Measurability.Gravity and Gauge Theories in Measurable Form at Low and High Energies

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Abstract

In this paper the author formulates a gauge field theory in terms of the **measurability** notion introduced in his previous works and performs a comparative analysis of passage to high energies for gravity and gauge theories. It has been found that **measurability** in gravity is in close association with *quantum fluctuations* of the space-time geometry (or at high energies with the **"space-time foam"**) introduced by J.A.Wheeler. It is demonstrated that at low energies $E \ll E_p$, in terms of **measurable** quantities, we can correctly define the *Least Action Principle* and *Noether's Theorem*.

1 Introduction

This paper is a continuation of the previous author's works devoted to the subject, especially [1]–[5]. The principal idea of the above-mentioned works is as follows. The majority of the researchers are of the opinion that at very high energies (Early Universe) there is the minimal length ℓ presumed to be on the order of the Planck length $\ell \propto l_p$, though not necessarily. Consequently, at the corresponding high energies the theory (understood as Quantum Theory of matter fields and Gravity) is discrete.
At the present time these theories are defined in the continuous space-time paradigm but are associated with serious problems, in particular with the

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(ultraviolet and infrared) divergences.
On the other hand, if the minimal length ℓ exists, it should be existent at all the energy (both low and high) scales, and the theories should be initially discrete. But the modern mathematical apparatus of these theories based on the use of the abstract infinitesimal variations $dt, dx_i, dp_i, dE, i = 1, ..., 3$ prevents from seeing this clearly.
It is obvious that, when ℓ exists, all variations in a physical system, irrespective of the energies, should be expressed in terms of ℓ and hence a theory should not involve the above-mentioned abstract infinitesimal variations. Though with the use of new terms, at low energies a theory becomes discrete, it is very close to the initial theory formulated in the continuous space-time. Actually, discreteness is revealed at high energies only. Besides, all infinitesimal variations in a system will be dependent on the existent energies.
The main instrument for realization of the idea put forward by the author is the notion of **measurability**, initially defined in [2]and also in Section 2 of this paper. In [3] in terms of this notion the author presents a detailed study of the spherically symmetric horizon spaces and black holes with the Schwarzschild metric at low energies $E \ll E_p$ and at high energies $E \approx E_p$. In [4],[5], within the scope of the **measurability** concept, gravity has been studied in the general case at low energies to show that in this case there is a possibility for the correct transition to high (Planck's) energies. Gravity in this case is understood as General Relativity.
The present paper contains the following recently obtained results.
In terms of the **measurability**notion, the author formulates a gauge field theory and performs a comparative analysis of the transition to high energies for gravity and gauge theories. By him, it has been found that the **measurability** in gravity is in close association with *quantum fluctuations* of the space-time geometry (or at high energies with the **"space-time foam"**)introduced by J.A.Wheeler. It is demonstrated that at low energies $E \ll E_p$, in terms of **measurable** quantities, the *Least Action Principle* and *Noether's Theorem* may be defined quite correctly.
The proposed approach is still in progress and, because of this, the author presents here some part of the earlier obtained results for better understanding: see Subsection 3.1 [4],[5], and Subsection 5.2 (beginning from this

subsection to the formula (68))[5]. In Section 2 the basic definitions and mathematical terms used are given which inevitably have intersections with other publications. It should be noted,however, that the section includes some new, more accurate definitions which are not at variance with the earlier results but clarify them. For example, in [1]–[3] the Uncertainty Principle was initially used for definition of the **measurability** notion. In subsequent papers (for instance, [4],[5]) the author has found the **measurability** definition without the use of this principle.
All other results in the present work are absolutely new.

2 Previous Information and Some Supplements

It is assumed that there is a minimal (universal) unit for measurement of the length ℓ corresponding to some maximal energy $E_\ell = \frac{\hbar c}{\ell}$ and a universal unit for measurement of time $\tau = \ell/c$. Without loss of generality, we can consider ℓ and τ at Plank's level, i.e. $\ell \propto l_p, \tau = \kappa t_p$, where the numerical constant κ is on the order of 1. Consequently, we have $E_\ell \propto E_p$ with the corresponding proportionality factor.
2.1. The **primarily measurable** space-time quantities (variations) are understood as the quantities Δx_i and Δt taking the form

$$\Delta x_i = N_{\Delta x_i}\ell, \Delta t = N_{\Delta t}\tau, \tag{1}$$

where $N_{\Delta x_i}, N_{\Delta t}$ are integer numbers. Further in the text we use both $N_{\Delta x_i}, N_{\Delta t}$ and the equivalent N_{x_i}, N_t.

2.2. Similarly, the **primarily measurable** momenta are considered as a subset of the momenta characterized by the property

$$p_{x_i} \doteq p_{N_{x_i}} = \frac{\hbar}{N_{x_i}\ell}, \tag{2}$$

where N_{x_i} is a nonzero integer number and p_{x_i} is the momentum corresponding to the coordinate x_i.

2.3. Finally, let us define any physical quantity as **primary or elementary measurable** when its value is consistent with point **1.1**,**1.2** of this Definition and formulae (1), (2).
Then we consider formula (2) and **Definition 1.** with the addition of the momenta $p_{x_0} \doteq p_{N_0} = \frac{\hbar}{N_{x_0}\ell}$, where N_{x_0} is an integer number corresponding to the time coordinate ($N_{\Delta t}$ in formula (1)).
For convenience, we denote **Primarily Measurable Quantities** satisfying **2.1**–**2.3** in the abbreviated form as **PMQ**. Also, for the **Primarily Measurable Momenta** we use the abbreviation **PMM.**

First, we consider the case of **Low Energies**, i.e. $E \ll E_p$.
It is obvious that all the nonzero integer numbers N_{x_i}, N_t (or same $N_{x_\mu}; \mu = 0, ..., 3$) from formulae (1),(2) should satisfy the condition $|N_{x_\mu}| \gg 1$. It is clear that all the momenta p_i at **low energies** $E \ll E_p$ meet the condition $p_i = \hbar/(N_i \ell)$, where $|N_i| \gg 1$ but is not necessarily an integer. With regard for smallness of ℓ and for the condition $|N_i| \gg 1$, we can easily show that the difference $1/(N_i\ell) - 1/([N_i]\ell), (\hbar/(N_i\ell) - \hbar/([N_i]\ell))$ is negligible and in this way all momenta in the region of low energies $E \ll E_p$ may be taken as **PMM** with a high accuracy.

Comment*.
Then it should be noted that, as all the experimentally involved energies E are low, they meet the condition $E \ll E_\ell$, specifically for LHC the maximal energies are $\approx 10 TeV = 10^4 GeV$, that is by 15 orders of magnitude lower than the Planck energy $\approx 10^{19} GeV$. But since the energy E_ℓ is on the order of the Planck energy $E_\ell \propto E_p$, in this case all the numbers N_i for the corresponding momenta will meet the condition $min|N_i| \approx 10^{15}$,i.e., the formula of (2).

It is assumed that a theory we are trying to resolve is a deformation of the initial continuous theory.

Comment.**
The deformation is understood as an extension of a particular theory by in-

clusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition [6].

