Generalized modular forms including the weak Maass forms, the Ramanujan's Theta functions and the Tau function

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Abstract

The modular forms are revisited from a geometric and an algebraic point of view leading to a geometric interpretation of the weak Maass forms connecting them to the Ramanujan Mock Theta functions and to the cusp forms generated from the Langlands global program.

Contents

Introduction

The theory of automorphic forms was initiated by H. Poincare with the Fuchsian functions, pursued by F. Klein in connection with the elliptic functions and developed in the frame of analytic number theory by many mathematicians among which E. Hecke [Hec1], G. Shimura, C.L. Siegel, A. Ogg and J.P. Serre. But, according to the opinion of R. Langlands [Lan], "**the theory of automorphic forms remains** in 1997 as it was in 1967: **a diffuse, disordered subject driven as much by the availability of techniques as by any high esthetic purpose**".

On the other hand, F. Dyson, referring to Mock theta functions, claimed in 1987: "**The Mock theta functions give us a tantalizing hints of a grand synthesis still to be discovered**. Somehow, it should be possible to build them into a coherent group-theoretical structure. This remains a challenge for the future".

Taking into account the great challenge of the program of Langlands, **the two-dimensional modular forms [Gel], [Ogg], are revisited here from a geometric and an algebraic point of view in the sense that**:

a) **the weight k is proved to refer to $k-$ dimensional modular forms**;

b) **the level N is the Galois (or transcendence) extension degree of algebraic (or transcendental) quanta** [Pie5].

The precedent advances in this field, developed in [Pie1], are:

1) the Fourier series development of a two-dimensional cusp form decomposes into **a tower of increasing embedded semitori: geometric interpretation**.

2) a cusp form is a function into $\mathbb{C}$ from a set of $\mathbb{R}^2/(\mathbb{Z}/N\mathbb{Z})^2$-lattices of transcendental quanta having extension degree N: algebraic interpretation.

3) **a cusp form is covered by a global elliptic semimodule, which is a real analytic cusp form**, in such a way that every two-dimensional semitorus is covered by a set of semicircles: **cuspidal representation**.

4) **a cosemialgebra of dual cusp forms exists** with respect to the semialgebra of cusp forms **leading to a bisemialgebra of cusp biforms** (sections 1.11 and 1.12).

All that constitutes the pieces of a theory of two-dimensional modular forms developed in chapters 1 and 2.

But, in order to enlarge this framework, I have also considered in chapter 3:

1) **the L-functions as nonperiodic transforms of modular forms, the nonperiodicity reflected by the different unit frequencies or quanta of energy of their nontrivial zeros**;

2) **the theta series as real analytic modular forms** in one-to-one correspondence with the respective complex modular forms;

3) **a geometric interpretation of the weak Maass forms** in the sense that **their series expansion consists of a sum of increasing embedded two-dimensional semitori with elliptic cross sections** in such a way that their nonholomorphic part corresponds to a shadow;

4) **the connections of the Ramanujan theta functions [Ram] with weak Maass forms, modular forms and global elliptic semimodules** in such a way that:

   a) the nonhomomorphic parts of the weak Maass forms correspond to the shadow of the Mock modular forms;

   b) the partitions in Ramanujan's theta functions are partitions of quanta.

5) **the (2-dimensional) cusp form of weight 12 and level 1 associated with the tau function and corresponding to a universal orthogonal cusp form**.

More concretely, **the theory of modular forms of weight k and level N is reviewed in chapter 1** with a special emphasis on:

a) the decomposition of the Poincare upper half plane H into $I\!R^2(\mathbb{Z}/N\mathbb{Z})^2$-lattices partitioning it into coset representatives $t^{(2)}_{\mathbb{Z}_N}[n]$ of the quotient (semi)group

$$T^2_{\mathbb{Z}_N}(H) = T_2(I\!R)\big/T_2(\mathbb{Z}/N\mathbb{Z}) , \qquad 1 \le n \le \infty , \quad n \in I\!N ,$$

   $T_2(I\!R)$ being the upper triangular group of matrices;

b) the action of $T_2(I\!R)$ on **the fundamental domain $D^{(2)}_{nN}$, being a unitary measurable set, i.e. a period**, of $t^{(2)}_{\mathbb{Z}_N}[n]$:

c) the action of the Hecke operator $T_k(n)$ on **a two-dimensional cusp form $f_k(z_N)$ of weight k and level N decomposing it into Fourier series of which n-th term**, corresponding to the coset representative $t^{(2)}_{\mathbb{Z}_N}[n]$, **is a two-dimensional semitorus T^2_{nN} generated by two orthogonal semicircles at n transcendental quanta**.

In order to be able to vary the weight of the two-dimensional cusp form $f_k(z_N)$ of weight k and level N, the origin of this weight k is assumed to be related

- **to a k-dimensional cusp form $\phi^{(k)}(z_{Nk})$**, having a decomposition in Fourier series

$$\phi^{(k)}(z_{Nk}) = \sum_{n=1}^{\infty} \lambda_n^{(k)} \; e^{2\pi i n z_{Nk}}$$

  where z_{Nk} is a k-tuple of complex numbers of order N as developed in the global program of Langlands over number fields [Pie2],

- **by a projective map**:

$$C\,P_{(k)\to 2}: \qquad \phi^{(k)}(z_{N_k}) \longrightarrow f_k(z_N)$$

  under the conditions of proposition 1.14,

- **in such a way that the fundamental domain $D_{nNk}^{(2)}$ of $f_k(z_N)$ be equal to the fundamental domain $D_{nN}^{(k)}$ of $\phi^{(k)}(z_{Nk})$**.

The morphism of projection from a two-dimensional cusp biform $(f_k(z_N^*) \times f_k(z_N))$ of weight k and level N to a cusp biform $(f_h(z_N^*) \times f_h(z_N))$ of weight h and some level N, $k > h$, k and h being even integers, depends on :

a) the morphism of projection $\mathrm{Phom}_{k\to h}$ from a k-dimensional cusp biform to a h-dimensional cusp biform;

b) the Langlands functoriality conjecture [Pie10] allowing to decompose 2-dimensional cusp biforms of weight k and h into 2-dimensional cusp biforms of weight 2;

as it was developed in proposition 1.17.

The global program of Langlands over number fields is used in chapter 2 to generate cusp (bi)forms from the Weil (bisemi)group of finite symmetric algebraic extensions characterized by degrees being integers module N of zero-th class, i.e. multiples of quanta.

Langlands global correspondences are then associated with the covariant (bi)-functor

$$\mathrm{FLGC}: \qquad \mathrm{CABG} \qquad \longrightarrow \qquad \mathrm{CBCF}$$

$$\mathrm{GL}_1(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega}) \quad \longrightarrow \quad \Pi(\mathrm{GL}_1(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega}))$$

$$\mathrm{FLGC}(\mathrm{Phom}_{k\to h}): \qquad \Pi_k(\mathrm{GL}_1(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega}) \quad \longrightarrow \quad \Pi_h(\mathrm{GL}_1(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega}))$$

which is a (bi)function assigning:

- to each algebraic bilinear semigroup $\mathrm{GL}_1(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega})$ over the product, right by left, of symmetric complex finite algebraic extensions $\widetilde{F}_{\overline{\omega}}$ and $\widetilde{F}_{\omega}$ of the bisemigroup category [Pie3] CABG;

- its cuspical representation $\Pi(\mathrm{GL}_1(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega}))$ of the bisemigroup category CBCF of complex cuspidal representations

in such a way that:

- to each map
$$\mathrm{Phom}_{k \to h}: \qquad \mathrm{GL}_k(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega}) \longrightarrow \mathrm{GL}_h(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega})$$
sending a bilinear semigroup of dimension k to a bilinear semigroup of dimension h, $k > h$, in the sense of what was abovementioned;

- corresponds a map $\mathrm{FLGC}(\mathrm{Phom}_{k \to h})$ sending the cuspidal representation $\Pi_k(\mathrm{GL}_1(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega})$ of $\mathrm{GL}_1(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega})$, which is a two-dimensional cusp (bi)form of weight k, to the cuspidal representation $\Pi_h(\mathrm{GL}_1(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega}))$ which is a two-dimensional cusp biform of weight h.

It then results that a (Weil) cusp form $^{(\omega)}f_k(z_N)$ of weight k and level N, generated from a Langlands global correspondence, can be identified with a (classical) cusp form $f_k(z_N)$ of weight k and level N if the Weil algebraic unitary fundamental domain $D^{(2)}_{F_{\omega_n};N;k}$ of $^{(\omega)}f_k(z_N)$ covers the fundamental classical domain $D^{(2)}_{nN;k}$ of $f_k(z_N)$.

Now, results on the local and global curvatures of two-dimensional tori are presented in proposition 2.9 in the light of the covering of the cusp form $f_k(z_N)$ by the global elliptic semimodule $\mathrm{ELLIP}_L(2, n, m_n)$: this leads to a new dynamical transition from global euclidean geometry to local hyperbolic and spherical geometries.

In chapter 3, a generalization of two-dimensional cusp forms towards the weak Maass forms, the theta series, the Mock modular forms and the tau function is proposed.
But, first, **the (Mellin) transform** [Bom] (which is a linear continuous map):

$$\phi_L: \qquad f_k(z_{N-k}) \longrightarrow L(f_k, s_{N-k_+}) = \sum_{n=1}^{\infty} c_{nk} \, n^{-s_{N-k_+}}$$

$$(\text{resp.} \quad \phi_R: \qquad f_k(z^*_{N-k}) \longrightarrow L(f^*_k, s_{N-k_+}) = \sum_{n=1}^{\infty} c^*_{nk} \, n^{-s_{N-k_-}})$$

from the two-dimensional cusp form of weight k and level N, $f_k(z_{N-k})$ (resp. the dual cusp form $f_k^*(z_{N-k}^*)$) to the corresponding L-function $L(f_k, s_{N-k_+})$ (resp. $L(f_k^*, s_{N-k_-})$) is pointed out to be **nonperiodic** because of the factors $n^{-s_{N-k_+}}$ (resp. $n^{-s_{N-k_-}}$) where s_{N-k_+} (resp. s_{N-k_-}) **is the complex variable conjugate to** z_{N-k} (resp. z_{N-k}^*) being a complex point of order $(N \times k)$.

The nonperiodicity of ϕ_L **(resp.** ϕ_R **) results from the map of the unique period** T **of** $f_k(z_{N-k})$ **(resp.** $f_k^*(z_{N-k}^*)$**) to the set** $\left\{ \dfrac{1}{T_{N-k}^{(n)}} \right\}_{n=1}^{\infty}$ **of inverse periods, i.e. unit complex frequencies, associated with the energies of one space quantum on the different levels "n" which can be evaluated from the consecutive spacings** $\delta\gamma_n = \gamma_{n+1} - \gamma_n$ **between the nontrivial zeros of** $\zeta(s)$ **[Pie8]**.

In sections 3.3 to 3.8, **the theta series**

$$\theta_{n^2 k/2}(z_{N-k}) = \underset{n}{\Sigma}\, d_{nk/2}\, e^{2\pi i n^2 z_{N-k}}\ ,$$

introduced from the quadratic form

$$Q(n) = \overset{k}{\underset{i=1}{\Sigma}}\,\underset{n}{\Sigma}\, a_{nn}^{(i)}\, n_i^2\ ,$$

are proved to be a real analytic modular form of weight $k/2$ **and level** N **in one-to-one correspondence with the two-dimensional modular form**

$$f_k(z_{N-k}) = \underset{n}{\Sigma}\, c_{nk}\, q_{N-k}^n$$

of weight k **and level** N in such a way that the two-dimensional semitori $c_{nk}\, q_{N-k}^n$, of which generators are two semicircles at n transcendental quanta, are sent into semicircles $d_{nk/2}\, e^{2\pi i n^2 x_{N-k/2}}$ at n^2 transcendental quanta.

This approach corresponds to the classical one developed by G. Shimura and J.P. Serre who proved that every modular form of weight $1/2$ on $\Gamma_1(N)$ is a linear combination of theta series with characters [Shi], [Ser], [S-S].

In sections 3.9 to 3.15, a geometric interpretation of weak Maass forms is proposed in terms of modular curves. As in the Fourier series expansion

$$f_k^{\omega M}(z) = \overset{\infty}{\underset{n=n_0}{\Sigma}}\, \gamma(f, n; y)\, q^{-n} + \overset{\infty}{\underset{n=n_1}{\Sigma}}\, a(f, n)\, q^n\ , \qquad q = e^{2\pi i z}$$

of a weak form $f_k^{\omega M}(z)$ of weight k, the imaginary dimension "y" is manifestly lowered with respect to the dimension "x" in $z = x + iy$, the weak Maass form $f_k^{\omega M}(z)$ will be decomposed in **series expansion**

$$f_k^{\omega M}(z_{N-k}) = \underset{n}{\Sigma}\, T_n^{2,(e\ell)}(S_{a_{n_{N-k/2}}}^1, e\ell_{ef, n_{N-k/2}}^1)$$

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consisting in the sum of two-dimensional semitori $T_n^{2,e\ell}(-)$ with semielliptic cross sections $e\ell^1_{ef,n_{N-k/2}}$ in such a way that:

a) the semicircular sections of the two-dimensional semitori $T_n^2(a_{nm|d^2}, d) \in f_k(z_{N-k}) \approx f_k^{\omega M}(z_{N-k})$ are transformed bijectively into semielliptic sections;

b) $f_k^{\omega M}(z_{N-k})$ **be periodic, holomorphic and weakly modular (i.e. deviated from circularity)**;

c) the holomorphic part $\sum\limits_{n} a(f, n)\, q^n$ of the Fourier series of $f_k^{\omega M}(z)$ corresponds to the sum of the products of two orthogonal semicircles of which the one at "imaginary" semicircular section is the equation of a semicircle inscribed in the ellipse $e\ell^1_{ef,n_{N-k/2}}$;

d) **the nonholomorphic part** $\sum\limits_{n} \gamma(f, n; y)\, q^{-n}$ **of** $f_k^{\omega M}(z)$ **corresponds to its shadow** as developed in proposition 3.11.

In section 3.14, generalized weak Maass forms are introduced as elliptic forms $f_k^{E\ell}(z_{N-k})$ **of weight** k **and level** N having a decomposition into the sum of surfaces of revolution of (semi)ellipses rotating around ellipses instead of circles as for the weak Maass forms.

The commutative diagramm

$$
\begin{array}{ccc}
f_k(z_{N-k}) & \xrightarrow{\;\sim\;} & f_k^{\omega M}(z_{N-k}) \\
& \searrow \qquad \swarrow & \\
& f_k^{E\ell}(z_{N-k}) &
\end{array}
$$

indicates **the possible transformation of cups forms** $f_k(z_{N-k})$ **of weight** k **and level** N **into the respective weak Maass forms** $f_k^{\omega M}(z_{N-k})$ **and elliptic forms** $f_k^{E\ell}(z_{N-k})$ **which are elliptic functions, i.e. doubly periodic meromorphic functions.**

In sections 3.16 to 3.20, the Ramanujan theta functions are analyzed in the light of the new geometric interpretation of weak Maass forms.

Each Ramanujan theta function [Ono1] is a Mock theta function [Wat] given by the q-series $H(q) = \sum\limits_{n} a_n\, q^n$ in such a way that $q^\lambda H(q)$, $\lambda \in \mathbb{Q}$, be a Mock modular form of weight $1/2$ whose shadow is a unary thera series of weight $3/2$. A Mock theta function is thus a Mock modular form [Fol] of the space $\mathbb{M}_k$ of such forms extending the space M_k of classical modular forms of weight k and characterized by a shadow $g = S[h]$ which is a modular form of weight $(2 - k)$.

Proposition 3.17 introduces a geometric interpretation of Ramanujan theta functions transformed into Mock theta functions related to weak Maass forms in such a way that:

a) **the nonholomorphic part of the weak Maass form $f_2^{\omega M}(\tau_{1-2})$ of weight 2 and level 1 corresponds to the shadow $g^*(\tau)$ of the Mock modular form $\hat{h}(\tau)$ of weight 1/2 resulting from the Ramanujan theta function $H(q)$** after the three step sequence transformation [Zag] recalled in section 3.16.

b) the space $\mathbb{M}_1$ of Mock theta functions $\hat{h}(\tau)$ and $\widehat{\mathbb{M}}_2$ of weak Maass forms of weight 2 are isomorphic.

Let

$$R(\omega; q) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(n, m) \, \omega^m \, q^n$$

be the partition function specializing the 17 Ramanujan's Mock theta functions $H(q)$ [B-O2].

As $N(n, m)$ is assumed to denote the number of partitions of $n^{(2)}$ transcendental quanta, **the Dyson's rank m of a partition of n must be the order of the maximal Galois group associated with the considered transcendental extension minus the number of Galois subgroups**: this result is the Galois interpretation of the rank of a partition introduced by Dyson as being its largest part minus the number of its parts.

In sections 3.21 to 3.23, the Ramanujan tau function

$$\sum_{n=1}^{\infty} \tau(n) \, q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \Delta(z) \, ,$$

which is a modular form of weight 12 and level 1, **is associated with the two-dimensional cusp form $f_2(z_{12-1})$ of weight 12 and level 1 which is proved to be a universal orthogonal cusp form corresponding throughout Langlands global correspondences to the sum of the cuspidal representations of six bilinear algebraic semigroups** generating three two-dimensional embedded toric bismisheaves as well as their orthogonal equivalents according to:

$$\Delta(z) \longrightarrow f_2(z_{12-1}) \longrightarrow \Pi^{(12)}(\mathrm{GL}_6(\widetilde{F_{\overline{\omega}}} \times_D \widetilde{F_\omega})) = \bigoplus_{i=1}^{6} \Pi^{(2_i)}(\mathrm{GL}_{1_i}(\widetilde{F_{\overline{\omega}}} \times_D \widetilde{F_\omega}))$$

where:

- $\Pi^{(12)}(\mathrm{GL}_6(\widetilde{F_{\overline{\omega}}} \times_D \widetilde{F_\omega}))$ is the twelve-dimensional cuspidal representation of $\mathrm{GL}_6(\widetilde{F_{\overline{\omega}}} \times_D \widetilde{F_\omega})$;

- $\Pi^{(2_i)}(\mathrm{GL}_{1_i}(\widetilde{F}_{\overline{\varpi}} \times_D \widetilde{F}_\omega))$ is the two-dimensional cuspidal representation of $\mathrm{GL}_{1_i}(\widetilde{F}_{\overline{\varpi}} \times_D \widetilde{F}_\omega)$.

This can be finally generalized to **universal nonorthogonal cuspidal representations including bilinear crossed cuspidal representations of interaction in such a way that** $\Pi^{(12)}(\mathrm{GL}_6(\widetilde{F}_{\overline{\varpi}} \times_D \widetilde{F}_\omega))$ **decomposes then nonorthogonally by means of the Langlands bilinear functoriality conjecture** according to:

$$\Pi^{(12)}_{re\ell}(\mathrm{GL}_6(\widetilde{F}_{\overline{\varpi}} \times_D \widetilde{F}_\omega)) = \overset{6}{\underset{i=1}{\oplus}} \Pi^{(2_i)}(\mathrm{GL}_{1_i}(\widetilde{F}_{\overline{\varpi}} \times_D \widetilde{F}_\omega))$$

$$\overset{6}{\underset{i\neq j=1}{\oplus}} \Pi^{(2_i)}(\mathrm{GL}_{1_i}(\widetilde{F}_{\overline{\varpi}})) \otimes_{OD} \Pi^{(2_j)}(\mathrm{GL}_{1_j}(\widetilde{F}_\omega))$$

where the second sum on the right hand side refers to the six relevant offdiagonal crossed cuspidal representations of interaction as developed in section 3.22.

This universal nonorthogonal cuspidal representation $\Pi^{(12)}_{re\ell}(\mathrm{GL}_6(\widetilde{F}_{\overline{\varpi}} \times_D \widetilde{F}_\omega))$ is then mapped injectively:

$$\mathcal{M}_{\Pi^{(12)} \to \underline{\times}^6} : \qquad \Pi^{(12)}_{re\ell}(\mathrm{GL}_6(\widetilde{F}_{\overline{\varpi}} \times_D \widetilde{F}_\omega)) \longrightarrow \overset{6}{\underset{i=1}{\times}}(f_{2_i}(z^*_{2-1}) \times f_{2_i}(z_{2-1}))$$

into the cross binary product

$$\overset{6}{\underset{i=1}{\times}}(f_{2_i}(z^*_{2-1}) \times f_{2_i}(z_{2-1})) = \left(\overset{6}{\underset{i=1}{\Sigma}} f_{2_i}(z^*_{2-1})\right) \times \left(\overset{6}{\underset{i=1}{\Sigma}} f_{2_i}(z_{2-1})\right)$$

between the six cusp biforms $f_{2_i}(z^*_{2-1}) \times f_{2_i}(z_{2-1})$ of dimension 2, weight 2 and level 1.

1 Cusp (bi)forms of weight k

1.1 Classical definitions of the modular forms

Let H be the Poincare upper half plane of complex numbers $z = x + iy$ with strictly positive imaginary parts y.

Let S_ω denote the set of pairs $\omega = (\omega_1, \omega_2)$ of complex numbers $\omega_i = x_i + iy_i$, $y_i > 0$, $i = 1, 2$.

Let $z \in H$ be given by $z = \dfrac{\omega_1}{\omega_2}$ in such a way that $\omega = (\omega_1, \omega_2)$ be sent into z by the $\mathbb{C}$-linear map $g_{\mathbb{C}} : \mathbb{C}^2 \to \mathbb{C}$ [Del].

The set $G_{\mathbb{Z}}^{(2)}(H)$ of lattices of H is the quotient $H / \mathrm{GL}(2, \mathbb{Z})$ of H by the group $\mathrm{GL}(2, \mathbb{Z})$ in such a way that $G_{\mathbb{Z}}^{(2)}(H)$ corresponds to $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{C})$.

Let $F(\omega) = \omega_2^{-k} f(\omega_1/\omega_2)$ define a holomorpic and homogeneous elliptic function of weight k in the upper half plane [God], k being fairly often an even integer.

$F(\omega)$ is a modular (or "automorphic") form of weight k if it is invariant under the substitution $\omega \to g_{SL_2}\omega$ where g_{SL_2} is the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $a, b, c, d \in \mathbb{Z}$, of the homogeneous modular group $SL(2, \mathbb{Z})$ of homographic transformations, verifying $ad - bc = 1$. The modular form $F(\omega)$ of weight k is then equivalent to a $\mathbb{C}$-valued function $f(z)$ of moderate growth on H verifying:

$$f(z) = (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) , \qquad \forall \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z}) ,$$

where $(cz + d)^{-k}$ is the weight factor.

The action of $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z}) / \pm I$ on $z \in \mathbb{C}$ then generates on $f(z)$:

a) **a "periodic" translation by $S : z \to z + 1$;**

b) **a "periodic" space inversion by $T : z \to -1/z$.**

The modular form of $f(z)$ of weight k is of level N if it is invariant under the congruence (sub)group $\Gamma_0(N)$ (resp. $\Gamma_1(N)$):

$$\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z}) : c \equiv 0 \ (\mathrm{mod}\ N)\}$$

$$(\text{resp.} \quad \Gamma_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z}) : c \equiv 0 \ (\mathrm{mod}\ N), a, d = 1 \ (\mathrm{mod}\ N),\}).$$

1.2 $\mathbb{R}^2/\mathbb{Z}^2$-lattices of the Poincare upper half plane

Let $G_{\mathbb{Z}}^{(2)}(H) = H/\operatorname{GL}(2,\mathbb{Z})$, or more precisely, $G_{\mathbb{Z}}^{(2)}(H) = \operatorname{GL}_2(\mathbb{R})/\operatorname{GL}_2(\mathbb{Z})$ **be the set of $\mathbb{R}^2/\mathbb{Z}^2$-lattices of the Poincare upper half plane H** generated by the group $\operatorname{GL}_2(\mathbb{R})$ isomorphic to the group $\operatorname{GL}(H)$ of automorphisms of H [Bor].
Similarly, let $G_{\mathbb{Z}_N}^{(2)}(H) = \operatorname{GL}_2(\mathbb{R})/\operatorname{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be the set of $\mathbb{R}^2/(\mathbb{Z}/N\mathbb{Z})^2$-lattices of H.

