The Crocco transformation: order reduction and construction of Bäcklund transformations and new integrable equations

A D Polyanin[1] **and A I Zhurov**[1,2]

[1] Institute for Problems in Mechanics, Russian Academy of Sciences,
101 Vernadsky Avenue, bldg 1, 119526 Moscow, Russia.
[2] School of Dentistry, Cardiff University, Heath Park, Cardiff CF14 4XY, UK.

E-mail: `polyanin@ipmnet.ru`

Abstract. Wide classes of nonlinear mathematical physics equations are described that admit order reduction through the use of the Crocco transformation, with a first-order partial derivative taken as a new independent variable and a second-order partial derivative taken as the new dependent variable. Associated Bäcklund transformations are constructed for evolution equations of general form (special cases of which are Burgers, Korteweg–de Vries, and many other nonlinear equations of mathematical physics). The results obtained are used for order reduction and constructing exact solutions of hydrodynamics equations (Navier–Stokes, Euler, and boundary layer). A number of new integrable nonlinear equations, inclusive of the generalized Calogero equation, are considered.

AMS classification scheme numbers: 35Q58, 35K55, 35K40, 35Q53, 35Q30

1. Preliminary remarks

The Crocco transformation is used in hydrodynamics for reducing the order of the plane boundary-layer equations [1–3]. It is a transformation in which a first-order partial derivative taken as a new independent variable and a second-order partial derivative taken as the new dependent variable (this applies to the equation for the stream function). So far, using the Crocco transformation has been limited solely to the theory of boundary layer.

The present paper reveals that the domain of application of the Crocco transformation is much broader. It can be successfully used for reducing the order of wide classes of nonlinear equations with mixed derivatives and constructing Bäcklund transformations for evolution equations of arbitrary order and quite general form, special cases of which include Burgers and Korteweg–de Vries type equations as well as many other nonlinear equations of mathematical physics. The Bäcklund transformations obtained with the Crocco transformation may, in turn, be used for constructing new integrable nonlinear equations. Examples of the generalized Calogero equation and a number of other integrable nonlinear second-, third-, and fourth-order equations are considered. A generalization of the Crocco transformation to the case of three independent variables is given.

It is noteworthy that various Bäcklund transformations and their applications to specific equations of mathematical physics can be found, for example, in [3–14].

In the present paper, the term *integrable equation* applies to nonlinear partial differential equations that admit solution in terms of quadratures or solutions to linear differential or linear integral equations.

2. Nonlinear equations that admit order reduction with the Crocco transformation

Consider the nth-order nonlinear equation with a mixed derivative

$$\frac{\partial^2 u}{\partial t\partial x}+[a(t)u+b(t)x]\frac{\partial^2 u}{\partial x^2}=F\left(t,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\frac{\partial^3 u}{\partial x^3},\dots,\frac{\partial^n u}{\partial x^n}\right). \tag{1}$$

1°. General property: if $\widetilde{u}(t,x)$ is a solution to equation (1), then the function

$$u=\widetilde{u}(t,x+\varphi(t))+\frac{1}{a(t)}[b(t)\varphi(t)-\varphi'_t(t)],\quad a(t)\not\equiv 0, \tag{2}$$

where $\varphi(t)$ is an arbitrary function, is also a solution to equation (1). If $a(t)\equiv 0$, then $u=\widetilde{u}(t,x)+\varphi(t)$ is another solution to (1).

2°. Denote

$$\eta=\frac{\partial u}{\partial x},\quad \Phi=\frac{\partial^2 u}{\partial x^2}. \tag{3}$$

Dividing (1) by $u_{xx}=\Phi$, differentiating with respect to x, and taking into account (3), we obtain

$$\frac{\Phi_t}{\Phi}-\frac{u_{tx}\Phi_x}{\Phi^2}+a(t)\eta+b(t)=\frac{\partial}{\partial x}\frac{F(t,\eta,\Phi,\Phi_x,\dots,\Phi_x^{(n-2)})}{\Phi}. \tag{4}$$