Then it should be noted that **PMQ** is inadequate for studies of the physical processes. In fact, among **PMQ**, we have no quantities «capable» to give the infinitesimal quantities $dx_\mu, \mu = 0, ..., 3$ in the limiting transition in a continuous theory.
Therefore, it is reasonable to use notion of **Generalized Measurability**
We define any physical quantity at all energy scales as **generalized measurable** *or, for simplicity,* **measurable** *if any of its values may be obtained in terms of* **PMQ** *specified by points* **1.1**–**1.3**.
The **generalized measurable** quantities will be denoted as **GMQ**.
Note that the space-time quantities

$$\frac{\tau}{N_t} = p_{N_t c}\frac{\ell^2}{c\hbar}$$
$$\frac{\ell}{N_i} = p_{N_i}\frac{\ell^2}{\hbar}, 1 = 1, ..., 3, \qquad (3)$$

where $p_{N_i}, p_{N_t c}$ are **Primarily Measurable** momenta, up to the fundamental constants, are coincident with $p_{N_i}, p_{N_t c}$ and they may be involved at any stage of the calculations but, evidently, they are not **PMQ**, but they are **GMQ**.
So, in the proposed paradigm at low energies $E \ll E_p$ a set of the **PMM** is discrete, and in every measurement of $\mu = 0, ..., 3$ there is the discrete subset $\mathbf{P}_{\mathbf{x}_\mu} \subset$ **PMM**:

$$\mathbf{P}_{\mathbf{x}_\mu} \doteq \{..., p_{N_{x_\mu}-1}, p_{N_{x_\mu}}, p_{N_{x_\mu}+1}, ...\}. \qquad (4)$$

In this case, as compared to the canonical quantum theory, in continuous space-time we have the following substitution:

$$\mathbf{\Delta p}_\mu \mapsto dp_\mu, \mathbf{\Delta p}_{\mathbf{N}_{\mathbf{x}_\mu}} = \mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}} - \mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1}} = \mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}(\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1})};$$
$$\frac{\mathbf{\Delta}}{\mathbf{\Delta p}_\mu} \mapsto \frac{\partial}{\partial \mathbf{p}_\mu}; \frac{\mathbf{\Delta F}(\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}})}{\mathbf{\Delta p}_\mu} = \frac{\mathbf{F}(\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}}) - \mathbf{F}(\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1}})}{\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}} - \mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1}}} = \frac{\mathbf{F}(\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}}) - \mathbf{F}(\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1}})}{\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}(\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1})}}. \qquad (5)$$

And

$$\frac{\ell}{\mathbf{N_{x_\mu}}} \mapsto dx_\mu;$$

$$\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}} \mapsto \frac{\partial}{\partial x_\mu}, \frac{\mathbf{\Delta F(x_\mu)}}{\mathbf{\Delta_{N_{x_\mu}}}} = \frac{\mathbf{F}(\mathbf{x}_\mu + \ell/\mathbf{N_{x_\mu}}) - \mathbf{F}(\mathbf{x}_\mu)}{\ell/\mathbf{N_{x_\mu}}}. \tag{6}$$

It is clear that for sufficiently high integer values of $|N_{x_\mu}|$, formulae (5),(6) reproduce a continuous paradigm in the momentum space to any preassigned accuracy. However, at low energies $E \ll E_\ell$ a set of **PMM** clearly is not a space. Considering this, the formulae at low energies offer **the Correspondence to Continuous Theory (CCT).**

It is important to make the following remarks in medias res:

Remark 2.1.
In this way any point $\{x_\mu\} \in \mathcal{M} \subset \mathbf{R^4}$ and any set of integer numbers high in absolute values $\{N_{x_\mu}\}$ are correlated with a system of the neighborhoods for this point $(x_\mu \pm \ell/N_{x_\mu})$. It is clear that, with an increase in $|N_{x_\mu}|$, the indicated system converges to the point $\{x_\mu\}$. In this case all the ingredients of the initial (continuous) theory the partial derivatives including are replaced by the corresponding finite differences.

Remark 2.2.
As long as ℓ is a minimal **measurable** length and τ is a minimal **measurable** time, values of all *observable quantities* should agree with this condition, i.e., their expressions should not involve the lengths $l < \ell$ and the times $t < \tau$ (and hence the momenta $p > p_\ell$ and the energies $E > E_\ell$). Because of this, values of the length ℓ/N_i and of the time ℓ/N_t from formula (3) could not appear in expressions for *observable quantities*, being involved only in intermediate calculations, especially at the summation for replacement of the infinitesimal quantities $dt, dx_i; i = 1, 2, 3$ on passage from a continuous theory to its measurable variant.
We can assume that at low energies $E \ll E_\ell$ all the *observable quantities* are **PMQ**.

At **High Energies**, $E \approx E_p$, the **primary measurable** momenta are *inadequate* for studies of the theory at these energies.
Indeed, as it has been shown in [3]– [5], the Generalized Uncertainty Principle (GUP) [7]–[14], that is generalization of the Heisenberg Uncertainty Principle (HUP)

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_p^2 \frac{\Delta p}{\hbar}, \tag{7}$$

where α' is a constant on the order of 1, leading to the minimal length ℓ on the order of the Planck length $\ell \doteq 2\sqrt{\alpha'} l_p$, at high energies inevitably results in the momenta $\Delta p(N_{\Delta x}, GUP)$ which are not **primarily measurable**:

$$\Delta p \doteq \Delta p(N_{\Delta x}, GUP) = \frac{\hbar}{1/2(N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1})\ell}. \tag{8}$$

It is clear that for $N_{\Delta x} \approx 1$ the momentum $\Delta p(N_{\Delta x}, GUP)$ is not a **primary measurable** momentum.

It should be noted that, using relations (7), it is easy to obtain a similar relation for the energy - time pair at least for the ultrarelativistic case $E \approx pc$ [15]. Indeed, (7) gives

$$\frac{\Delta x}{c} \geq \frac{\hbar}{\Delta pc} + \alpha' l_p^2 \frac{\Delta p}{c\hbar}, \tag{9}$$

then

$$\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' \frac{l_p^2}{c^2} \frac{\Delta pc}{\hbar} = \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar}. \tag{10}$$

Here, from $E \approx pc$ it follows that the difference between ΔE and $\Delta(pc)$ can be neglected, i.e. $\Delta E = \Delta(pc) = \Delta pc$. Inequality (10) gives analogously to (7) the lower boundary for time $\Delta t \geq 2t_p$ determining the fundamental time

$$\Delta t_{min} = 2\sqrt{\alpha'} t_p = \tau. \tag{11}$$

Starting in (10) at the **primarily measurable** "small times" $\Delta t = N_{\Delta} t \tau$ (N_{Δ} – small integer number), we can, in analogy with (8), derive a formula for ΔE at high energies $E \approx E_p$.

$$\Delta E \doteq \Delta E(N_{\Delta t}, GUP) = \frac{\hbar c}{1/2(N_{\Delta t} + \sqrt{N_{\Delta t}^2 - 1})\tau}. \tag{12}$$

Naturally, formula (8) represents only a particular case of variations in the **generalized measurable** momenta at high energies $E \approx E_p$. Suppose, we know that in the general case at high energies $E \approx E_p$ minimal variations of momenta are given by a set of the **generalized measurable** quantities $p_{N_{x_\mu}}$, where we have the integer numbers $N_{x_\mu}, |N_{x_\mu}| \approx 1$. Then it is reasonable to assume that minimal variations of "coordinates" at high energies are given by the following formula:

$$l_H(p_{N_{x_\mu}}) \doteq \frac{\ell^2}{\hbar} p_{N_{x_\mu}}, \tag{13}$$

where $p_{N_{x_\mu}}$ are the above-mentioned **generalized measurable** momenta at high energies.

The main target of the author is to form a quantum theory and gravity only in terms of **generalized measurable** quantities (or of **PMQ**).
In conclusion of this Section it may be stated its the principal result is as follows.

Remark 2.3 At low energies far from the Planck energies $E \ll E_p$ we replace the space-time manifold $\mathcal{M} \subseteq \mathcal{R}^4$ by the lattice-like model (denoted by $Latt^{LE}_{\{N_{x_\mu}\}}\mathcal{M}$, where the upper index ^{LE} is the abbreviation for "Low Energies"), with the nodes taken at the points $\{x_\mu\} \in \mathcal{M}$ so that all the edges belonging to $\{x_\mu\}$ have the size ℓ/N_{x_μ},where N_{x_μ} - integers having the property $|N_{x_\mu}| \gg 1$. As the edge lengths ℓ/N_{x_μ}, within a constant factor, are coincident with the **primarily measurable** momenta (formula (3)),the model $Latt^{LE}_{\{N_{x_\mu}\}}\mathcal{M}$ is dynamic and dependent on the existing energies. In this case all the main attributes of a Quantum Theory in the manifold $\mathcal{M}$ have their adequate analogs on the above-mentioned lattice-like model $Latt^{LE}_{\{N_{x_\mu}\}}\mathcal{M}$, giving the low-energy deformation of Quantum Theory in terms of paper [6].