As the subgroup $\operatorname{GL}_2(\mathbb{Z})$ (resp. the congruence subgroup $\operatorname{GL}_2(\mathbb{Z}/N\mathbb{Z})$) leads to the generation of the set $\{\Lambda_{\mathbb{Z}}^{(2)}[n]\}$ (resp. $\{\Lambda_{\mathbb{Z}_N}^{(2)}[n]\}$) of $\mathbb{Z}^2$- (resp. $(\mathbb{Z}/N\mathbb{Z})^2$-) lattices in H, **the quotient group $G_{\mathbb{Z}}^{(2)})(H)$ (resp. $G_{\mathbb{Z}_N}^{(2)})(H)$) is given by the coset representatives**:

$$\left\{ g_{\mathbb{Z}}^{(2)}[n] \right\}_{n=1}^{t} \qquad (\text{resp.} \quad \left\{ g_{\mathbb{Z}_N}^{(2)}[n] \right\}_{n=1}^{t}), \qquad t \leq \infty .$$

The order of the n-th coset representative $g_{\mathbb{Z}}^{(2)}[n]$ (resp. $g_{\mathbb{Z}_N}^{(2)}[n]$) is the integer n (resp. $n\,N$), corresponding to the number of its elements or automorphisms
On the other hand, as $\Lambda_{\mathbb{Z}}^{(2)}[n]$ is included into $\Lambda_{\mathbb{Z}_N}^{(2)}[n]$, the quotient subgroup $\Lambda_{\mathbb{Z}_N}^{(2)}[n]/\Lambda_{\mathbb{Z}}^{(2)}[n]$ is finite and its order is N: $\Lambda_{\mathbb{Z}_N}^{(2)}[n]$ is thus of index N in $\Lambda_{\mathbb{Z}}^{(2)}[n]$ [Ser].
Let H_n (resp. H_{nN}) denote the Poincare upper half plane restricted to the n-th coset representative $g_{\mathbb{Z}}^{(2)}[n]$ (resp. $g_{\mathbb{Z}_N}^{(2)}[n]$), then $H_n/\Lambda_{\mathbb{Z}}^{(2)}[n]$ is included into $H_{nN}/\Lambda_{\mathbb{Z}_N}^{(2)}[n]$.
For every integer n labeling a coset representative $g_{\mathbb{Z}}^{(2)}[n]$ (resp. $g_{\mathbb{Z}_N}^{(2)}[n]$) of $G_{\mathbb{Z}}^{(2)}(H)$ (resp. $G_{\mathbb{Z}_N}^{(2)}(H)$), we have the monomorphism

$$h_{\Lambda \to g}: \quad \Lambda_{\mathbb{Z}}^{(2)}[n] \to g_{\mathbb{Z}}^{(2)}[n] \qquad (\text{resp.} \quad h_{\Lambda_N \to g_N}: \quad \Lambda_{\mathbb{Z}_N}^{(2)}[n] \to g_{\mathbb{Z}_N}^{(2)}[n]), \quad 1 \leq n \leq t \leq \infty .$$

1.3 Fundamental domains

A fundamental domain $D_n^{(2)}$ (resp. $D_{nN}^{(2)}$) of H_n with respect to the sublattice $\Lambda_{\mathbb{Z}}^{(2)}[n]$ (resp. $\Lambda_{\mathbb{Z}_N}^{(2)}[n]$) is a unitary measurable set (i.e. a period) of H_n in such a way that its translations by the vectors of $\Lambda_{\mathbb{Z}}^{(2)}[n]$ (resp. $\Lambda_{\mathbb{Z}_N}^{(2)}[n]$) are a partition of H_n.
As a modular form is periodic, all the fundamental domains $D_n^{(2)}$ (resp. $D_{nN}^{(2)}$), $\forall\, n$, are equal:

$$D_1^{(2)} = D_2^{(2)} = \ldots = D_n^{(2)} = \cdots = D_t^{(2)}$$

$$(\text{resp.} \quad D_{1N}^{(2)} = D_{2N}^{(2)} = \ldots = D_{nN}^{(2)} = \cdots = D_{tN}^{(2)})).$$

The fundamental domain $D_n^{(2)}$ (resp. $D_{nN}^{(2)}$) corresponds then to the isotropy group $I_{\mathbb{Z}}^{(2)} = \{i_{\mathbb{Z}} \in I_{\mathbb{Z}}^{(2)} \mid i_{\mathbb{Z}}\, z\, i_{\mathbb{Z}}^{-1} = z\}$ (resp. $I_{\mathbb{Z}_N}^{(2)} = \{i_{\mathbb{Z}_N} \in I_{\mathbb{Z}_N}^{(2)} \mid i_{\mathbb{Z}_N}\, z\, i_{\mathbb{Z}_N}^{-1} = z\}$) of z.

1.4 Proposition

The isotropy subgroup $I_{\mathbb{Z}}^{(2)}$ (resp. $I_{\mathbb{Z}_N}^{(2)}$) acts on $\mathrm{GL}_2(\mathbb{R})$ by conjugation in such a way that the coset representatives $\{g_{\mathbb{Z}}^{(2)}[n]\}_n$ (resp. $\{g_{\mathbb{Z}_N}^{(2)}[n[\}_n)$ of $G_{\mathbb{Z}}^{(2)}(H)$ (resp. $G_{\mathbb{Z}_N}^{(2)}(H)$) are the conjugacy classes of $\mathrm{GL}_2(\mathbb{R})$.

Proof: The isotropy subgroup $I_{\mathbb{Z}}^{(2)}$ (resp. $I_{\mathbb{Z}_N}^{(2)}$) has order 1 (resp. order N) according to section 1.2.

As the isotropy subgroup $I_{\mathbb{Z}}^{(2)}$ (resp. $I_{\mathbb{Z}_N}^{(2)}$) is defined with respect to the set $\{\Lambda_{\mathbb{Z}}^{(2)}[n]\}_{n=1}^t$ (resp. $\{\Lambda_{\mathbb{Z}_N}^{(2)}[n]\}_{n=1}^t$) of sublattices of $\mathrm{GL}_2(\mathbb{R})$ which is isomorphic to the group of automorphisms of H, it is clear that there are t classes of automorphisms in $\mathrm{GL}_2(\mathbb{R})$ generated by the translation vectors of the sublattices $\{\Lambda_{\mathbb{Z}}^{(2)}[n]\}_n$ (resp. $\{\Lambda_{\mathbb{Z}_N}^{(2)}[n]\}_n$) on the fundamental domain $D_n^{(2)}$ (resp. $D_{nN}^{(2)}$). As every automorphism of H is induced by a conjugation of $\mathrm{GL}_2(\mathbb{R})$, there are "$t$" classes of conjugation of $\mathrm{GL}_2(\mathbb{R})$ given by their orders which are integers $1 \leq n \leq t \leq \infty$ (resp. $N \leq n \cdot N \leq t \cdot N \leq \infty$).

And, thus, the cosets of $G_{\mathbb{Z}}^{(2)}(H)$ (resp. $G_{\mathbb{Z}_N}^{(2)}(H)$) are in one-to-one correspondence with the conjugacy classes of $\mathrm{GL}_2(\mathbb{R})$. ∎

1.5 Corollary

The action of $\mathrm{GL}_2(\mathbb{R})$ on the fundamental domain $D_n^{(2)}$ (resp. $D_{nN}^{(2)}$) induces a homomorphism [Hun]

$$h_{G \to P_n}: \quad \mathrm{GL}_2(\mathbb{R}) \longrightarrow P(D_n^{(2)}) \qquad (resp. \quad h_{G \to P_{nN}}: \quad \mathrm{GL}_2(\mathbb{R}) \longrightarrow P(D_{nN}^{(2)}))$$

into the group $P(D_n^{(2)})$ (resp. $P(D_{nN}^{(2)})$) of all permutations of $D_n^{(2)}$ (resp. $D_{nN}^{(2)}$).

Proof: The map $h_{G \to P_n}$ (resp. $h_{G \to P_{nN}}$) being a bijection implies that $P(D_n^{(2)})$ (resp. $P(D_{nN}^{(2)})$) is a group of permutations. ∎

1.6 Actions of Hecke operators

The sum of the functions on the cosets of $T_{\mathbb{Z}}^{(2)}(H) = T_2(\mathbb{R})/T_2(\mathbb{Z})$ (resp. $T_{\mathbb{Z}_N}^{(2)}(H) = T_2(\mathbb{R})/T_2(\mathbb{Z}/N\mathbb{Z})$) ($T_2(\mathbb{R})$ being the subgroup of upper triangular matrices of $\mathrm{GL}_2(\mathbb{R})$) **can be reached by the action of the Hecke operator $T_k(n)$ (resp. $T_k(n; N)$) on the modular form $f_k(z)$ (resp. $f_k(z)_N$)** of weight k and level 1

(resp. N) given by:

$$T_k(n) \, f_k(z) = n^{k-1} \, T_k(n) \, F_k(G^{(2)}_{\mathbb{Z}}(H))$$

$$= n^{k-1} \sum_{\substack{ad=n \\ 0 \le b < d}} d^{-k} \, f_k \left(\frac{az+b}{d} \right)$$

$$(\text{resp.} \quad T_k(n;N) \, f_k(z)_N = (n \, N)^{k-1} \, T_k(n;N) \, F_k(G^{(2)}_{\mathbb{Z}_N}(H))$$

$$= (n \, N)^{k-1} \sum_{ad=nN} d_N^{-k} \, f_k \left(\frac{az+b}{d} \right)_N \,)$$

according to a complete set of upper triangular representatives $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right)$ (resp. $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right)_N$), $ad = 0$ modulo N, $b = *$ modulo N of the group M^n (resp. $\Gamma_1(N)$, $\Gamma_0(N), \dots$)) of integral matrices with determinant n (resp. $n \, N \dots$) with respect to $SL_2(\mathbb{Z})$ (resp. $SL_2(\mathbb{Z}/N\mathbb{Z})$).

1.7 Proposition

The sum $\sum\limits_{ad=n} d^{-k} \, f_k \left(\dfrac{az+b}{d} \right)$ (resp. $\sum\limits_{ad=nN} d_N^{-k} \, f_k \left(\dfrac{az+b}{d} \right)_N$) of the functions on the coset representatives $t^{(2)}_{\mathbb{Z}})[n]$ (resp. $t^{(2)}_{\mathbb{Z}_N})[n]$) of $T_2(\mathbb{R})/T_2(\mathbb{Z})$ (resp. $T_2(\mathbb{R})/T_2(\mathbb{Z}/N\mathbb{Z})$) is in one-to-one correspondence with the action

$$T_k(n): \qquad T^{(2)}_{\mathbb{Z}}(H) \longrightarrow \sum_{(T^{(2)}_{\mathbb{Z}}(H):T^{(2)}_{\mathbb{Z}}(H))=n} t^{(2)}_{\mathbb{Z}}(H)$$

$$(\text{resp.} \quad T_k(n;N): \qquad T^{(2)}_{\mathbb{Z}_N}(H) \longrightarrow \sum_{(T^{(2)}_{\mathbb{Z}_N}(H):T^{(2)}_{\mathbb{Z}_N}(H))=nN} t^{(2)}_{\mathbb{Z}_N}(H) \,)$$

of the Hecke operator $T_k(n)$ (resp. $T_k(n;N)$) associating (by a correspondence) with the lattice $T^{(2)}_{\mathbb{Z}}(H)$ (resp. $T^{(2)}_{\mathbb{Z}_N}(H)$) the sum of its sublattices $t^{(2)}_{\mathbb{Z}}(H)$ (resp. $t^{(2)}_{\mathbb{Z}_N}(H)$) of index n (resp. $n \, N$) in $T^{(2)}_{\mathbb{Z}}(H)$ (resp. $T^{(2)}_{\mathbb{Z}_N}(H)$).

Proof: This is immediate if we take into account the monomorphism

$$h_{\Lambda \to g}: \quad \Lambda^{(2)}_{\mathbb{Z}}[n] \longrightarrow t^{(2)}_{\mathbb{Z}}[n] \qquad (\text{resp.} \quad h_{\Lambda_N \to g_N}: \quad \Lambda^{(2)}_{\mathbb{Z}_N}[n] \longrightarrow t^{(2)}_{\mathbb{Z}_N}[n] \,)$$

of section 1.2 mapping the n-th $\mathbb{Z}^2$-sublattice $\Lambda^{(2)}_{\mathbb{Z}}[n]$ (resp. $(\mathbb{Z}/N\mathbb{Z})^2$-sublattice $\Lambda^{(2)}_{\mathbb{Z}_N}[n]$) into the n-th coset representative $t^{(2)}_{\mathbb{Z}}[n]$ (resp. $t^{(2)}_{\mathbb{Z}_N}[n]$) of $T_2(\mathbb{R})/T_2(\mathbb{Z})$ (resp. $T_2(\mathbb{R})/T_2(\mathbb{Z}/N\mathbb{Z})$).

Then, the functions on the cosets $t^{(2)}_{\mathbb{Z}}[n]$ (resp. $t^{(2)}_{\mathbb{Z}_N}[n]$) are in one-to-one correspondence with the functions $f_k\left(\dfrac{az+b}{d}\right)$ (resp. $f_k\left(\dfrac{az+b}{d}\right)_N$) on the sublattices $\Lambda^{(2)}_{\mathbb{Z}}[n]$ (resp. $\Lambda^{(2)}_{\mathbb{Z}_N}[n]$) described by the coset representatives $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)_N$) of M^n (resp. $\Gamma_1(N), \Gamma_0(N), \dots$). $\blacksquare$

1.8   Fourier series development of cusp forms

Let

$$T_k(n): \qquad F_k(T^{(2)}_{\mathbb{Z}}(H)) \longrightarrow \sum_{ad=n} d^{-k}\, f_k\left(\frac{az+b}{d}\right)$$

$$(\text{resp.} \quad T_k(n;N): \qquad F_k(T^{(2)}_{\mathbb{Z}_N}(H)) \longrightarrow \sum_{ad=nN} d_N^{-k}\, f_k\left(\frac{az+b}{d}\right)_N \ )$$

be the action of the Hecke operator $T_k(n)$ (resp. $T_k(n;N)$) on the function $F_k(T^{(2)}_{\mathbb{Z}}(H))$ (resp. $F_k(T^{(2)}_{\mathbb{Z}_N}(H))$) over the two-dimensional lattice $T^{(2)}_{\mathbb{Z}}(H)$ (resp. $T^{(2)}_{\mathbb{Z}_N}(H)$) given by the Laurent series:

$$F_k(T^{(2)}_{\mathbb{Z}}(H)) = \sum_{n=0}^{\infty} a'_n\, h^n \qquad (\text{resp.} \quad F_k(T^{(2)}_{\mathbb{Z}_N}(H)) = \sum_{n=0}^{\infty} b'_n\, k^n \ )$$

with $h = H_k(z)$ (resp. with $k = K_k(z_N)$) being a function of z (resp. z_N) **which is a complex point of order $(1 \times k)$ (resp. $(N \times k)$) to which a period "$1 \times k$" (resp. $N \times k$) corresponds**.
Let

$$\tau^{(k)}_{h\to q}: \quad h = H_k(z) \longrightarrow q = e^{2\pi i z} \qquad (\text{resp.} \quad \tau^{(k)}_{k\to q_N}: \quad k = K_k(z_N) \longrightarrow q_N = e^{2\pi i z_N} \ )$$

be **the toroidal mapping sending h (resp. k) into q (resp. q_N) in such a way that the Laurent series $F_k(T^{(2)}_{\mathbb{Z}}(H))$ (resp. $F_k(T^{(2)}_{\mathbb{Z}_N}(H))$) be transformed into the Fourier series**:

$$f_k(z) = \sum_{n=1}^{\infty} a_n\, q^n \qquad (\text{resp.} \quad f_k(z_N) = \sum_{n=1}^{\infty} b_n\, q_N^n \ ).$$

Taking into account the main property

$$T_k(m)\, T_k(n) = \sum_{d|m;n} T_k(mn/d^2)\, d^{k-1}$$

of the Hecke operators, we find that [Lan]

$$T_k(m)\, f_k(z) = \sum_{n=1}^{\infty} a_{nk}(m)\, q^n \qquad (\text{resp.} \quad T_k(m;N)_\varepsilon\, f_k(z_N) = \sum_{n=1}^{\infty} c_{nk}\, q_N^n \ )$$

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with $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ a Dirichlet character mod N, where

$$a_{nk}(m) = \sum_{d \mid m,n} d^{k-1}\, a_{nm \mid d^2} \qquad (\text{resp.} \quad c_{nk} = \sum_{d \mid (m,n)} \varepsilon(d)\, d^{k-1}\, a_{nm \mid d^2}\,).$$

If $f_k(z)$ (resp. $f_k(z_N)$) is an eigenfunction of all the Hecke operators $T_k(m)$ (resp. $T_k(m;N)$), $1 \le m \le \infty$, according to:

$$T_k(n)\, f_k(z) = \lambda(n)\, f_k(z) \qquad (\text{resp.} \quad T_k(m;N)_\varepsilon\, f_k(z_N) = \lambda(n)_N\, f_k(z_N)\,),$$

we have that $a_{nk} = \lambda(n)$ for $f_k(z)$ normalized with $a_1 = 1$ (resp. $c_{nk} = \lambda(n)_N = \sum_{d \mid n} \varepsilon(d)\, d^{k-1}$) where $\boldsymbol{\lambda(n)}$ (resp. $\boldsymbol{\lambda(n)_N}$) **is a Hecke eigencharacter of $\boldsymbol{f_k(z)}$** (resp. $\boldsymbol{f_k(z_N)}$).

If $a_1 = 0$ (resp. $c_1 = 0$), then $T_k(n)\, f_k(z)$ (resp. $T_k(n;N)\, f_k(z_N)$) is a cusp form of weight k and level 1 (resp. N).

1.9 Proposition (Geometric interpretation of cusp forms)

Let

$$T_2(n)\, f_2(z) = \sum_{n=1}^{n} a_n(m)\, q^n \qquad (\text{resp.} \quad T_2(n;N)\, f_2(z_N) = \sum_{n=1}^{n} c_n\, q_N^n\,)$$

be a weight 2 ($k = 2$) cusp form of level 1 (resp. N).

Then, we have that **the n-th term $a_n(m)\, q^n$ (resp. $c_n\, q_N^n$) given by**

$$T_n^2(a_{nm \mid d^2}, d)$$
$$\simeq a_n(m)\, q^n = r_{S_{a_n}^1}\, e^{2\pi i n x} \times r_{S_{d_n}^1}\, e^{2\pi i n (iy)}$$

$$(\text{resp.} \quad T_{nN}^2(a_{nm \mid d^2}, d\varepsilon(d))$$
$$\simeq c_n\, q_N^n = r_{S_{a_{nN}}^1}\, e^{2\pi i n x_N} \times r_{S_{d_{nN}}^1}\, e^{2\pi i n (iy_N)})$$

is a two-dimensional semitorus $T_n^2(a_{nm \mid d^2}, d)$ (resp. $T_{nN}^2(a_{nm \mid d^2}, d\varepsilon(d))$):

a) **generated by two orthogonal semicircles $S_{a_n}^1$ (resp. $S_{a_{nN}}^1$) and $S_{d_n}^1$ (resp. $S_{d_{nN}}^1$) respectively at a_n and d_n (transcendental) quanta (see proposi-tion 2.8)** *in such a way that the Hecke eigencharacter* $\lambda(n) = a_{nk}$ *(resp.* $\lambda(n)_N = c_{nk}$*) decomposes according to* $\lambda(n) = \lambda_a(n) \times \lambda_d(n)$ *(resp.* $\lambda(n)_N = \lambda_a(n)_N \times \lambda_d(n)_N$*) where* $\lambda_a(n)$ *(resp.* $\lambda_a(n)_N$*) refers to the radius* $r_{S_{a_n}^1}$ *(resp.* $r_{S_{a_{nN}}^1}$*) and* $\lambda_d(n)$ *(resp.* $\lambda_d(n)_N$*) refers to the radius* $r_{S_{d_n}^1}$ *(resp.* $r_{S_{d_{nN}}^1}$*);*

b) **whose area results from n (resp. nN) permutations of the fundamental domain $D_n^{(2)}$ (resp. $D_{nN}^{(2)}$) of $T_2(n)\, f_2(z)$ (resp. $T_2(n;N)\, f_2(z_N)$).**

Proof:

1) The n-th term $f_2^{(n)}(z) = a_n(m) \, q^n$ (resp. $f_2^{(n)}(z_N)$) of the cusp form $T_2(n) \, f_2(z)$ (resp. $T_2(n; N) \, f_2(z_N)$)corresponds to the n-th cuspidal sublattice on products $(\omega_1' \times \omega_2')$ (resp. $(\omega_{1_N}' \times \omega_{2_N}')$) of complex numbers with positive imaginary parts and results from the map:

$$z = \tfrac{\omega_1}{\omega_2} \longrightarrow f_2^{(n)}(z) = \omega_1' \times \omega_2' \qquad (\text{resp.} \quad z_N = \tfrac{\omega_{1N}}{\omega_{2N}} \longrightarrow f_2^{(n)}(z_N) = \omega_{1N}' \times \omega_{2N}' \,)$$

in such a way that ω_1' and ω_2' (resp. ω_{1N}' and ω_{2N}') are respectively the orthogonal semicircles

$$S_{a_n}^1 = r_{S_{a_n}^1} \, e^{2\pi i n x} \qquad (\text{resp.} \quad S_{a_{nN}}^1 = r_{S_{a_{nN}}^1} \, e^{2\pi i n x_n} \,), \qquad x > 0 \,,$$

$$\text{and} \quad S_{d_n}^1 = r_{S_{d_n}^1} \, e^{2\pi i n (iy)} \qquad (\text{resp.} \quad S_{d_{nN}}^1 = r_{S_{d_{nN}}^1} \, e^{2\pi i n (iy_n)} \,), \qquad y > 0 \,,$$

of radii $r_{S_{a_n}^1}$ (resp. $r_{S_{a_{nN}}^1}$) and $r_{S_{d_n}^1}$ (resp. $r_{S_{d_{nN}}^1}$).

Indeed, as it was proved in [Pie1] and [Pie2],

$$\boldsymbol{f_2^{(1)}(z) \simeq q = e^{2\pi i z} = e^{2\pi i x} \cdot e^{2\pi i (iy)}}$$

corresponds to the product of two orthogonal unitary (semi)circles because $e^{2\pi i (iy)} \neq e^{-2\pi y}$ where (iy) is an "imaginary" angle which cannot be "assimilated" to the function $\exp(i\theta) = \cos\theta + i\sin\theta$. And, thus, this "imaginary" angle is the angle of a circle orthogonal to $e^{2\pi i x}$.

2) As the n-th cusp subform $f_2^{(n)}(z)$ (resp. $f_2^{(n)}(z_N)$) is a function on the n-th coset representative $t_{\mathbb{Z}}^{(2)}[n]$ (resp. $t_{\mathbb{Z}_N}^{(2)}[n]$) of $T_{\mathbb{Z}}^{(2)}(H) = T_2(\mathbb{R})/T_2(\mathbb{Z})$ (resp. $T_{\mathbb{Z}_N}^{(2)}(H) = T_2(\mathbb{R})/T_2(\mathbb{Z}/N\mathbb{Z})$), the action of $t_{\mathbb{Z}}^{(2)}[n]$ (resp. $t_{\mathbb{Z}_N}^{(2)}[n]$) on the fundamental domain $D_n^{(2)}$ (resp. $D_{nN}^{(2)}$) induces the map:

$$h_{f_2^{(n)}(z) \to P_n} : \qquad f_2^{(n)}(z) \quad \longrightarrow \quad P_n(f_2^{(1)}(z))$$

$$(\text{resp.} \quad h_{f_2^{(n)}(z_N) \to P_{nN}} : \qquad f_2^{(n)}(z_N) \quad \longrightarrow \quad P_{nN}(f_2^{(1)}(z_N)) \,)$$

of $f_2^{(n)}(z)$ (resp. $f_2^{(n)}(z_N)$) into the subgroup $P_n(f_2^{(1)}(z))$ (resp. $P_{nN}(f_2^{(1)}(z_N))$) of n (resp. nN) permutations of the unitary cusp subform $f_2^{(1)}(z) = a_1 \, q^1$ (resp. $f_2^{(1)}(z_N) = c_1 \, q_N^1$) on $D_n^{(2)}$ (resp. $D_{nN}^{(2)}$) according to corollary 1.5. ∎

1.10 Proposition

The weight **2** cusp form of level **1** (resp. N)

$$T_2(n)\ f_2(z) = \sum_{n=1}^{\infty} a_n(m)\ q^n \qquad (\text{resp.} \quad T_2(n; N)\ f_2(z_N) = \sum_{n=1}^{\infty} c_n\ q_N^n\ )$$

is a function

a) composed of the sum of increasing embedded two-dimensional semitori localized in the upper half plane:

$$a_1\ q^1 \subset a_2\ q^2 \subset \cdots \subset a_n\ q^n \qquad (\text{resp.} \quad c_1\ q_N^1 \subset c_2\ q_N^2 \subset \cdots \subset c_n\ q_N^n\ );$$

b) which is periodic in the sense that each cusp subform $f_2^{(n)}(z)$ (resp. $f_2^{(n)}(z_N)$) has an area being n (resp. nN) times the area of the unitary cusp subform $f_2^{(1)}(z) = a_1\ q^1$ (resp. $f_2^{(1)}(z_N) = c_1\ q_N^1$).

