Feigenbaum graphs at the onset of chaos

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Abstract –We analyze the properties of the self-similar network obtained from the trajectories of unimodal maps at the transition to chaos via the horizontal visibility (HV) algorithm. We first show that this network is uniquely determined by the encoded sequence of positions in the dynamics within the Feigenbaum attractor and it is universal in that it is independent of the shape and nonlinearity of the maps in this class. We then find that the network degrees fluctuate at all scales with an amplitude that increases as the size of the network grows. This suggests the definition of a graph-theoretical Lyapunov exponent that measures the expansion rate of trajectories in network space. On good agreement with the map's counterpart, while at the onset of chaos this exponent vanishes, the subexponential expansion and contraction of network degrees can be fully described via a Tsallis-type scalar deformation of the expansion rate, that yields a discrete spectrum of non-null generalized exponents. We further explore the possibility of defining an entropy growth rate that describes the amount of information created along the trajectories in network space. Making use of the trajectory distributions in the map's accumulation point and the scaling properties of the associated network, we show that such entropic growth rate coincides with the spectrum of graph-theoretical exponents, what constitutes a set of Pesin-like identities in the network.

Introduction. – Pesin's theorem [1] prescribes the equality of the Kolmogorov-Sinai (KS) entropy h_{KS} with the sum of the positive Lyapunov exponents $\lambda_i > 0$ of a dynamical system, i.e. $h_{KS} = \sum_i \lambda_i$. The former is a rate of entropy growth that is a metric invariant of the dynamical system, while the latter is the total asymptotic expansion rate present in the chaotic dynamics. This relation provides a deep connection between equilibrium statistical mechanics and chaos. In the limiting case of vanishing Lyapunov exponent, as in the onset of chaos in one-dimensional nonlinear maps, Pesin's theorem is still valid but there are important underlying circumstances that are not expressed by the trivial identity $h_{KS} = \lambda = 0$. At the transition to chaos phase space is no longer visited in an ergodic way and trajectories within the attractor show self-similar temporal structures, they preserve memory of their previous locations and do not have the mixing property of chaotic trajectories [2,3]. Exponential separation of trajectories no longer occurs and the sensitivity to initial conditions does not converge to any single-valued function but displays fluctuations that grow indefinitely. For initial positions on the attractor the sensitivity develops a universal self-similar temporal structure and its envelope grows with iteration time t as a power law [2,3]. For unimodal maps it has been shown [2,3] that this rich borderline condition accepts a description via a spectrum of generalized Lyapunov exponents that matches a spectrum of generalized entropy growth rates obtained from a scalar deformation of the ordinary entropy functional, the so-called Tsallis entropy expression [4,5]. The entropy rate also differs from the KS entropy in that it is local in iteration time. Here we present evidence that this general scenario of unimodal maps at the onset of chaos is captured by a special type of complex network.

Very recently [6,7], the horizontal visibility (HV) algorithm [8,9] that transforms time series into networks has offered a view of chaos and its genesis in low-dimensional maps from an unusual perspective favorable for the discovery of novel features and new understanding. We focus here on networks generated by unimodal maps at their

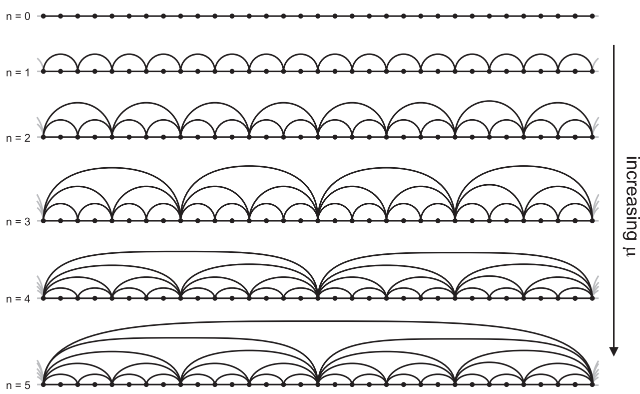

Fig. 1: Feigenbaum graphs for periodic series of increasing period 2^n undergoing a period-doubling cascade. The resulting patterns follow from the universal order with which an orbit visits the positions of the attractor.

period-doubling accumulation points and characterize the fluctuations in connectivity as the network size grows. We show that the expansion of connectivity fluctuations admits the definition of a graph-theoretical Lyapunov exponent. Furthermore, the entropic functional that quantifies the amount of information generated by the expansion rate of trajectories in the original map appears to translate, in the light of the scaling properties of the resulting network, into a generalized entropy that surprisingly coincides with the spectrum of graph-theoretical Lyapunov exponents. This suggests that Pesin-like identities valid at the onset of chaos could be found in complex networks that possess certain scaling properties.