Remark 2.4 At high Planck's energies $E \propto E_p$, the lattice-like model $Latt^{LE}_{\{N_{x_\mu}\}}\mathcal{M}$ is replaced by the lattice-like model $Latt^{HE}_{\{N_{x_\mu}\}}\mathcal{M}$ (the upper index ^{HE} is the abbreviation for "High Energies"), the edges with the lengths ℓ/N_{x_μ} are replaced by those with the lengths $l_H(p_{N_{x_\mu}})$ from formula (13) which, within a constant factor, are coincident with the **generalized measurable** momenta $p_{N_{x_\mu}}$, where N_{x_μ}-integer numbers having the property $|N_{x_\mu}| \approx 1$. In this way $Latt^{HE}_{\{N_{x_\mu}\}}\mathcal{M}$ also represents a dynamic model that is dependent on the existing energies and may be the basis for the construction of a correct variant of the high-energy deformation in Quantum Theory.

Let us call the lattice-like model $Latt^{LE}_{\{N_{x_\mu}\}}\mathcal{M}$ from **Remark 2.3** the low-energy **ℓ/N_{x_μ}-deformation** of space-time manifold $\mathcal{M}$.
Correspondingly let us call the lattice-like model $Latt^{HE}_{\{N_{x_\mu}\}}\mathcal{M}$ from **Remark 2.4** the high-energy **$l_H(p_{N_{x_\mu}})$-deformation** of space-time manifold $\mathcal{M}$.

Remark 2.5
Finally, when at low energies $E \ll E_p$ we lift restrictions on integrality of N_{x_μ}, from formulae (5),(6) it directly follows that in this case we have a continuous analog of the well-known theory with the only difference: all the used small quantities become dependent on the existent energies and we can correlate them.
In this way formula (6) may be written as

$$\begin{aligned} dx_\mu \leftrightarrow \frac{\ell}{\mathbf{N_{x_\mu}}} &\rightarrow \frac{\ell}{[\mathbf{N_{x_\mu}}]}, \\ \frac{\partial}{\partial x_\mu} \leftrightarrow \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}} &\rightarrow \frac{\mathbf{\Delta}}{\mathbf{\Delta_{[N_{x_\mu}]}}} \end{aligned} \tag{14}$$

where $|N_{x_\mu}| \gg 1$ is a sufficiently large number that varies continuously. It is clear that in formula (14) the first arrow corresponds to the continuous theory with a specific selection of values of the infinitesimal quantities dx_μ. As noted above, the difference $\ell/N_{x_\mu} - \ell/[N_{x_\mu}]$ is negligible and hence the second arrow corresponds to passage from the initial continuous theory to a similar discrete theory. Of course, formula (5) may be rewritten in the like manner.
In what follows, formula (14) plays a crucial part in derivation of the results and is greatly important for their understanding.

3 Coordinate Transformations and Poincare Group in Measurable Case

3.1 General Form of Coordinate Transformations in Measurable Format

According to the results from the previous section, the **measurable** variant of gravity at low energies $E \ll E_p$ should be formulated in terms of the **measurable** space-time quantities $\ell/N_{\Delta x_\mu}$ or **primary measurable** momenta $p_{N_{\Delta x_\mu}}$.
Let us consider the case of the random metric $g_{\mu\nu} = g_{\mu\nu}(x)$ [16],[17], where $x \in R^4$ is some point of the four-dimensional space R^4 defined in **measurable** terms. Now, any such point $x \doteq \{x_\chi\} \in R^4$ and any set of integer numbers $\{N_{x_\chi}\}$ dependent on the point $\{x_\chi\}$ with the property $|N_{x_\chi}| \gg 1$ may be correlated to the "bundle" with the base R^4 as follows:

$$B_{N_{x_\chi}} \doteq \{x_\chi, \frac{\ell}{N_{x_\chi}}\} \mapsto \{x_\chi\}. \tag{15}$$

It is clear that $\lim\limits_{|N_{x_\chi}|\to\infty} B_{N_{x_\chi}} = R^4$.
As distinct from the normal one, the "bundle" $B_{N_{x_\chi}}$ is distinguished only by the fact that the mapping in formula (15) is not continuous (smooth) but discrete in fibers, being continuous in the limit $|N_{x_\chi}| \to \infty$.
Then as a *canonically measurable prototype* of the infinitesimal space-time interval square [16],[17]

$$ds^2(x) = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{16}$$

we take the expression

$$\Delta s^2_{N_{x_\chi}}(x) \doteq g_{\mu\nu}(x, N_{x_\chi})\frac{\ell^2}{N_{x_\mu}N_{x_\nu}}. \tag{17}$$

Here $g_{\mu\nu}(x, N_{x_\chi})$ – metric $g_{\mu\nu}(x)$ from formula (16) with the property that minimal **measurable** variation of metric $g_{\mu\nu}(x)$ in point x has form

$$\Delta g_{\mu\nu}(x, N_{x_\chi})_\chi = g_{\mu\nu}(x + \ell/N_{x_\chi}, N_{x_\chi}) - g_{\mu\nu}(x, N_{x_\chi}), \tag{18}$$

Let us denote by $\Delta_\chi g_{\mu\nu}(x, N_{x_\chi})$ quantity

$$\Delta_\chi g_{\mu\nu}(x, N_{x_\chi}) = \frac{\Delta g_{\mu\nu}(x, N_{x_\chi})_\chi}{\ell / N_{x_\chi}}. \tag{19}$$

It is obvious that in the case under study the quantity $\Delta g_{\mu\nu}(x, N_{x_\chi})_\chi$ is a **measurable** analog for the infinitesimal increment $dg_{\mu\nu}(x)$ of the χ-th component $(dg_{\mu\nu}(x))_\chi$ in a continuous theory, whereas the quantity $\Delta_\chi g_{\mu\nu}(x, N_{x_\chi})$ is a **measurable** analog of the partial derivative $\partial_\chi g_{\mu\nu}(x)$.
In this manner we obtain the (15)-formula induced bundle over the metric manifold $g_{\mu\nu}(x)$:

$$B_{g,N_{x_\chi}} \doteq g_{\mu\nu}(x, N_{x_\chi}) \mapsto g_{\mu\nu}(x). \tag{20}$$

Referring to formula (3), we can see that (17) may be written in terms of the **primary measurable** momenta $(p_{N_i}, p_{N_t}) \doteq p_{N_\mu}$ as follows:

$$\Delta s^2_{N_{x_\mu}}(x) = \frac{\ell^4}{\hbar^2} g_{\mu\nu}(x, N_{x_\chi}) p_{N_{x_\mu}} p_{N_{x_\nu}}. \tag{21}$$

Considering that $\ell \propto l_P$ (i.e., $\ell = \kappa l_P$), where $\kappa = const$ is on the order of 1, in the general case (21), to within the constant $\ell^4/\hbar^2$, we have

$$\Delta s^2_{N_{x_\mu}}(x) = g_{\mu\nu}(x, N_{x_\chi}) p_{N_{x_\mu}} p_{N_{x_\nu}}. \tag{22}$$

As follows from the previous formulae, the **measurable** variant of General Relativity should be defined in the bundle $B_{g,N_{x_\chi}}$.
Let us consider any coordinate transformation $x^\mu = x^\mu(\bar{x}^\nu)$ of the space–time coordinates in continuous space—time. Then we have

$$dx^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\nu} d\bar{x}^\nu. \tag{23}$$

As mentioned at the beginning of this section, in terms of **measurable** quantities we have the substitution

$$dx^\mu \mapsto \frac{\ell}{N_{\Delta x_\mu}}; d\bar{x}^\nu \mapsto \frac{\ell}{\bar{N}_{\Delta \bar{x}_\nu}}, \tag{24}$$

where $N_{\Delta x_\mu}, \bar{N}_{\Delta \bar{x}_\nu}$ – integers ($|N_{\Delta x_\mu}| \gg 1, |\bar{N}_{\Delta \bar{x}_\nu}| \gg 1$) sufficiently high in absolute value, and hence in the **measurable** case (23) is replaced by

$$\frac{\ell}{N_{\Delta x_\mu}} = \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu})\, \frac{\ell}{\bar{N}_{\Delta \bar{x}_\nu}}. \tag{25}$$

