Counterpropagating Wavepacket Solutions of the Time-Dependent Schrödinger Equation for a Decaying Potential Field

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Abstract

We investigate wavepacket solutions for time-dependent Schödinger equation in the presence of an exponentially decaying potential. Assuming for travelling wave solutions the phase to be a linear combination of the space and time coordinates, we obtain two distinct wavepacket solutions for the Schrödinger equation. The wavepackets counterpropagate in space at a constant velocity without any distortion or spreading thus retain their initial form at arbitrarily large distances.

1 Introduction

Wavepacket solutions of the Schrödinger equation and their interpretation has been investigated for diverse cases of physical interest. In non-relativistic quantum theory attempts have been made to find wavepacket solutions for such potential fields as the Morse and the generalized Morse potentials [1,2], harmonic and pseudoharmonic potentials [3], the Woods-Saxon potentials [4,5], and in the presence of a Coulomb field [6]. In these cases a complete

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correspondence with the classical theory can be established in general [7], although for time-dependent potentials this correspondence is usually more complicated and in most cases not possible in the sense of Ehrenfest theorem [8]. A persistent feature of wavepackets, both in case of time independent as well as time dependent potentials, is their spatial spreading with time. Propagating free wavepackets based on Fourier construction methods exhibit this spreading property especially. However there are cases, such as the free Bessel wavepacket solutions [9-11], where the spreading does not occur. These wavepackets have been demonstrated in lab (see for example [12,13] and references therein). In general nonspreading Bessel wavepackets are not normalizable so that the Ehrenfest theorem cannot applied.

In this paper we investigate new exact wavepacket solutions of the Schödinger equation in the presence of an exponentially decaying potential field. Such potentials are of interest due to the fact that exponentially decaying fields are easily set up and maintained in practice. In the asymptotic limit an exponentially decaying potential becomes constant hence the wavepacket can be considered as free at a sufficiently large distance from the source. Also the exponentially decaying potential considered here can be reduced to some special cases of time-independent potentials, such as the generalized Morse potential or the inverse Wood-Saxon potential. We solve the time-dependent Schrödinger equation by splitting it into the quantum Hamilton-Jacobi equation, and the continuity equation. We assume a propagating wave solution by considering a linear combination of the space and time variables for the phase function. We thus obtain two possible wavepacket solutions, one for the time variable t and another with the time inversion $-t$ propagating in the opposite directions. These solutions are expressed as the zeroth order Bessel function, and posses the nonspreading property.

The paper is organized as follows. In section 2 we solve the time-dependent Schrödinger equation for the potential using the corresponding quantum Hamilton-Jacobi and the continuity equations. Travelling wave form of the solutions is then obtained by requiring that the phase function is a linear combination of the space and time coordinates. In section 3 we discuss the physical properties of the wavepacket solutions. It is shown that the wavepackets can be considered as a continuous superposition of travelling waves that propagate in opposite directions at each instance of time with a constant velocity. It is also observed that the wavepackets retain their shape throughout their motion without any dissipation. In the last section we summarize the main results of the study.

2 The One Dimensional Bessel Wavepackets

We consider the decaying potential of the form Ce^{-x-kt} where k is a dimensional constant later fixed. The potential function rapidly decreases as x increases so that at a sufficiently large distance the particle can be considered as free. The time-dependent Schrödinger equation is then given as

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} = i\hbar\frac{\partial\Psi(x,t)}{\partial t} + Ce^{-x-kt}\Psi(x,t). \tag{1}$$

Let the solution $\Psi(x,t)$ be given as

$$\Psi(x,t) = R(x,t)e^{iS(x,t)/\hbar}. \tag{2}$$

This leads to the equation for the real part:

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + Ce^{-x-kt} - \frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{\partial^2 R}{\partial x^2}\right), \tag{3}$$

where the last terms is the quantum potential. Corresponding to the imaginary part we have the continuity equation

$$\frac{\partial R}{\partial t} + \frac{1}{m}\left(\frac{\partial R}{\partial x}\right)\left(\frac{\partial S}{\partial x}\right) + \frac{1}{2m}\left(R\frac{\partial^2 S}{\partial x^2}\right). \tag{4}$$

For a travelling wave solution we assume that the phase function $S(x,t)$ is expressed as $\alpha(x-\beta t)$ where α and β are constants. This gives in equation (3):

$$\frac{\alpha^2}{2m} + Ce^{-x-kt} - \frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{\partial^2 R}{\partial x^2}\right) = \alpha\beta. \tag{5}$$