Let us pass in (4) from the old variables to the Crocco variables:

$$t,\ x,\ u=u(t,x)\quad\Longrightarrow\quad t,\ \eta,\ \Phi=\Phi(t,\eta), \tag{5}$$

where η and Φ are defined by (3). The derivatives are transformed as follows:

$$\frac{\partial}{\partial x}=\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}=u_{xx}\frac{\partial}{\partial\eta}=\Phi\frac{\partial}{\partial\eta},\qquad \frac{\partial}{\partial t}=\frac{\partial}{\partial t}+\frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta}=\frac{\partial}{\partial t}+u_{tx}\frac{\partial}{\partial\eta}.$$

As a result, equation (4), and hence the original equation (1), is reduced to the $(n-1)$st-order equation

$$\frac{a(t)\eta+b(t)}{\Phi}-\frac{\partial}{\partial t}\frac{1}{\Phi}=\frac{\partial}{\partial\eta}\left[\frac{1}{\Phi}F\left(t,\eta,\Phi,\Phi\frac{\partial\Phi}{\partial\eta},\ldots,\frac{\partial^{n-2}\Phi}{\partial x^{n-2}}\right)\right]. \qquad (6)$$

The higher derivatives are calculated by the formulas

$$\frac{\partial^k u}{\partial x^k}=\frac{\partial^{k-2}\Phi}{\partial x^{k-2}}=\Phi\frac{\partial}{\partial\eta}\frac{\partial^{k-3}\Phi}{\partial x^{k-3}},\quad \frac{\partial}{\partial x}=\Phi\frac{\partial}{\partial\eta},\quad k=3,\ldots,n.$$

Given a solution to the original equation (1), formulas (3) define a solution to equation (6) in parametric form.

Let $\Phi=\Phi(t,\eta)$ be a solution to equation (6). Then, in view of (3), the function $u(t,x)$ satisfies the equation

$$u_{xx}=\Phi(t,u_x), \qquad (7)$$

which can be treated as an ordinary differential equation in x with parameter t. The general solution to equation (7) may be written in parametric form as

$$x=\int_{\eta_0}^{\eta}\frac{\mathrm{d}s}{\Phi(t,s)}+\varphi(t),\quad u=\int_{\eta_0}^{\eta}\frac{s\,\mathrm{d}s}{\Phi(t,s)}+\psi(t), \qquad (8)$$

where $\varphi(t)$ $\psi(t)$ are arbitrary functions and η_0 is an arbitrary constant. Since the derivation of (6) is based on differentiating (1), one of the arbitrary functions in solution (8) is redundant. In order to remove this redundancy, it suffices to substitute (8) into (1). However, it more convenient to take advantage of the solution property (2) and note that solution (8) must also possess this property. In view of this, the general solution to the original equation (1) can be rewritten in the parametric form

$$x=\int_{\eta_0}^{\eta}\frac{\mathrm{d}s}{\Phi(t,s)}+\varphi(t),\quad u=\int_{\eta_0}^{\eta}\frac{s\,\mathrm{d}s}{\Phi(t,s)}+\frac{1}{a(t)}[b(t)\varphi(t)-\varphi'_t(t)], \qquad (9)$$

where $\varphi(t)$ is an arbitrary function.

Example 1 (generalized Calogero equation). With $F=f(t,u_x)u_{xx}+g(t,u_x)$, which corresponds to the nonlinear second-order equation

$$u_{tx}=[f(t,u_x)-a(t)u-b(t)x]u_{xx}+g(t,u_x), \qquad (10)$$

passing to the Crocco variables (5), (3) leads to the first-order equation

$$\frac{a(t)\eta+b(t)}{\Phi}-\frac{\partial}{\partial t}\frac{1}{\Phi}=\frac{\partial}{\partial\eta}\left[f(t,\eta)+\frac{g(t,\eta)}{\Phi}\right],$$

which becomes linear with the substitution $\Phi=1/\Psi$.

