Quantum string motion due to periodic local force

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Abstract

We consider the string under local periodic force. We derive the quantum internal motion of this system.

1 Introduction

According to Nielsen and Olesen (1973) there is parallelism between the Higgs model of broken gauge invariance and the Landau-Ginzburg superconductivity theory on the one hand and the dual string model being the Abrikosov flux lines in superconductors II. So, dual string is mathematical realization of magnetic flux tube in equilibrium againts the pressure of the surrounding charged superfluid. Only strings with no ends were considered by them (Nambu, 1974). The internal quantum motion of strings is not considered by the authors.

We consider here the string of the length l, the left and right ends of which are fixed and the string is under local periodic force $A\varrho \sin \omega t$. We derive the quantum internal motion of this system using the so called the oscillator quantization of the string.

The non-relativistic quantization of the equation for the energy of a free particle

$$\frac{p^2}{2m} = E \tag{1}$$

consists in replacing classical quantities by operators. We get the non-relativistic Schrödinger equation for a free particle. The operator replacings are $E \to i\hbar\partial/\partial t$, $\mathbf{p} \to -i\hbar\nabla$.

The Schrödinger equation suffers from not being relativistically covariant, meaning it does not take into account Einstein's special relativity.

It is natural to perform the special relativity generalization of the energy relation describing the energy:

$$E = \sqrt{p^2c^2 + m^2c^4}. \tag{2}$$

Then, just inserting the quantum mechanical operators for momentum and energy yields the equation

$$i\hbar\frac{\partial}{\partial t} = \sqrt{(-i\hbar\nabla)^2c^2 + m^2c^4}. \tag{3}$$

This, however, is a cumbersome expression to work with because the differential operator cannot be evaluated while under the square root sign.

Klein and Gordon instead began with the square of the above identity, i.e. $E^2 = p^2c^2 + m^2c^4$, which, when quantized, gives

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 = (-i\hbar\nabla)^2c^2 + m^2c^4. \tag{4}$$

So, we have seen that the quantization of classical mechanics is the simple replacing classical quantities by operators. We use here the novel quantization method where classical oscillators forming the classical systems are replaced simply by the quantum solution of quantum oscillators. The natural step is to apply the method to motion of the classical string.

2 The classical derivation of the string motion

The differential equation of motion of string elements can be derived by the following way (Tikhonov et al., 1977). We suppose that the force acting on the element dx of the string is given by the law:

$$T(x,t) = ES\left(\frac{\partial u}{\partial x}\right), \tag{5}$$

where E is the modulus of elasticity, S is the cross section of the string. We easily derive that

$$T(x+dx) - T(x) = ESu_{xx}dx. \tag{6}$$

The mass dm of the element dx is $\varrho ESdx$, where $\varrho = const$ is the mass density of the string matter and the dynamical equilibrium gives

$$\varrho Sdxu_{tt} = ESu_{xx}dx. \tag{7}$$

So, we get

$$\frac{1}{a^2}u_{tt} - u_{xx} = 0; \quad a = \left(\frac{E}{\varrho}\right)^{1/2}. \tag{8}$$

Now, let us consider the following problem of the mathematical physics. The string is under the local periodic force $F = \varrho A \sin \omega t$ at point $0 < c < l$. With regard to the fact that we consider the additional periodical force $F = \varrho A \sin \omega t$ at point $0 < c < l$, we must reformulate the standard string problem of mathemtical physics as it follows. The motion of the string is u_1 in the interval $0 < x < c$ and u_2 in the interval $c < x < l$.

So, we solve the mathematical problem:

$$(u_1)_{tt} = a^2(u_1)_{xx}; \quad 0 < x < c \tag{9a}$$

$$(u_2)_{tt} = a^2(u_2)_{xx}; \quad c < x < l \tag{9b}$$

with the conditions

$$u_1(x=0) = 0; \quad u_2(x=l) = 0 \tag{10a}$$

$$u_1(x=c) = u_2(x=c) \tag{10b}$$

$$E(u_1)_x(x=c) - E(u_2)_x(x=c) = \varrho A \sin \omega t. \tag{11}$$

We look for the solution in the form (Koshlyakov, et al., 1962)

$$u_1(x,t) = C_1 \sin \frac{\omega x}{a} \sin \omega t \tag{12a}$$

$$u_2(x,t) = C_2 \sin \frac{\omega(l-x)}{a} \sin \omega t. \tag{12b}$$

For the determination of the arbitrary constants C_1, C_2 with regard to the conditions $(10a), (10b), (11)$, we get the solution of our problem in the folowing form (Koshljakov, et al., 1962):

$$u_1(x,t) = \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\sin\frac{\omega x}{a}\sin\omega t; \quad 0 < x < c \tag{13a}$$

$$u_2(x,t) = \frac{A}{a\omega}\frac{\sin\frac{\omega c}{a}}{\sin\frac{\omega l}{a}}\sin\frac{\omega(l-x)}{a}\sin\omega t; \quad c < x < l. \tag{13b}$$