The rest of the Letter is as follows: We first recall the HV algorithm that converts a time series into a network and focus on the so-called Feigenbaum graphs [6, 7] as the subfamily of HV graphs generated by iterated nonlinear one-dimensional maps. We then expose the universal scale-invariant structure of the Feigenbaum graphs that arise at the period-doubling accumulation points. We subsequently show how to define a graph-theoretical Lyapunov exponent in this context. In agreement with the known dynamics of unimodal maps at the transition to chaos the graph-theoretical Lyapunov exponent vanishes, and proceed to define generalized exponents that take into account the subexponential expansion of connectivity fluctuations. We finally show that the Feigenbaum graph that represents the onset of chaos admits a spectrum of graph-theoretical generalized exponents that coincide with a spectrum of deformed entropies.

The Feigenbaum graph at the onset of chaos. – The horizontal visibility (HV) algorithm is a general method to convert time series data into a graph [8, 9] and is concisely stated as follows: assign a node i to each datum x_i of the time series $\{x_i\}_{i=1,...,N}$ of N real data, and then connect any pair of nodes i, j if their associated data fulfill the criterion $x_i, x_j > x_n$ for all n such that $i < n < j$. The capability of the method to transfer properties of different types of time series into their resultant graphs has been demonstrated in recent works [10, 11]. When the series under study are the trajectories within the attractors generated by unimodal or circle maps the application of the HV algorithm yield subfamilies of visibility graphs, named Feigenbaum graphs, that render the known low-dimensional routes to chaos in a new setting [6, 7, 12]. For illustrative purposes, in Fig. 1 we show a hierarchy of Feigenbaum graphs obtained along the period-doubling bifurcation cascade of unimodal maps. At the accumulation point of the cascade the trajectories within its non-chaotic multifractal attractor become aperiodic, and in analogy with the original Feigenbaum treatment [14], the associated Feigenbaum graphs evidence scaling properties that can be exploited by an appropriate graph-theoretical Renormalization Group transformation [6, 7]. Because the order of visits of positions of periodic attractors or of bands of chaotic attractors in unimodal maps are universal, a relevant consequence of the HV criterion is that the resulting Feigenbaum graph at the onset of chaos is the same for every unimodal map. That is, it is independent of the shape and nonlinearity of the map [6, 7]. This permits us to concentrate our study on a specific unimodal map, that for simplicity will be the logistic map, and claim generality of the results.

Consider the logistic map $x_{t+1} = f(x_t) = \mu x_t(1 - x_t)$, $0 \leq x \leq 1$, $0 \leq \mu \leq 4$, at the period-doubling accumulation point $\mu_\infty = 3.5699654...$ and generate a trajectory $\{x_t\}_{t=1,2,3,...}$ within the attractor. Since the position of the maximum of the map belongs to the attractor we can chose $x_0 = f(1/2)$ as the initial condition. The corresponding time series is shown in the top panel of figure 2 and in the left panel of the same figure we represent in logarithmic scales the positions of the rescaled variable $z_t = |x_t - 1/2|$, as usual in functional renormalization group theory. The panel shows the striped intertwined self-similar structure of the trajectory positions as they are visited sequentially and reflects the multifractal structure of the attractor [2,3]. The degree $k(N)$ of the Feigenbaum graph generated by this trajectory is shown in the right panel of figure 2 in semi-logarithmic scales as a function of node N. It can be observed that the network inherits the distinctive band pattern of the attractor although in a simplified manner where the fine structure is replaced by single lines of constant degree. The HV algorithm transforms the multifractal attractor into a discrete set of connectivities, whose evolution mimics the intertwined fluctuations of the map (see the labels in both panels that indicate successive values of t or N). It is convenient to write the node index $N \equiv t$ in the form $N \equiv m2^j$, where $j = 0, 1, 2, ...$ and $m = 1, 3, 5, ...$, so that running over the index j with m fixed selects subsequences of data placed along lines with fixed slope shown in both panels of figure 2. This representation of N generates a one-to-one