Proof: This results immediately from proposition 1.9. ∎

1.11 Cosemialgebra of dual cusp forms

Let $S_L(k,1)$ (resp. $S_L(k,N)$) denote the (semi)algebra of cusp forms $f_k(z)$ (resp. $f_k(z_N)$) of weight k and level 1 (resp. N) which, expanded in Fourier series

$$f_k(z) = \sum_{n=1}^{\infty} a_{nk}(m)\ q^n \qquad (\text{resp.} \quad f_k(z_N) = \sum_{n=1}^{\infty} c_{nk}(m)\ q_N^n\ )$$

are eigenfunctions of Hecke operators $T_k(m)$ (resp. $T_k(m; N)$) and are holomorphic in the Poincare upper half plane H.

The dual (semi)algebra of $S_L(k,1)$ (resp. $S_L(k,N)$) is the cosemialgebra $S_R(k,1)$ (resp. $S_R(k,N)$) of dual cusp forms $f_k(z^*)$ (resp. $f_k(z_N^*)$) of weight k and level 1 (resp. N) which:

- are holomorphic in the Poincare lower half plane H^* of complex numbers $z^* = x - iy$ with negative imaginary parts $-y$;

- are eigenfunctions of Hecke operators $T_k^*(m)$ (resp. $T_k^*(m; N)$);

- are **expanded in Fourier series**

$$f_k(z^*) = \sum_{n=1}^{\infty} a_{nk}^*(m)\ q^{*n}$$

$$(\text{resp.} \quad f_k(z_N^*) = \sum_{n=1}^{\infty} c_{nk}^*(m)\ q_N^{*n}\ ), \qquad q^* = e^{2\pi i z^*},$$

according to the complete set of lower triangular coset representatives $\left(\begin{smallmatrix} a & 0 \\ b & d \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} a & 0 \\ b & d \end{smallmatrix}\right)_N$ $ad = 0 \bmod N$, $b = *\bmod N$) of the group M^n (resp. the congruence subgroup $\Gamma_1(N), \Gamma_0(N), \dots$) of integral matrices with determinant n (resp. nN) with respect to $SL_2(\mathbb{Z})$ (resp. $SL_2(\mathbb{Z}/N\mathbb{Z})$).

We thus have that:

a) the n-th dual cusp subform $f_k^{(n)}(z^*)$ (resp. $f_k^{(n)}(z_N^*)$) of weight k with respect to $f_k^{(n)}(z)$ (resp. $f_k^{(n)}(z_N)$) is a cofunction on the n-th coset representative $t_{k,\mathbb{Z}}^{*(2)}[n]$ (resp. $t_{k,\mathbb{Z}_N}^{*(2)}[n]$) of

$$T_{k,\mathbb{Z}}^{(2)}(H^*) = T_2^t(\mathbb{R})/T_2^t(\mathbb{Z}) \qquad (\text{resp.} \quad T_{k,\mathbb{Z}_N}^{2)}(H^*) = T_2^t(\mathbb{R})/T_2^t(\mathbb{Z}/N\mathbb{Z}))$$

where:

- $T_{k,\mathbb{Z}}^{(2)}(H^*)$ (resp. $T_{k,\mathbb{Z}_N}^{(2)}(H^*)$) is the set of $\mathbb{R}^2/\mathbb{Z}^2$ (resp. $\mathbb{R}^2/(\mathbb{Z}/N\mathbb{Z})^2$)- lattices in the Poincare lower half plane H^*;

- $T_2(\mathbb{R})$ (resp. $T_2^t(\mathbb{R})$) is the two-dimensional subgroup of upper (resp. lower) triangular matrices.

b) $D_n^{(k)}$ (resp. $D_{nN}^{(k)}$) is a fundamental domain of H_n^* with respect to the n-th $\mathbb{Z}^2$-sublattice $\Lambda_{k,\mathbb{Z}}^{*(2)}[n]$ (resp. $(\mathbb{Z}/N\mathbb{Z})^2$-sublattice $\Lambda_{k,\mathbb{Z}_N}^{*(2)}[n]$).

From now on, we shall only consider the semialgebra $S_L(k, N)$ (resp. the cosemialgebra $S_R(k, N)$) of weight k cusp forms $f_k(z_N)$ (resp. dual cusp forms $f_k(z_N^*)$) of level N, the level $N = 1$ being a particular case of it.

1.12 Bisemialgebra of cusp biforms

Let
$$R_{f_{k,N}} = \{f_k^{(n)}(z_N)\}_n \qquad (\text{resp.} \quad R_{f_{k,N}^*} = \{f_k^{(n)}(z_N^*)\}_n)$$
denote the left (resp. right) semiring of cusp subforms $f_k^{(n)}(z_N)$ (resp. dual cusp subforms $f_k^{(n)}(z_N^*)$) as introduced in section 1.11.

Then, $S_L(k, N)$ (resp. $S_R(k, N)$) is a left $R_{f_{k,N}}$-semialgebra (resp. right $R_{f_{k,N}^*}$-cosemi algebra).

Let $R_{f_{k,N}^*} \times R_{f_{k,N}}$ be the associated bisemiring [Pie3].

Then, **the $R_{f_{k,N}^*} \times R_{f_{k,N}}$-bisemialgebra $S_{R \times L}(k, N)$ of cusp biforms $f_k(z_N^*) \times f_k(z_N)$ of weight k and level N is a bisemiring such that:**

a) $S_{R \times L}(k, N)$ is a unitary $R_{f_{k,N}^*} \times R_{f_{k,N}}$-bisemimodule:

b) $\mu_{R \times L} : S_{R \times L}(k, N) \underline{\times} S_{R \times L}(k, N) \to S_{R \times L}(k, N)$ is a bilinear homomorphism where $\underline{\times}$ **is the cross binary operation acting on the cusp biforms** as follows:

$$\left[\left({}^1 f_k(z_N^*) \times {}^1 f_k(z_N) \right) \underline{\times} \left({}^2 f_k(z_N^*) \times {}^2 f_k(z_N) \right) \right]$$
$$= \left[\left({}^1 f_k(z_N^*) + {}^2 f_k(z_N^*) \right) \times \left({}^1 f_k(z_N) + {}^2 f_k(z_N) \right) \right]$$

allowing cross products $\left({}^1 f_k(z_N^*) \times {}^2 f_k(z_N) \right)$ **and** $\left({}^2 f_k(z_N^*) \times {}^1 f_k(z_N) \right)$ **between left and right cusp forms** **1** **and** **2**.

c) $\eta_{R \times L} : R_{f_{k,N}^*} \times R_{f_{k,N}} \to S_{R \times L}(k, N)$ is an injective homomorphism [Pie1].

The existence of the bisemialgebra $S_{R \times L}(k, N)$ of cusp biforms is especially important due to the homomorphism:

$$\psi : \quad S_{R \times L}(k, N) \longrightarrow \mathrm{End}(S_L(k, N))$$

allowing to compute the endomorphism $\mathrm{End}(S_L(k, N))$ of the semialgebra $S_L(k, N)$ of cusp forms $f_k(z_N)$.

As the semialgebra $S_L(k, N)$ (resp. the cosemialgebra $S_R(k, N)$) of cusp forms $f_k(z_N)$ (resp. dual cusp forms $f_k(z_N^*)$) is defined on the quotient semigroup

$$T_{\mathbb{Z}_H}^{(2)}(H) = T_2(\mathbb{R}) / T_2(\mathbb{Z}/N\mathbb{Z}) \qquad \text{(resp.} \quad T_{\mathbb{Z}_H}^{t(2)}(H^*) = T_2^t(\mathbb{R}) / T_2^t(\mathbb{Z}/N\mathbb{Z}) \text{)}$$

of lattices of H (resp. H^*), the bisemialgebra $S_{R \times L}(k, N)$ of cusp biforms $f_k(z_N^*) \times f_k(z_N)$ will be defined on the quotient bisemigroup

$$G_{\mathbb{Z}_H}^{(2)}(H^* \times H) = \mathrm{GL}_2(\mathbb{R} \times \mathbb{R}) / \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})^2$$

where

$$G_{\mathbb{Z}_H}^{(2)}(H^* \times H) = T_{\mathbb{Z}_H}^{t(2)}(H^*) \times T_{\mathbb{Z}_H}^{(2)}(H) \ ,$$
$$\mathrm{GL}_2(\mathbb{R} \times \mathbb{R}) = T_2^t(\mathbb{R}) \times T_2(\mathbb{R}) \ ,$$
$$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})^2 = T_2^t(\mathbb{Z}/N\mathbb{Z}) \times T_2(\mathbb{Z}/N\mathbb{Z}) \ .$$

1.13 Varying the weight of cusp (bi)forms

The bisemigroup category CAF of cusp biforms $f_k(z_N^*) \times f_k(z_N)$ is then given by:

1) the cusp biforms $(f_k(z_N^*) \times f_k(z_N))$, k and N varying;

2) **the (bi)morphisms of projection**

$$\mathbf{Hom}\left[(f_k(z_N^*) \times f_k(z_N)), (f_h(z_N^*) \times f_h(z_N))\right] , \qquad k > h ,$$

sending cusp biforms $(f_k(z_N^*) \times f_k(z_N))$ of weight k into corresponding biforms $(f_h(z_N^*) \times f_h(z_N))$ of weight h.

But, in order to understand the (bi)morphism of projection:

$$\mathrm{Phom}^{(2)}_{k \to h} : \qquad f_k(z_N^*) \times f_k(z_N) \quad \longrightarrow \quad f_h(z_N^*) \times f_h(z_N)$$

from a cusp biform of weight k onto a cusp biform of weight h, we have to give **a new meaning to the weight of cusp forms**: this will be done in the next sections.

1.14 Proposition (Origin of the weight of cusp forms)

Let

$$\phi_k^{(2)}(z_N) \equiv T_k(n; N)_\varepsilon \; f_k(z_N) = \sum_{n=1}^{\infty} c_{nk} \; q_N^n$$

denote a weight k two-dimensional cusp form of level N as described in section 1.8, where

$$c_{nk} = \sum_{d|(m,n)} \varepsilon(d) \; d^{k-1} \; a_{mn|d^2} \qquad and \qquad q_N^n = e^{2\pi i n z_N} ,$$

Let

$$\phi^{(k)}(z_{Nk}) = \sum_{n=1}^{\infty} \lambda_n^{(k)} \; e^{2\pi i n z_{Nk}}$$

be the Fourier series development of a cusp form of real dimension k (k being an even integer) [Pie2] and level N on $\mathbb{C}^{k/2}$ above the weight k two-dimensional cusp form

$$\phi_k^{(2)}(z_N) \equiv f_k(z_N)$$

where:

- $\lambda_n^{(k)} = \prod_{d=1}^{k/2} \lambda_{nd}^{(2)}$ *with* $\lambda_{nd}^{(2)} = r_{S_{d_1}^1} \times r_{S_{d^2}^1}$ *the product of the radii of two circles $S_{d_1}^1$ and $S_{d^2}^1$;*

- $z_{Nk} = z_N^{(1)} + z_N^{(2)} + \cdots + z_N^{(k)}$ *is the sum of the elements of a k-tuple of complex numbers $z_N^{(\cdot)}$ of order N.*

Then, **there exists a map***:*

$$CP_{(k)\to 2}: \qquad \phi^{(k)}(z_{Nk}) \quad \longrightarrow \quad \phi_k^{(2)}(z_N)$$

in such a way that*,* $\forall\, n\,,\; 1 \le n \le \infty:$

1) **every two-dimensional semitorus** $T_{nN}^{(2)}(a, d\varepsilon(d)_k) = c_{nk}\; q_N^n$ **of** $\phi_k^{(2)}(z_N)$ **be the inverse image of a** k**-dimensional real semitorus** $T_{nN}^{(k)}(\lambda_n^{(k)}) = \lambda_n^{(k)}\; e^{2\pi i n z_{Nk}}$ **of** $\phi^{(k)}(z_{Nk})\,;$

2) the product $\lambda_n^{(k)} = \prod\limits_{d=1}^{k/2} \lambda_d^{(2)}$ *, representing the product of the radii of the generators of* $T_{nN}^{(k)}(\lambda_n^{(k)})\,,$ *be sent by the map*

$$CP_{(k)\to 2}(n): \qquad \lambda_n^{(k)} \quad \longrightarrow \quad c_{nk}$$

into the coefficient c_{nk} *of* $\phi_k^{(2)}(z_N)$ *implying that* $\|\lambda_n^{(k)}\| = \|c_{nk}\|\,.$

Proof: The weight k of the two-dimensional cusp form of level N, $\phi_k^{(2)}(z_N)$ then originates from the map $CP_{(k)\to 2}$ of a k-dimensional cusp form of level N, $\phi^{(k)}(z_{Nk})$ into $\phi_k^{(2)}(z_N)$ at the conditions of this proposition.

The Fourier series development of $\phi^{(k)}(z_{Nk})$ was introduced in [Pie2] from which it results that it is the sum of increasing embedded k-dimensional real semitori $T_{nN}^{(k)}(\lambda_n^{(k)})$.

The conditions given by the map

$$\boldsymbol{CP_{(k)\to 2}(n):} \qquad \boldsymbol{\lambda_n^{(k)}} \quad \to \quad \boldsymbol{c_{nk}}$$

on the coefficients imply the inflation of the coefficients $\boldsymbol{c_n}$ **of a weight** $\boldsymbol{2}$ **cusp form to the corresponding coefficients** $\boldsymbol{c_{nk}}$ **of a weight** $\boldsymbol{k}$ **cusp form** (according to section 1.8 and proposition 1.9) associated with the inflation map

$$I_{T_2\to k}: \qquad T_2(n, N)\; f_2(z_N) \quad \longrightarrow \quad T_k(m, N)_\varepsilon\; f_k(z_N)$$

of a weight 2 cusp form into a weight k cusp form.

The origin of this inflation map $I_{T_2\to k}$ proceeds mainly from the factor $d^{(k-1)}$ of c_{nk} and, similarly, from the "normalization factor" $(cz + d)^{-k}$ in the automorphic condition

$$f(z) = (cz + d)^{-k}\; f\left(\frac{az + b}{cz + d}\right)\,. \qquad \blacksquare$$

1.15 Proposition

The fundamental domain $D_{nNk}^{(2)}$ of the weight k two-dimensional cusp form $\phi_k^{(2)}(z_N) \equiv f_k(z_N)$ is equal to the fundamental domain $D_{nN}^{(k)}$ of the corresponding k-dimensional cusp form $\phi^{(k)}(z_{Nk})$ implying that

$$T_{nN}^{(2)}(a, d, \varepsilon(d)_k) \simeq T_{nN}^{(k)}(\lambda_n^{(k)}) , \qquad \forall \, n .$$

Proof:

1) The fundamental domain $D_{nNk}^{(2)}$ of the weight k two-dimensional cusp form $\phi_k^{(2)}(z_N)$ is given by $q_N^1 = e^{2\pi i 1 z_N}$ where z_N is a complex point of order $(k \times N)$ according to propositions 1.4 and 1.14 while the fundamental domain $D_{nN}^{(k)}$ of the corresponding k-dimensional cusp form $\phi^{(k)}(z_{Nk})$ is given by $q_{Nk}^1 = e^{2\pi i 1 z_{Nk}}$ where z_{Nk} is also a complex point of order $(k \times N)$ according to proposition 1.14.

   Consequently, we have that:
   $$D_{nNk}^{(2)} = D_{nN}^{(k)} .$$

2) The domain $A_{nNk}^{(2)}$ of the n-th periodic cusp subform $T_{nN}^{(2)}(a, d\varepsilon(d)_k) = f_k^{(n)}(z_N)$ is evidently given by:
   $$A_{nNk}^{(2)} = n \, D_{nNk}^{(2)}$$
   while the domain $A_{nN}^{(k)}$ of the n-th periodic cusp subform $T_{nN}^{(k)}(\lambda_n^{(k)})$ of $\phi^{(k)}(z_{Nk})$ is:

   $$A_{nN}^{(k)} = n \, D_{nN}^{(k)} .$$

   Consequently, we have that:
   $$A_{nNk}^{(2)} = A_{nN}^{(k)}$$
   and, then, that
   $$T_{nN}^{(2)}(a, d\varepsilon(d)_k) = T_{nN}^{(k)}(\lambda_n^{(k)}) . \qquad \blacksquare$$

1.16 k-dimensional cusp biforms

The k-dimensional real cusp biform of level N associated with the k-dimensional real cusp form $\phi^{(k)}(z_{Nk})$ of level N is:

$$\phi^{(k)}(z_{Nk}^*) \times \phi^{(k)}(z_{Nk}) .$$

As $\phi^{(k)}(z_{Nk})$ (resp. $\phi^{(k)}(z_{Nk}^*)$) is a function on the sum of the cosets of $T_k(\mathbb{R})/T_k(\mathbb{Z}/N\mathbb{Z})$ (resp. $T_k^t(\mathbb{R})/T_k^t(\mathbb{Z}/N\mathbb{Z})$) according to section 1.6, $\phi^{(k)}(z_{Nk}^*) \times \phi^{(k)}(z_{Nk})$ will be a bi-function on the sum of the cosets of

$$\mathrm{GL}_k(\mathbb{R} \times \mathbb{R})\big/ \mathrm{GL}_k(\mathbb{Z}/N\mathbb{Z})^2 \equiv T_k^t(\mathbb{R}) \times T_k(\mathbb{R})/T_k^t(\mathbb{Z}/N\mathbb{Z}) \times T_k(\mathbb{Z}/N\mathbb{Z})$$

and will be written as follows:

$$\phi_{z_{Nk}^*}^{(k)} \times \phi_{z_{Nk}}^{(k)} (\mathrm{GL}_k(\mathbb{R} \times \mathbb{R})\big/ \mathrm{GL}_k(\mathbb{Z}/N\mathbb{Z})^2) \ .$$

1.17 Proposition (Changing the weight of cusp (bi)forms)

Let

$$\phi_{z_{Nh}^*}^{(h)} \times \phi_{z_{Nh}}^{(h)} (\mathrm{GL}_h(\mathbb{R} \times \mathbb{R})\big/ \mathrm{GL}_h(\mathbb{Z}/N\mathbb{Z})^2)$$

be the h-dimensional real cusp biform of level N above the 2-dimensional cusp biform $(f_h(z_N^*) \times f_h(z_N))$ (also written $(\phi_h^{(2)}(z_N^*) \times \phi_h^{(2)}(z_N))$) of weight h and level N. Then, the bimorphism (of projection) between 2-dimensional cusp biforms of weights k and h (introduced in section 1.13):

$$\mathrm{Phom}_{k \to h}^{(2)}: \qquad f_k(z_N^*) \times f_k(z_N) \quad \longrightarrow \quad f_h(z_N^*) \times f_h(z_N) \ , \quad k > h \ ,$$

directly depends on:

  a) *the bimorphism (of projection)*

$$\mathrm{Phom}_{k \to h}: \qquad \phi_{z_{Nk}^*}^{(k)} \times \phi_{z_{Nk}}^{(k)} (\mathrm{GL}_k(\mathbb{R} \times \mathbb{R})\big/ \mathrm{GL}_k(\mathbb{Z}/N\mathbb{Z})^2)$$
$$\longrightarrow \quad \phi_{z_{Nh}^*}^{(h)} \times \phi_{z_{Nh}}^{(h)} (\mathrm{GL}_h(\mathbb{R} \times \mathbb{R})\big/ \mathrm{GL}_h(\mathbb{Z}/N\mathbb{Z})^2)$$

  between k and h-dimensional cusp biforms;

  b) *the decomposition*

$$f_k(z_N^*) \times f_k(z_N) = \overset{k/2}{\underset{\ell=1}{\oplus}} (f_{2\ell}(z_N^*) \times f_{2\ell}(z_N)) \ ,$$

$$f_h(z_N^*) \times f_h(z_N) = \overset{h/2}{\underset{\ell=1}{\oplus}} (f_{2\ell}(z_N^*) \times f_{2\ell}(z_N)) \ ,$$

  h andf k being even integers,

  of weight k and h 2-dimensional cusp biforms $(f_k(z_N^*) \times f_k(z_N))$ and $(f_h(z_N^*) \times f_h(z_N))$ into 2-dimensional cusp biforms of weight 2 according to the Langlands functoriality conjecture.

This can be summarized in the following diagram:

$$
\begin{array}{ccccc}
\begin{array}{c} \mathrm{GL}_2(I\!R \times I\!R) \\ \big/\, \mathrm{GL}_2(Z\!\!\!Z/N Z\!\!\!Z)^2 \\ = \mathbb{GL}_2 \end{array} & \longrightarrow & \begin{array}{c} \mathrm{GL}_k(I\!R \times I\!R) \\ \big/\, \mathrm{GL}_k(Z\!\!\!Z/N Z\!\!\!Z)^2 \\ = \mathbb{GL}_k \end{array} & \longrightarrow & \begin{array}{c} \mathrm{GL}_h(I\!R \times I\!R) \\ \big/\, \mathrm{GL}_h(Z\!\!\!Z/N Z\!\!\!Z)^2 \\ = \mathbb{GL}_h \end{array}
\end{array}
$$

$$
\Big\downarrow \Pi^{(2)} \qquad\qquad \Big\downarrow \Pi^{(k)} \qquad\qquad \Big\downarrow \Pi^{(h)}
$$

$$
\phi^{(2)}_{z_N^*} \times \phi^{(2)}_{z_N}(\mathbb{GL}_2) \xrightarrow{\ \mathrm{Phom}_{2\to k}\ } \phi^{(k)}_{z_{Nk}^*} \times \phi^{(k)}_{z_{Nk}}(\mathbb{GL}_k) \xrightarrow{\ \mathrm{Phom}_{k\to h}\ } \phi^{(h)}_{z_{Nh}^*} \times \phi^{(h)}_{z_{Nh}}(\mathbb{GL}_h)
$$

$$
\Big\downarrow \mathrm{Id} \qquad\qquad \Big\downarrow CP_{(k)\to 2_{R\times L}} \qquad\qquad \Big\downarrow CP_{(h)\to 2_{R\times L}}
$$

$$
f_2(z_N^*) \times f_2(z_N) \xrightarrow{\ \mathrm{Phom}^{(2)}_{2\to k}\ } f_k(z_N^*) \times f_k(z_N) \xrightarrow{\ \mathrm{Phom}^{(2)}_{k\to h}\ } f_h(z_N^*) \times f_h(z_N)
$$

where:

- $\phi^{(2)}_{z_N^*} \times \phi^{(2)}_{z_N}(\mathrm{GL}_2(I\!R \times I\!R)/\mathrm{GL}_2(Z\!\!\!Z/N Z\!\!\!Z)^2) \equiv f_2(z_N^*) \times f_2(z_N)$ *is a two-dimensional cusp biform of weight* 2 *;*

- *the map* $\Pi^{(k)}$ *is a cuspidal representation of the quotient bisemigroup* $(\mathrm{GL}_k(I\!R \times I\!R)/\mathrm{GL}_k(Z\!\!\!Z/N Z\!\!\!Z)^2)$ *given by the corresponding cusp biform and depending on a toroidal compactification [Pie2].*

Proof: **The bimorphism (of projection) $\mathrm{Phom}_{k\to h}$ is directly related to the Langlands functoriality conjecture** transposed from cuspidal representations of bilinear algebraic semigroups [Pie4] to cuspidal representation of quotient bisemigroups $\mathrm{GL}_k(I\!R \times I\!R)/\mathrm{GL}_k(Z\!\!\!Z/N Z\!\!\!Z)^2$ and asserting that the cuspidal representation $\Pi^{(k)}(\mathrm{GL}_k(I\!R \times I\!R)/\mathrm{GL}_k(Z\!\!\!Z/N Z\!\!\!Z)^2)$ of the quotient bisemigroup $(\mathrm{GL}_k(I\!R \times I\!R)/\mathrm{GL}_k(Z\!\!\!Z/N Z\!\!\!Z)^2)$ is orthogonally completely reducible if it decomposes diagonally according to [Pie4]:

$$
\Pi^{(k)}(\mathrm{GL}_k(I\!R \times I\!R)/\mathrm{GL}_k(Z\!\!\!Z/N Z\!\!\!Z)^2) = \overset{k/2}{\underset{\ell=1}{\oplus}} \Pi^{(2\ell)}(\mathrm{GL}_{2\ell}(I\!R \times I\!R)/\mathrm{GL}_{2\ell}(Z\!\!\!Z/N Z\!\!\!Z)^2) \ .
$$

This corresponds to the injective map $\mathrm{Phom}_{2\to k}$ which is above the injective map $\mathrm{Phom}_{2\to k}$ leading to **the decomposition**

$$
\boldsymbol{f_k(z_N^*) \times f_k(z_N) = \overset{k/2}{\underset{\ell=1}{\oplus}} (f_{2\ell}(z_N^*) \times f_{2\ell}(z_N))}
$$

of weight k cusp biforms into weight 2 cusp biforms.