In the special case of $a(t)=-1$, $b(t)=0$, $f(t,u_x)=0$, and $g(t,u_x)=g(u_x)$, equation (10) reduces to the Calogero equation, which was considered in [15, 16] (see also [3, p. 433–434]).

Example 2 (equation arising in gravitation theory). The nonlinear third-order equation

$$u_{txx}=kuu_{xxx},$$

which is cross-disciplinary between projective geometry and gravitation theory [16, 17], can be reduced, by integrating with respect to x, to the form

$$u_{tx}=kuu_{xx}-\tfrac{1}{2}kw_x^2+\psi(t), \qquad (11)$$

where $\psi(t)$ is an arbitrary function. Equation (11) is a special case of equation (10), and hence can be reduced to a linear first-order equation.

Example 3 (Navier–Stokes and Euler equations). An unsteady three-dimensional flow of a viscous incompressible fluid may be described by the Navier–Stokes and continuity equations

$$\begin{aligned}&\frac{\partial V_n}{\partial t}+V_1\frac{\partial V_n}{\partial x}+V_2\frac{\partial V_n}{\partial y}+V_3\frac{\partial V_n}{\partial z}\\&\qquad=-\frac{1}{\rho}\nabla_n P+\nu\left(\frac{\partial^2 V_n}{\partial x^2}+\frac{\partial^2 V_n}{\partial y^2}+\frac{\partial^2 V_n}{\partial z^2}\right),\quad n=1,\,2,\,3,\\&\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}=0,\end{aligned}\tag{12}$$

where x, y, and z are Cartesian coordinates, t time, V_1, V_2, and V_3 the fluid velocity components, P pressure, and ρ the fluid density; also $\nabla_1 P=\partial P/\partial x$, $\nabla_2 P=\partial P/\partial y$, and $\nabla_3 P=\partial P/\partial z$. Equations (12) are obtained under the assumption that the bulk forces are potential and included into pressure. In the degenerate case of $\nu=0$, equations (12) become the Euler equations for an ideal (inviscid) fluid.

The equations of motion of a viscous incompressible fluid, (12), admit exact three-dimensional solutions of the form

$$\begin{aligned}&V_1=u,\quad V_2=-\frac{1}{2}y\frac{\partial u}{\partial x},\quad V_3=-\frac{1}{2}z\frac{\partial u}{\partial x},\\&\frac{P}{\rho}=\frac{1}{4}p(t)(y^2+z^2)+s(t)-\frac{1}{2}u^2+\nu\frac{\partial u}{\partial x}-\int\frac{\partial u}{\partial t}\,\mathrm{d}x,\end{aligned}$$

where $p(t)$ and $s(t)$ are arbitrary functions of time t, and $u=u(t,x)$ satisfies the nonlinear third-order equation

$$\frac{\partial^2 u}{\partial t\partial x}+u\frac{\partial^2 u}{\partial x^2}-\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2=\nu\frac{\partial^3 u}{\partial x^3}+p(t),\tag{13}$$

which is a special case of equation (1) with $a(t)=1$, $b(t)=0$, and $F=\nu u_{xxx}+\frac{1}{2}u_x^2+p(t)$.

The Crocco transformation (5) brings (13) to the nonlinear second-order equation

$$\frac{\partial\Phi}{\partial t}+[\tfrac{1}{2}\eta^2+p(t)]\frac{\partial\Phi}{\partial\eta}=\nu\Phi^2\frac{\partial^2\Phi}{\partial\eta^2},\tag{14}$$

which can be rewritten in the form of a nonlinear equation of convective thermal conduction with a parabolic, Poiseuille-type velocity profile:

$$\frac{\partial\Psi}{\partial t}+[\tfrac{1}{2}\eta^2+p(t)]\frac{\partial\Psi}{\partial\eta}=\nu\frac{\partial}{\partial\eta}\left(\frac{1}{\Phi^2}\frac{\partial\Psi}{\partial\eta}\right),\qquad\Psi=\frac{1}{\Phi}.$$

It should be noted that in the special case of inviscid fluid ($\nu=0$), the original nonlinear equation (13) is reducible to the linear first-order partial differential equation (14), which can be solved by the method of characteristics.