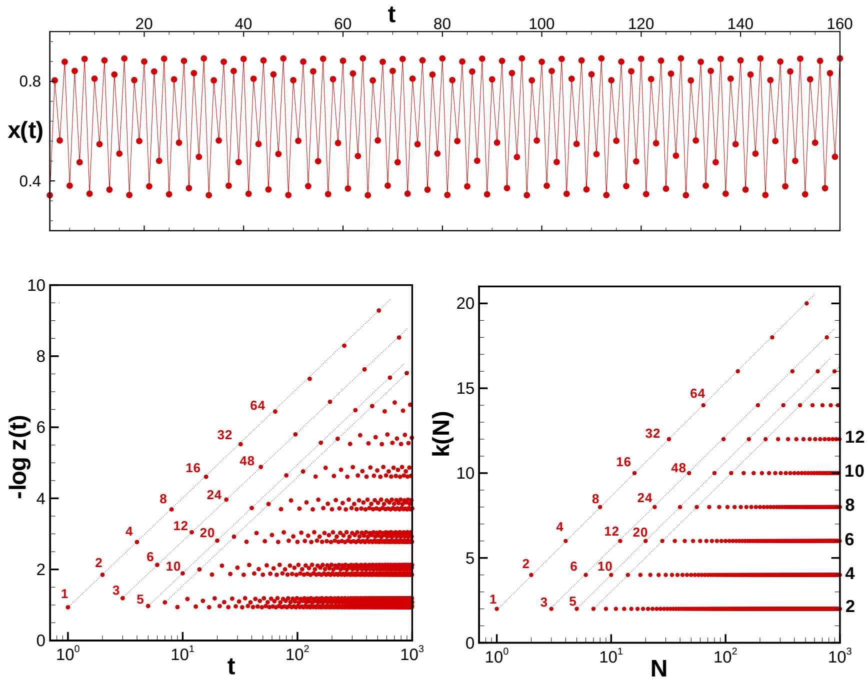

Fig. 2: *Up:* Series $x(t)$ as a function of time t for the first 10^6 data generated from a logistic map at the period-doubling accumulation point (only the first 160 data are shown). *Left:* Semilog plot of $-\log(z(t))$, $z_t = |x_t - 1/2|$, as a function of t. *Right:* Semilog plot of the degree $k(N)$ as a function of the node N, of the Feigenbaum graph generated from the same time series as for the left panel. The labels indicate successive values of t or N and lines characterize certain subsequences (see the text).

tiling of the natural numbers. The aforementioned constant degree and fixed-slope lines in the figure are, respectively, $k(N \equiv m2^j) = 2j + 2$, $m = 1, 3, 5, ...,$ j fixed, and $k(N \equiv m2^j) = 2j + 2$, $j = 0, 1, 2, ...,$ m fixed. The scaling property associated with the former case is that all horizontal lines with constant degree $k = 2j + 2$, j fixed, can be overlapped into the bottom line $k = 2$, $j = 0$, via consecutive translations each consisting of a shift of 2 in k and a shift of $\log 2$ in $\log N$, that is, $k(N/2) = k(N) - 2$. This scaling property has a parallel for the trajectory within the Feigenbaum attractor where the shift of 2 in k is replaced by a shift of $2 \log \alpha$, where α is the absolute value of the Feigenbaum constant [2, 3]. The scaling property associated with the latter case is seen via the collapse of all fixed-slope lines of data into a single sequence of values aligned along the first $m = 1$ subsequence $k(2^j) = 2j + 2$, $j = 0, 1, 2, ...,$ when the numbers of nodes for each of the other subsequences are rescaled consecutively by N/m, $m = 3, 5, 7, ...$