Equivalently, in terms of the **primary measurable** momenta we have

$$p_{N_{\Delta x_\mu}} = \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu})\, p_{\bar{N}_{\Delta \bar{x}_\nu}}, \tag{26}$$

where $\Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \doteq \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, p_{N_{\Delta x_\mu}}, p_{\bar{N}_{\Delta \bar{x}_\nu}})$ – corresponding matrix represented in terms of **measurable** quantities.
It is clear that, in accordance with formula (3), in passage to the limit we get

$$\lim_{|N_{\Delta x_\mu}| \to \infty} \frac{\ell}{N_{\Delta x_\mu}} = dx^\mu =$$

$$= \lim_{|\bar{N}_{\Delta \bar{x}_\nu}| \to \infty} \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu})\, \frac{\ell}{\bar{N}_{\Delta \bar{x}_\nu}} = \frac{\partial \bar{x}^\mu}{\partial x^\nu}\, dx^\nu. \tag{27}$$

Equivalently, passage to the limit (27) may be written in terms of the **primary measurable** momenta $p_{N_{\Delta x_\mu}}, p_{\bar{N}_{\Delta \bar{x}_\nu}}$ multiplied by the constant $\ell^2/\hbar$.
How we understand formulae (24)–(27)?
The initial (continuous) coordinate transformations $x^\mu = x^\mu(\bar{x}^\nu)$ gives the matrix $\frac{\partial x^\mu}{\partial \bar{x}^\nu}$. Then, for the integers sufficiently high in absolute value $\bar{N}_{\Delta \bar{x}_\nu}, |\bar{N}_{\Delta \bar{x}_\nu}| \gg 1$, we can derive

$$\frac{\ell}{N_{\Delta x_\mu}} = \frac{\partial x^\mu}{\partial \bar{x}^\nu}\, \frac{\ell}{\bar{N}_{\Delta \bar{x}_\nu}}, \tag{28}$$

where $|N_{\Delta x_\mu}| \gg 1$ but the numbers for $N_{\Delta x_\mu}$ are not necessarily integer.
Then using the formula (14) from **Remark 2.5** and substitution of $[N_{\Delta x_\mu}]$ for $N_{\Delta x_\mu}$ in the left-hand side of (28) leads to replacement of the initial matrix $\frac{\partial x^\mu}{\partial \bar{x}^\nu}$ by the matrix $\Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu})$ represented in terms of **measurable** quantities and, finally, to the formula (25). Clearly, for sufficiently high $|N_{\Delta x_\mu}|, |\bar{N}_{\Delta \bar{x}_\nu}|$, the matrix $\Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu})$ may

be selected no matter how close to $\frac{\partial x^{\mu}}{\partial \bar{x}^{\nu}}$.
Similarly, in the **measurable** format we can get the formula

$$d\bar{x}^{\mu}=\frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}}\,dx^{\nu}. \tag{29}$$

and correspondingly the matrix $\widetilde{\Delta_{\mu\nu}}(x^{\mu},\bar{x}^{\nu},1/N_{\Delta x_{\mu}},1/\bar{N}_{\Delta \bar{x}_{\nu}})$ instead of the matrix $\Delta_{\mu\nu}(x^{\mu},\bar{x}^{\nu},1/N_{\Delta x_{\mu}},1/\bar{N}_{\Delta \bar{x}_{\nu}})$.
Thus, any coordinate transformations may be represented, to however high accuracy, by the **measurable** transformation (i.e., written in terms of **measurable** quantities), where the principal components are the **measurable** quantities $\ell/N_{\Delta x_{\mu}}$ or the **primary measurable** momenta $p_{N_{\Delta x_{\mu}}}$.

3.2 Poincare Invariance and Its Specialities in Measurable Consideration

It is obvious that all the derivations for general coordinate transformations and for a random metric are valid for the Lorentz transformations and Minkowskian metric.
Actually, according to the preceding subsection, a *canonically measurable prototype* of the **relativistic** infinitesimal space-time interval square

$$ds^2=\eta_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{30}$$

is given by

$$\Delta s^2_{N_{x_{\chi}}}(x)\doteq \eta_{\mu\nu}(x,N_{x_{\chi}})\frac{\ell^2}{N_{x_{\mu}}N_{x_{\nu}}}, \tag{31}$$

where $\eta_{\mu\nu}$ is the Minkowskian metric

$$||\eta_{\mu\nu}||=||\eta^{\mu\nu}||=Diag\,(1,-1,-1,-1)\,. \tag{32}$$

Here the integers $N_{x_{\chi}}$ naturally satisfy the condition $|N_{x_{\chi}}|\gg 1$, components of the **measurable** Minkowskian metric $\eta_{\mu\nu}(x,N_{x_{\chi}})$ are "close" to $\eta_{\mu\nu}$,i.e. we have

$$\lim_{(|N_{x_{\chi}}|)\rightarrow\infty}\eta_{\mu\nu}(x,N_{x_{\chi}})=\eta_{\mu\nu}. \tag{33}$$

Without loss of generality, we can assume that $\eta_{\mu\nu}(x, N_{x_\chi}) = 0, \mu \neq \nu$.
Returning to Subsection 3.1, we suppose that $\mathrm{g} \in LG$ is a random element of the Lorentz Group (LG) acting linearly in space time with the coordinates $\bar{x}$. g is represented by the matrix $(\mathrm{g}_{\mu\nu})$.
Applying to the case of plane geometry under consideration all argumentations from Subsection 3.1., specifically **Remark 2.5** and hence formulae (28) and (25), we get the following:

$$\frac{\ell}{N_{\Delta x_\mu}} = \mathrm{g}_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu})\,\frac{\ell}{\bar{N}_{\Delta \bar{x}_\nu}}. \tag{34}$$

Here, with the symbols used, we have $N_{\Delta x_\chi} \doteq N_{x_\chi}, \bar{N}_{\Delta \bar{x}_\chi} \doteq \bar{N}_{\bar{x}_\chi}$ and

$$\lim_{|\bar{N}_{\bar{x}_\nu}| \to \infty} \mathrm{g}_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{x_\mu}, 1/\bar{N}_{\bar{x}_\nu}) = \mathrm{g}_{\mu\nu}. \tag{35}$$

From formula (34) it follows that large integer numbers $|\bar{N}_{\bar{x}_\nu}|$ generate large integer $|N_{x_\mu}|$. As follows from (35) and **Remark 2.5**, at sufficiently large integers $|\bar{N}_{\bar{x}_\nu}|, |N_{x_\mu}|$, however the accuracy, we have the equality

$$\mathrm{g}_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{x_\mu}, 1/\bar{N}_{\bar{x}_\nu}) = \mathrm{g}_{\mu\nu}, \tag{36}$$

and also the equality

$$\eta_{\mu\nu}(\bar{x}, \bar{N}_{\bar{x}_\chi})\frac{\ell^2}{\bar{N}_{\bar{x}_\mu}\bar{N}_{\bar{x}_\nu}} = \eta_{\mu\nu}(x, N_{x_\chi})\frac{\ell^2}{N_{x_\mu}N_{x_\nu}}, \tag{37}$$

where

$$\lim_{|\bar{N}_{\bar{x}_\chi}| \to \infty} \eta_{\mu\nu}(\bar{x}, \bar{N}_{\bar{x}_\chi}) = \lim_{|N_{x_\chi}| \to \infty} \eta_{\mu\nu}(x, N_{x_\chi}) = \eta_{\mu\nu}. \tag{38}$$

In this way we can obtain the *relativistic invariance* in a **measurable** form for flat case, i.e. for Minkowskian space-time.
It is clear that translations in time and space add nothing new to these calculations and hence all the above arguments are valid for the Poincare group as well.

Remark 3.1. Any space-time coordinate x_μ e can express in terms of

measurable quantities, no matter how high the accuracy. This trivially follows from the fact that any real number may be approximated by rational numbers to the accuracy however high.