Example 4 (system of hydrodynamic-type equations). Consider the system of equations

$$\frac{\partial^2 u}{\partial t\partial x}+u\frac{\partial^2 u}{\partial x^2}-\left(\frac{\partial u}{\partial x}\right)^2=\nu\frac{\partial^3 u}{\partial x^3}+q(t)\frac{\partial u}{\partial x}+p(t),\tag{15}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial x}=\nu\frac{\partial^2 v}{\partial x^2},\tag{16}$$

which describes several classes of exact solutions to the Navier–Stokes equations in two and three dimensions [3, 18–20]. The nonlinear equation (15) is independent of v and can be treated separately. Although linear in v, equation (16) involves the function u, which is governed by equation (15).

The Crocco transformation (5) brings system (15)–(16) to the form

$$\frac{\partial\Phi}{\partial t}+(\eta^2+q\eta+p)\frac{\partial\Phi}{\partial\eta}=(\eta+q)\Phi+\nu\Phi^2\frac{\partial^2\Phi}{\partial\eta^2}, \tag{17}$$

$$\frac{\partial v}{\partial t}+(\eta^2+q\eta+p)\frac{\partial v}{\partial\eta}=\eta v+\nu\Phi^2\frac{\partial^2 v}{\partial\eta^2}. \tag{18}$$

Here and henceforth, the arguments of $p(t)$ and $q(t)$ are omitted for brevity. Equation (18) was obtained using the representation of the mixed derivative u_{tx} obtained from (15).

Equation (18) has exact solutions of the form

$$v=A\eta+B\Phi+C, \tag{19}$$

where $A=A(t)$, $B=B(t)$, and $C=C(t)$ are unknown functions determined from an appropriate system of ordinary differential equations. This fact can be proved by substituting (19) into (18) and taking into account (17).

Formula (19) allows one to arrive at the following important result with regard to solutions of the original equation (16). Let $u=u(t,x)$ be a solution to equation (15). Then equation (16) admits the solution

$$v=A'_t+Aq+A\frac{\partial u}{\partial x}+B\frac{\partial^2 u}{\partial x^2}, \tag{20}$$

where $A=A(t)$ and $B=B(t)$ satisfy the ordinary differential equations

$$A''_{tt}+qA'_t+(p+q'_t)A=0, \tag{21}$$

$$B'_t+qB=0. \tag{22}$$

The general solution to (22) is $B=C_1\exp\left(-\int q\,\mathrm{d}t\right)$, where C_1 is an arbitrary constant.

Listed below are some exact solutions to equation (15) representable in terms of elementary functions and suitable for finding exact solutions to equation (16) using formulas (20).

1°. Generalized separable solution rational in x:

$$u=-\alpha'_t(t)+\beta(t)[x+\alpha(t)]-\frac{6\nu}{x+\alpha(t)},\quad q=-4\beta,\quad p=\beta'_t+3\beta^2,$$

where $\alpha=\alpha(t)$ and $\beta=\beta(t)$ are arbitrary functions.

2°. Generalized separable solution exponential in x:

$$u=\alpha(t)\mathrm{e}^{-\sigma x}+\beta(t),\quad p=0,\quad q=\frac{\alpha'_t}{\alpha}-\sigma\beta-\sigma^2\nu,$$

where $\alpha=\alpha(t)$ and $\beta=\beta(t)$ are arbitrary functions and σ is an arbitrary constant. By choosing periodic functions as $\alpha(t)$ and $\beta(t)$, one obtains time-periodic solutions.

3°. Multiplicative separable solution periodic in x:

$$u=\alpha(t)\sin(\sigma x+C_1),\quad \alpha(t)=C_2\exp\left[-\nu\sigma^2 t+\int q(t)\,\mathrm{d}t\right],$$

$$p=-\sigma^2\alpha^2(t),\quad q=q(t)\text{ is an arbitrary function,}$$

where C_1, C_2, and σ are arbitrary constants. By setting $q(t)=\nu\sigma^2+\varphi'_t(t)$ with periodic $\varphi(t)$, one obtains a periodic solution in both x and t.