The organization of connectivities in the right panel of figure 2 can be understood by observing how the HV algorithm assigns links between nodes from position trajectories shown in the left panel of the same figure. One half of all trajectory positions x_t, $t = m2^0 = 1, 3, 5, ...,$ are located close to the initial position $x_0 = f(1/2)$, they have the smallest values of $-\log\ z_t$ and are situated in the bottom band of the figure. Because their nearest neighbor positions $x_{t\pm1}$ are larger than x_t, with $-\log\ z_{t\pm1}$ located in the 2nd or higher bands of the panel, the HV criterion assigns a degree $k = 2$ to all nodes $N = m2^0 = 1, 3, 5, ...$ One fourth of all trajectory positions x_t, $t = m2^1 = 2, 6, 10, ...,$ have values larger than their nearest neighbors $x_{t\pm1}$ but lie below their 2nd nearest neighbors $x_{t\pm2}$, with $-\log\ z_{t\pm2}$ placed in the 3rd or higher bands of the panel, the HV criterion assigns a degree $k = 4$ to all nodes $N = m2^1 = 2, 6, 10, ...$ In general, $1/2^{j+1}$-th of all trajectory positions x_t, $t = m2^j$, j fixed, have values larger than their nearest neighbors $x_{t\pm1}$ up to the j-th order but lie below their $(j + 1)$-th nearest neighbors $x_{t\pm(j+1)}$, with $-\log\ z_{t\pm(j+1)}$ located in the j-th or higher bands of the panel, the HV criterion assigns a degree $2j+2$ to all nodes $N = m2^j$, $m = 1, 3, 5, ...,$ j fixed.

From the above it is clear that all trajectory positions within each band in the left panel of figure 2 become nodes with the same degree in the right panel of the same figure. This degeneracy applies also to positions of trajectories that are initiated off, but close to, the attractor positions. In particular, if an ensemble of uniformly-distributed initial positions is placed around a small interval around $x_0 = f(1/2)$ their trajectories remain uniformly-distributed at later iteration times [2, 3] and the HV algorithm assigns to all of them the same Feigenbaum graph. This degeneracy accounts for the universal feature that only a single Feigenbaum graph represents the transition to chaos for all unimodal maps. It also determines the way in which fluctuations in trajectory separations translate into their respective degree fluctuations. The bounded phase space of a generic unimodal map $[a, b]$ transforms directly under the application of the HV method into an unbounded, yet discrete, network phase space $\{\exp(2), \exp(4), ...\}$. The rescaled trajectory positions z_t appear in the network as exponentials of the degree $\exp(k(N))$.

Fluctuating dynamics and graph-theoretical Lyapunov exponents. – We recall that the standard Lyapunov exponent λ accounts for the degree of exponential separation of nearby trajectories, such that two trajectories whose initial separation is $d(0)$ evolve with time distancing from each other as $d(t) \sim d(0)\exp(\lambda t)$, $t \gg 1$, where $d(t) = |x_t - x'_t|$. The sensitivity to initial conditions is $\xi(t) = d(t)/d(0)$ and the general definition of the Lyapunov exponent is

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \xi(t). \tag{1}$$

Deterministic fluctuations about $\lambda > 0$ are permanently put out when $t \to \infty$, but, as mentioned, at the period-doubling accumulation points of unimodal maps $\lambda = 0$ and trajectories display successive subexponential separation and convergence for all t. Two possible extensions of the standard Lyapunov exponent have been used when $\lambda = 0$ while the amplitude of the fluctuations grows as a power law, or, equivalently, expansion and contraction rates of trajectories are logarithmic in time: Mori *et al* [13] suggested the expression

$$\lambda_M = \frac{1}{\log t} \log \xi(t). \tag{2}$$

Although this expression captures subexponential fluctuation rates, this quantity is not well defined if we want to measure the magnitude of rates *per unit time*, in which case we need a quantity that grows linearly in time. This is important if we want to relate the growth of fluctuations to some kind of entropic rates that derive from an extensive entropy expression. In this respect a better-suited alternative for capturing subexponential fluctuations is to deform the logarithm in equation 1 by an amount that recovers the linear growth present for $\lambda > 0$. This is

$$\lambda_q = \frac{1}{t} \log_q \xi(t), \tag{3}$$

with $\log_q x \equiv (x^{1-q} - 1)/(1 - q)$ ($\log x$ is restored in the limit $q \to 1$) where the extent of deformation q is such that while the expansion $\xi(t)$ is subexponential, $\log_q \xi(t)$ grows linearly with t [2, 3]. Notice that we have eliminated the $t \to \infty$ limit in the definition of the generalized Lyapunov exponents since fluctuations are present for all values of t and the objective is to characterize them via a spectrum of generalized exponents [13], [2, 3].

