitrary because the XYZ frame is, now, considered local in the construction.

3.3 Filling of $\mathbf{R}^3$ using the halo

Perhaps it is good to provide a manner in which disjoint elements $G_R(O_{loc})$ from $\mathcal{G}$ can be added to $\mathbf{R}^3$. The case of $\vec{x} = O'_{loc}$ for a new object is clear, but how can a $\vec{x} \neq O'_{loc}$ be construed? In the first place let us note that $G_R(O)$ is a subset of $E_R(O)$ as defined in equation (12). The 'halo' object can be defined by

$$H_R(O) = (E_R(O) \backslash G_R(O)) \cup \left(\bigcup_{\alpha \in A} \partial C_{||V_\alpha - W_\alpha||}(V_\alpha) \right) \tag{18}$$

Secondly, space can be filled through entering a new $G_R(O)$ by constructing this object requiring that the object 'plus' halo $H_R(O) \cup G_R(O)$ has for the cardinality of the intersection of halo with collective: $|H_R(O) \cap \mathcal{G}| = 1$. If the halo $H_R(O)$ is subsequently removed the new $G_R(O)$ is disjoint from the other elements in $\mathcal{G}$ and can be added to this collective. The next object can be construed similarly and note that the space that has been left open by eliminating $H_R(O)$ can be filled with a properly fitting of another $G_{R'}(O')$ object using its $H_{R'}(O')$ to fit it in,..., ad infinitum.

4 Conclusion

From the previous it follows that $\mathbf{R}^3$ can completely be covered with $G_R(O_{loc})$ objects. Each $G_R(O_{loc})$ object consists out of disjoint circles, hence the collective $\mathcal{G}$ consist out of disjoint circles. This implies we finally may conclude that a proof has been delivered with intuitive geometric means that $\mathbf{R}^3$ is composed out of disjoint circles.

Note that a $\vec{x} = O''_{loc}$ implies that this point lies on a disjoint circle. So the construction implies that if $\vec{x} \notin G_R(O_{loc})$ and $\vec{x} \notin G_{R'}(O'_{loc})$.... then selecting $\vec{x} = O''_{loc}$ for a $G_{R''}(O''_{loc})$ means it is constructed to be on a disjoint circle. This is so because any O_{loc} of a given $G_R(O_{loc})$ lies on a disjoint circle. Hence, if $\vec{x}$ is not on any object of a collection, $\mathcal{G}$, of disjoint $G_R(O_{loc})$ objects then there is an object in $\mathcal{G}$ to which $\vec{x}$ has the smallest distance. This measure of distance is taken to construe a new disjoint $G_R(O_{loc})$ object and $\vec{x}$ the (local) origin would be on a disjoint circle. If it somehow would not be sufficient to have $\vec{x} = O$ then another neighboring point $\vec{y} \notin \mathcal{G}$ can be $\vec{y} = O$ and the new disjoint $G_R(O_{loc})$ object, using the halo construction method, can be such that $\vec{x} \in G_R(O_{loc})$. Hence, the object can be added to the collection $\mathcal{G}$.

It is conjectured that the $|\mathcal{G}|$ i.e. the cardinality of the collection, must be countably infinite in order to completely fill $\mathbf{R}^3$ with disjoint circles. To this we may add that the cardinality of the object $G_R(O)$ itself is equal to the continuum.

Finally, the proof that $\mathbf{R}^3$ is construed out of disjoint circles seems to allow a distance measure where parts of circles instead of straight lines can be traveled to define the length of a journey.

References

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