More complicated solutions to equation (15) can be found in [20].

3. Some generalizations

Consider the nonlinear nth-order equation

$$c(t)u_{tx}+[a(t)u+b(t)x]u_{xx}+d(t)(u_x u_{tx}-u_t u_{xx})=F(t,u_x,u_{xx},\dots,u_x^{(n)}), \tag{23}$$

which becomes (1) for $c(t)=1$ and $d(t)=0$.

1°. General property: if $\widetilde{u}(t,x)$ is a solution to equation (23), then the function

$$u=\widetilde{u}(t,\,x+\varphi(t))+\psi(t),$$

where $\varphi=\varphi(t)$ and $\psi=\psi(t)$ are related by $d(t)\psi'_t-a(t)\psi=c(t)\varphi'_t-b(t)\varphi$ (either function can be chosen arbitrarily), is also a solution to (23).

2°. Let us divide (23) by u_{xx}, differentiate the resulting equation with respect to x, and then pass to the Crocco variables (5), (3) to obtain the $(n-1)$st-order equation

$$\frac{a(t)\eta+b(t)}{\Phi}-[d(t)\eta+c(t)]\frac{\partial}{\partial t}\frac{1}{\Phi}=\frac{\partial}{\partial\eta}\left[\frac{1}{\Phi}F\left(t,\eta,\Phi,\Phi\frac{\partial\Phi}{\partial\eta},\dots,\frac{\partial^{n-2}\Phi}{\partial x^{n-2}}\right)\right].$$

Example. In the special case of $n=3$, $a(t)=b(t)=c(t)=0$, $d(t)=1$, and $F=[f(u_{xx})]_x$, (23) is a general boundary layer equation for a non-Newtonian fluid [3], with u being the stream function. By the Crocco transformation (5), this equation can be reduced to the second-order equation $\eta\Phi_t=\Phi^2[f(\Phi)]_{\eta\eta}$, which can be linearized by the substitution $\Psi=1/\Phi$ if $f(\Phi)=1/\Phi$.

Remark. Equation (23) can be generalized by adding the arguments J_x, …, $J_x^{(m)}$, with $J=u_{xx}u_{txx}-u_{tx}u_{xxx}$, to the function F.

4. Using the Crocco transformation for constructing RF-pairs and Bäcklund transformations

Consider a fairly general nth-order evolution equation

$$u_t+[a(t)u+b(t)x]u_x=F(t,u_x,u_{xx},u_{xxx},\dots,u_x^{(n)}). \tag{24}$$

General property: if $\widetilde{u}(t,x)$ is a solution to equation (24), then the function

$$u=\widetilde{u}(t,x+\psi(t))+C,$$

where C is an arbitrary constant and $\psi=\psi(t)$ satisfies the linear ordinary differential equation $\psi'_t-b(t)\psi+Ca(t)=0$, is also a solution to (24).

Differentiating (24) with respect to x yields an $(n+1)$st-order equation with a mixed derivative of the form (1):

$$u_{tx}+[a(t)u+b(t)x]u_{xx}=-a(t)u_x^2-b(t)u_x+\frac{\partial}{\partial x}F(t,u_x,u_{xx},u_{xxx},\dots,u_x^{(n)}). \tag{25}$$

By passing in (25) from t, x, u to the Crocco variables (5), we one arrives at the nth-order equation

$$\frac{3a(t)\eta+2b(t)}{\Phi}-\frac{\partial}{\partial t}\frac{1}{\Phi}+[a(t)\eta^2+b(t)\eta]\frac{\partial}{\partial\eta}\frac{1}{\Phi}=\frac{\partial^2}{\partial\eta^2}F\left(t,\eta,\Phi,\Phi\frac{\partial\Phi}{\partial\eta},\dots,\frac{\partial^{n-2}\Phi}{\partial x^{n-2}}\right). \tag{26}$$