The surjective morphism $\mathrm{Phom}^{(2)}_{k\to h}$ is then directly associated with:

a) the deflation of the product $(c_{nh}^* \times c_{nh})$ of the coefficients of the two-dimensional cusp biforms $f_h(z_N^*) \times f_h(z_N)$ of weight h from the product $(c_{nk}^* \times c_{nk})$ of the coefficients

of the cusp biform $f_k(z_N^*) \times f_k(z_N^*)$ of weight k, $1 \le n \le t \le \infty$, according to proposition 1.14 and giving the increasing sequence

$$c_{n2}^* \times c_{n2} \longrightarrow c_{nh}^* \times c_{nh} \longrightarrow c_{nk}^* \times c_{nk} \ ;$$

b) the increasing sequence

$$(z_N^* \times z_N)_2 \longrightarrow (z_N^* \times z_N)_h \longrightarrow (z_N^* \times z_N)_k$$

of cusp bipoints, where $(z_N^* \times z_N)$ is a complex bipoint of order $(N \times 2)$, $(z_N^* \times z_N)_h$ is a complex bipoint of order $(N \times h)$ and $(z_N^* \times z_N)_k$ is a complex bipoint of order $(N \times k)$, with $k > h$. ∎

2 Cusp (bi)forms from the Langlands global program

Cusp (bi)forms are characterized by the two integer numbers k and N where k refers to the geometric dimension of a cusp (bi)form above a cusp (bi)form of weight 2 while N will be recalled to refer to an algebraic dimension.

Indeed, we shall at first prove that the quotient bisemigroup $\mathrm{GL}_k(I\!\!R \times I\!\!R)/\mathrm{GL}_k(Z\!\!\!Z/NZ\!\!\!Z)^2$, on which a cusp biform can be defined, is covered by the (algebraic) bilinear semigroup $\mathrm{GL}_k(F_{\overline{v}} \times F_v)$ over the product $(F_{\overline{v}} \times F_v)$ of sets $F_{\overline{v}}$ and F_v of pseudoramified transcendental extensions referring respectively to the lower and upper half space.

2.1 Algebraic and transcendental quanta

a) Let then $\widetilde{F}$ denote a set of finite algebraic extensions of a number field k of characteristic 0: $\widetilde{F}$ is assumed to be a set of **symmetric splitting fields composed of the left and right algebraic extension semifields $\widetilde{F_L}$ and $\widetilde{F_R}$** being respectively the set of complex and conjugate complex simple roots of the polynomial ring $k[x]$.

In the real case, the symmetric splitting fields are the left and right symmetric splitting semifields $\widetilde{F}_L^+$ and $\widetilde{F}_R^+$ composed of the set of positive and symmetric negative simple real roots.

b) The left and right equivalence classes of infinite Archimedean completions of $\widetilde{F}_L$ and $\widetilde{F}_R$ are the left and right symmetric infinite complex places

$$\omega = \{\omega_1, \ldots, \omega_n, \ldots, \omega_t\} \quad \text{and} \quad \overline{\omega} = \{\overline{\omega}_1, \ldots, \overline{\omega}_n, \ldots, \overline{\omega}_t\}, \qquad t \leq \infty .$$

In the real case, the infinite places are similarly

$$v = \{v_1, \ldots, v_n, \ldots, v_t\} \quad \text{and} \quad \overline{v} = \{\overline{v}_1, \ldots, \overline{v}_n, \ldots, \overline{v}_t\},$$

covering the respective infinite complex places as described subsequently.

c) All these (pseudoramified) completions, corresponding to the transcendental extensions, proceed from the associated algebraic extensions by a suitable isomorphism of compactification [Pie5] and are built from irreducible real subcompletions $F_{v_n^1}$ (resp. $F_{\overline{v}_n^1}$) characterized by a transcendence degree

$$\mathrm{tr} \cdot d \cdot F_{v_n^1}/k = \mathrm{tr} \cdot d \cdot F_{\overline{v}_n^1}/k = N \quad \text{equal to} \quad [\widetilde{F}_{v_n^1} : k] = [\widetilde{F}_{\overline{v}_n^1} : k] = N ,$$

which is the Galois extension degree of the associated irreducible algebraic closed subsets $\widetilde{F}_{v_n^1}$ and $\widetilde{F}_{\overline{v}_n^1}$.

All these irreducible (unitary) subcompletions (resp. subextensions) are assured to be transcendental (resp. algebraic) quanta [Pie6].

d) The pseudoramified real extensions are characterized by degrees:

$$[\widetilde{F}_{v_n} : k] = [\widetilde{F}_{\overline{v}_n} : k] = * + n\,N\ , \qquad 1 \le n \le t \le \infty$$

which are integers modulo N, $\mathbb{Z}/N\mathbb{Z}$, where:

- $\widetilde{F}_{v_n}$ and $\widetilde{F}_{\overline{v}_n}$ are algebraic extensions corresponding to the infinite completions F_{v_n} and $F_{\overline{v}_n}$, respectively at the v_n-th and $\overline{v}_n$-th symmetric real pseudoramified places;

- $*$ denotes an integer inferior to N, taken generally to be "0".

Similarly, the pseudoramified complex extensions $\widetilde{F}_{\omega_n}$ and $\widetilde{F}_{\overline{\omega}_n}$, corresponding to the completions F_{ω_n} and $F_{\overline{\omega}_n}$ at the infinite places ω_n and $\overline{\omega}_n$, are characterized by extension degrees:

$$[\widetilde{F}_{\omega_n} : k] = [\widetilde{F}_{\overline{\omega}_n} : k] = (* + n\,N)\,m_n\ ,$$

where m_n is the multiplicity of the n-th real extension F_{v_n} and $F_{\overline{v}_n}$ covering its n-th complex equivalent.

Let then $\widetilde{F}_{v_n,m_n}$ (resp. $\widetilde{F}_{\overline{v}_n,m_n}$) denote a left (resp. right) pseudoramified real extension equivalent to $\widetilde{F}_{v_n}$ (resp. $\widetilde{F}_{\overline{v}_n}$).

The corresponding pseudounramified real extensions $\widetilde{F}^{nr}_{v_n,m_n}$ and $\widetilde{F}^{nr}_{\overline{v}_n;m_n}$ are characterized by their global residue degree (case $N = 1$):

$$f_{v_n} = [\widetilde{F}^{nr}_{v_n,m_n} : k] = j \quad \text{and} \quad f_{\overline{v}_n} = [\widetilde{F}^{nr}_{\overline{v}_n,m_n} : k] = j\ .$$

e) If the orders of the Galois sub(semi)groups correspond to the class zero of the integers modulo N, then these Galois sub(semi)groups $\mathrm{Gal}(\widetilde{F}_{v_n}/k)$ (resp. $\mathrm{Gal}(\widetilde{F}_{\overline{v}_n}/k)$) are global Weil sub(semi)groups of extensions $\widetilde{F}_{v_n}$ (resp. $\widetilde{F}_{\overline{v}_n}$) constructed from n algebraic quanta.

By an isomorphism of compactification, these n noncompact algebraic quanta of $\widetilde{F}_{v_n}$ (resp. $\widetilde{F}_{\overline{v}_n}$) are sent into the corresponding compactified n transcendental compact quanta F_{v_n} (resp. $F_{\overline{v}_n}$).

As in the Galois case, there is a one-to-one correspondence between all transcendence extension subfields:

$$F_{v_1} \subset \cdots \subset F_{v_n} \cdots \subset F_{v_t} \qquad (\text{resp.} \quad F_{\overline{v}_1} \subset \cdots \subset F_{\overline{v}_n} \cdots \subset F_{\overline{v}_t})$$

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and the set of all (normal) sub(semi)groups of these:

$$\operatorname{Aut}_k(F_{v_1}) \supset \cdots \supset \operatorname{Aut}_k(F_{v_n}) \supset \cdots \supset \operatorname{Aut}_k(F_{v_t})$$

$$(\text{resp.} \quad \operatorname{Aut}_k(F_{\overline{v}_1}) \supset \cdots \supset \operatorname{Aut}_k(F_{\overline{v}_n}) \supset \cdots \supset \operatorname{Aut}_k(F_{\overline{v}_t}))$$

taking into account that $\operatorname{Gal}(\widetilde{F}_{v_n}/k) \simeq \operatorname{Aut}_k(F_{v_n})$ (resp. $\operatorname{Gal}(\widetilde{F}_{\overline{v}_n}/k) \simeq \operatorname{Aut}_k(F_{\overline{v}_n})$).

2.2 Abstract bisemivarieties

a) Let

$$\operatorname{GL}_k(\widetilde{F}_{\overline{v}} \times \widetilde{F}_v) \equiv T_k^t(\widetilde{F}_{\overline{v}}) \times T_k(\widetilde{F}_v)$$

be **the algebraic bilinear semigroup of matrices** over the product of sets $\widetilde{F}_v = \{\widetilde{F}_{v_1}, \ldots, \widetilde{F}_{v_n}, \ldots, \widetilde{F}_{v_t}\}$ and $\widetilde{F}_{\overline{v}} = \{\widetilde{F}_{\overline{v}_1}, \ldots, \widetilde{F}_{\overline{v}_n}, \ldots, \widetilde{F}_{\overline{v}_t}\}$ of pseudoramified algebraic extensions in such a way that its representation bisemispace is given by the tensor product $\widetilde{M}_{v_R} \otimes \widetilde{M}_{v_L}$ of a right $T_k^t(\widetilde{F}_{\overline{v}})$-semimodule $\widetilde{M}_{v_R}$ by a left $T_k(\widetilde{F}_v)$-semimodule $\widetilde{M}_{v_L}$.

Considering the monomorphism:

$$\widetilde{\sigma}_{v_R} \times \widetilde{\sigma}_{v_L}: \qquad W^{ab}_{\widetilde{F}_{\overline{v}}} \times W^{ab}_{\widetilde{F}_v} \quad \longrightarrow \quad \operatorname{GL}_k(\widetilde{F}_{\overline{v}} \times \widetilde{F}_v)$$

from the product of the global Weil semigroup $W^{ab}_{\widetilde{F}_{\overline{v}}} \times W^{ab}_{\widetilde{F}_v}$ into $\operatorname{GL}_k(\widetilde{F}_{\overline{v}} \times \widetilde{F}_v)$ and the isomorphism:

$$W^{ab}_{\widetilde{F}_{\overline{v}}} \times W^{ab}_{\widetilde{F}_v} \quad \xrightarrow{\sim} \quad \operatorname{Aut}_k(F_{\overline{v}}) \times \operatorname{Aut}_k(F_v)$$

of $(W^{ab}_{\widetilde{F}_{\overline{v}}} \times W^{ab}_{\widetilde{F}_v})$ with respect to the product $\operatorname{Aut}_k(F_{\overline{v}}) \times \operatorname{Aut}_k(F_v)$ of the automorphisms (semi)groups of the sets $F_{\overline{v}}$ and F_v of increasing transcendental extensions, we have the commutative diagram:

$$
\begin{array}{ccc}
W^{ab}_{\widetilde{F}_{\overline{v}}} \times W^{ab}_{\widetilde{F}_v} & \xrightarrow{\widetilde{\sigma}_{v_R} \times \widetilde{\sigma}_{v_L}} & G^{(k)}(\widetilde{F}_{\overline{v}} \times \widetilde{F}_v) \\
\downarrow \wr & & \downarrow \wr \\
\operatorname{Aut}_k(F_{\overline{v}}) \times \operatorname{Aut}_k(F_v) & \xrightarrow{\sigma_{v_R} \times \sigma_{v_L}} & G^{(k)}(F_{\overline{v}} \times F_v)
\end{array}
$$

where $G^{(k)}(\widetilde{F}_{\overline{v}} \times \widetilde{F}_v) \equiv \widetilde{M}_{v_R} \otimes \widetilde{M}_{v_L}$ and $G^{(k)}(F_{\overline{v}} \times F_v)$ is an abstract bisemivariety covered by the affine bisemigroup $G^{(k)}(\widetilde{F}_{\overline{v}} \times \widetilde{F}_v)$.

b) Let $\boldsymbol{G^{(k)}(F^{nr}_{\overline{v}} \times F^{nr}_v)}$ **be the abstract bisemivariety of dimension** $\boldsymbol{k}$ over the product $(F^{nr}_{\overline{v}} \times F^{nr}_v)$ of the sets $F^{nr}_{\overline{v}} = \{F^{nr}_{\overline{v}_1}, \ldots, F^{nr}_{\overline{v}_n}\}$ and $F^{nr}_v = \{F^{nr}_{v_1}, \ldots, F^{nr}_{v_n}\}$ of increasing pseudounramified (case $N = 1$) extensions.

Then, the kernel $\mathrm{Ker}(G^{(k)}_{F\to F^{nr}})$ of the map:

$$G^{(k)}_{F\to F^{nr}}: \qquad G^{(k)}(F_{\bar{v}}\times F_v) \quad\longrightarrow\quad G^{(k)}(F^{nr}_{\bar{v}}\times F^{nr}_v)$$

is the minimal (unitary) bilinear parabolic subsemigroup $P^{(k)}(F_{\bar{v}^1}\times F_{v^1})$ over the product of the sets $F_{\bar{v}^1}=\{F_{\bar{v}^1_1},\ldots,F_{\bar{v}^1_n},\ldots\}$ and $F_{v^1}=\}F_{v^1_1},\ldots,F_{v^1_n},\ldots\}$ of unitary Archimedean pseudoramified completions.

$G^{(k)}(F_{\bar{v}}\times F_v)$ acts on the unitary bilinear parabolic subsemigroup $P^{(k)}(F_{\bar{v}^1}\times F_{v^1})$ by conjugation [Pie5].

c) At every infinite Archimedean biplace $(\bar{v}_n\times v_n)$ corresponds **a conjugacy class** $\boldsymbol{cg^{(k)}_{v_{R\times L}}[n]}$ **of the abstract bisemivariety** $\boldsymbol{G^{(k)}(F_{\bar{v}}\times F_v)}$ whose number of representatives corresponds to the number of equivalent transcendental extensions of $F_{\bar{v}_n}\times F_{v_n}$.

The n-th conjugacy class representative $g^{(k)}_{v_{R\times L}}[n]$ is defined over n transcendental biquanta (a biquantum being the product of a right quantum by its symmetric left equivalent) in such a way that the number of biquanta in $g^{(k)}_{v_{R\times L}}[n]$ is

$$n_{\mathrm{biq}}(g^{(k)}_{v_{R\times L}}[n]) = n^k \ .$$

d) Let $G^{(k)}(F_{\bar{\omega}}\times F_\omega)$ denote the complex abstract bisemivariety which is a $\mathrm{GL}_{k/2}(F_{\bar{\omega}}\times F_\omega)$-bisemimodule $M_{\omega_R}\otimes M_{\omega_L}$ and the representation space of the bilinear semigroup of matrices $\mathrm{GL}_{k/2}(F_{\bar{\omega}}\times F_\omega)$ over the product $F_{\bar{\omega}}\times F_\omega$ of sets of complex pseudoramified increasing transcendental extensions.

Assume that each conjugacy class representative $g^{(k)}_{\omega_{R\times L}}[n]$ of $G^{(k)}(F_{\bar{\omega}}\times F_\omega)$ is unique and can be covered by m_n real conjugacy class representatives $\{g^{(k/2)}_{v_{R\times L}}[n,m_n]\}$ of $G^{(k/2)}(F_{\bar{v}}\times F_v)$, $1\le n\le t\le\infty$.

So, the complex bipoints of $G^k(F_{\bar{\omega}}\times F_\omega)$ are in one-to-one correspondence with the real bipoints of $G^k(F_{\bar{v}}\times F_v)$ and we have the inclusion:

$$G^{(k)}(F_{\bar{v}}\times F_v)/G^{(k/2)}(F_{\bar{v}}\times F_v) \quad\hookrightarrow\quad G^{(k)}(F_{\bar{\omega}}\times F_\omega)\ .$$

2.3 Cuspidal representation of complex (algebraic) bilinear semigroups

a) **Providing a cuspidal representation of the complex abstract bisemivariety $\boldsymbol{G^{(k)}(F_{\bar{\omega}}\times F_\omega)}$ consists in finding a cusp biform of dimension $\boldsymbol{k}$ on**

$G^{(k)}(F_{\overline{\omega}} \times F_{\omega})$ by summing the cuspidal subrepresentations of its conjugacy class representatives [Pie2].

Let then

$$\gamma^{T}_{F_{\omega_n}} : \quad F_{\omega_n} \longrightarrow F^{T}_{\omega_n} \qquad (\text{resp.} \quad \gamma^{T}_{F_{\overline{\omega}_n}} : \quad F_{\overline{\omega}_n} \longrightarrow F^{T}_{\overline{\omega}_n}) , \quad \forall\, n ,$$

be **the toroidal isomorphism** mapping each left (resp. right) complex transcendental extension F_{ω_n} (resp. $F_{\overline{\omega}_n}$) into its toroidal equivalent $F^{T}_{\omega_n}$ (resp. $F^{T}_{\overline{\omega}_n}$) which is a complex one-dimensional semitorus localized in the upper (resp. lower) half space.

b) **Each left (resp. right) function on the conjugacy class representative** $g^{(k)}_{\omega^T_L}[n] \in T^{(k)}(F^T_\omega) \subset G^{(k)}(F^T_{\overline{\omega}} \times F^T_\omega)$ (resp. $g^{(k)}_{\omega^T_R}[n] \in T^{(k)}(F^T_{\overline{\omega}})$) **is a function** (resp. cofunction) $\phi_L(T^k_L[n])$ (resp. $\phi_R(T^k_R[n])$), $g^{(k)}_{\omega^T_R}[n] \equiv T^k_L[n]$, **on the even k-dimensional real semitorus $T^k_L[n]$** (resp. $T^k_R[n]$) having the analytic development:

$$\phi_L(T^k_L[n]) = \lambda(k,n)\, e^{2\pi i n z_{Nk}} \qquad (\text{resp.} \quad \phi_R(T^k_R[n]) = \lambda^*(k,n)\, e^{2\pi i n z^*_{Nk}})$$

where:

- $z_{Nk} = z^{(1)}_N + z^{(2)}_N + \cdots + z^{(k)}_N$ (resp. $z^*_{Nk} = z^{*(1)}_N + z^{*(2)}_N + \cdots + z^{*(k)}_N$) is a complex (resp. conjugate complex) point of order $(k \times N)$ according to proposition 1.14;

- $\lambda(k,n) \times \lambda^*(k,n)$ is the product of the eigenvalues of the n-th coset representative of the product, right by left, of Hecke operators [Pie2].

c) **This left (resp. right) function $\phi_L(T^k_L[n])$ (resp. $\phi_R(T^k_R[n])$) constitutes the cuspidal representation $\Pi^{(n)}(g^{(k)}_{\omega_L}[n])$ (resp. $\Pi^{(n)}(g^{(k)}_{\omega_R}[n])$) of the n-th conjugacy class representative** of $G^{(k)}(F_\omega)$ (resp. $G^{(k)}(F_{\overline{\omega}})$) **in such a way that the cusp biform of $\mathrm{GL}_{k/2}(F_{\overline{\omega}} \times_D F_\omega)$ is given by the Fourier biseries**:

$$\Pi(\mathrm{GL}_{k/2}(F_{\overline{\omega}} \times_D F_\omega) = \overset{t}{\underset{n=1}{\oplus}}\, \Pi^{(n)}(g^{(k)}_{\omega_{R \times L}}[n])$$

$$= \left(\underset{n}{\Sigma}\, \lambda^*(k,n)\, e^{2\pi i n z^*_{Nk}} \right) \times_D \left(\underset{n}{\Sigma}\, \lambda(k,n)\, e^{2\pi i n z_{Nk}} \right)$$

where $\times_D$ is the diagonal product.

2.4 Cuspidal representation of real (algebraic) bilinear semigroups

A real cuspidal representation, covering the complex cuspidal representation $\Pi(\mathrm{GL}_{k/2}(F_{\overline{\omega}} \times_D F_\omega))$, can be obtained for the real diagonal bilinear semigroup $G^k(F_{\overline{v}} \times_D F_v)$ by summing the cuspidal subrepresentations $\Pi^{(n)}(g_{v_{R \times L}}^{(k)}[n, m_n])$ of its conjugacy class representatives $g_{v_{R \times L}}^{(k)}[n, m_n]$.

Every left (resp. right) function on the set of conjugacy class representatives $\{g_{v_L^T}^{(k)}[n, m_n]\}_{m_n}$ (resp. $\{g_{v_R^T}^{(k)}[n, m_n]\}_{m_n}$) is a function (resp. cofunction) $\psi_L(T_L^k[n, m_n])$ (resp. $\psi_R(T_R^k[n, m_n])$) on the k-dimensional real semitorus $T_L^k[n, m_n]$ (resp. $T_R^k[n, m_n]$) localized in the upper (resp. lower) half space, covered by m_n semitori of dimension $k/2$ and having the analytic development:

$$\psi_L(T_L^k[n, m_n]) = \underset{m_n}{\Sigma}\, \lambda(k/2, n, m_n)\, e^{2\pi i n m_n x_{Nk/2}}$$

$$(\text{resp.}\quad \psi_R(T_R^k[n, m_n]) = \underset{m_n}{\Sigma}\, \lambda^*(k/2, n, m_n)\, e^{-2\pi i n m_n x_{Nk/2}}\ )$$

where x_{Nk} is a real point of order $N \times (k/2)$.

The real cuspidal representation $\Pi(\mathrm{GL}_k(F_{\overline{v}} \times_D F_v)$ is then given by the k-dimensional global elliptic $\Pi(\mathrm{GL}_k(F_{\overline{v}} \times_D F_v)$-bisemimodule

$$\mathrm{ELLIP}_{R \times L}(k, n, m_n) = \underset{n}{\Sigma}\, \underset{m_n}{\Sigma}\, (\lambda^*(k/2, n, m_n)\, e^{-2\pi i n m_n x_{Nk/2}} \times_D (\lambda(k/2, n, m_n)\, e^{2\pi i n m_n x_{Nk/2}}\ ,$$

covering the k-dimensional cusp biform $\Pi(\mathrm{GL}_{k/2}(F_{\overline{\omega}} \times_D F_\omega))$ in the sense that

$$\mathrm{ELLIP}_{R \times L}(k, n, m_n) \quad \hookrightarrow \quad \Pi(\mathrm{GL}_{k/2}(F_{\overline{\omega}} \times_D F_\omega))\ .$$

2.5 (Bi)functor FLGC of the Langlands global correspondence(s)

Let **CABG** denote the bisemigroup category of algebraic bilinear semigroups $\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_\omega})$ given by

a) the algebraic bilinear semigroups $\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_\omega})$, the geometric dimension k varying;

b) the (bi)morphisms of projection

$$\mathrm{Hom}\left(\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_\omega}), \mathrm{GL}_{h/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_\omega})\right)\ ,\qquad k > h\ ,$$

sending $\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_\omega})$ into the algebraic bilinear semigroup $\mathrm{GL}_{h/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_\omega})$.

Similarly, let **CBCF denote the bisemigroup category of (complex) cuspidal representations** $\Pi(\mathbf{GL_{k/2}}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}}))$ of the algebraic bilinear semigroup $\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}})$ whose (bi)morphisms of projection are:

$$\mathrm{Hom}\left(\Pi(\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}})), \Pi(\mathrm{GL}_{h/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}}))\right) .$$

Then, **there exists a covariant (bi)functor FLGC associated with the Langlands global correspondences** [Pie2]:

$$\begin{array}{rcl}
\mathrm{FLGC}: & \mathrm{CABG} \longrightarrow & \mathrm{CBCF} \\
\\
& \mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}}) \longrightarrow & \Pi(\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}})) \\
\\
\mathrm{FLGC(Phom}_{k \to h}): & \Pi(\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}})) \longrightarrow & \Pi(\mathrm{GL}_{h/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}}))
\end{array}$$

which is a (bi)function assigning:

a) to each algebraic bilinear semigroup $\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}}) \in \mathrm{CABG}$ its cuspidal representation $\Pi(\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}})) \in \mathrm{CBCF}$;

b) to each morphism $\mathrm{Phom}_{k \to h}^{(\mathrm{alg})} : \mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}}) \to \mathrm{GL}_{h/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}})$ a morphism $\mathrm{FLGC(Phom}_{k \to h}^{(\mathrm{alg})}) : \Pi(\mathrm{GL}_{k/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}})) \to \Pi(\mathrm{GL}_{h/2}(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_{\omega}}))$.