Remark 3.2. Note that in this section we have studied *only* the problem of actions associated with the group of general coordinate transformations and the Poincare group in space-time at low energies $E \ll E_p$ in terms of **measurable** quantities, without reference to the invariance problem.

4 Remark on Least Action Principle and Noether's Theorem in Measurable Form

Considerations of Section 2 point to the fact that the Least Action Principle and Noether's Theorem at low energies $E \ll E_p$ are valid in a **measurable** form with substitution of the measurable analogs defined in Section 2 for all the components involved in proof of these arguments. For the canonical (continuous) case we use the notation of Section 3 in [18].
Let φ be a set of all the considered fields $\varphi \doteq (\varphi_1, \varphi_2$. Then the action S in the continuous case taking the form

$$S = \int \mathcal{L}(\varphi, \partial_\mu \varphi) d^4 x \tag{39}$$

is replaced by the **measurable** action $S_{meas,N}$

$$S_{meas,\{N\}} = \sum \mathcal{L}_{meas,\{N\}}(\varphi, \frac{\boldsymbol{\Delta}\varphi}{\boldsymbol{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}) \prod \frac{\ell}{N_{x_\mu}}, \tag{40}$$

where N_{x_μ} – integers with the property $|N_{x_\mu}| \gg 1$,$\mathcal{L}_{meas,N}$–Lagrangian density of the **measurable** fields φ and of their **measurable** analogs for partial derivatives in formula (6) $\frac{\boldsymbol{\Delta}\varphi}{\boldsymbol{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}$. This means that all variations of these functions are expressed in terms of only **measurable** quantities. In the product $\prod$ the index μ takes the values $\mu = 0, ..., 3$, and $\{N\}$–collection of all N_{x_μ},i.e. $\{N\} \doteq \{N_{x_\mu}\}$. Further, where this causes no confusion, for the **measurable** quantities corresponding to the set $\{N\}$ we can equally use

both the lower index $\{N\}$ and N.
According to **Remark 2.1.** and **Remark 2.5.**, for the integer numbers N_{x_μ} sufficiently high in absolute value we, to a high accuracy, have

$$S = S_{meas,\{N\}}. \tag{41}$$

Then it is assumed that all the considered functions are **measurable**, i.e. all variations of these functions are expressed in terms of only **measurable** quantities.
In this case the ordinary variations $\delta x_\mu, \delta\varphi$ going to zero at the boundary $\partial\mathcal{R}$ of the four-dimensional region $\mathcal{R}$ are replaced by **measurable** variations $(\delta x_\mu)_{meas}, (\delta\varphi)_{meas}$ with the same property. The **measurable** complete field variation φ denoted as $(\Delta\varphi)_{meas}$ in the first-order approximation for $(\delta x_\mu)_{meas}$ takes the form

$$(\Delta\varphi)_{meas} = (\delta\varphi)_{meas} + \frac{\mathbf{\Delta}\varphi}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}(\delta x_\mu)_{meas}. \tag{42}$$

As follows from **Remark 2.5.**, for N_{x_μ} sufficiently large in absolute value formula (42) correlates (to a high accuracy) with the well-known formula $\Delta\varphi$ in the case of complete variation in a continuous variant

$$\Delta\varphi = \delta\varphi + (\partial_\mu\varphi)\delta x_\mu. \tag{43}$$

Similarly, we can find the **measurable** variation $(\delta S_{meas,\{N\}})_{meas}$ for the action $S_{meas,\{N\}}$ from formula (40), making substitutions relative to the continuous pattern as in formula (40)

$$\int \mapsto \sum; \partial_\mu \mapsto \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}; d^4x \mapsto \prod \frac{\ell}{N_{x_\mu}}, ... \tag{44}$$

and replacing the expression $d^4x^{'} = J(x^{'}/x)d^4x$, where $J(x^{'}/x)$ –Jacobian transformations of $x \to x^{'} = x + \delta x$ in the continuous case, by the formula $\prod \frac{\ell}{N_{x^{'}_\mu}} = J_{meas}(x^{'}/x) \prod \frac{\ell}{N_{x_\mu}}$, where $J_{meas}(x^{'}/x)$ – **"measurable"** Jacobian corresponding to the matrix $(\Delta_{\mu\nu})$ of the transformation $x \to x^{'} = x + (\delta x)_{meas}$ in **measurable** consideration from formula (25). With regard

to **Remark 2.5.**, we can see that in this way in **measurable** consideration one can reproduce the results of a continuous picture for the integer numbers N_{x_μ} sufficiently high in absolute value to any preassigned accuracy.
In this manner, using the infinitesimal quantities dx_μ of the form ℓ/N_{x_μ}, where N_{x_μ} – real numbers sufficiently high in absolute value, and then **Remark 2.5**, we can take all the steps to the proof of the Variance Principle (including Gauss theorem) to any accuracy and obtain the canonical Euler-Lagrange equations of the **measurable** form

$$\frac{\partial \mathcal{L}_{meas,N}}{\partial \varphi} - \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}[\frac{\partial \mathcal{L}_{meas,N}}{\partial(\frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}\varphi)}] = 0. \tag{45}$$

For the above-mentioned conditions, these equations give very exact approximation of Euler-Lagrange equations in the continuous paradigm

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial x_\mu}[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}] = 0. \tag{46}$$

Noether's Theorem may be represented in the **measurable** form in a similar way.
In this case the energy-momentum tensor Θ

$$\Theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi - \delta^\mu_\nu \mathcal{L} \tag{47}$$

in the **measurable** format, similar to (45), takes the form

$$(\Theta_{meas,N})^\mu_\nu = [\frac{\partial \mathcal{L}_{meas,N}}{\partial(\frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}\varphi)}]\frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\nu}}}\varphi - \delta^\mu_\nu \mathcal{L}_{meas,N}. \tag{48}$$

If the action S from formula (39) is invariant by some transformation group $\mathcal{G}$ involving x^μ and φ, then $S_{meas,\{N\}}$ from formula (40) for the components of the set $\{N\}$ sufficiently large in absolute value are invariant by the action $\mathcal{G}$ at the accuracy however high. This is obvious if we naturally suppose that the action $\mathcal{G}$ for the fields φ in the general and in the **measurable** *considerations is identical, whereas for the coordinates x^μ,with regard to*

Remark 3.1. *and* **Remark 2.5.**, *the action may be considered identical too for the components of the set* $\{N\}$ sufficiently large in absolute value.
Proceeding from the paragraph indicated by italics, we can repeat all the steps of the proof for Noether's Theorem in the **measurable** form with the corresponding substitutions form formula refS3.m).
Then for the **"measurable"** currents $(J_{meas,N})^{\mu}_{\nu})$, to a high accuracy, we have

$$\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_t}}}\sum(J_{meas,N})^0_\nu \prod_{i=1}^{3}\frac{\ell}{N_{x_i}} = \frac{\mathbf{\Delta}(\mathbf{Q_{meas,N}})_\nu}{\mathbf{\Delta_{N_t}}} = 0. \quad (49)$$