Equations (24) and (26) are linked by the Bäcklund transformation

$$\begin{aligned}&u_t+[a(t)u+b(t)x]\eta=F(t,\eta,\Phi,\Phi_x,\dots,\Phi_x^{(n-2)}),\\&u_x=\eta,\quad u_{xx}=\Phi.\end{aligned} \tag{27}$$

Remark. Sometimes, it is convenient to rewrite (26) in the form

$$\Psi_t - [a(t)\eta^2 + b(t)\eta]\Psi_\eta - [3a(t)\eta + 2b(t)]\Psi = -\frac{\partial^2}{\partial\eta^2}F, \quad \Psi = \frac{1}{\Phi}.$$

Example 1. The unnormalized Burgers equation

$$u_t + auu_x = \beta u_{xx} \tag{28}$$

is a special case of (24) with $a(t) = a = \text{const}$, $b(t) = 0$, and $F = \beta u_{xx} = \beta\Phi$. By the Bäcklund transformation (27), equation (28) can be reduced to

$$\Phi_t - a\eta^2\Phi_\eta + 3a\eta\Phi = \beta\Phi^2\Phi_{\eta\eta}.$$

Example 2. The nonlinear second-order equation

$$u_t + [a(t)u + b(t)x]u_x = \frac{f(t,u_x)}{u_{xx}} + g(t,u_x) \tag{29}$$

is a special case of (24). The Bäcklund transformation (27) reduces (29) to the equation

$$\frac{3a(t)\eta + 2b(t)}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} + [a(t)\eta^2 + b(t)\eta]\frac{\partial}{\partial\eta}\frac{1}{\Phi} = \frac{\partial^2}{\partial\eta^2}\left[\frac{f(t,\eta)}{\Phi} + g(t,\eta)\right],$$

which becomes linear after substituting $\Phi = 1/\Psi$.

Example 3. The unnormalized Burgers Korteweg–de Vries equation

$$u_t + auu_x = \beta u_{xxx} \tag{30}$$

is a special case of (24) $a(t) = \text{const}$, $b(t) = 0$, and $F = \beta u_{xxx} = \beta\Phi\Phi_\eta$. The Bäcklund transformation (27) reduces (30) to the equation

$$\Phi_t - a\eta^2\Phi_\eta + 3a\eta\Phi = \beta\Phi^2(\Phi\Phi_\eta)_{\eta\eta},$$

which, after submitting $\Phi = \theta^{1/2}$, becomes

$$\theta_t - a\eta^2\theta_\eta + 6a\eta\theta = \beta\theta^{3/2}\theta_{\eta\eta\eta}.$$

Example 4. The nonlinear third-order equation

$$u_t + auu_x = \frac{f(t,u_x)}{u_{xx}^3}u_{xxx} \tag{31}$$

can be reduced, using the Bäcklund transformation (27) with $b(t) \equiv 0$ and $F = f(t,u_x)u_{xx}^{-3}u_{xxx} = f(t,\eta)\Phi^{-2}\Phi_\eta$ followed by substituting $\Phi = 1/\Psi$, to the linear equation

$$\Psi_t - a\eta^2\Psi_\eta - 3a\eta\Psi = [f(t,\eta)\Psi_\eta]_{\eta\eta}.$$

Example 5. The linear third-order equation

$$u_t = \alpha u_{xxx} + \beta u_{xx} \tag{32}$$

is a special case of (24) with $F = \alpha u_{xxx} + \beta u_{xx} = \alpha\, Phi\Phi_\eta + \beta\Phi$, $a(t) \equiv 0$, and $b(t) \equiv 0$. By applying to (32) the Bäcklund transformation (27) and then substituting $\Phi = 1/\Psi$, one arrives at the nonlinear equation

$$\Psi_t = \alpha(\Psi^{-3}\Psi_\eta)_{\eta\eta} + \beta(\Psi^{-2}\Psi_\eta)_\eta. \tag{33}$$

The special cases of (33) with $\alpha = 0$, $\beta \neq 0$ and $\beta = 0$, $\alpha \neq 0$ were copnsidered in [21] and [3], respectively.