2.6   Hecke eigenbivalues as Galois representation

Consider now the two-dimensional case $\mathbf{GL_2(F_{\overline{v}} \times F_v)}$.
The ring of the endomorphisms of the $\mathbf{GL_2(F_{\overline{v}} \times F_v)}$**-bisemimodule** $(\mathbf{M_{v_R}} \otimes \mathbf{M_{v_L}})$, decomposing it into the set of subbisemimodules $\{M_{F_{\overline{v}_{n,m_n}}} \otimes M_{F_{v_{n,m_n}}}\}_{n,m_n}$ or conjugacy class representatives $\{g_{v_{R \times L}}^{(2)}[n]\}_{n,m_n}$ (see section 2.2), according to the $(\mathbb{Z}/N\mathbb{Z})^2$-bisemilattices $(\Lambda_{\overline{v}_{n,m_n}}^{(2)} \otimes \Lambda_{v_{n,m_n}}^{(2)})$ [Pie1], **is generated over** $(\mathbb{Z}/N\mathbb{Z})^2$ **by the product** $(\mathbf{T_{n_R}} \otimes \mathbf{T_{n_L}})$ **of Hecke operators** T_{n_R} over T_{n_L} for $n \nmid N$ and by the product $(U_{n_R} \times U_{n_L})$ of Hecke operators U_{n_R} and U_{n_L} for $n \mid N$: it is noted $T_H(N)_R \otimes T_H(N)_L$ (weight two case).
The coset representative of U_{n_L} (resp. U_{n_R}), referring to the upper (resp. lower) half plane, can be chosen to be upper (resp. lower) triangular and given by the integral matrix $\left(\begin{smallmatrix} 1 & b_N \\ 0 & n_N \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} 1 & 0 \\ b_N & n_N \end{smallmatrix}\right)$) of the congruence subgroup $\Gamma_L(N)$ (resp. $\Gamma_R(N)$) in $\mathrm{GL}_2(\mathbb{Z})$ in such a way that:

- $n_N = * + n\,N$ where n_N is the number of transcendental subfields in the n-th conjugacy class representative $g_N^{(2)}[n]$;

- $b_N = * + b \cdot N$ refers to a phase shift;

where $*$ denotes an integer inferior to N.

The coset representative of $(U_{n_R} \times U_{n_L})$ will be given by

$$U_{n_R} \times U_{n_L} = \left[\begin{pmatrix} 1 & b_N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_N & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 0 & n_N^2 \end{pmatrix}$$

where $\begin{pmatrix} 1 & b_N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_N & 1 \end{pmatrix}$ is an element of the nilpotent group of matrices $u_2(b_N) \cdot u_2(b_N)^t$ and of the unimodular decomposition group $D_{n_N^2}$ acting on the split Cartan subgroup element $\alpha_{n_N^2} = \begin{pmatrix} 1 & 0 \\ 0 & n_N^2 \end{pmatrix}$.

$(U_{n_R} \times U_{n_L})$ then factorizes according to the bilinear Gauss decomposition into nilpotent and diagonal parts [Pie1].

Let $\boldsymbol{\lambda_+^2(n_N^2, b_N^2)}$ and $\boldsymbol{\lambda_-^2(n_N^2, b_N^2)}$ be the eigenvalues of $\boldsymbol{(D_{n_N^2; b_N^2} \times \alpha_{n_N^2})} \equiv \boldsymbol{GL_2(T_H(N)_R \otimes T_H(N)_L)}$ corresponding to the irreducible semisimple (pseudo)-ramified representation:

$$\rho_{\lambda_\pm^2}: \qquad \mathrm{Gal}(\widetilde{F}_{\overline{v}_n}/k) \times \mathrm{Gal}(\widetilde{F}_{v_n}/k) \qquad \longrightarrow \qquad \mathrm{GL_2}(T_H(N)_R) \otimes T_H(N)_L)$$

associated with a weight two cusp form in such a way that λ_+^2 and λ_-^2 verify [Pie1]:

$$\mathrm{trace}\ \rho_{\lambda_\pm^2} = 1 + b_N^2 + n_N^2 \ ,$$

$$\det \rho_{\lambda_\pm^2} = \lambda_+^2(n_N^2, b_N^2) \times \lambda_-^2(n_N^2, b_N^2) = n_N^2 \ .$$

Indeed, the eigenvalues $\lambda_\pm^2$ are:

$$\lambda_\pm^2 = \frac{(1 + b_N^2 + n_N^2) \pm [(1 + b_N^2 + n_N^2) - 4n_N^2]^{1/2}}{2}$$

and the characteristic polynomial of $\rho_{\lambda_\pm^2}$ has the form:

$$X^2 - \mathrm{trace}\ \rho_{\lambda_\pm^2} X + \det \rho_{\lambda_\pm^2} = 0 \ .$$

All that can be summarized in the commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Gal}(\widetilde{F}_{\overline{v}}/k) \times \mathrm{Gal}(\widetilde{F}_v/k) & \longrightarrow & \mathrm{GL_2}(F_{\overline{v}} \times F_v) \\
\downarrow & & \uparrow \\
\mathrm{GL_2}(T_H(N)_R \otimes T_H(N)_L) & \longrightarrow & \mathrm{GL_2}(F_R^+ \times F_L^+) \big/ \mathrm{GL_2}(\mathbb{Z}/N\mathbb{Z})^2)
\end{array}
$$

Note that, in the weight k case, the Cartan subgroup element $\alpha_{n_N^2} = \begin{pmatrix} 1 & 0 \\ 0 & n_N^2 \end{pmatrix}$ is then given by $\alpha_{n_{N,k}^2} = \begin{pmatrix} a_N^2 & 0 \\ 0 & d_N^{2k-2} \end{pmatrix}$ where $n_{N,k}^2 = \sum\limits_{d|n} a_N^2 \cdot d_N^{2k-2} \cdot \epsilon^2(d)$ denotes the number of transcendental quanta, i.e. $((k/2) \times n)^2$ referring to sections 1.8 and 2.2.

2.7 Two-dimensional cusp biforms and global elliptic bisemi-modules

The k-dimensional bilinear cuspidal representation $\Pi(\mathrm{GL}_{k/2}(F_{\overline{\omega}} \times_{(D)} F_\omega)$ of the complex bilinear semigroup $\mathrm{GL}_{k/2}(F_{\overline{\omega}} \times_D F_\omega)$ is a k-dimensional cusp biform whose Fourier biseries are given by:

$$\Pi(\mathrm{GL}_{k/2}(F_{\overline{\omega}} \times_{(D)} F_\omega) = \left(\sum_n \lambda^*(k,n) \, e^{2\pi i n z^*_{Nk}} \right) \times_D \left(\sum_n \lambda(k,n) \, e^{2\pi i n z_{Nk}} \right)$$

according to section 2.3.

The corresponding 2-dimensional ("Weil") cusp biform of weight k can be reached by considering the projective (bi)map:

$$CP_{(k)\to(2)}: \qquad \Pi(\mathrm{GL}_{k/2}(F_{\overline{\omega}} \times_D F_\omega) \quad \longrightarrow \quad {}^{(\omega)}f_k(z^*_N) \times_D {}^{(\omega)}f_k(z_N) \,,$$

introduced in proposition 1.14,

where ${}^{(\omega)}f_k(z^*_N) \times_D {}^{(\omega)}f_k(z_N)$ are given by

$$ {}^{(\omega)}f_k(z^*_N) \times_D {}^{(\omega)}f_k(z_N) = \left(\sum_n \lambda^*_k(2,n) \, e^{2\pi i n z^*_N} \right) \times_D \left(\sum_n \lambda_k(2,n) \, e^{2\pi i n z_N} \right)$$

where:

- $\lambda_k(2,n) \equiv [\lambda^2_{k_\pm}(n^2_N, b^2_N)]^{1/2}$ is the product of the radii of two orthogonal circles at $(k/2) \times n$ transcendental quanta referring to proposition 1.9 and section 2.6;

- z_N is a complex point of order $(N \times k)$ written now z_{N-k}.

Similarly, the real k-dimensional bilinear cuspidal representation $\Pi(\mathrm{GL}_k(F_{\overline{v}} \times_{(D)} F_v))$ of the bilinear semigroup $\mathrm{GL}_k(F_{\overline{v}} \times_{(D)} F_v)$ is a k-dimensional elliptic $\Pi(\mathrm{GL}_k(F_{\overline{v}} \times_{(D)} F_v))$-bisemimodule:

$$\mathrm{ELLIP}_{R \times L}(k,n,m_n) = \sum_n \sum_{m_n} (\lambda^*(k/2,n,m_n) \, e^{-2\pi i n m_n x_{Nk/2}} \times_D (\lambda(k/2,n,m_n) \, e^{2\pi i n m_n x_{Nk/2}}$$

which is a real analytic k-dimensional cusp biform.

The corresponding 2-dimensional global elliptic bisemimodule of weight k will be reached by the projective (bi)map:

$$CP^{\mathrm{ELLIP}}_{(k)\to(2)}: \qquad \mathrm{ELLIP}_{R \times L}(k,n,m_n) \quad \longrightarrow \quad \mathrm{ELLIP}_{k_{R \times L}}(2,n,m_n)$$

where:

$$\mathrm{ELLIP}_{k_{R \times L}}(2,n,m_n)$$
$$= \sum_n \sum_{m_n} (\lambda^*_{k/2}(1,n,m_n) \, e^{-2\pi i n m_n x_{N-k/2}} \times_D \lambda_{k/2}(1,n,m_n) \, e^{2\pi i n m_n x_{N-k/2}})$$

with:

- $\lambda_{k/2}^2(1, n, m_n)$ being eigenvalues of the coset representative $(U_{n_{Rk}} \times U_{n_{Lk}})$ of the product of Hecke operators where $n_{N;k}^2$, denoting the number of transcendental quanta, is now written according to:

$$n_{N;k}^2 / d^2 = a_{N;k}^2 = a_N^{2k-2} \ \epsilon^2(a_N^{2k-2})$$

referring to section 2.6.

So, $\lambda_{k/2}(1, n, m_n)$ corresponds to the radius of a semicircle equal to the respective coefficient $\lambda(k/2, n, m_n)$ of $\mathrm{ELLIP}_L(k, n, m_n)$ by considering proposition 1.14.

- x_N being a real point of order $(N \times k/2)$ written now $x_{N-k/2}$.

2.8 Proposition

The **2-dimensional elliptic bisemimodule** $\mathbf{ELLIP}_{k_{R \times L}}(2, n, m_n)$ *of weight* k, *which is a real analytic cusp biform,* **covers the corresponding cusp biform** $^{(\omega)}f_k(z_{N-k}^*) \times {}^{(\omega)}f_k(z_{N-k})$ *according to:*

$$\mathrm{ELLIP}_{k_{R \times L}}(2, n, m_n) \quad \overset{\sim}{\hookrightarrow} \quad {}^{(\omega)}f_k(z_{N-k}^*) \times_D {}^{(\omega)}f_k(z_{N-k})$$

in the sense that:

a) **every 2-dimensional semitorus**

$$\boldsymbol{T_L^2[n] = \lambda_k(2, n) \ e^{2\pi i n z_{N-k}}} \qquad (\text{resp.} \quad \boldsymbol{T_R^2[n] = \lambda_k^*(2, n) \ e^{2\pi i n z_{N-k}^*}})$$

of class "n" of $^{(\omega)}f_k(z_{N-k})$ **(resp.** $^{(\omega)}f_k(z_{N-k}^*)$**) is covered by** $\boldsymbol{m_n =} \underset{d|n}{\Sigma} \boldsymbol{d \cdot N \cdot nu}$ **semicircles**

$$\boldsymbol{T_L^1[n, m_n] = \lambda_{k/2}(1, n, m_n) \ e^{2\pi i n m_n x_{N-k/2}}}$$

$$(\text{resp.} \quad \boldsymbol{T_R^1[n, m_n] = \lambda_{k/2}^*(1, n, m_n) \ e^{-2\pi i n m_n x_{N-k/2}}})$$

at $(\frac{k}{2} \times a_{N;k})$ **transcendental quanta of** $\mathrm{ELLIP}_{k_L}(2, n, m_n)$ *(resp.* $\mathrm{ELLIP}_{k_R}(2, n, m_n)$*) localized in the upper (resp. lower) half plane, where* nu *is the number of nonunits of Galois extensions;*

b) **the parameter** $\boldsymbol{b_N}$ **of the nilpotent group of matrices** $\boldsymbol{u_2(b_N) \cdot u_2(b_N)^t}$ **is a "phase shift"** *in the first dimension of the considered* $(\mathbb{Z}/N\mathbb{Z})^2$ *-bisemilattice.*

When this phase shift "b_N" increases, the difference between the two eigenvalues $\lambda_{k/2_+}(1, n, m_n)$ *and* $\lambda_{k/2_-}(1, n, m_n)$ *of* $(U_{n_{R_k}} \times U_{n_{L_k}})$ *also increases but verifies* $\lambda_{k/2_+}^2(1, n, m_n) \times \lambda_{k/2_-}^2(1, n, m_n) = n_{N;k}^2$.

Proof:

a) As the n-th semitorus $T_L^2[n] = \lambda_k(2,n)\, e^{2\pi i n z_{N-k}}$ of the cusp form ${}^{(\omega)}f_k(z_{N-k})$ is generated by the product of two orthogonal semicircles at $(k/2 \times n)$ transcendental quanta whose product of radii is $\lambda_{k/2}(2,n)$ which is an eigenvalue of $(U_{n_{R_k}} \times U_{n_{L_k}})$ with $\alpha_{n_N^2} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & n_N^2 \end{smallmatrix} \right)$ and $n_{N;k}^2 = \sum\limits_{d|n} a_N^2 \cdot d_N^{2k-2}\, \epsilon^2(d)$, we have that the number "m_n" of semicircles $T_L^1[n, m_n] = \lambda_{k/2}(1, n, m_n)\, e^{2\pi i n_{m_n} x_{N-k/2}}$ covering $T_L^2[n[$ is the integer $d_{N \cdot nu} = \sum\limits_{d|n} d^{(k-1)} \cdot N \cdot nu$ since, in this real case, $a_{N;k}^2 = a_N^{2k-2} \cdot \epsilon^2(a_N^{2k-2})$.

b) It is easy to calculate that when the phase shift $b_N = b \cdot N$ increases, $\left| \lambda_{k/2_+}(1, n, b_N) - \lambda_{k/2_-}(1, n, b_N) \right|$ also increases, reflecting a deformation of one radius $\lambda_{k/2_+}(1, n, b_N)$ with respect to the order $\lambda_{k/2_-}(1, n, b_N)$. ∎

2.9 Proposition (Local curvature of a torus)

Let

$$\{ T_L^1(n, m_n) = \lambda_{k/2}(1, n, m_n)\, e^{2\pi i n_{m_n} x_{N-k/2}} \}_{m_n=1}^{d \cdot N \cdot nu}$$

be the set of semicircles covering the n-th two-dimensional semitorus $T_L^2[n] \in {}^{(\omega)}f_k(z_{N-k})$.
Then, **the increase of the length of the semicircles with respect to m_n depends on the twisting of "opposite" semicircles in function of their curvature leading to degenerate singularities** *of fold type (or possible of cusp type) on them.*
The **blowups of the versal deformations of these singularities in codimension 1 and 2 are then sent on opposite semicircles increasing then their lengths.**

Proof: Assume that the semitorus $T_L^2[n]$ is generated from a cylinder covered by $d \cdot N \cdot nu$ line segments having the same length, i.e. by "d" transcendental quanta.
Then, by bending this cylinder in order to get a semitorus, degenerate singularities of fold type are generated on the most bent line segments.
Let $f(x) = x^3$ be one of these fold singularities.
Its versal unfolding in codimension 1 is

$$F(x, a_1) = x^3 + a_1\, x^1 .$$

Possible degenerate singularities of cusp type $f(x) = x^4$ can be generated of which versal unfolding in codimension 2 are

$$F(x, a_1, a_2) = x^4 + a_1\, x^1 + a_2\, x^2 .$$

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Blowups of these versal deformations, consisting in the extensions of their quotient algebras, are introduced in [Pie7]. They are smooth endomorphisms based on Galois antiautomorphisms disconnecting the monomials $x^1 \in F(x, a_1)$ and $x^1, x^2 \in F(x, a_1, a_2)$ which are then sent on the opposite least bent line segments. ∎

This dynamical process is then responsible for the transition from the global euclidean geometry of the semitorus to local hyperbolic and spherical geometries [Pie12].

2.10   Proposition

1) Let

$$T_{\mathbb{Z}_N}^{(2)}(H) = T_2(I\!R)\big/T_2(\mathbb{Z}, N\mathbb{Z})$$

be the set of $I\!R^2\big/(\mathbb{Z}/N\mathbb{Z})^2$ -sublattices of the Poincare upper half plane H .

Let $g_{\mathbb{Z}_N}^{(2)}[n]$ denote the n -th coset representative of $T_{\mathbb{Z}_N}^{(2)}(H)$ whose fundamental domain is $D_{nN,k}^2$ with respect to the sublattices $\Lambda_{\mathbb{Z}_N}^{(2)}[n]$.

Let $f_k(z_{N-k})$ be a cusp form of weight k and level N which is:

a) invariant under a congruence subgroup $\Gamma(N)$ of $SL_2(\mathbb{Z})$;

b) periodic and expanded in Fourier series $f_k(z_{N-k}) = \sum\limits_{n=0}^{\infty} c_{n_k}\, q_{N-k}^n$;

c) eigenfunction of Hecke operators $T_k(n; N)$ acting on the function $F_k(T_{\mathbb{Z}_N}^{(2)}(H))$ on $T_{\mathbb{Z}_N}^{(2)}(H)$ according to a complete set of upper triangular coset representatives of $\Gamma(N)$.

2) Let

$$\left(T_1(F_\omega) = T_1(F_L)\Big/T_1(\mathbb{Z}/N\mathbb{Z})\right) \simeq \left(T_2(F_v) = T_2(F_L^+)\Big/T_2(\mathbb{Z}/N\mathbb{Z})\right)$$

be the set of $F_\omega\big/(\mathbb{Z}/N\mathbb{Z})^2 \simeq (F_v)\big/(\mathbb{Z}/N\mathbb{Z})^2$ -(sub)lattices of Galois on the Poincare upper half plane H .

Let $D_{F_{\omega n},N;k}^{(2)}$ (resp. $D_{F_{v n},N;k}^{(2)}$) be the fundamental unitary domain of the n -th conjugacy class representative $g_{\omega_L}^{(2)}[n] \in T_1(F_\omega)$ (resp. $g_{v_L}^{(2)}[n] \in T_2(F_v)$) with respect to the $(\mathbb{Z}/N\mathbb{Z})^2$ -sub(semi)lattice $\Lambda_{\omega n}^{(2)}$ (resp. $\Lambda_{v n}^{(2)}$).

Let $^{(\omega)}f_k(z_{N-k})$ be a "Weil" cusp form of weight k and level N on $T_1(F_\omega)$ or on $T_2(F_v)$ which is:

a) invariant under a suitable congruence subgroup $\Gamma(N)$;

b) periodic and expanded in Fourier series $^{(\omega)}f_k(z_{N-k}) = \sum\limits_{n} \lambda_k(2,n) \; e^{2\pi i n z_{N-k}} ;$

c) eigenfunction of the Hecke operator $T_{H-k}(N)$ *according to section 2.6.*

If the Weil unitary fundamental domain $D^{(2)}_{F_{\omega_n};N;k}$ (covered by $D^{(2)}_{F_{v_n};N;k}$) covers the fundamental classical domain $D^{(2)}_{nN,k}$, then the Weil cusp form $^{(\omega)}f_k(z_{N-k})$ of weight k and level N can be identified with the classical cusp form $f_k(z_{N-k})$ of weight k and level N in such a way that:

$$^{(\omega)}f_k(z_{N-k}) \approx f_k(z_{N-k}) ,$$

the weight k referring to a geometric dimension while the level N is an algebraic dimension related to the degree of extension of transcendental quanta: this constitutes the main objective of the Langlands program.

Proof: As the cusp form $f_k(z_{N-k})$ of weight k originates from a k-dimensional cusp form $\phi^{(k)}(z_{N_k})$ according to proposition 1.14, the two-dimensional fundamental domain $D^{(2)}_{nN;k}$ of weight k is given by:

$$D^{(2)}_{nN;k} = \left(D^{(2)}_{nN;k=2} \right)^{k/2}$$

where $D^{(2)}_{nN;k=2}$ is the corresponding two-dimensional fundamental domain of weight 2 expanded by a power $k/2$.

Similarly, let $D^{(2)}_{F_{\omega_n};N;k=2}$ be the two-dimensional Weil unitary fundamental domain of weight 2: its surface $S(D^{(2)}_{F_{\omega_n};N;k})$ is the square of the one transcendental quantum of level N;

$$S(D^{(2)}_{F_{\omega_n};N;k=2}) = N^2 \times (nu)^2$$

where nu is the number of nonunits.

So, the two-dimensional unitary domain $D^{(2)}_{F_{\omega_n};N;k}$ of weight k is:

$$D^{(2)}_{F_{\omega_n};N;k} = \left(S(D^{(2)}_{F_{\omega_n};N;k=2}) \right)^{k/2} = \left(N^2 \times (nu)^2 \right)^{k/2} .$$

Now, assume by hypothesis that:

$$D^{(2)}_{F_{\omega_n};N;k} \simeq D^{(2)}_{\omega_n;N;k} .$$

By periodicity of cusp forms, the fundamental unitary domains are the same in all coset representatives of $T^{(2)}_{\mathbb{Z}_N}(H)$ which is covered by $T_1(F_\omega) \simeq T_2(F_v)$.

The unitary fundamental classical (resp. Galois or Weil) domain(s) $D^{(2)}_{nN;k}$ (resp. $D^{(2)}_{F_{\omega_n};N;k}$) generate(s), under the action of the Hecke operator having a representation in the upper triangular group of matrices $T_2(\mathbb{Z}/N\mathbb{Z})$ (or $\Gamma(N)$), the classes of automorphisms in $T_2(\mathbb{R})$ (resp. $T_1(F_\omega)$ or in $T_2(F_v)$) according to the $(\mathbb{Z}/N\mathbb{Z})^2$-sublattices in the Poincare upper half plane H.

As the $(\mathbb{Z}/N\mathbb{Z})^2$-lattice is the same in the classical and Weil (or Galois) case, the classical and Galois two-dimensional cusp forms of weight k and level N correspond, and, thus, we have:

$$^{(\omega)}f_k(z_{N-k}) \approx f_k(z_{N-k}) \ . \qquad\blacksquare$$

3 Zeta functions, Theta series, weak Maass forms, Mock Theta functions and the Tau function

This chapter will be devoted to a generalization of classical (or Weil) cusp forms in order to take into account Maass forms and Mock modular forms.

But, first, the generation of

    a) L-series from cusp forms [Hec1] by a globally nonperiodic transform map,

    b) theta series from modular forms with integer weights

will be introduced.

3.1 Proposition (L-functions as nonperiodic transforms of cusp forms)

Let

$$f_k(z_{N-k}) = \sum_{n=1}^{\infty} c_{nk}\ q_{N-k}^n \qquad (resp. \quad f_k(z_{N-k}^*) = \sum_{n=1}^{\infty} c_{nk}^*\ q_{N-k}^{*n}\ ), \quad q_{N-k}^n = e^{2\pi i n z_{N-k}}\ ,$$

be a left (resp. right dual) cusp form of weight k and level N as developed in sections 1.11 to 1.14 and 2.6 to 2.8.

Let

$$s_+ = \sigma + i\tau \qquad (resp. \quad s_- = \sigma - i\tau\ )$$

be a complex variable conjugate to $z = x + iy$ (resp. $z^* = x - iy$).