And formula (49)for the components of the set $\{N\}$ sufficiently high in absolute value reproduces Noether's Theorem in the canonical form to any preassigned accuracy

$$\frac{d}{dt}\int J^0_\nu d^3x = \frac{dQ_\nu}{dt} = 0. \quad (50)$$

5 Measurability,Gauge Fields,Gravity and Transition to High Energies

5.1 Measurability for Gauge Theories at Low Energies

In this section we use the formalism from [18],[19].
It is easily seen that at low energies $E \ll E_p$ for the gauge theories written in the **measurable** form all formula of the canonical (continuous) theory are valid with the corresponding substitution according to formulae (5),(6),(44). Indeed, let **G** – gauge group and $\{N\} \doteq \{N_{x_\mu}\}$, similar to formulae from the preceding section,– fixed set of the integers $|N_{x_\mu}| \gg 1$ sufficiently large in absolute value.
As **G** - group of the local internal symmetries of a physical system and the definition of **measurability** refers only to the space-time indexes, we can

get the following correspondences:

$$\mathbf{W}'_\mu = U\,\mathbf{W}_\mu\,U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \mapsto \mathbf{W}'_{\mu,\{N\}} \doteq$$
$$\doteq U\,\mathbf{W}_{\mu,\{N\}}, U^{-1} - \frac{i}{g}(\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}}U)U^{-1},$$
$$D_\mu = \partial_\mu - ig\,\mathbf{W}_\mu \mapsto D_{\mu,\{N\}} \doteq$$
$$\doteq \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}} - ig\,\mathbf{W}_{\mu,\{N\}},$$
$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu - i\,g\,[\mathbf{W}_\mu, \mathbf{W}_\nu] \mapsto \mathbf{F}_{\mu\nu,\{N\}} \doteq$$
$$\doteq \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}}\mathbf{W}_{\nu,\{N\}} - \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\nu}}}}\mathbf{W}_{\mu,\{N\}} - i\,g\,[\mathbf{W}_{\mu,\{N\}}, \mathbf{W}_{\mu,\{N\}}]. \quad (51)$$

And, similarly, we have

$$\overline{\Psi}\,(i\gamma^\mu D_\mu - m)\Psi \mapsto \overline{\Psi}_{\{N\}}\,(i\gamma^\mu D_{\mu,\{N\}} - m)\Psi_{\{N\}}. \quad (52)$$

Here g is a coupling constant,$\mathbf{W}_\mu$ – space-time components of gauge fields, $\Psi, \overline{\Psi}$–corresponding material fields (in this case fermion),D_μ–covariant derivative and U - element of the gauge group $\mathbf{G}$.
Passage in formulae (51),(52) from the left- to the right-hand side is associated with the transition from the canonical (continuous) consideration to the representation in terms of **measurable** quantities for the fixed set $\{N\} \doteq \{N_{x_\mu}\}$. It is clear that in this case all the transformable quantities in the right-hand sides of these formulae should depend on $\{N\}$,that is indicated by the additional lower index $\{N\}$. In a similar way, **the "measurable"** metric $g_{\mu\nu}(x, N_{x_\chi}) \equiv g_{\mu\nu}(x, \{N\})$ from formula (17) is dependent on $\{N\}$.
However, considering that the energies are low and the numbers $|N_{x_\mu}| \gg 1$ are sufficiently high, the above-mentioned relationship is very weak.
As follows from formulae (51),(52) and from the paragraph preceding these formulae, if $\mathcal{L}$ – gauge-invariant Lagrangian associated with the left-hand sides of these formulae, the corresponding Lagrangian given in terms of **measurable** quantities $\mathcal{L}_{meas,\{N\}}$ is also gauge-invariant by $\mathbf{G}$ and we have

$$\mathcal{L} \approx \mathcal{L}_{meas,\{N\}}. \quad (53)$$

Besides, from the above formulae it follows that all the known relations for the gauge theory with the group $\mathbf{G}$ are valid, to a high accuracy, at low energies for a **measurable** variant of this theory on replacement of all basic quantities in the initial theory by the corresponding quantities with the additional lower index $\{N\}$.
Specifically, the "gauge" analog *Bianchi identity*

$$D_\rho \mathbf{F}_{\mu\nu} + D_\mu \mathbf{F}_{\nu\rho} + D_\nu \mathbf{F}_{\rho\mu} = 0 \tag{54}$$

in the **measurable** form is replaced, to a high accuracy, by the identity

$$D_{\rho,\{N\}} \mathbf{F}_{\mu\nu,\{N\}} + D_{\mu,\{N\}} \mathbf{F}_{\nu\rho,\{N\}} + D_{\nu,\{N\}} \mathbf{F}_{\rho\mu,\{N\}} = 0. \tag{55}$$

Obviously, this accuracy is the higher the greater the absolute values of the numbers from the set $\{N\}$.
Similar to the canonical case, formula (54) is equivalent to the *Jacoby identity*

$$\sum_{cyclic\, permutations} [D_\rho, [D_\mu, D_\nu]] = 0, \tag{56}$$

in the **measurable** consideration formula (55) to a high accuracy is equivalent to the **measurable** form of *Jacoby identity*

$$\sum_{cyclic\, permutations} [D_{\rho,\{N\}}, [D_{\mu,\{N\}}, D_{\nu,\{N\}}]] = 0. \tag{57}$$

5.2 General Relativity in Terms of Measurable Quantities and Its High-Energy Deformations

At low energies $E \ll E_p$ for connectivity coefficients in gravity, i.e. *Christoffel symbols*, and for the fixed set $\{N\}$ in his papers [4],[5] the author has derived their expressions in the **measurable** form (formula (50) in [5]):

$$\Gamma^\alpha_{\mu\nu}(x,\{N\}) = \frac{1}{2} g^{\alpha\beta}(x,\{N\}) \left(\Delta_\nu g_{\beta\mu}(x,\{N\}) + \Delta_\mu g_{\nu\beta}(x,\{N\}) - \right.$$
$$\left. - \Delta_\beta g_{\mu\nu}(x,\{N\})\right). \tag{58}$$

Here, to make it short, the author denotes the operator $\mathbf{\Delta}/\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\chi}}$ from formula (6) as Δ_χ, and N_{x_χ}–corresponding element from the set $\{N\}$.
In [4],[5] it is shown that, with the use of (58) in the **measurable** form, one can obtain all the base quantities of General Relativity (GR), in particular the *Riemann tensor* $R^{\mu}{}_{\nu\alpha\beta}(x,\{N\})$ and, finally, the **measurable** form of Einstein Equations, for short denoted as $(\mathcal{EEM})$ (abbreviation for *Einstein Equations Measurable*) (formula (57) in [5]):

$$R_{\mu\nu}(x,\{N\}) - \frac{1}{2}\,R(x,N_{x_\chi})\,g_{\mu\nu}(x,\{N\}) - \frac{1}{2}\,\Lambda(x,\{N\})\,g_{\mu\nu}(x,\{N\}) = \\ = 8\,\pi\,G\,T_{\mu\nu}(x,\{N\}). \quad (59)$$

Considering the properties of $\{N\}$, for the **measurable** form of GR the *Bianchi identity* may be written, to a high accuracy, as follows:

$$\widetilde{D}_{\rho,\{N\}}R^{\chi}_{\lambda\mu\nu}(x,\{N\}) + \widetilde{D}_{\mu,\{N\}}R^{\chi}_{\lambda\mu\rho}(x,\{N\}) + \widetilde{D}_{\nu,\{N\}}R^{\chi}_{\nu\alpha\beta}(x,\{N\}) = 0, \quad (60)$$

where $\widetilde{D}_{\alpha,\{N\}} = \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\alpha}}} + \Gamma^{\mu}_{\nu\alpha}(x,\{N\})$ and $N_{x_\alpha} \in \{N\}$.

Thus, at low energies in **measurable** consideration, as in the canonical case, there is correlation between gauge theories and gravity.
But, in principle, the understanding of "high energies" in gravity and in gauge theories is different. According to the current knowledge, in gravity these energies are at a level of the Planck energies $E \approx E_p$ (or same $E \approx E_\ell$) which are associated with origination of the quantum-gravitational effects.
In [4],[5], using the definitions given in **Comment****, the author has constructed a high-energy (Planck) deformation of GR of the form

$$\begin{aligned} \mathcal{EEM}[N_q] \doteq R_{\mu\nu}(x,\{N_q\}) - \frac{1}{2}\,R(x,\{N_q\})\,g_{\mu\nu}(x,\{N_q\}) - \\ - \frac{1}{2}\,\Lambda(x,\{N_q\})\,g_{\mu\nu}(x,\{N_q\}) = \\ = 8\,\pi\,G\,T_{\mu\nu}(x,\{N_q\}). \end{aligned} \quad (61)$$