Example 6. The linear fourth-order equation $u_t = \alpha u_{xxxx}$ is reduced, using the same transformation as in the preceding example and substituting $\Phi = \theta^{1/2}$, to the nonlinear fourth-order equation

$$\theta_t = \alpha\theta^{3/2}(\theta^{1/2}\theta_{\eta\eta})_{\eta\eta}.$$

Remark. Equation (26) remains unchanged if the sum $p(t)u+q(t)x+s(t)$, with arbitrary functions $p(t)$, $q(t)$, and $s(t)$, is added to the right-hand side of (24) and that of the first equation in (27).

Corollary. If equation (24) is integrable for some right-hand side F, then the equation with the more complicated right-hand side $F+p(t)u+q(t)x+s(t)$ is also integrable.

Example 1. Since the Burgers equation $u_t + auu_x = bu_{xx}$ is integrable, the more complicated equation

$$u_t + auu_x = bu_{xx} + p(t)u + q(t)x + s(t)$$

is also integrable.

Example 2. Likewise, since the Korteweg–de Vries equation $u_t + auu_x = bu_{xxx}$ is integrable, the more complicated equation

$$u_t + auu_x = bu_{xxx} + p(t)u + q(t)x + s(t)$$

is also integrable.

5. Extension of the Crocco transformation to the case of three independent variables. Application to unsteady boundary-layer equations

Transformation (5) can be extended to the cases of more independent variables. In particular, it can be shown that the Crocco transformation

$$t,\ x,\ y,\ u=u(t,x,y) \quad\Longrightarrow\quad t,\ x,\ \eta,\ \Phi=\Phi(t,x,\eta), \quad \text{where} \quad \eta=u_y,\ \Phi=u_{yy}, \tag{34}$$

reduces the order of the nth-order equation

$$\begin{aligned}[a(t,x)u+b(t,x)y]u_{yy} + c_1(t,x)u_{ty} + c_2(t,x)u_{xy} + d_1(t,x)(u_y u_{ty} - u_t u_{yy})\\ + d_2(t,x)(u_y u_{xy} - u_x u_{yy}) = F(t,x,u_y,u_{yy},\ldots,u_y^{(n)}).\end{aligned} \tag{35}$$

Example. Consider the Prandtl system

$$\begin{aligned} u_t + uu_x + vu_y &= \nu u_{yy} + f(t,x),\\ u_x + v_y &= 0, \end{aligned} \tag{36}$$

which describes a flat unsteady boundary layer with pressure gradient (u and v the fluid velocity components) [1–3]. Equations (36) can be reduced, by introducing a stream function w such that $u=w_y$ and $v=-w_x$, to a single third-order equation [1, 3]:

$$w_{ty} + w_y w_{xy} - w_x w_{yy} = \nu w_{yyy} + f(t,x). \tag{37}$$

This equation is a special case of (35) (up to the obvious renaming $u \rightleftarrows w$).

Dividing (37) by w_{yy} followed by differentiating with respect to y and passing from t, x, y, w to the Crocco variables t, x, $\eta=w_y$, $\Phi=w_{yy}$, one arrives at the second-order equation

$$\frac{\partial\Phi}{\partial t} + \eta\frac{\partial\Phi}{\partial x} + f(t,x)\frac{\partial\Phi}{\partial\eta} = \nu\Phi^2\frac{\partial^2\Phi}{\partial\eta^2}, \tag{38}$$

which is reduced, with the substitution $\Phi=1/\Psi$, to the nonlinear heat equation

$$\frac{\partial\Psi}{\partial t} + \eta\frac{\partial\Psi}{\partial x} + f(t,x)\frac{\partial\Psi}{\partial\eta} = \nu\frac{\partial}{\partial\eta}\left(\frac{1}{\Psi^2}\frac{\partial\Psi}{\partial\eta}\right). \tag{39}$$

Remark. In the steady-state case with $\partial/\partial t=0$ and $f(t,x)=0$, equation (38) reduces to one considered in [1, 3].