Then, there exists a linear continuous map (Mellin transform):

$$\Phi_L : f_k(z_{N-k}) \longrightarrow L(f_k, s_{N-k_+}) \qquad (resp. \quad \Phi_R : f_k^*(z_{N-k}^*) \longrightarrow L(f_k^*, s_{N-k_-})\ )$$

in such a way that the $\boldsymbol{L}$-function $\boldsymbol{L(f_k, s_{N-k_+})}$ (resp. $\boldsymbol{L(f_k^*, s_{N-k_-})}$) be the nonperiodic transform of $\boldsymbol{f_k(z_{N-k})}$ (resp. $\boldsymbol{f_k^*(z_{N-k})}$) given by:

$$L(f_k, s_{N-k_+}) = \sum_{n=1}^{\infty} \int f_k^{(n)}(z_{N-k})\ e^{-2\pi i n z_{N-k}} \cdot n^{-s_{N-k_+}}\ dz_{N-k}$$

$$= \sum_{n=1}^{\infty} c_{n_k}\ n^{-s_{N-k_+}}$$

$$(resp. \quad L(f_k^*, s_{N-k_-}) = \sum_{n=1}^{\infty} \int f_k^{(n)}(z_{N-k}^*)\ e^{-2\pi i n z_{N-k}^*} \cdot n^{-s_{N-k_-}}\ dz_{N-k}$$

$$= \sum_{n=1}^{\infty} c_{n_k}^*\ n^{-s_{N-k_-}}\ )$$

and corresponding to the Mellin transform

$$L(f_k, s_{N-k_+}) = (2\Pi)^{s_+} \ \Gamma(s_+)^{-1} \int_0^\infty (-i \ z_{N-k})^{s_+} \ f_k(z_{N-k} \ \frac{dz_{N-k}}{z_{N-k}}$$

where:

- $$f_k^{(n)}(z_{N-k}) \ = c_{nk} \ q_{N-k}^n = c_{nk} \ e^{2\pi i n z_{N-k}}$$

  (resp. $f_k^{(n)}(z_{N-k}^*) \ = c_{nk}^* \ q_{N-k}^{*n} = c_{nk}^* \ e^{2\pi i n z_{N-k}^*}$ *)*

  is the n *-th term of* $f_k(z_{N-k})$ *(resp.* $f_k(z_{N-k}^*)$ *);*

- z_{N-k} *(resp.* z_{N-k}^* *) is a point of order* $(N \times k)$ *conjugate to the point* s_{N-k_+} *(resp.* s_{N-k_-} *) of order* $(N \times k)$ *.*

Proof: Referring to the literature of L-functions, we must admit that s_+ (resp. s_-) is conjugate to z (resp. z^*) in the sense that $s_+ \approx \frac{1}{z}$ (resp. $s_- \approx \frac{1}{z^*}$) is a complex inverse space variable restricted to the upper (resp. lower) half plane.

s_{N-k_+} (resp. s_{N-k_-}) is a complex point of order $(N \times k)$, usually called period $(N \times k)$.

ϕ_L (resp. ϕ_R) is the nonperiodic equivalent of the Fourier transform. Indeed, if ϕ_L (resp. ϕ_R) was globally periodic, $L(f_k, s_{N-k_+})$ (resp. $L(f_k^*, s_{N-k_-})$) would be given by:

$$L(f_k, s_{N-k_+}) = \sum_{n=1}^\infty \int f_k^{(n)}(z_{N-k}) \ e^{-2\pi i n z_{N-k}} \ dz_{N-k} = \sum_n c_{nk}$$

(resp. $\quad L(f_k^*, s_{N-k_-}) = \sum_{n=1}^\infty \int f_k^{(n)}(z_{N-k}^*) \ e^{-2\pi i n z_{N-k}^*} \ dz_{N-k} = \sum_n c_{nk}^*) \ .$

The factor $n^{-s_{N-k_+}}$ **(resp.** $n^{-s_{N-k_-}}$ **) is thus responsible for the global nonperiodicity of** ϕ_L **(resp.** ϕ_R **)** which can be analyzed by the Poisson formula [E-K].

$f_k(z_N)$ is a periodic function of period $T = D_{nk}^{(2)}$ (resp. $T_N = D_{nNk}^{(2)}$), being a unitary fundamental domain according to section 1.3 and proposition 1.15, which can be expressed generally as [Rod]:

$$f_k(z_{N-k}) = f_k(z_{N-k}) \star \sum_{n=1}^\infty \delta(z_{N-k} - nNkT)$$

$$= \sum_n f_k(z_{N-k} - nNkT) \ ,$$

the period T being explicitly written.

Its Fourier transform is:

$$\frac{1}{T}\widehat{f}_k(\nu_{N-k}) \sum_n \delta\left(\nu_{N-k} - \frac{nNk}{T}\right) = \frac{1}{T} \sum_n \widehat{f}\left(\frac{nNk}{T}\right) \delta\left(\nu_{N-k} - \frac{nNk}{T}\right) ,$$

ν_{N-k} having the "periodic" variable conjugate to z_{N-k}, i.e. a series of Dirac equidistant distributions with interval $\frac{1}{T}$.

By taking into account the inverse Fourier transform, we get:

$$\sum_n f_k(z_N - nNkT) = \frac{1}{T} \sum_n \widehat{f}_k\left(\frac{nNk}{T}\right) e^{2\pi i n z_{N-k}} .$$

That can be summarized by the Fourier transform

$$\sum_n \delta\left(z_{N-k} - nNkT\right) \quad \xrightarrow{\text{FT}} \quad \sum_n \delta\left(T\nu_{N-k} - nNk\right) \simeq \frac{1}{T} \sum_n \left(\nu_{N-k} - \frac{nNk}{T}\right)$$

of Dirac distributions which is periodic [Rod] while the nonperiodic transform Φ_L would lead to:

$$\sum_n \delta\left(\nu_{N-k} - nNkT\right) \quad \xrightarrow{\Phi_L} \quad \sum_n \frac{1}{T^{(n)}} \beta\left(s_{N-k_+} - \frac{nNk}{T^{(n)}}\right)$$

where $\beta\left(s_{N-k_+} - \dfrac{nNk}{T^{(n)}}\right)$ would be the nonperiodic equivalent of the Dirac distribution $\delta\left(\nu_{N-k} - \dfrac{nNk}{T}\right)$.

The nonperiodicity of Φ_L (resp. Φ_R) can then be evaluated from the map of the unique period T of $f_k(z_{N-k})$ (resp. $f_k(z^*_{N-k})$), or more exactly T_N associated with one space quantum, to the set $\left\{1/T^{(n)}\right\}_{n=1}^{\infty}$ of inverse periods, i.e. unit periods, or more exactly $\left\{1/T^{(n)}_{N-k}\right\}_{n=1}^{\infty}$ associated with the energies of one space quantum on the different packets "n" [Pie8]. ∎

3.2 Corollary

If $k = 2$ and $N = 1$ (pseudounramified case), let the weight 2 cusp biform $f_2(z^*_{N=1}) \otimes_D f_2(z_{N=1})$ be reduced to the double global elliptic bisemimodule

$$\text{ELLIP}_{R\times L}(1,n) = \sum_n \left(2\lambda^{nr}(1,n) \, e^{-2\pi i n x} \otimes_D 2\lambda^{nr}(1,n) \, e^{2\pi i n x}\right)$$

where, according to sections 2.7 and [Pie1]:

$$\lambda^{nr}(1,n) = 4n^2 , \qquad x \in \mathbb{R} .$$

Then, the nonperiodic map

$$\Phi_R \otimes_D \Phi_L : \qquad \text{ELLIP}_{R\times L}(1,n) \quad \longrightarrow \quad \zeta_R(s_-) \otimes_D \zeta_L(s_+)$$

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sends the global elliptic bisemimodule $\mathrm{ELLIP}_{R \times L}(1, n)$ *into the (diagonal) product* $\zeta_R(s_-)$
$\otimes_D \zeta_L(s_+)$ *of the classical zeta functions:*

$$\zeta_R(s_-) = \sum_n n^{-s_-} \,, \qquad s_- = \sigma - i\tau \,, \qquad \text{and} \qquad \zeta_L(s_+) = \sum_n n^{-s_+} \,, \qquad s_+ = \sigma + i\tau \,.$$

The nonperiodicity of Φ_L (resp. Φ_R) can then be evaluated from the Gaussian distribution of the consecutive spacings $\delta\gamma_n = \gamma_{n+1} - \gamma_n$ between the nontrivial zeros of $\zeta_R(s_-)$, $\zeta_L(s_+)$ and $\zeta(s)$.

Proof: Referring to [Pie8], the kernel $\mathrm{Ker}[\Phi_R \otimes_D \Phi_L]$ of $(\Phi_R \otimes_D \Phi_L)$ maps the product, right by left, of degeneracies of $(2\lambda^{nr}(1, n) \, e^{-2\pi i n x} \otimes_D 2\lambda^{nr}(1, n) \, e^{2\pi i n x})$ into the product "$4n^2$" of the trivial zeros of $\zeta_R(s_-)$ and $\zeta_L(s_+)$. And, these products of trivial zeros are mapped into the products of the corresponding pairs of nontrivial zeros under the action of $D_{4n^2; i^2} \, \epsilon_{4n^2}$ which is a coset representative of the Lie bisemialgebra of the decomposition (bisemi)group restricted to $4n^2$: this corresponds to the solution of the Riemann hypothesis [Pie1].

So, **the consecutive spacings $\delta\gamma_n = \gamma_{n+1} - \gamma_n$ between the nontrivial zeros of $\zeta(s)$ are the energies of one free transcendental quantum in subsemilattices of $(n + 1)$ quanta as proved in [Pie8].**

As the energies $\delta\gamma_n$ vary from one level "n" to another, for example "$n + j$" the zeta functions $\zeta_R(s_-)$ and $\zeta_R(s_+)$ are non periodic, which is also the case for the map $\Phi_R \otimes_D \Phi_L$. ∎

3.3 Theta series

Let $Q(r) = \langle n, n \rangle = \sum_{i=1}^{k} \sum_n a_{nn}^{(i)} \, n_i^2$ be a $\mathbb{Z}$-valued positive definite quadratic form on $\mathbb{Z}^k$ reduced to its diagonal form which is always possible.

As envisaged classically [Ser], the diagonal terms $a_{nn}^{(i)}$ of the matrix A of rank $(t \times k)$, $1 \le n \le t \le \infty$, $1 \le i \le k$, are equal to 1 and the $n_i \in \mathbb{Z}$ are the generators of $\mathbb{Z}$-subsemilattices $\{\Lambda_{\mathbb{Z}}^{(2)}[n]\}$ in the Poincare upper half plane.

From the quadratic form $Q(n)$, **we introduce the theta series $\Theta(z_{N-k})$ by:**

$$\Theta(z_N) = \sum_i \sum_n (q_N)^{\sum_i n_i^2} \,, \qquad\qquad 1 \le n \le t \,, \quad 1 \le i \le k \,,$$

$$= \sum_{i=1}^{k} \sum_n e^{2\pi i \left(\sum_i n_i^2 z_N \right)} \,, \qquad z_N \text{ being a point of period } N \,,$$

$$= \sum_n e^{2\pi i n^2 z_{N-k}} \,, \qquad\qquad z_{N-k} \text{ being a point of period } (N \times k) \,,$$

or, more generally as suggested by D. Zagier in [Zag]:

$$\Theta_{a,b}(z_{N-k}) = \sum_{n+a} e^{i\pi i(n+a)^2 z_{N-k}} \sum_{i=1}^{k} e^{2\pi i b(n_i^2 + a^2)} \ , \qquad a,b \in \mathbb{Q}^k \ .$$

3.4 Theta series as real analytic modular forms

Let $\Theta_{n^2 k/2}(z_{N-k}) = \sum_n d_{nk/2} \ e^{2\pi i n^2 z_{N-k}}$ be a theta series introduced from the quadratic form

$$Q(n) = \sum_{i=1}^{k} \sum_{n} a_{nn}^{(i)} \ n_i^2 \ .$$

Then, $\Theta_{n^2 k/2}(z_{N-k})$ is a real analytic form of weight $k/2$, in one-to-one correspondence with the weight k modular form $f_k(z_{N-k}) = \sum_n c_{nk} \ q_{N-k}^n$.

Proof: Let

$$f_k(z_{N-k}) = \sum_{n=1}^{\infty} c_{nk} \ q_{N-k}^n$$

be a (left) cusp form of weight k and level N such that, in $q_{N-k} = e^{2\pi i z_{N-k}}$, z_{N-k} is a point of order $(N \times k)$ according to proposition 1.15.

Assume that the theta series $\Theta_{n^2 k/2}(z_{N-k})$ originates from the cusp form $f_k(z_{N-k})$ by the map:

$$f\Theta_{kN} : \qquad f_k(z_{N-k}) \quad \longrightarrow \quad \Theta_{n^2 k/2}(z_{N-k})$$

in such a way that **the two-dimensional semitori $c_{nk} \ q_{N-k}^n$, of which generators are two semicircles at n transcendental quanta** (see proposition 1.9), **are sent into semicircles $d_{nk/2} \ e^{2\pi i n^2 x_{N-k/2}}$ at n^2 transcendental quanta by the bijective maps:**

$$T - c_{nk} : \qquad c_{nk} \ e^{2\pi i n z_{N-k}} \quad \longrightarrow \quad d_{nk/2} \ e^{2\pi i n^2 x_{N-k/2}} \ , \qquad \forall \ n \ ,$$

at the condition that the complex point of order $(k \times N)$ be sent into the real point $x_{N-k/2}$ of order $\left(\dfrac{k}{2} \times N\right)$, i.e. that, in $z_{N-k} = x_{N-k/2} + iy_{N-k/2}$, $iy_{N-k/2} \to 0$ which is suggested in [D-S].

As $f_k(z_{N-k})$ is a modular form of weight k, generated from the k-dimensional geometric cusp form $\Pi(\mathrm{GL}_{k/2}(F_\omega))$ according to sections 2.7 and 2.3, and as the theta series $\Theta_{n^2 k/2}(z_{N-k})$ results from $f_k(z_{N-k})$ by the isomorphic maps $\{T - c_{nk}\}_n$ stretching two-dimensional semitori into semicircles, $\Theta_{n^2 k/2}(z_{N-k})$ is a modular form of weight $k/2$ since the maps $T - c_{nk}$ divide the geometric dimension by two. ∎

3.5 Proposition

Let $f_k(z_{N-k}) = \sum_n c_{nk}\, q_{N-k}^n$ *be a cusp form of weight* k *and level* N.
Let

$$\Theta_{n^2 k/2}(z_{N-k/2}) = \sum_n d_{nk/2}\, e^{2\pi i n^2 z_{N-k/2}}$$

be a theta series generated from $f_k(z_{N-k})$ *by the bijective maps* $\{T - c_{nk}\}_n$.
Let

$$\mathrm{ELLIP}_{k/2}(2, n, m_n) = \sum_n \sum_{m_n} \lambda_{k/2}(1, n, m_n)\, e^{2\pi i n m_n x_{N-k/2}}$$

be the global elliptic semimodule covering the cusp form $f_k(z_{N-k})$.
Then, **we have the following commutative diagram***:*

in such a way that $\Theta\mathrm{EL}_{kN} = f\Theta_{kN} \circ f\mathrm{EL}_{kN}^{-1}$ *be a bijective map.*

Proof: The map

$$\Theta\mathrm{EL}_{kN}: \qquad \Theta_{n^2 k/2}(z_{N-k/2}) \quad \longrightarrow \quad \mathrm{ELLIP}_{k/2}(2, n, m_n)$$

sends semicircles of $\Theta_{n^2 k/2}(z_{N-k/2})$ at $\left(\frac{k}{2} \times n\right)^2$ transcendental quanta into semicircles of $\mathrm{ELLIP}_{k/2}(2, n, m_n)$ at $\left(n \times \frac{k}{2}\right)$ transcendental quanta.
As $f\mathrm{EL}_{kN}$ and $f\Theta_{kN}$ are bijective maps, $\Theta\mathrm{EL}_{kN}$ is clearly bijective, reorganizing the transcendental quanta of the theta series $\Theta_{n^2 k/2}(z_{N-k/2})$ according to the underlying $\mathbb{Z}/N\mathbb{Z}$-subsemilattices of $\Theta_{n^2 k/2}(z_{N-k/2})$ and $\mathrm{ELLIP}_{k/2}(2, n, m_n)$:

3.6 Locally compact global elliptic semimodule

Assume that the compact two-dimensional global elliptic semimodule $\mathrm{ELLIP}_{1_L}(2, n, m_n)$ of weight 1 and level N covering the weight 2 cusp form $f_2(z_{N-2})$ is transformed by the map:

$$C - Lc_{\mathrm{ELL}_2}: \qquad \mathrm{ELLIP}_{1_L}(2, n, m_n) \quad \longrightarrow \quad \mathrm{ELLIP}_{1_L}^{(Lc)}(1, n, m_n)$$

into the locally compact one-dimensional global elliptic semimodule $\mathrm{ELLIP}_{1_L}^{(Lc)}(1, n, m_n)$ in such a way that the m_n- semicircles $\{\lambda_{1_L}(2, n, m_n)\, e^{2\pi i n m_n x_{N-2}}\}_{m_n}$ of $\mathrm{ELLIP}_{1_L}(2, n, m_n)$, covering compactly the n-th semitorus of $f_2(z_{N-2})$, are no more connected.

Thus, $\mathrm{ELLIP}_{1_L}^{(Lc)}(1, n, m_n)$ becomes one-dimensional and its weight, related to the geometric dimension, can be considered as being $1 \cdot \mathrm{ELLIP}_{1_L}^{(Lc)}(1, n, m_n)$ is then a modular form of weight 1 and will be written according to $\mathrm{ELLIP}_{1_L}^{(Lc)}(1, n, m_n)$.

So, we have the commutative diagram:

3.7   Proposition (The Theta series of weight $1/2$)

$$\Theta_{n^2 1/2}(z_{N-1/2}) = \sum_n d_{n 1/2}\, e^{2\pi i n^2 x_{N-1/2}}$$

is generated either from the weight 2 modular form $f_2(z_{N-2})$ or from the weight 1 real analytic modular form $\mathrm{ELLIP}_{1_L}^{(Lc)}(1, n, m_n)$.

Proof: This results directly from the commutative diagram of section 3.6 by taking into account that the locally compact modular form of weight 1: $\mathrm{ELLIP}_{1_L}^{(Lc)}(1, n, m_n)$ covers the weight 2 cusp form $f_2(z_{N-2})$. ∎

3.8   Remark (Classical approach of modular forms of weight $1/2$)

This way of generating modular forms of weight $1/2$ generalizes the classical approach of G. Shimura [Shi] and J.P. Serre [Ser] in which it is proved that every modular form of weight $1/2$ on $\Gamma_1(N)$ is a linear combination of theta series with characters.

3.9   Weak Maass forms: a quick review

Maass forms [Maa] of weight 2, now called generally weak Maass forms, are functions $f_2^{WM}(z)$, on the Poincare upper half plane, equipped with the Riemannian metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$ and being modular under the action of $SL(2, \mathbb{Z})$ implying that

$$f_2^{WM}\left(\frac{az + b}{cz + d}\right) = f_2^{WM}(z), \qquad \forall\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z}).$$

$F_2^{WM}(z)$ has a Fourier series expansion of the form [Boo]:

$$f_2^{WM}(z) = \sum_{n=1}^{\infty} \lambda(n) \sqrt{y} \, K_{ir}(2\pi ny) \, \cos(2\pi nx) \,, \qquad z = x + iy \,,$$

where $K_{ir}(2\pi ny)$ is the classical Bessel function, and is an eigenfunction of the hyperbolic Laplace operator $\Delta = y^2 \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right)$ having eigenvalue $\frac{1}{4} + r^2$.

The L-function associated with $f_2^{WM}(z)$ is the series $L(f_2^{WM}, s) = \sum_{n=1}^{\infty} \lambda(n) \, n^{-s}$ converging for $\mathrm{Re}(s) > 1$.

A **weak Maass form of weight k on a subgroup $\Gamma \subset \Gamma_0(4)$** verifies [Ono1]:

$$f_k^{WM}(Az) = \left(\frac{c}{d} \right)^{2k} \epsilon_d^{-2k}(cz+d)^k \, f_k^{WM}(z) \qquad \text{for } A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$$

where $\left(\frac{c}{d} \right)$ is the extended Legendre symbol
and **has a Fourier expansion of the type**:

$$f_k^{WM}(z) = \sum_{n=n_0}^{\infty} \gamma(f, n; y) \, q^{-n} + \sum_{n=n_1}^{\infty} a(f, n) \, q^n \,, \qquad q = e^{2\pi iz} \,,$$

where $\gamma(f, n; y)$ are functions in y and $a(f, n)$ are complex numbers. $\sum_{n=n_0}^{\infty} \gamma(f, n; y) \, q^n$ **refers to the nonholomorphic part of $f_k^{WM}(z)$ while** $\sum_{n=n_1}^{\infty} a(f, n) \, q^n$ **is its holomorphic part**.

As no explicit examples of the Maass forms are known at this day [Boo], **a geometric interpretation of weak Maass forms will be given to them in terms of modular curves**.

A first elementary fundamental remark is that, in the Fourier series expansion $f_2^{WM}(z)$ of a weak Maass form, **the "imaginary" dimension "y" is lowered with respect to the dimension "x"** in $z = x + iy$.

Taking into account that the Fourier series expansion of a modular curve decomposes into two-dimensional semitori $\{T_n^2(a_{nm|d^2}, d)\}_{n=1}^{t}$, $t \leq \infty$, generated by two orthogonal semicircles referring to propositions 1.9, 1.10 and 2.8, we are led to the following proposition.

3.10 Proposition (Geometric interpretation of weak Maass forms)

The series expansion $f_2^{WM}(z)$ of a weak Maass form consists of a sum of two-dimensional semitori $\{T_n^{2,(e\ell)}(S_{a_n}^1 \times e\ell_{ef,n}^1)\}_{n=1}^{t}$, $t \leq \infty$, with elliptic cross sections $e\ell_{ef,n}^1$ in such a way that:

- *the elliptic section $e\ell_{ef,n}^1$ is a bijective deformation of the semicircle $S_{d_n}^1$ generator of the n-th semitorus $T_n^2(a_{nm|d^2}, d) \simeq S_{a_n}^1 \times S_{d_n}^1$ of the modular curve (cusp form);*

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- *the equation of the ellipse $e\ell^1_{ef,n}$ corresponding to the circle $S^1_{d_n}$ having radius $r_{S^1_{d_n}}$ is given by:*

$$z_{e\ell^1_n} = r_{S^1_{d_n}} \left(f \cos 2\pi(iy) + ie \sin 2\pi(iy) \right)$$

where e and f are the half lengths of the axis of the ellipse $e\ell^1_{ef,n}$.

Proof: As the imaginary dimension "y" is lowered with respect to the dimension "x", the semicircular sections $\mathbf{S^1_{d_n}}$ of the two-dimensional semitori $\mathbf{T^2_n(a_{nm|d^2}, d)}$ are transformed into semielliptic sections $\mathbf{e\ell^1_{df,n}}$ according to the bijective map:

$$Se: \qquad S^1_{d_n} \qquad \longrightarrow \qquad e\ell^1_{cf,n}, \qquad \forall\, n \in I\!N,$$

in such a way that:

1) the length (or area) of $S^1_{d_n}$ is equal to the length of $e\ell^1_{ef,n}$;

2) the number of transcendental quanta on $S^1_{d_n}$ is equal to that on $e\ell^1_{ef,n}$.

The canonical equation

$$\mathbf{f^2\, X^2 + e^2\, Y^2 = f^2\, e^2}$$

of the ellipse $\mathbf{e\ell^1_{ef,n}}$, having the same length as the circle $S^1_{a_n}$, **is transformed under the bijective map "Se"** into:

$$f^2\, r^2_{S^1_{d_n}} \cos^2 2\pi iy + e^2\, r^2_{S^1_{d_n}} \sin^2 2\pi iy = e^2\, f^2$$

or

$$r^2_{S^1_{d_n}} \left(f^2 \cos^2 2\pi iy + e^2 \sin^2 2\pi iy \right) = e^2\, f^2$$

or

$$\mathbf{z_{e\ell^1_n} = r_{S^1_{d_n}} \left(f \cos 2\pi iy + i\, e \sin 2\pi iy \right)}$$

by the change of variables $\begin{cases} X &= r_{S^1_{d_n}} \cos 2\pi iy, \\ Y &= r_{S^1_{d_n}} \sin 2\pi iy, \end{cases}$

where:

- the "imaginary angle" iy refers to a circle localized in a plane orthogonal to the circle $S^1_{a_n}$ [Pie2];

- e and f are the half lengths of the axis of the ellipse $e\ell\ell^1_{ef,n}$. ∎

3.11 Proposition (Holomorphic and nonholomorphic part of the new series expansion of a weak Maass form)

Let

$$f_2^{WM}(z) = \sum_n \left[\left(r_{S^1_{d_n}} (f \cos 2\pi n(iy) + ie \sin 2\pi n(iy)) \right) \times \left(r_{S^1_{a_n}} e^{2\pi i n x} \right) \right]$$

$$= \sum_n T_n^{2,(e\ell)}(S^1_{a_n}, e\ell^1_{ef,n})$$

be **the new series expansion of the weak Maass form** $f_2^{WM}(z)$ **of weight** 2.
If

$$f_2^{WM,hol}(z) = \sum_n c(n) \, e^{2\pi i n z} \; ,$$

where $c(n) = r_{S^1_{a_n}} \times r_{S^1_{d_n,inscr}}$ **denotes the holomorphic part of the "Fourier"**
series expansion of $f_2^{WM}(z)$ *in such a way that*

$$S^1_{d_n,inscr} = r_{S^1_{d_n,inscr}} \, e^{2\pi i n(iy)}$$

be the equation of the circle inscribed in the elliptic $e\ell^1_{ef,n}$,
then, **the nonholomorphic part** $f_2^{WM,nonhol}(z)$ **of the weak Maass form** $f_2^{WM}(z)$,
is given by*:*

$$f_2^{WM,nonhol}(z) = \sum_n T_n^{2,(e\ell)}(S^1_{a_n}, e\ell^1_{ef,n}) - \sum_n \left(r_{S^1_{a_n}} \times r_{S^1_{d_n,inscr}} \right) e^{2\pi i n z}$$

and **corresponds to the shadow of** $f_2^{WM}(z)$.