Here $\{N_q\} \doteq \{N_{x_\chi}\}, \chi = 0,...,3$ is a set of the integer numbers N_{x_χ} the absolute values of which are close to 1.
The small quantity $\ell/N_{x_\chi} = \frac{\ell^2}{\hbar}p_{N_{x_\chi}}$,where $p_{N_{x_\chi}}$ is a **primarily measurable**

momentum and $|N_{x_\chi}|\gg 1$, at low energies $E\ll E_\ell$ in the case under study has its analog–the quantity $l_H(p_{N_{x_\mu}})$that is given by formula (13) in the present paper (or formula 113) in [5]).
As absolute values of the integers N_{x_μ} are small, the quantities $l_H(p_{N_{x_\mu}})$ are varying discretely (similar to the denominator in the right-hand side of formula (8)) and hence the high-energy deformation of GR specified by $\mathcal{EEM}[N_q]$ (formula (61)) is in fact a *discrete* theory.
It is clear that in this case the limit

$$p_{N_{x_\chi}},(|N_{x_\chi}|\approx 1)\stackrel{|N_{x_\chi}|\approx 1\rightarrow|N_{x_\chi}|\gg 1}{\Rightarrow}p_{N_{x_\chi}},(|N_{x_\chi}|\gg 1), \tag{62}$$

where momenta in the right-hand side of formula (62), i.e. $p_{N_{x_\chi}},(|N_{x_\chi}|\gg 1)$, are the **primarily measurable** momenta at low energies $E\ll E_p$ and $p_{N_{x_\chi}},(|N_{x_\chi}|\approx 1)$ – corresponding **generalized measurable** momentum from formula (13, should be valid. Obviously, the momentum from formula (8)for $N_{\Delta x}\doteq N_{x_\chi}$ satisfies this condition.
Then formula (17) for the *canonically measurable prototype* of the infinitesimal space-time interval at low energies $E\ll E_p$ is replaced by its quantum analog or the *canonically measurable quantum prototype* for $E\approx E_p$ taking the form

$$\Delta s^2_{\{N\}}(x,\mathbf{q})\doteq g_{\mu\nu}(x,\{N\},\mathbf{q})l_H(p_{N_{x_\mu}})l_H(p_{N_{x_\nu}})=\frac{\ell^4}{\hbar^2}g_{\mu\nu}(x,\{N\},\mathbf{q})p_{N_{x_\mu}}p_{N_{x_\nu}}. \tag{63}$$

Here there is no doubt that the numbers N_{x_μ},N_{x_ν} belong to the set $\{N\}$, all the components of this set are integers with small absolute values, $p_{N_{x_\chi}}$ are the **generalized measurable** momenta at high energies corresponding to formula (62) and $g_{\mu\nu}(x,\{N\},\mathbf{q})$ meets the condition

$$g_{\mu\nu}(x,\{N\},\mathbf{q}),(|\{N\}|\approx 1)\stackrel{|\{N\}|\approx 1\rightarrow|\{N\}|\gg 1}{\Rightarrow}g_{\mu\nu}(x,\{N\}),(|\{N\}|\gg 1), \tag{64}$$

where $g_{\mu\nu}(x,\{N\})=g_{\mu\nu}(x,N_{x_\chi})$ is a metric in the **measurable** form at low energies (formula (17)).
From formula (62) we have

$$l_H(p_{N_{x_\chi}})\doteq\frac{\ell^2}{\hbar}p_{N_{x_\chi}};|N_{x_\chi}|\approx 1. \tag{65}$$

Then by the substitution $\ell/N_{x_\chi} \mapsto l_H(p_{N_{x_\chi}})$ in formulae (18),(19) we can have quantum analogs of *minimal measurable variations* of the metric and of the partial derivative

$$\Delta_{\mathbf{q}} g_{\mu\nu}(x, N_{x_\chi}, \mathbf{q})_\chi \doteq g_{\mu\nu}(x + l_H(p_{N_{x_\chi}}), N_{x_\chi}, \mathbf{q}) - g_{\mu\nu}(x, N_{x_\chi}, \mathbf{q}),$$
$$\Delta_{\chi,\mathbf{q}} g_{\mu\nu}(x, N_{x_\chi}, \mathbf{q}) \doteq \frac{\Delta_{\mathbf{q}} g_{\mu\nu}(x, N_{x_\chi}, \mathbf{q})_\chi}{l_H(p_{N_{x_\chi}})}. \qquad (66)$$

Using the substitution in formula (6)

$$\frac{\ell}{\mathbf{N}_{\mathbf{x}_\mu}} \mapsto l_H(p_{N_{x_\mu}}); \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}} \mapsto \frac{\mathbf{\Delta}_{\mathbf{q}}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu},\mathbf{q}}},$$
$$\frac{\mathbf{\Delta}_{\mathbf{q}}\mathbf{F}(\mathbf{x}_\mu)}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu},\mathbf{q}}} = \frac{F(x_\mu + l_H(p_{N_{x_\mu}})) - F(x_\mu)}{l_H(p_{N_{x_\mu}})} \qquad (67)$$

and applying this substitution to all corresponding formulae in the **measurable** format of GR at low energies, we can derive at planck energies $E \approx E_p$ all the components high-energy deformation of Einstein Equations in the **measurable** form $\mathcal{EEM}[N_q]$ (61) (or formula (117) in [5])
As a result, we have

$$\lim_{E \ll E_p} \mathcal{EEM}[N_q] = \mathcal{EEM} \quad or \lim_{|\{N_q\}| \gg 1} \mathcal{EEM}[N_q] = \mathcal{EEM}. \qquad (68)$$

For $\mathcal{EEM}[N_q]$, the metrics $g_{\mu\nu}(x, N_{x_\chi}, \mathbf{q})$ (formula (63)) represent the solution.

It should be noted that the proposed approach can be considered as a development of the idea of *quantum fluctuations* in the space-time geometry (**"space-time foam"**) [20]–[22]but for the case of **discrete consideration**. Really, at low energies $E \ll E_p$ the canonical metric components in a continuous consideration $g_{\mu\nu}(x)$ may be taken as components of the metric in the **measurable** form $g_{\mu\nu}(x, N_{x_\chi})$ (formula (17) for $N_{x_\chi} = \infty$, i.e. we have $g_{\mu\nu}(x) = g_{\mu\nu}(x, \infty)$). But, as at low energies $|N_{x_\chi}| \gg 1$, the theory may be considered continuous to a high accuracy due to **Remark 2.5**. Then, expanding the quantity $g_{\mu\nu}(x, N_{x_\chi})$ into a series in terms of the small parameter $1/N_{x_\chi}$ close to the point $g_{\mu\nu}(x)$ and retaining only the zero- or first-order terms (due to obvious smallness of all the remaining terms), in fact, we arrive at the formula for fluctuation of the metric g in a region with the size L ([22],formula (43.29)):

$$\Delta g \sim \frac{l_p}{L}. \tag{69}$$

Indeed, as $l_p \propto \ell$, considering that the energies are low and with due regard for **Remark 2.2**, L represents **PMQ**. Then, setting $L = N_{x_\chi}\ell$ and substituting it into (69),we get the following:

$$\Delta g \sim \frac{l_p}{L} \sim \frac{\ell}{N_{x_\chi}\ell} = \frac{1}{N_{x_\chi}}. \tag{70}$$

So, at low energies the indicated *quantum fluctuations* are very small, actually being coincident with the basic parameters in the **measurable** approach (parameters of the corresponding deformation).
But, as demonstrated by formulae (61)–(67), at high energies $E \approx E_p$ this is not the case, and *quantum fluctuations*
$g_{\mu\nu}(x, \{N\}, \mathbf{q}), (|\{N\}| \approx 1)$ of the metric $g_{\mu\nu}(x, \{N\}), (|\{N\}| \gg 1)$ are great.
In this case in the **measurable** form the notion **"space-time foam"**) is absolutely adequate because the only restriction imposed on $g_{\mu\nu}(x, \{N\}, \mathbf{q}), (|\{N\}| \approx 1)$ is (64). It is clear that in this case there is a great deal of different $g_{\mu\nu}(x, \{N\}, \mathbf{q}), (|\{N\}| \approx 1)$. As the **measurable** analogs of Einstein Equations at low energies $\mathcal{EEM}$ (59) and at high energies $\mathcal{EEM}[N_q]$ (61), according to the above formulae, are determined by the quantities $p_{N_{x_\chi}}$, where $|N_{x_\chi}| \gg 1, |N_{x_\chi}| \approx 1$, respectively, at low energies for the given metric $g_{\mu\nu}(x, \{N\}, \mathbf{q}), (|\{N\}| \gg 1)$ its *quantum fluctuations* in the general case are determined by the functions $\mathcal{G}_\mu(N_{x_\mu}), \mu = 0, ..., 3$ which are dependent on integer values of N_{x_μ} so that

$$p_{N_{x_\mu}} \doteq \frac{\hbar}{\mathcal{G}_\mu(N_{x_\mu})\ell}, \tag{71}$$

and

$$\lim_{|N_{x_\mu}| \to \infty} \mathcal{G}_\mu(N_{x_\mu}) = N_{x_\mu}. \tag{72}$$