1°. In the special case $f(t,x) = f(t)$, equation (39) admits an exact solution of the special form

$$\Psi = Z(\xi,\tau), \quad \xi = x - \eta t + \int t f(t)\,\mathrm{d}t, \quad \tau = \frac{1}{3}t^3.$$

Hence we arrive at the integrable equation

$$\frac{\partial Z}{\partial \tau} = \nu \frac{\partial}{\partial \xi}\left(\frac{1}{Z^2}\frac{\partial Z}{\partial \xi}\right), \tag{40}$$

which can be reduced to the linear heat equation [3, 21].

2°. In the more general case $f(t,x) = f(t)x + g(t)$, we have solutions of the special form

$$\Psi = Z(\xi,\tau), \quad \xi = \varphi(t)x + \psi(t)\eta + \theta(t), \quad \tau = \int \psi^2(t)\,\mathrm{d}t,$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\theta = \theta(t)$ are determined by the linear system of ordinary differential equations

$$\varphi'_t + f\psi = 0, \quad \psi'_t + \varphi = 0, \quad \theta'_t + g\psi = 0.$$

As a result, we arrive at an integrable equation (40).

Acknowledgments

The work was carried out under partial financial support of the Russian Foundation for Basic Research (grants No. 08-01-00553, No. 08-08-00530 and No. 09-01-00343).

References

[1] Loitsyanskiy L G 1995 *Mechanics of Liquids and Gases* (New York: Begell House) p 971.
[2] Schlichting H and Gersten M 2000 *Boundary Layer Theory, 8th Edition* (Heidelberg: Springer) p 801
[3] Polyanin A D and Zaitsev V F 2004 *Handbook of Nonlinear Partial Differential Equations* (Boca Raton: Chapman & Hall/CRC Press) p 840
[4] Ibragimov N H 1994, 1995 *CRC Handbook of Lie Group Anlysis of Differential Equations, Vols 1, 2* (Boca Raton: CRC Press) p 448, 576
[5] Miura R M (ed) 1975 *Bäcklund Transformations* (Berlin: Springer-Verlag)
[6] Rogers C and Shadwick W F 1982 *Bäcklund Transformations and Their Applications* (New York: Academic Press) p 174
[7] Nimmo J J C and Crighton D J 1982 *Proc. Royal Soc. London* A **384** 381–401
[8] Kingston J G, Rogers C and Woodall D 1984 *J. Physics A: Math. Gen.* **17** L35–8
[9] Weiss J 1986 *J. Math. Phys.* **27** 1293–305
[10] Kudryashov N A 1991 *Phys. Let.* A **155** 269–75
[11] Nucci M C 1993 *J. Math. Anal. Appl.* **178** 294–300
[12] Karasu A and Sakovich S Yu 2001 *J. Phys.* A: *Math. Gen.* **34** 7355–8
[13] Wang M, Wang Y and Zhou Y 2002 *Phys. Let.* A **303** 45–51
[14] Yonga X, Zhangb Z and Chen Y 2008 *Phys. Let.* A **372** 6273–9
[15] Calogero F 1984 *Stud. Appl. Math.* **70** 189–99
[16] Pavlov M V 2001 *Theor. & Math. Phys.* **128** 927–32
[17] Dryuma V S 2000 *Proc. Workshop on Nonlinearity, Integrability, and All That: Twenty years after NEEDS'79 (Lecce, Italy)* ed M Boiti, L Martina *et al* (Singapore: World Scientific) 109–16
[18] Polyanin A D 2001 *Doklady Physics* **46**(10) 726–31
[19] Pukhnachev V V 2006 *Uspekhi Mekhaniki* [in Russian] **4**(1) 6–76
[20] Aristov S N and Polyanin A D 2009 *Doklady Physics* **54**(7) 316–21
[21] Bluman G W and Kumei S 1980 *J. Math. Phys.* **21** 1019–23