Proof: As the new series expansion of the weak Maass form $f_2^{WM}(z)$ is the sum, over the integers $n \in I\!\!N$, of the products of the equation of the circle $S^1_{a_n}$ over "x" by the equation of the ellipse $e\ell^1_{ef,n}$ over "$i \cdot y$", which is a bijection with the equation of the circle having as radius $r_{S^1_{d_n}}$, it appears clearly that the holomorphic part $f_2^{WM,hol}(z)$ of $f_2^{WM}(z)$ must be given by

$$f_2^{WM,hol}(z) = \sum_n r_{S^1_{a_n}} e^{2\pi i n x} \times r_{S^1_{d_n,incr}} e^{2\pi i n(iy)} = \sum_n c(n) \, e^{2\pi i n z}$$

where $r_{S^1_{d_n,incr}} e^{2\pi i n(iy)}$ is the equation of the circle inscribed in the ellipse $e\ell^1_{ef,n}$.
So, the **nonholomorphic part** $f_2^{WM,nonhol}(z)$ of $f_2^{WM}(z)$ will be given by the series expansion:

$$f_2^{WM,nonhol}(z)$$

$$= \sum_n T_n^{2,(el)}(S_{a_n}^1, el_{ef,n}^1) - \sum_n c(n)\ e^{2\pi inz}$$

$$= \sum_n \left[S_{a_n}^1 \times (el_{ef,n}^1 - S_{d_n,incr}^1) \right]$$

$$= \sum_n \left\{ r_{S_{a_n}^1}\ e^{2\pi inx} \times \left[\left(r_{S_{d_n}^1}\left(f\cos 2\pi iy + ie\sin 2\pi iy \right) \right) - \left(r_{S_{d_n,incr}^1}\ e^{2\pi in(iy)} \right) \right] \right\}$$

which clearly corresponds to the nonholomorphic part $\sum_{n=n_0}^{\infty} \gamma(f,n;y)\ q^{-n}$ of the Fourier expansion of the weak Maass form $f_2^{WM}(z)$ as developed in section 3.9.

This nonholomorphic part $f_2^{WM,nonhol}(z)$ is then the "shadow" of the weak Maass form $f_2^{WM}(z)$ with respect to its holomorphic part $f_2^{WM,hol}(z)$ in the sense that the shadow introduced for describing the Mock modular forms [Zag] is a unary theta series of weight $3/2$, i.e. a function of the form $\sum_n \varepsilon(n)\ n\ q^{\kappa n^2}$ with $\kappa \in \mathbb{Q}$ and ε and odd periodic function. Taking into account propositions 3.4 and 3.5, it can be shown that $f_2^{WM,nonhol}(z)$ can be transformed in such a unary theta series of weight $3/2$. ∎

3.12   Weak Maass forms of weight k and level N

Referring to proposition 1.14 and section 2.7, we infer that **the weak Maass form $f_2^{WM}(z)$ of weight 2 and level 1 can be extended to the weak Maas form $f_k^{WM}(z_{N-k})$ of weight k and level N** defined by the series:

$$f_k^{WM}(z_{N-k})$$

$$= \sum_n T_n^{2,(el)}(S_{a_{n_{N-k/2}}}^1, el_{ef,n_{N-k/2}}^1)$$

$$= \sum_n \left[\left(r_{S_{d_{n-k/2}}^1}\left(f\cos 2\pi n(iy)_{N-k/2} \right) + ie\sin 2\pi n(iy)_{N/2} \right) \times r_{S_{a_{n-k/2}}^1}\ e^{2\pi inx_{N-k/2}} \right],$$

i.e. **by the sum over the integers n of two-dimensional semitori $T_n^{2,(el)}(S_{a_{n_{N-k/2}}}^1,$ $el_{ef,n_{N-k/2}}^1)$ with elliptic cross sections $el_{ef,n_{N-k/2}}^1$ of weight $k/2$ and level N** where:

- $S_{a_{n_{N-k/2}}}^1 = r_{S_{a_{n-k/2}}^1}\ e^{2\pi inx_{N-k/2}}$ is the equation of a semicircle over " x " at $(a \times k/2)$ transcendental quanta with radius $r_{S_{a_{n-k/2}}^1}$ and real points $x_{N-k/2}$ of order $(N \times k/2)$, $n = ad$;

- $e\ell^1_{ef,n_{N-k/2}} = r_{S^1_{d_{n-k/2}}}(f \cos 2\pi n(iy_{N-k/2}) + ie \sin 2\pi n(iy_{N-k/2}))$ is the equation of an ellipse over "$i \cdot y$" at $(d \times k/2)$ transcendental quanta with $r_{S^1_{d_{n-k/2}}}$ being the radius of the circle in one-to-one correspondence with $e\ell^1_{ef,n_{N-k/2}}$ and "real" points "$iy_{N-k/2}$" of order $(N \times k/2)$.

3.13 Proposition

The weak Maass form $f_k^{WM}(z_{N-k})$ of weight k and level N defined by the series:

$$f_k^{WM}(z_{N-k}) = \sum_n T_n^{2,(el)}(S^1_{a_{n_{N-k/2}}}, e\ell^1_{ef,n_{N-k/}})$$

is:

a) periodic;

b) holomorphic and weakly modular.

Proof:

a) $f_k^{WM}(z_{N-k})$ is periodic because it verifies:

$$f_k^{WM}(z_{N-k}) = f_k^{WM}(z_{N-k} + 1) ,$$

b) $f_k^{WM}(z_{N-k})$ **is holomorphic but not modular with respect to the elliptic cross section** $e\ell^1_{ef,n_{N-k/2}}$ **deviation from circularity**. On the other hand, the circle factor $S^1_{a_{n_{N-k/2}}}$ of $T_n^{2,(el)}$ is modular.

Consequently, $f_k^{WM}(z_{N-k})$ will be said to be weakly modular, the modularity given by:

$$f_k^{WM}\left(\frac{az_{N-k} + b}{cz_{N-k} + d}\right) = f_k^{WM}(z_{N-k}) \qquad \text{for } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z}) .$$

Remark that the modularity can be checked by using the formula:

$$\cos(a \cdot b) = \frac{1}{2}(\exp(iab) + \exp(-iab)) = \frac{1}{2}\left((\exp ia)^b + (\exp -ia)^b\right)$$

and

$$\cos\left(\tfrac{a}{b}\right) = \tfrac{1}{2}\left((\exp(ia)^{1/b} + (\exp -ia)^{1/b}\right) . \qquad \blacksquare$$

3.14 Generalized weak Maass forms: Elliptic forms

The weak Maass form $f_k^{WM}(z_{N-k})$ of weight k and level N, having a decomposition into the sum over the n sublattices, $1 \le n \le \infty$, of two-dimensional semitori $T_n^{2,(el)}$ with elliptic cross section, **can be generalized to elliptic forms of weight k and level N introduced as having a decomposition into the sum over n sublattices of surfaces of revolution of (semi)ellipses rotating around ellipses instead of circles** as for the weak Maass forms.

Thus, **a two-dimensional elliptic form $f_k^{EL}(z_{N-k})$ of weight k and level N will be defined by the series**:

$$f_k^{EL}(z_{N-k}) = \sum_n EL_n^{2,(el)}(el^1_{ab_{n_{N-k/2}}}, el^1_{ef,n_{N-k/2}})$$

$$= \sum_n \left[\left(r_{S^1_{a_{n-k/2}}} (b \cos 2\pi n x_{N-k/2} + ia \sin 2\pi n x_{N-k/2}) \right) \right.$$

$$\left. \times \left(r_{S^1_{d_{n-k/2}}} (f \cos 2\pi n(iy_{N-k/2}) + ie \sin 2\pi n(iy_{N-k/2})) \right) \right]$$

where:

- $EL_n^{2,(el)}(el^1_{ab_{n_{N-k/2}}}, el^1_{ef,n_{N-k/2}})$ is the n-th surface of revolution of the ellipse $el^1_{ef,n_{N-k/2}}$ rotating around the ellipse $el^1_{ab_{n_{N-k/2}}}$;

- the equation of the ellipse $el^1_{ab_{n_{N-k/2}}}$ is given by:

$$el_{ab_{n_{N-k/2}}} = r_{S^1_{a_{n-k/2}}} (b \cos 2\pi n x_{N-k/2} + ia \sin 2\pi n x_{N-k/2})$$

  with $r_{S^1_{a_{n-k/2}}}$ being the radius of the circle in bijection with $el^1_{ab_{n_{N-k/2}}}$ referring to section 3.12 and $x_{N-k/2}$ being a point or order $(N \times k/2)$.

The elliptic form $f_k^{EL}(z_{N-k})$, defined over a doubly periodic lattice, is thus an elliptic function characterized by four parameters "a, b, e, f" which are the half lengths of the axis of the two ellipses $el^1_{ab_{n_{N-k/2}}}$ and $el^1_{ef,n_{N-k/2}}$.

3.15 Proposition

Let $f_k(z_{N-k})$, $f_k^{WM}(z_{N-k})$ and $f_k^{EL}(z_{N-k})$ be respectively a cusp form, a weak Maass form and an elliptic form of weight k and level N.

Then **the commutative diagram**

$$f_k(z_{N-k}) \xrightarrow{\quad\sim\quad} f_k^{WM}(z_{N-k})$$

$$\searrow \sim \qquad \tilde{\sim} \swarrow$$

$$f_k^{EL}(z_{N-k})$$

indicates the possible transformation of one of these forms into another.

Proof: The above commutative diagram results from the one-to-one correspondence between these forms referring to proposition 3.10. ∎

3.16 Mock Theta functions of Ramanujan: A brief synthesis

Each of the 17 Ramanujan's theta functions belongs to one of the three families of functions:

a) Lerch sums,

b) quotients of binary theta series by unary theta series,

c) Fourier coefficients of Jacobi forms,

as discovered by S. Zwegers [Zwe].
These theta functions are q-hypergeometric series of the form $\sum\limits_{n=0}^{\infty} A_n(q)$, $A_n(q) \in \mathbb{Q}(q)$. The first of these is:

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)\ldots(1+q^n)^2} \ .$$

Each Ramanujan theta function is a Mock theta function given by the q-series $H(q) = \sum\limits_{n>0} a_n\, q^n$ in such a way that $q^\lambda\, H(q)$, $\lambda \in \mathbb{Q}$, be a Mock modular form of weight $1/2$ whose shadow is a unary theta series of weight $3/2$ given by a function of the form $\sum\limits_{n} \varepsilon(n)\, n\, q^{Kn^2}$, $K \in \mathbb{Q}$, where $\varepsilon(n)$ is an odd periodic function. **A Mock theta function is thus a Mock modular form of weight k of the space $\mathbb{M}_k$ of such forms extending the space M_k of classical modular forms** of weight k and characterized **by a shadow $g = S[h]$** which is a modular form of weight $(2-k)$. Zagier then deduced that **the Ramanujan's Mock theta function $H(q)$ acquire modularity after carrying out the following three steps [Zag]**:

a) multiply $H(q)$ by a rational power q^λ of q;

b) change the variable $q = e^{2\pi i \tau}$ by τ, setting $h(\tau) = e^{2\pi i \lambda \tau}\, H(e^{2\pi i \tau})$;

c) add a nonholomorphic correction term $g^*(\tau)$ to $h(\tau)$ in such a way that $\widehat{h}(\tau) = h(\tau) + g^*(\tau)$ transforms like a modular form of weight $1/2$ with $g^*(\tau)$ associated to a theta series $g(\tau) = \sum\limits_{n \in \mathbb{Z}+d} n\, q^{Kn^2}$ being a modular form of weight $3/2$ and the shadow of $h(\tau)$.

Finally, it appears that the space $\mathbb{M}_k$ of Mock theta functions is in one-to-one correspondence with the space $\widetilde{\mathbb{M}}_k$ of weak Maas forms.

Having proposed in the preceding sections a geometric and algebraic approach to theta series and weak Maass forms, **we will try to give in the next proposition a geometric interpretation of Ramanujan's theta functions transformed into Mock theta functions related to weak Maass forms**.

3.17 Proposition (Connection of Ramanujan Theta functions with weak Maass forms, modular forms and global elliptic semimodules)

Let $\mathcal{M}_{H \to \widehat{h}} : H(q) \longrightarrow \widehat{h}(\tau)$ denote the map corresponding to the three step sequence recalled in section 3.16 and transforming each Ramanujan theta function $H(q)$ into a Mock modular form $\widehat{h}(\tau)$ of weight $1/2$ characterized by a shadow $g^*(\tau)$ being a modular form of weight $3/2$.

*Then, the map $\mathcal{M}_{H \to \widehat{h}}$ corresponds to the composition of maps $(\mathcal{M}_{f_2^{WM;hol} \to \widehat{h}} \circ \mathcal{M}_{H \to f_2^{WM;hol}})$ of **the commutative diagram***:
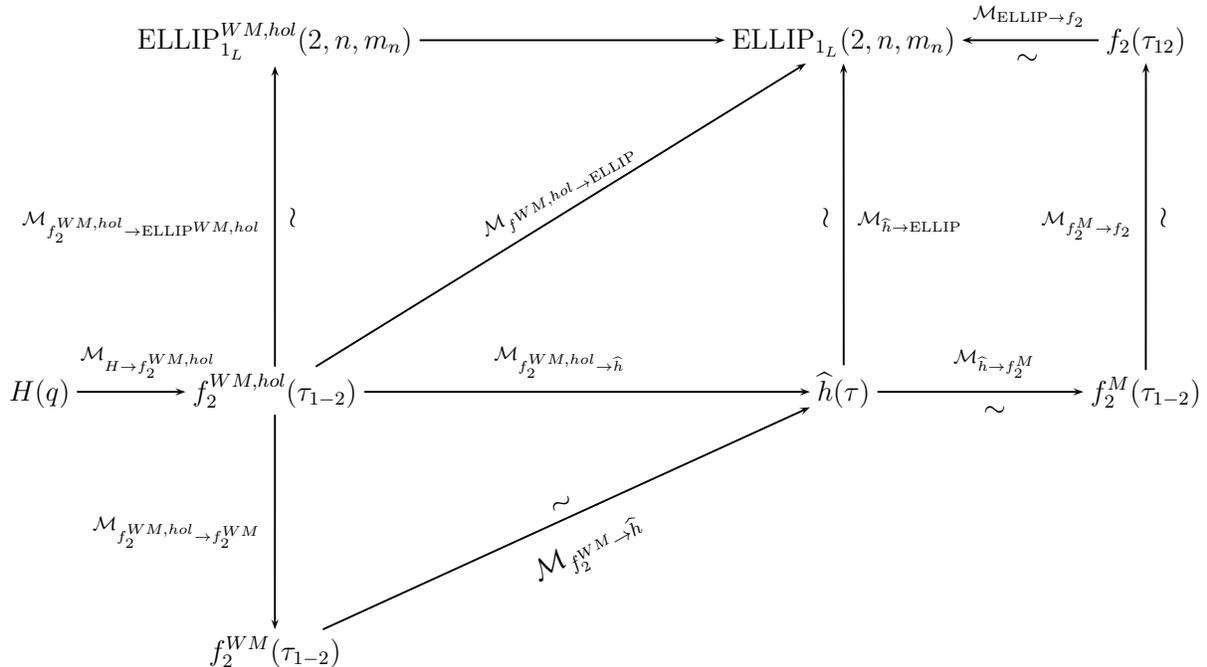
where:

- $f_2^{WM,hol}(\tau_{1-2})$ *is the holomorphic part of the weak Maass form* $f_2^{WM}(\tau_{1-2})$ *of the space* $\widehat{\mathbb{M}}_2$ *of weight* 2 *and level* 1 *;*

- $f_2^M(\tau_{1-2})$ *is the Mock modular form of weight* 2 *and level* 1 *of the space* $\mathbb{M}_2$ *extending the space* $SL(2,1)$ *of classical (or Weil) modular forms* $f_2(\tau_{1-2})$ *of weight* 2 *and level* 1 *;*

- $\text{ELLIP}_{1_L}(2,n,m_n)$ *and* $\text{ELLIP}_{1_L}^{WM,hol}(2,n,m_n)$ *are the global elliptic semimodules of weight* 1 *and level* 1 *covering respectively the modular forms* $f_2(\tau_{1-2})$ *and* $f_2^{WM,hol}(\tau_{1-2})$ *;*

- $\mathcal{M}_{H \to f_2^{WM,hol}}$ *and* $\mathcal{M}_{f_2^{WM,hol} \to \widehat{h}}$ *are the maps respectively of the steps ((a) – b) and c)) of the three-step sequence recalled in section 3.16;*

- $\mathcal{M}_{\widehat{h} \to f_2^M}$ *is the covering map of the Mock modular form* $\widehat{h}(\tau)$ *of weight* 1/2 *and level* 1 *by the Mock modular form* $f_2^M(\tau_{1-2})$ *of weight* 2 *and level* 1 *,*

in such a way that:

a) **the nonholomorphic part** $\boldsymbol{f_2^{WM,nonhol}(\tau_{1-2})}$ **of the weak Maass form** $\boldsymbol{f_2^{WM}(\tau_{1-2})}$ **corresponds to the shadow** $\boldsymbol{g^*(\tau)}$ **of the Mock modular form** $\boldsymbol{\widehat{h}(\tau)}$ **;**

b) *the spaces* $\mathbb{M}_1$ *of Mock theta functions* $\widehat{h}(\tau)$ *and* $|\widehat{M}|_2$ *of weak Maass forms* $f_2^{WM}(\tau_{1-2})$ *are isomorphic;*

c) *the bijective map* $\mathcal{M}_{\widehat{h} \to \text{ELLIP}} : \widehat{h}(\tau) \xrightarrow{\sim} \text{ELLIP}_{1_L}(2,n,m_n)$ *is an extension of the bijective map:*

$$\mathcal{M}_{f_2^{WM,hol} \to \text{ELLIP}} : \qquad f_2^{WM,hol}(\tau_{1-2}) \quad \xrightarrow{\sim} \quad \text{ELLIP}_{1_L}^{WM,hol}(2,n,m_n)$$

by the shadow $\widehat{h}(\tau)$ *.*

Proof: The map

$$\mathcal{M}_{H \to f_2^{WM,hol}} : \qquad H(q) \quad \longrightarrow \quad f_2^{WM,hol}(\tau_{1-2}) \,,$$

corresponding to the steps a) and b) of section 3.16, transforms each Ramanujan's q-hypergeometric series $H(q) = \sum_n a_n \, q^n$ into a holomorphic function which may be the holomorphic part $f_2^{WM,hol}(\tau_{1-2})$ of a weak Maass form $f_2^{WM}(\tau_{1-2})$ of weight 2 and level 1, referring to proposition 3.10.

Then, the map

$$\mathcal{M}_{f_2^{WM,hol} \to \widehat{h}} : \qquad f_2^{WM,hol}(\tau_{1-2}) \quad \longrightarrow \quad \widehat{h}(\tau)$$

corresponds to the map

$$\mathcal{M}_{f_2^{WM,hol} \to f_2^{WM}} : \qquad f_2^{WM,hol}(\tau_{1-2}) \quad \longrightarrow \quad f_2^{WM}(\tau_{1-2})$$

extending $f_2^{WM,hol}(\tau_{1-2})$ by the shadow $g^*(\tau)$ which may be the nonholomorphic part $f_2^{WM,nonhol}(\tau_{1-2})$ of the weak Maass form $f_2^{WM}(\tau_{1-2})$ according to proposition 3.11.

It then results that the global elliptic semimodule $\mathrm{ELLIP}_{1_L}(2, n, m_n)$, covering $\widehat{h}(t\tau)$, is an extension by the shadow $g^*(\tau)$ of the global elliptic semimodule $\mathrm{ELLIP}_{1_L}^{WM,hol}(2, n, m_n)$ covering $f_2^{WM,hol}(\tau_{1-2})$.

And, thus, the spaces $\mathbb{M}_1$ of Mock theta functions $\widehat{h}(\tau)$, which are real analytic modular forms [Wat], and $\widehat{\mathbb{M}}_2$ of weak Maass forms $f_2^{MW}(\tau_{1-2})$ are isomorphic (see proposition 3.7). ∎

3.18 Corollary (Mock modular forms of weight k)

The weight 2 case of Mock modular forms studied in proposition 3.17 can be easily generalized to the weight k by considering that the shadow of weight $(2 - k)$ of a Mock modular form of weight k corresponds to the nonholomorphic part of a weak Maass form $f_2^{MW}(\tau_{1-k})$ of weight k and level 1.

3.19 Partitions of n and Dyson's rank

In order to provide a combinatorial explanation of Ramanujan's congruence for the number of partitions $p(n)$ of an integer n [B-O1] of which generating function is

$$\sum_{n=0}^{\infty} p(n) \, q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \ ,$$

F. Dyson [Dys] introduced the rank of a partition defined to be its largest part minus the number of its parts.

In this respect, let $N(n, m)$ denote the number of partitions of n having rank m congruent to $r \bmod s$.

The generating function giving the number of partitions of n with rank m is:

$$R(\omega, q) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} N(n, m) \, \omega^m \, q^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{m=1}^{n} (1 - \omega \, q^m)(1 - \omega^{-1} \, q^m)}$$

where $\omega = e^{2\pi ix/s}$ is a s-th root of unity.

According to D. Zagier [Zag], knowing the functions $n \to N(n,m)$ for all $r \pmod s$ is equivalent to knowing the specializations of $R(\omega; q)$ to all s-th root of unity ω. For $\omega = -1$, $R(\omega, q)$ specializes to the first Ramanujan's Mock theta function $f(q)$ given in section 3.16. Bringmann and Ono [B-O2] generalize this to other roots of unity.

3.20 Proposition (Partitions of quanta in Ramanujan's Theta functions)

Let $R(\omega, q)$ be the partition function specializing to the (17) Ramanujan's Mock theta functions $H(q)$ of weight $1/2$ and level 1.

As $N(n,m)$ denotes the number of partitions of $n^{(2)}$ transcendental quanta, it follows that:

1) **the Dyson's rank m of a partition of n must be the order of the maximal Galois sub(bisemi)group associated with the considered transcendental biextension minus the number of Galois sub(bisemi)groups**;

2) **ω^m is a phase factor** *related to $N(n,m)$;*

3) **$N(n,m) \times nu$ may be the multiplicity of the n-th global elliptic subsemimodule $ELLIP_{1_L}(1,n)$ covering the n-th term of the Ramanujan's Mock theta function $H(q)$** *specialized from $R(\omega; q)$ where nu is the number of nonunits of Galois extensions;*

4) *there is a map:*
$$\mathcal{M}_{h \to f_{(2)}}: \qquad H(q) \quad \longrightarrow \quad f_2(\tau_{1-2})$$
from the theta function $H(q)$ to the corresponding Hecke cusp form $f_2(\tau_{1-2})$ of weight 2 and level 1 in such a way that $N(n,m)\,\omega^m$ maps into the n-th coefficient c_{n2} (being a global Hecke character) of the cusp form $f_2(\tau_{1-2}) = \sum_n c_{n2} \cdot q_{1-2}^n$.

Proof:

1) As the integer n in the Mock theta series refers to the n-th $\mathbb{Z}^2$-sublattice (see proposition 1.7) of the cusp form $f_2(\tau_{1-2})$ extending $H(q)$, as developed in proposition 3.17, and as $f_2(\tau_{1-2})$ is generated from the Weil (or Galois) group $W_{F_\omega}^{ab}$ according to section 2.2 by means of a Langlands global correspondence (see section 2.5), it (i.e. the integer n) must correspond to the order of a Weil (or Galois) subgroup.

Taking into account the definition of the rank m of a partition of n, it is clear that the **largest part of partition is the order of the maximal Galois sub(bisemi)-group associated with the considered partition and that the number of parts of the partition is the number of intermediate Galois (or Weil) sub(bisemi)groups**.

2) Referring to the diagram of proposition 3.17, we see that the global elliptic semimodule $\mathrm{ELLIP}_{1_L}^{WM,hol}(2, n, m_n)$ covers the holomorphic part $f_2^{WM,hol}(\tau_{1-2})$ of the weak Maass form $f_2^{WM}(\tau_{1-2})$ and thus also $H(q)$.