So, we see that the string motion is a such that at every point $X \in (0,l)$ there is an oscillator with an amplitudes

$$A_1 = \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\sin\frac{\omega x}{a}; \quad 0 < x < c \tag{14a}$$

$$A_2 = \frac{A}{a\omega}\frac{\sin\frac{\omega c}{a}}{\sin\frac{\omega l}{a}}\sin\frac{\omega(l-x)}{a}; \quad c < x < l. \tag{14b}$$

3 Quantization of the string motion by harmonic oscillators

It is well known that harmonic oscillator equation

$$\ddot{x} + \omega^2 x = 0; \quad \omega = \sqrt{k/m} \tag{15a}$$

has the solution

$$x(t) = A\cos(\omega t + \varphi). \tag{15b}$$

In case of the quantum mechanical oscillator motion, the solution for the stationary sates is (Grashin, 1974)

$$\psi_n = N_n H_n \exp\left(-\xi^2/2\right); \quad \xi = x\sqrt{m\omega/\hbar}, \tag{16}$$

where N_n is the normalization constant

$$N_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{1}{2^n n!}} \tag{17}$$

and H_n are the Hermite polynomials defined by the following relation

$$H_n = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}\exp\left(-\xi^2/2\right). \tag{18}$$

So, the wave function of the one string oscillator of the string with the periodic force at point c in the form:

$$\psi_i(x,t) = \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\sin\frac{\omega(x_i - x)}{a}N_{n_i}\frac{\cos\omega l}{c}H_{n_i}; \quad 0 < x < c \qquad (19a)$$

and

$$\psi_i(x,t) = \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\sin\frac{\omega(l - x_i + x)}{a}N_{n_i}\frac{\cos\omega l}{c}H_{n_i}; \quad c < x < l. \quad (19b)$$

The total wave function of the string system of oscillators is then

$$\Psi_1(x,t) = \Pi_i^\infty \psi_i(x,t) =$$

$$\Pi_i^\infty \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\sin\frac{\omega(x_i - x)}{a}N_{n_i}\frac{\cos\omega l}{c}H_{n_i}; \quad 0 < x < c \qquad (20a)$$

and

$$\Psi_2(x,t) = \Pi_i^\infty \psi_i(x,t) =$$

$$\Pi_i^\infty \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\sin\frac{\omega(l - x_i + x)}{a}N_{n_i}\frac{\cos\omega l}{c}H_{n_i}; \quad c < x < l. \qquad (20b)$$

Or,

$$\Psi_1(x,t) = \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\Pi_i^\infty \sqrt{\frac{1}{2^{n_i}n_i!}}\frac{\cos\omega l}{c}(-1)^{n_i}e^{\xi^2}\frac{d^{n_i}}{d\xi^{n_i}}\exp\left(-\xi^2/2\right); \quad 0 < x < c \qquad (21a)$$

with

$$\xi = (x_i - x)\sqrt{m\omega/\hbar}. \qquad (21a\xi)$$

and

$$\Psi_2(x,t) = \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\Pi_i^\infty\sqrt{\frac{1}{2^{n_i}n_i!}}\frac{\cos\omega l}{c}(-1)^{n_i}e^{\xi^2}\frac{d^{n_i}}{d\xi^{n_i}}\exp\left(-\xi^2/2\right); \quad c < x < l. \tag{21b}$$

with

$$\xi = (l - x_i + x)\sqrt{m\omega/\hbar}. \tag{21$b\xi$}$$

So, the quantization of string is possible only if we devide the string into elementary discrete points supposing that in every point of string $X \in (0, l)$, there is a quantum oscillator with the stationary states described by eq. (19). There is an analogue representation to eq. (21), which was applied by Feynman for determination of the quantum theory of the Mössbauer effect (Feynman, 1972).

4 Discussion

The starting point for string theory is the idea that the point-like particles are modeled by one-dimensional objects called strings. Strings propagate through space and interact with each other. In a given version of string theory, there is only one kind of string, which may look like a small loop, or, segment of ordinary string, and it can vibrate in different ways. On distance scales larger than the string scale, a string will look just like an ordinary particle, with its mass, charge, and other properties determined by the vibrational state of the string. In this way, all of the different elementary particles may be viewed as vibrating strings. In string theory, one of the vibrational states of the string gives rise to the graviton, a quantum mechanical particle that carries gravitational force. Thus string theory is also theory of quantum gravity and replaces the quantum gravity with the gravitons with spin 2.

The string theory can be extended to the quark-quark interaction by the string potential, defined as the quark mass correction to the string potential, which was performed by Lambiase and Nesterenko (1996). The calculation of the interquark potential generated by a string with massive ends was performed by Nesterenko and Pirozhenko (1997), and others. The propagation of a pulse in the real strings and rods which can be applied to the two-quark system as pion and so on, was calculated by author (Pardy, 2005). So, it is not excluded that our oscillator quantization of the string can be extended to generate the new way of the string theory of matter and space-time.

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