We define now a connectivity expansion rate for the Feigenbaum graph under study. Since the graph is connected by construction (all nodes have degree $k \geq 2$), the uncertainty is only associated with a rescaled degree $k_+ \equiv k - 2$. To keep notation simple we make use of this variable and drop the subindex + from now on. The formal network analog of the sensitivity to initial conditions $\xi(t)$ has as a natural definition $\xi(N) \equiv \delta(N)/\delta(0) = \exp k(N)$, the ratio of the distance $\delta(N) = \exp k(N)$ in the network phase space at node N to the initial distance $\delta(0) = \exp k(0) = 1$, the shortest distance that the HV algorithm yields for nearby trajectory positions. Accordingly, the standard network Lyapunov exponent is defined as

$$\lambda \equiv \lim_{N\to\infty} \frac{1}{N} \log \xi(N). \tag{4}$$

The value of $k(N)$ oscillates with N (see figure 2) but its bounds grow slower than N, as $\log N$, and therefore in network context $\lambda = 0$, in parallel to the ordinary Lyapunov exponent at the onset of chaos. The logarithmic growth of the bounds of $\log \xi(N) = k(N)$ is readily seen by writing $k(N = m2^j) = 2j$ as

$$k(N) = \frac{2}{\log 2} \ln \frac{N}{m}. \tag{5}$$

A first approach to study the subexponential fluctuations is to proceed *à la Mori* and define the following graph-theoretical generalized exponent

$$\lambda_M = \frac{1}{\log N} \log \xi(N) = \frac{k(N)}{\log N}. \tag{6}$$

To visualize the network growth paths $N = m2^j$, where $m = 1, 3, 5, ...$ is fixed to a constant value and $j = 1, 2, 3, ...$, the expression above for λ_M is written as

$$\lambda_M = \frac{2j}{\log(m2^j)} = \frac{2j}{\log m + j \log 2}. \tag{7}$$

A constant $\lambda_M = 2/\log 2$ for all j is obtained only when $m = 1$, otherwise the same value is reached for all other m when $j \to \infty$. In general, a spectrum of Mori-like generalized exponents is obtained by considering all paths (m, j) such that $N \to \infty$.

As indicated, the preceding generalization of the Lyapunov exponent is not time extensive and it is therefore not useful if we ultimately wish to relate the network link fluctuations to an entropy growth rate. With this purpose in mind we deform the the ordinary logarithm in $\log \xi(N) = k(N)$ into $\log_q \xi(N)$ by an amount $q > 1$ such that $\log_q \xi(N)$ depends linearly in N, and define the associated generalized graph-theoretical Lyapunov exponent as

$$\lambda_q = \frac{1}{N - m} \log_q \xi(N), \tag{8}$$

where for the sake of generality we suppose that the initial time is of the form $m \cdot 2^0$ (see left panel of figure 2, where the expansion along the m lines start always for $j = 0$). One obtains

$$\lambda_q = \frac{2}{m \log 2}, \tag{9}$$

with $q = 1 - \log 2/2$. Let us explain in some detail equations 8 and 9, as these results are due to a special property of the trajectories within the attractor at the period-doubling accumulation point. This property is that *all* such trajectories, with generic initial position x_0, can be referred to the trajectory with initial position at the extremum of the unimodal map, $x = 1/2$ for the logistic map. A trajectory initiated at any x_0 is related to the trajectory we use here, initiated at $x = f(1/2)$, only one iteration away from $x = 1/2$, via a time shift equal to the number of iterations t_w required by the latter trajectory to reach position x_0, *i.e.* $x_{t_w} = x_0$. This property is fully inherited by the networks obtained via the HV method and here it is of interest to consider the particular cases where $t_w = m = 3, 5, 7, \ldots$ The shifted node number $N - m$ in the denominator of equation 8 matches the equivalences in connectivity $k(N)$ along any node sequence $N = m2^j$, $j = 0, 1, 2, \ldots, m > 1$.
According to Eq. (9) a spectrum of exponents is spanned by running over the values of m, in parallel with the spectrum of generalized exponents previously found at the transition to chaos in unimodal maps [3], where the value of the parameter, $q = 1 - \log 2/2$ in the Feigenbaum graph is to be compared with $q = 1 - \log 2/\log \alpha$ in the unimodal map for trajectories originating at the most compact region of the multifractal attractor that are seen to expand at prescribed times when the least compact region of the multifractal is visited [3].