Then, $(N(n, m) \times nu)$ must refer to the multiplicity "m_n" of the n-th global elliptic subsemimodule

$$\mathrm{ellip}_{1_L}(1, n) = \underset{m_n}{\Sigma}\, \lambda_1(1 \text{ or } 2, n)\, e^{2\pi i n m_n x_{1-1}}$$

according to section 2.7.

... in contrast with the result of proposition 2.8 referring to the multiplicity with respect to Hecke cusp forms in the sense of:

3) the map

$$\mathcal{M}_{H \to f(2)}: \qquad H(q) \quad \longrightarrow \quad f_2(\tau_{1-2})$$
$$N(n, m)\, \omega^m \quad \longrightarrow \quad c_{n2}\,, \qquad \forall\, n \in I\!N\,,$$

is such that $N(n, m)\, \omega^m$ be sent into the n-th coefficient of $f_2(\tau_{1-2})$ which is a product of radii of two orthogonal circles (see proposition 1.9). ∎

3.21 The Tau function: A brief summary

The Ramanujan tau function is the function defined by:

$$\overset{\infty}{\underset{n=1}{\Sigma}}\, \tau(n)\, q^n = q\, \overset{\infty}{\underset{n=1}{\Pi}}(1 - q^n)^{24} = \Delta(z)\,, \qquad q = e^{2\pi i z}\,;, \quad z = x + iy\,, \quad y > 0\,.$$

As we have the equality

$$\Delta(z) = \eta(z)^{24}\,,$$

where $\eta(z) = q^{1/24}\, \overset{\infty}{\underset{n=1}{\Pi}}(1 - q_n)$ is the Dedekind's eta function of weight $1/2$ and level 1 due to transformation laws:

$$\eta(-1/z) = (-iz)^{1/2}\, \eta(z) \qquad \text{and} \qquad n(z + 1) = e^{\pi i/12}\, n(z)\,,$$

and as $\Delta(z)$ satisfies the symmetry condition:

$$\Delta(-1/z) = z^{12}\,\Delta(z)\,,$$

$\Delta(z)$ is a modular form of weight 12 and level 1.
The coefficients $\tau(n)$ satisfy:

   a) $\tau(nm) = \tau(n)\,\tau(m)$ if $(n,m) = 1$,

   b) $\tau(p^{n+1}) = \tau(p)\,\tau(p^n) - p^{11}\,\tau(p^{n-1})$, $n > 1$,

   c) $|\tau(p)| \le 2\,p^{11/2}$, $\forall$ prime

which is the Ramanujan's conjecture proved by P. Deligne.
The $\tau(n)$ enjoy many congruence relations, for example:

$$\tau(n) = \sigma_{11}(n)\bmod 2^{11} \qquad \text{for } n = 1\bmod 8\,,$$

and a very interesting one is:

$$\tau(n) = \sum_{0<d|n} d^{11}\bmod 691$$

in such a way that the Eisenstein series

$$E_{12}(q) = -B_{12}/24 + \sum_{n=1}^{\infty}\left(\sum_{0<d|n} d^{11}\right) q^n\,,$$

where

$$B_{12}/24 = -691/65520 = 0\bmod 691\,,$$

has the same Fourier expansion coefficients as $\Delta(z)$ as analyzed by B. Mazur in [Maz].
The Dirichlet series associated with the Ramanujan tau function is:

$$L_\tau(s) = \prod_p \frac{1}{1 - \tau(p)\,p^{-s} + p^{11-2s}}$$

and the function $(2\pi)^{-s}\,\Gamma(s)\,L_\tau(s)$ is invariant under the substitution $s \to 12 - s$.
The Ramanujan tau function $\Delta(z) = \sum_n \tau(n)\,q^n = q\,\prod_n(1-q^n)^{24}$ is a generating function
on numbers of inverse partitions of quanta (corresponding perhaps to partitions in the dual
(semi)space with respect to the generating function

$$\sum_n p(n)\,q^n = \prod_n \frac{1}{1-q^n}$$

of numbers of partitions $p(n)$ of quanta as analyzed in section 3.19.

Referring to proposition 3.20, **the map**

$$\mathcal{M}_{\Delta \to f_{(12)}} : \qquad \Delta(z) \quad \longrightarrow \quad f_2(z_{12-1})$$

from the tau function $\Delta(z)$ of weight 12 to the associated two-dimensional (Hecke) Weil cusp form $f_2(z_{12-1})$ of weight 12 and level 1 will be studied in the next proposition and proved to correspond to a map from this tau function to a cusp form $\phi^{(12)}(z_{12-1})$ in 12 real dimensions and level 1 (on $\mathbb{C}^6$) **which is an orthogonal universal structure**. This may be a clue to know how to compute the congruence relations of $\tau(n)$.

3.22 Proposition (Universal (orthogonal) cusp form in dimension 2 and weight 12)

*Let $f_2(z_{12-1})$ be a two-dimensional Weil cusp form of weight 12 and level 1.
Let*

$$CP_{(12) \to 2} : \qquad \phi^{(12)}(z_{12-1}) \quad \longrightarrow \quad f_2(z_{12-1})$$

be the map from the cusp form $\phi^{(12)}(z_{12-1})$ on $\mathbb{C}^6$ of level 1 projecting it into the two-dimensional cusp form $f_2(z_{12-1})$ and satisfying the commutative diagram:

Then, **the two-dimensional cusp form $f_2(z_{12-1})$ of weight 12 and level 1 is a universal "orthogonal" cusp form corresponding throughout Langlands global correspondences to the sum of the cuspidal representations of six bilinear algebraic semigroups generating three two-dimensional embedded toric bisemisheaves as well as their orthogonal equivalents** *according to:*

$$\Pi^{(12)}(\mathrm{GL}_6(\widetilde{F_{\varpi}} \times_D \widetilde{F_{\omega}})) = \bigoplus_{i=1}^{6} \Pi^{(2_i)}(\mathrm{GL}_{1_i}(\widetilde{F_{\varpi}} \times_D \widetilde{F_{\omega}})) \xrightarrow{CP_{R \times L \to L}^{(12) \to 2}} f_2(z_{12-1})$$

where:

- $\Pi^{(2i)}(\mathrm{GL}_{1_i}(\widetilde{F_{\varpi}} \times_D \widetilde{F_{\omega}}))$, *being the two-dimensional cuspidal representation of the bilinear algebraic semigroup $\mathrm{GL}_{1_i}(\widetilde{F_{\varpi}} \times_D \widetilde{F_{\omega}})$, is in one-to-one correspondence with the two-dimensional cusp form $f_{2_i}(z_{2-1})$ of weight 2 and level 1, $1 \leq i \leq 6$;*

- $\Pi^{(12)}(\mathrm{GL}_6(\widetilde{F_{\overline{\omega}}} \times_D \widetilde{F_\omega}))$, *being the twelve-dimensional cuspidal representation of* $\mathrm{GL}_6(\widetilde{F_{\overline{\omega}}} \times_D \widetilde{F_\omega})$, *is in one-to-one correspondence with the cusp form* $\phi^{(12)}(z_{12-1})$.

This is summarized in the commutative diagram:

$$
\begin{array}{ccc}
\{\Pi^{(2i)}(\mathrm{GL}_{1_i}(\widetilde{F_{\overline{\omega}}} \times_D \widetilde{F_\omega}))\}_{i=1}^6 & \longrightarrow & \Pi^{(12)}(\mathrm{GL}_6(\widetilde{F_{\overline{\omega}}} \times_D \widetilde{F_\omega})) \\
\Big\downarrow & & \Big\downarrow {\scriptstyle \phi^{(12)}(z_{12-1})} \\
& & \Big\downarrow {\scriptstyle CP_{(12)\to 2}} \\
\{f_{2_i}(z_{2-1})\}_{i=1}^6 & \longrightarrow & f_2(z_{12-1})
\end{array}
$$

Proof:

1) Let $\{\phi_R(g^{(2)}_{\overline{\omega}_R}[n]) \otimes \phi_L(g^{(2)}_{\omega_L}[n])\}_{n=1}^{t\le\infty}$ denote the set of two-dimensional bisections of a bisemisheaf $(M_R^{(2)}(F_{\overline{\omega}}) \otimes M_L^{(2)}(F_\omega))$ of differentiable bifunctions constituting the functional representation space $\mathrm{FREPSP}(\mathrm{GL}_1(\widetilde{F_{\overline{\omega}}} \times \widetilde{F_\omega}))$ of the bilinear algebraic semigroup $\mathrm{GL}_1(F_{\overline{\omega}} \times F_\omega)$ over sets of complex transcendental extensions $F_{\overline{\omega}}$ and F_ω.

   Let $(M_R^{(2)\perp}(F_{\overline{\omega}}) \otimes M_L^{(2)\perp}(F_\omega))$ be the orthogonal complement bisemisheaf of $(M_R^{(2)}(F_{\overline{\omega}}) \otimes M_L^{(2)}(F_\omega))$.

   Degenerate singularities on these bisemisheaves can give rise, by versal deformations and blowups of these, to one or two new covering bisemisheaves according to the kind of considered singularities as developed in [Pie9], [Pie7].

   If there are degenerate singularities of corank 1 and codimension 3 on these bisemisheaves, we get, after a process of versal deformations, blowups of these, desingularizations and toroidal compactifications, **the three shell embedded bisemisheaves**:

   $$\Pi^{(2_1)}(\mathrm{GL}_{1_1}(F_{\overline{\omega}} \times F_\omega)) \oplus \Pi^{(2_2)}(\mathrm{GL}_{1_2}(F_{\overline{\omega}} \times F_\omega)^\perp)$$
   $$\subset \Pi^{(2_3)}(\mathrm{GL}_{1_3}(F_{\overline{\omega}_{cov(1)}} \times F_{\omega_{cov(1)}})) \oplus \Pi^{(2_4)}(\mathrm{GL}_{1_4}(F_{\overline{\omega}_{cov(1)}} \times F_{\omega_{cov(1)}})^\perp)$$
   $$\subset \Pi^{(2_5)}(\mathrm{GL}_{1_5}(F_{\overline{\omega}_{cov(2)}} \times F_{\omega_{cov(2)}})) \oplus \Pi^{(2_6)}(\mathrm{GL}_{1_6}(F_{\overline{\omega}_{cov(2)}} \times F_{\omega_{cov(2)}})^\perp)$$

   where:

- $\Pi^{(2_i)}(\mathrm{GL}_{1_i}(-\times-))$ is the cuspidal representation of the i-th bilinear (algebraic) semigroup $\mathrm{GL}_{1_i}(-\times-))$, $1 \leq i \leq 6$ in such a way that the sum of its conjugacy class representatives, which are products, right by left, of two-dimensional semitori according to sections 2.6 to 2.8, is the two-dimensional cusp biform $f_{2_i}(z_{2-1}) \times f_{2_i}(z_{2-1})$ of weight 2 and level 1 referring to sections 2.5 and 1.12;

- $\Pi^{(2_4)}(\mathrm{GL}_{1_4}(F_{\overline{\omega}_{cov(1)}} \times F_{\omega_{cov(1)}}))$ is the cuspidal representation of the blinear (algebraic) semigroup constituting the first shell covering of $\Pi^{(2_2)}(\mathrm{GL}_{1_2}(F_{\overline{\omega}} \times F_{\omega}))$ with "$\perp$" referring to the orthogonal complement cuspidal representation.

2) Taking into account the Langlands global functoriality conjecture introduced in [Pie4], we have that the sum $\sum_{i=1}^{6} \Pi^{(2_i)}(\mathrm{GL}_{1_i}(F_{\overline{\omega}} \times F_{\omega}))$ of the six above cuspidal representations is equal to the 12-dimensional cuspidal representation $\Pi^{(12)}(\mathrm{GL}_6(F_{\overline{\omega}} \times_D F_{\omega}))$ of the bilinear (abstract) complex semigroup $\mathrm{GL}_6(F_{\overline{\omega}} \times_D F_{\omega})$.

Indeed, the Langlands global functoriality conjecture states that **the 12-dimensional cuspidal representation $\Pi^{(12)}(\mathrm{GL}_6(F_{\overline{\omega}} \times_D F_{\omega}))$ is nonorthogonally reducible it if decomposes diagonally according to the direct sum of irreducible cuspidal representations of the (algebraic) bilinear semigroups $\mathrm{GL}_{1_i}(F_{\overline{\omega}} \times F_{\omega})$ and offdiagonally according to the direct sum**

$$\bigoplus_{i \neq j=1}^{6} (\Pi^{(2_i)}(\mathrm{GL}_{1_i}(F_{\overline{\omega}})) \times \Pi^{(2_j)}(\mathrm{GL}_{1_j}(F_{\omega})))$$

of the (tensor) products of irreducible cuspidal representations of cross (algebraic) linear semigroups

$$\mathrm{GL}_{1_i}(F_{\overline{\omega}})) \times (\mathrm{GL}_{1_j}(F_{\omega})) \equiv T_{1_i}^{t}(F_{\overline{\omega}}) \times T_{1_j}(F_{\omega}) .$$

So,

$$\Pi^{(12)}(\mathrm{GL}_6(F_{\overline{\omega}} \times F_{\omega})) = \bigoplus_{i=1}^{6} \Pi^{(2_i)}(\mathrm{GL}_{1_i}(F_{\overline{\omega}} \times F_{\omega}))$$

$$\bigoplus_{i \neq j=1}^{6} \Pi^{(2_i)}(\mathrm{GL}_{1_i}(F_{\overline{\omega}})) \otimes \Pi^{(2_j)}(\mathrm{GL}_{1_j}(F_{\omega}))$$

in such a way that, if $\Pi^{(12)}(\mathrm{GL}_6(F_{\overline{\omega}} \times F_{\omega}))$ is orthogonally completely reducible, then the crossed cuspidal representations $\bigoplus_{i \neq j=1}^{6} \Pi^{(2_i)}(\mathrm{GL}_{1_i}(F_{\overline{\omega}})) \otimes \Pi^{(2_j)}(\mathrm{GL}_{1_j}(F_{\omega}))$ are equal to zero.

3) Taking into account the sum of the 12-dimensional cuspidal conjugacy class representatives of $\Pi^{(12)}(\mathrm{GL}_6(F_{\overline{\omega}} \times F_{\omega}))$, we get **the 12-dimensional cusp biform**

$\phi^{(12)}(z^*_{12-1}) \times \phi^{(12)}(z_{12-1})$ of level 1 which, by the map $CP_{(12)\to 2}$, sends it into the two-dimensional cusp biform $f_2(z^*_{12-1}) \times f_2(z_{12-1})$ of weight **12 and level 1** referring to propositions 1.14 and 1.17:

$$\Pi^{(12)}(\mathrm{GL}_6(F_{\overline{\varpi}} \times_D F_\omega)) \longrightarrow \phi^{(12)}(z^*_{12-1}) \times_D \phi^{(12)}(z_{12-1})$$

with vertical map $CP_{(12)\to 2}$ to

$$f_2(z^*_{12-1}) \times_D f_2(z_{12-1})$$

and down to

$$f_2(z_{12-1})$$

As, by hypothesis, $f_2(z^*_{12-1}) \times_D f_2(z_{12-1})$ and $f_{2_i}(z^*_{2-1}) \times_D f_{2_i}(z_{2-1})$ are diagonal cusp biforms, they are in one-to-one correspondence respectively with their left equivalents $f_2(z_{12-1})$ and $f_{2_i}(z_{2-1})$.

Then, **we get the equality**:

$$f_2(z_{12-1}) = \overset{6}{\underset{i=1}{\oplus}} f_{2_i}(z_{2-1})$$

resulting from the above commutative diagram and **corresponding to the announced universal "orthogonal" cusp form $f_2(z_{12-1})$ in dimension 2, weight 12 and level 1**.

4) Finally, the map

$$\mathcal{M}^{-1}_{\Delta\to\phi^{(12)}} : \qquad \phi^{(12)}(z_{12-1}) \quad \longrightarrow \quad \Delta(z)$$

directly results from the maps

$$CP_{(12)\to 2} : \quad \phi^{(12)}(z_{12-1}) \longrightarrow f_2(z_{12-1}) \quad \text{and} \quad \mathcal{M}_{\Delta\to f(12)} : \quad \Delta(z) \longrightarrow f_2(z_{12-1}) .$$

Then, $\mathbf{\Delta(z)}$, defined by

$$\Delta(z) = (\eta(z)^4)^6 = (q^{1/6} \prod_n (1 - q^n)^4)^6 ,$$

i.e. by the 6-th power of $(\eta(z))^4$ which is a modular form of weight 2, **is directly related to a 12-dimensional cusp form $\phi^{(12)}(z_{12-1})$ of level 1**. ∎

3.23 Universal nonorthogonal bilinear cuspidal representations

The generalization of proposition 3.22 to universal "nonorthogonal" cuspidal representations including crossed cuspidal representations of interaction will now be envisaged.

Referring to the Langlands functoriality conjecture [Pie4], **the bilinear cuspidal representation $\Pi^{(12)}(\mathrm{GL}_6(F_{\overline{\varpi}} \times F_\omega))$ of the (algebraic) bilinear semigroup $\mathrm{GL}_6(F_{\overline{\varpi}} \times F_\omega)$ decomposes essentially according to**:

$$\Pi^{(12)}_{\mathrm{rel}}(\mathrm{GL}_6(F_{\overline{\varpi}} \times F_\omega)) = \overset{6}{\underset{i=1}{\oplus}} \Pi^{(2_i)}(\mathrm{GL}_{1_i}(F_{\overline{\varpi}} \times_D F_\omega))$$

$$\overset{6}{\underset{i \neq j = 1}{\oplus}} \Pi^{(2_i)}(\mathrm{GL}_{1_i}(F_{\overline{\varpi}})) \otimes_{OD} \Pi^{(2_j)}(\mathrm{GL}_{1_j}(F_\omega))$$

where the second sum on the right hand side refers to crossed cuspidal representations leading to a nonorthogonally reducible representation of $\Pi^{(12)}(\mathrm{GL}_6(F_{\overline{\varpi}} \times F_\omega))$.

According to [Pie9], **the six diagonal cuspidal representations $\Pi^{(2_i)}(\mathrm{GL}_{1_i}(F_{\overline{\varpi}} \times_D F_\omega))$, $1 \leq i \leq 6$, correspond to an embedded three-shell universal physical structure of space-time ("ST"), middle-ground ("MG") and mass ("M")**:

$$\Pi^{(2_1)}(\mathrm{GL}_{1_1}^{ST}(F_{\overline{v}} \times_D F_v))_T \oplus \Pi^{(2_2)}(\mathrm{GL}_{1_1}^{ST}(F_{\overline{\varpi}} \times_D F_\omega))_S$$

$$\subset \Pi^{(1_3)}(\mathrm{GL}_{1_3}^{MG}(F_{\overline{v}} \times_D F_v))_T \oplus \Pi^{(2_4)}(\mathrm{GL}_{1_4}^{MG}(F_{\overline{\varpi}} \times_D F_\omega))_S$$

$$\subset \Pi^{(1_5)}(\mathrm{GL}_{1_5}^{M}(F_{\overline{v}} \times_D F_v))_T \oplus \Pi^{(2_6)}(\mathrm{GL}_{1_6}^{M}(F_{\overline{\varpi}} \times_D F_\omega))_S$$

in such a way that:

- $\Pi^{(1_1)}(\mathrm{GL}_{1_1}^{ST}(F_{\overline{v}} \times_D F_v))_T \approx \Pi^{(2_1)}(\mathrm{GL}_{1_1}(F_{\overline{\varpi}} \times_D F_\omega))$,

  $\Pi^{(1_3)}(\mathrm{GL}_{1_3}^{MG}(F_{\overline{v}} \times_D F_v))_T \approx \Pi^{(2_3)}(\mathrm{GL}_{1_3}(F_{\overline{\varpi}} \times_D F_\omega))$,

  and $\Pi^{(1_5)}(\mathrm{GL}_{1_5}^{M}(F_{\overline{v}} \times_D F_v))_T \approx \Pi^{(2_5)}(\mathrm{GL}_{1_5}(F_{\overline{\varpi}} \times_D F_\omega))$,

  are **the one-dimensional "time" diagonal cuspidal representations respectively of the space-time, middle-ground and mass shells**;

- $\Pi^{(2_2)}(\mathrm{GL}_{1_2}^{ST}(F_{\overline{\varpi}} \times_D F_\omega))_S$, $\Pi^{(2_4)}(\mathrm{GL}_{1_4}^{MG}(F_{\overline{\varpi}} \times_D F_\omega))_S$ and $\Pi^{(2_6)}(\mathrm{GL}_{1_6}^{M}(F_{\overline{\varpi}} \times_D F_\omega))_S$ are similarly **the two-dimensional "space" diagonal cuspidal representations respectively of the space-time, middle-ground and mass shells**.

In this context, **the six relevant off diagonal or crossed cuspidal representations $\Pi^{(2_i)}(\mathrm{GL}_{1_i}(F_{\overline{\varpi}})) \otimes_{OD} \Pi^{(2_j)}(\mathrm{GL}_{1_j}(F_\omega))$, being interaction crossed cuspidal representations $\Pi^{(2_i)}(\mathrm{GL}_{2_i}(F_{\overline{v}})) \otimes_{int} \Pi^{(2_j)}(\mathrm{GL}_{2_j}(F_v))$, are for [Pie10]**:

- **$i \neq j = 1$, 2 and 3 the crossed cuspidal representations of the interacting fields respectively of the space-time, middle-ground and mass shells**;

- $i \neq j = 4$, 5 and 6 the interacting crossed cuspidal representations between respectively the three different right and left semifields of the "$ST_R - MG_L$", "$MG_R - M_L$" and "$ST_R - M_L$", mixed shells:

$$\Pi^{(2_4)}(\mathrm{GL}_{2_4}^{ST}(F_{\overline{v}}))_T \otimes_{int} \Pi^{(2_5)}(\mathrm{GL}_{2_5}^{MG}(F_v))_S \ , \ \Pi^{(2_5)}(\mathrm{GL}_{2_5}^{MG}(F_{\overline{v}}))_T \otimes_{int} \Pi^{(2_6)}(\mathrm{GL}_{2_6}^{M}(F_v))_S$$

and $\Pi^{(2_4)}(\mathrm{GL}_{2_4}^{ST}(F_{\overline{v}}))_T \otimes_{int} \Pi^{(2_6)}(\mathrm{GL}_{2_6}^{M}(F_v))_S$.

3.24   Proposition

The cross binary product

$$\underset{i=1}{\overset{6}{\times}} (f_{2_i}(z_{2-1}^*) \times f_{2_i}(z_{2-1})) = \left(\underset{i=1}{\overset{6}{\Sigma}} f_{2_i}(z_{2-1}^*) \right) \times \left(\underset{i=1}{\overset{6}{\Sigma}} f_{2_i}(z_{2-1}) \right)$$

between the six cusp biforms $f_{2_i}(z_{2-1}^*) \times f_{2_i}(z_{2-1})$ *of dimension* 2 *, weight* 2 *and level* 1 *is the image of the injective map:*

$$\mathcal{M}_{\Pi^{(12)} \to \underline{6}} : \qquad \Pi_{\mathrm{rel}}^{(12)}(\mathrm{GL}_6(F_{\overline{\omega}} \times F_\omega)) \quad \longrightarrow \quad \underset{i=1}{\overset{6}{\times}} (f_{2_i}(z_{2-1}^*) \times f_{2_i}(z_{2-1}))$$

from **the relevant universal nonorthogonal bilinear cuspidal representation** $\Pi_{\mathrm{rel}}^{(12)}(\mathrm{GL}_6(F_{\overline{\omega}} \times F_{\omega}))$ **decomposing into six relevant cuspidal representations of interaction** *by taking into account the section 3.23.*

Proof: The cross binary product between cusp biforms was introduced in [Pie3] and recalled in section 1.12.

The six diagonal cusp biforms $\{f_{2_i}(z_{2-1}^*) \times_D f_{2_i}(z_{2-1}))\}_{i=1}^6$ are in one-to-one correspondence with the six time and space diagonal cuspidal representations of the embedded three shell universal physical structures ST, MG and M mentioned in section 3.23.

The thirty off diagonal bilinear cuspidal representations $\{f_{2_i}(z_{2-1}^*) \times_{OD} f_{2_j}(z_{2-1}))\}_{\substack{i \neq j=1 \\ i > j}}^6$ reduce by symmetry, $i > j$, to fifteen ones among which the six in one-to-one correspondence with the interacting crossed cuspidal representations taken into account in section 3.23 are relevant. The other nine remaining off diagonal bilinear cuspidal representations are diagonal crossed cuspidal representations of gravitational fields between mixed shells $ST-MG$, $ST-M$ and $MG-M$. Knowing that gravitational fields can be transformed into electromagnetic fields referring to [Pie11], we understand why these off diagonal bilinear cuspidal representations are not really relevant. ∎

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