Denoting the constant $a = (\alpha^2/2m) - \alpha\beta$ we have the differential equation

$$\frac{\partial^2 R}{\partial x^2} = \frac{2m}{\hbar^2}\left(Ce^{-x-kt} + a\right)R. \tag{6}$$

This equation can be solved by separating the variables for $R(x,t)$, which gives

$$R(x,t) = C_1 J_{-q}\left(\sqrt{\frac{8mC}{\hbar^2}}e^{-(x+kt)/2}\right)\Gamma(1-q) + C_2 J_q\left(\sqrt{\frac{8mC}{\hbar^2}}e^{-(x+kt)/2}\right)\Gamma(1+q), \tag{7}$$

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where $q = \sqrt{-2am/\hbar}$, $J_q(z)$ is the qth order Bessel function and $\Gamma(z)$ is the gamma function.

We now consider equation (4). With the phase function $\alpha(x - \beta t)$ we have

$$\frac{\partial R}{\partial t} = -\frac{\alpha}{m}\left(\frac{\partial R}{\partial x}\right) \tag{8}$$

Equation (8) is satisfied by $R(x,t)$ given in equation (7) provided that $q = 0$, $\alpha = 2m$ and $k = -2$. Alternatively equation (8) holds for $q = 0$, when $\alpha = -2m$ and $k = 2$. We thus have for the $\Psi(x,t)$ function two solutions:

$$\Psi_1(x,t) = cJ_0\left(\sqrt{\frac{8mC}{\hbar^2}}e^{-(x-2t)/2}\right)e^{2mi(x-t)/\hbar}, \text{ when } k = -2; \tag{9}$$

and

$$\Psi_2(x,t) = cJ_0\left(\sqrt{\frac{8mC}{\hbar^2}}e^{-(x+2t)/2}\right)e^{-2mi(x-t)/\hbar}, \text{ when } k = 2; \tag{10}$$

where $c = C_1 + C_2$ is a constant. Both Ψ_1 and Ψ_2 satisfy the Schödinger equation in the limiting case of a free particle that is $V = 0$. Although Ψ_1 and Ψ_2 are two distinct solutions of the time-dependent Schödinger equation, a linear superposition of these does not satisfy (1). Notice that with $k = \pm 2$ the potential is $Ce^{-x\pm 2t}$.

3 Physical Properties of the Wavepacket Solutions

The integral representation of the Bessel function

$$J_0\left(\sqrt{\frac{8mC}{\hbar^2}}\exp(\pm t - \frac{x}{2})\right) = \frac{1}{\pi}\int_0^\pi \cos(\sqrt{\frac{8mC}{\hbar^2}}\exp(\pm t - \frac{x}{2})\sin t)dt \tag{11}$$

shows that the wavepackets can be interpreted as a continuous superposition of plane wave solutions $\cos(\sqrt{8mC/\hbar^2}\exp(\pm t - \frac{x}{2})\sin t)$.

Figure (1) and (2) show the probability densities corresponding to the Ψ function solutions (9) and (10) at different time instances. We notice that

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the wavepackets propagate in opposite directions. The propagation speed can be calculated using $v = (\partial S/\partial x)/m$. This shows that both wavepackets counterpropagate with equal velocity. Another feature of the wavepacket solutions is nonspreading (exhibited in figure (3) and (4) in three dimensions). This property is retained by the wavepacket even at very large distances where the effects of the potential function are negligible. Thus in the asymptotic limit both wavepackets are free, and move in opposite directions while retaining their initial shapes.

4 Conclusions

In this paper we have obtained wavepacket solutions of the Schödinger equation in the presence of an exponentially decaying potential field. Two coupled differential equations, namely the Hamilton-Jacobi equation and the equation of continuity, were obtained by making a substitution of the form $R\ e^{iS/\hbar}$. These equation lead to two independent solutions representing continuous superpositions of travelling plane waves collectively forming a wavepacket. These wavepackets counterpropagate with a constant velocity and posses the property of nonspreading hence retain their form at very large distances.

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FIGURE CAPTIONS:

Figure 1: Probability density plots for the wavepacket Ψ_1 at different times with $8mC/\hbar^2 = 1$, exhibiting nonspreading while propagating in the negative x-direction.

Figure 2: Probability density plots for the wavepacket Ψ_2 at different times with $8mC/\hbar^2 = 1$, exhibiting nonspreading while propagating in the positive x-direction.

Figure 3: 3D plot for the probability density for one dimensional Bessel wavepacket Ψ_1 with $8mC/\hbar^2 = 1$.

Figure 4: 3D plot for the probability density for one dimensional Bessel wavepacket Ψ_2 with $8mC/\hbar^2 = 1$.

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