Entropic functionals and Pesin-like identities. – To complete our arguments we summon up the persistency property of trajectory distributions of unimodal maps at the period-doubling onset of chaos. That is, for a small interval of length l_0 with $\mathcal{N}$ uniformly-distributed initial conditions around the extremum of a unimodal map (*e.g.* $x = 1/2$ for the logistic map we use), all trajectories behave similarly, remain uniformly-distributed at later times and follow the concerted pattern shown in the left panel of Fig. 2. (See [2] [3] for details). We denote this distribution by $\pi(t) = 1/W(t)$ where $W(t) = \mathcal{N}/l_t$ and l_t is the length of the interval that contains the trajectories at time t. As stated, all such trajectories give rise to the same Feigenbaum graph, and at iteration times, say, of the form $t = 2^j$, $j = 0, 1, 2, \ldots$, the HV criterion assigns $k = 2j$ links to the common node $N = 2^j$. While π is defined in the map, we inquire about the j-dependence of $\pi(2^j)$, i.e. how π scales along the set of nodes $N = 2^j$ with $k = 2j$ links, $j = 0, 1, 2, ...$, and it what follows we will argue that the concrete scaling properties of the network generate a key scaling property in π. First, as noticed above, we know that when $m = 1$ the node-connectivity pairs (N, k) associated with any two consecutive values of j can be matched by a shift of 2 in k and a shift of $\log 2$ in $\log N$. This suggests that the uniform distributions π for the consecutive pairs $(2^j, 2j)$ and $(2^{j+1}, 2(j+1))$ scale with the same factors and this leads us to conclude that the j-dependence for these distributions is

$$\pi(2^j) = W_j^{-1} = 2^{-(2/\ln 2)j} = \exp(-2j). \tag{10}$$

By extension of the argument the same expression holds for all other values of m. Since

$$W_j = \exp(2j) = \left(\frac{N}{m}\right)^{2/\log 2}, \tag{11}$$

the ordinary entropy associated with π grows logarithmically with the number of nodes N, $S_1[\pi(N)] = \log W_j \sim \log N$. However, the q-deformed entropy

$$S_q[\pi(N)] = \ln_q W_j = \frac{1}{1-q}\left[W_j^{1-q} - 1\right], \tag{12}$$

where the amount of deformation q of the logarithm has the same value as before, grows linearly with N, as W_j can be rewritten as

$$W_j = \exp_q[\lambda_q(N - m)], \tag{13}$$

with $q = 1 - \log 2/2$ and $\lambda_q = 2/(m \log 2)$. Therefore, if we define the entropy growth rate

$$h_q[\pi(N)] \equiv \frac{1}{N-m} S_q[\pi(N)] \tag{14}$$

we obtain

$$h_q[\pi(N)] = \lambda_q, \tag{15}$$

a Pesin-like identity at the onset of chaos (effectively one identity for each subsequence of node numbers given each by a value of $m = 1, 3, 5, ...$).

In conclusion, the transcription into a network of a special class of time series, the trajectories associated with the attractor at the period-doubling transition to chaos of unimodal maps, via the HV algorithm has proved to be a valuable enterprise [6] as it has led to the uncovering of a new property related to the Pesin identity in nonlinear dynamics. The HV method leads to a self-similar network with a structure illustrated by the related networks of periods 2^n, $n = 0, ..., 5$, shown in Fig. 1. Under the HV algorithm many nearby trajectory positions lead to the same network node and degree, all positions within one band in the left panel of Fig. 1 lead to the same line of constant degree in the right panel of the figure. Only when trajectory positions cross a gap between bands in the left panel the corresponding node increases its degree by two new links. Also trajectories off the attractor but close to it transform into the same network structure. The degrees of the nodes span all even numbers $k = 2j$, $j = 0, 1, 2, ...$, and we have studied how these fluctuate as the number of nodes increases. The fluctuations of the degree capture the core behavior of the fluctuations of the sensitivity to initial conditions at the transition to chaos and they are universal for all unimodal maps. The graph-theoretical analogue of the sensitivity was identified as $\exp(k)$ while the amplitude of the variations of k grows logarithmically with the number of nodes N. These deterministic fluctuations are described by a discrete spectrum of generalized graph-theoretical Lyapunov exponents that appear to relate to an equivalent spectrum of generalized entropy growths, yielding a set of Pesin-like identities. The definitions of these quantities involve a deformation of the ordinary logarithmic function that ensures their linear growth with the number of nodes. Therefore the entropy expression involved is extensive, and of the Tsallis type with a precisely defined index q. The capability of the methodology employed in this study to reveal the occurrence of structural elements of this nature in real networked systems that grow in time [15, 16] appears to be viable.

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