We can see that the functions $\mathcal{G}(\Delta x) \doteq 1/2(N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1})\ell; N_{\Delta x} \doteq N_{x_i}, i = 1,2,3$ from the right side of formula (8) and $\mathcal{G}(\Delta t) \doteq 1/2(N_{\Delta t} + \sqrt{N_{\Delta t}^2 - 1})\tau; N_{\Delta \tau} \doteq N_{x_0}$ from the right side of formula (12) satisfy the condition of (72).
In [5] at low energies $E \ll E_p$ for the **measurable** form of gravity $\mathcal{EEM}$ (59) the author has derived the Least Action Principle and the Lagrangian formalism (particular case in the first part of Section 4 in the present paper). The action for GR in the **measurable** format can be derived from the action for the canonical GR in continuous space-time

$$S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{|g|}\ (R + \Lambda) \tag{73}$$

with substitution in formula (44), leading to the **"measurable"** action

$$\begin{aligned} S_{EH}(N_{x_\chi}) = -\frac{1}{16\pi G} \sum \Delta_{(N_{x_\chi})}\Omega \sqrt{|g(N_{x_\chi})|}\, \cdot \\ \cdot \left(R(x, N_{x_\chi}) + \Lambda(x, N_{x_\chi})\right), |N_{x_\chi}| \gg 1, \end{aligned} \tag{74}$$

where $\Delta_{(N_{x_\chi})}\Omega$ is the volume element in a **measurable** variant of GR (formula (44)-(46) in [5]).
It is obvious that at high energies $E \approx E_p$, due to real discreteness of the theory, the Least Action Principle in the general case is no longer valid for this theory. We can note only the Planck deformation $S_{EH}(N_{x_\chi}, q)$ of the **"measurable"** action $S_{EH}(N_{x_\chi})$ (74):

$$\begin{aligned} S_{EH}(N_{x_\chi}, \mathbf{q}) \doteq -\frac{1}{16\pi G} \sum \Delta_{(N_{x_\chi}),\mathbf{q}}\Omega \sqrt{|g(N_{x_\chi}, \mathbf{q})|}\, \cdot \\ \cdot \left(R(x, N_{x_\chi}) + \Lambda(x, N_{x_\chi}, \mathbf{q})\right), |N_{x_\chi}| \approx 1, \end{aligned} \tag{75}$$

with substitution of all components in formula (74) in accordance with the formulae in this subsection.
Of course, in this case the condition

$$S_{EH}(N_{x_\chi}, \mathbf{q}), (|N_{x_\chi}| \approx 1) \stackrel{|N_{x_\chi}| \approx 1 \rightarrow |N_{x_\chi}| \gg 1}{\Rightarrow} S_{EH}(N_{x_\chi}), (|N_{x_\chi}| \gg 1) \tag{76}$$

must be fulfilled. It should be noted that the above-mentioned results may be applied for the derivation of a **measurable** variant of gravitational thermodynamics for horizon spaces [3].

Specifically, T.Padmanbhan has obtained, for space with static spherically-symmetric horizon at the horizon $r = a$, Einstein field equations ([23], eq.(117)) taking the form

$$\frac{c^4}{G}\left[\frac{1}{2}f'(a)a-\frac{1}{2}\right]=4\pi Pa^2, \tag{77}$$

where $P = T^r_r$ is the trace of the momentum-energy tensor and radial pressure. And

$$k_BT=\frac{\hbar cf'(a)}{4\pi}. \tag{78}$$

where T – corresponding temperature for the horizon spaces [23].
Naturally, in terms of **measurable** quantities it is assumed that the radius $r = a$ is a **primarily measurable** quantity,i.e. $a = N_a\ell$. Then Einstein equations for the spherically-symmetric horizon spaces [23],derived in the **measurable** form in [3] and written at low energies ($N_a \gg 1$) in terms of the parameter $\alpha_a(HUP) \doteq 1/N_a^2$ or at high energies ($N_a \approx 1$) in terms of the parameter $\alpha_a(GUP) \doteq 1/[1/4(N_a + \sqrt{N_a^2-1})^2]$, completely comply with their general form (59),(61).
Besides, in accordance with **Remark 2.2**, the condition $N_a \geq 2$ should be fulfilled. This fact was also noted in [24],[15], however, on the basis of another approach. Besides, in terms of **measurable** quantities and in [3], some implications for gravitational thermodynamics of black holes [25] at all the energy scales have been suggested.

5.3 Gauge Theories in Measurable Consideration and Transition to High Energies

We assume that at high energies E (close to the Planck energy $E \approx E_p$ or same maximal $E \approx E_\ell$) space-time is always **curved**. Because of this, we should consider three different possibilities.

5.3.1. Low energies $E \ll E_p$ and flat space-time.
In the well-known Quantum Field Theory (QFT) [19],[18] and, specifically, in its part used for the collider computations, in the general case space-time is assumed to be flat, i.e. to be Minkowskian.

Besides, as noted in **Comment***, actually all the energies considered experimentally meet the condition $E \ll E_p$ and hence (see the end of **Remark 2.2** in **measurable** consideration all *observable quantities* are **PMQ**.
In this case we have a *discrete* QFT that is *almost-continuous* due to **Remark 2.5**. As such a theory in the momentum representation has the upper limit cut-off, it is not Lorentz-invariant from the start. This is not surprising because it is known that, if a theory involves the minimal length ℓ, in the general case Lorentz invariance is violated (for example, see most known work of KMM [14]). As distinct from other works involving ℓ, in the proposed approach the wave function is considered separately at high energies $E \approx E_p$ and at low energies $E \ll E_p$, with the imposed restriction that the first function is a high-energy deformation of the second function [2]. In other works (for example, in [14]) the wave function is common for all the energy scales. But, considering the assumption in the beginning of this subsection, this is impossible because the indicated functions belong to spaces of different geometries: curved and flat.
It is clear that the above-mentioned *discrete* (*almost-continuous*) (QFT), with a cut-off at a certain upper limit of the momenta which are considerably lower than the Planck's, should be ultraviolet-finite. In this case passage to higher energies means going from the momenta $p_N, |N| \gg 1$ to the momenta $p_{N'}, |N| > |N'| \gg 1$ and, vice versa, passage to lower energies is going in the last equality from the integers N' to the integers N.
For further resolution of the indicated QFT, along with formula (44), we should "translate" correctly the mathematical apparatus of the Dirac δ-function into the **measurable** representation.
Note that at the present time there is a strong belief that Lorentz-invariance is violated on passage to higher energies even for the particular quantum-field models without involvement of the minimal length ℓ, i.e. in the continuous space-time paradigm(for example, [26]).

5.3.2. Low energies $E \ll E_p$ and curved space-time.
In this case it is assumed that a **measurable** Lagrangian, containing a quantum gauge field in the **measurable** form $\mathbf{W}_{\mu,\{N\}}$ from formula (51) and the terms including material fields $\Psi_{\{N\}}$ (formula (52)),is considered in the space-time geometry generated by the **measurable** metric $g_{\alpha\beta}(x, \{N\})$.

Such consideration corresponds to the *semiclassical approximation* in the canonical (continuous) form. In fact, as $E \ll E_p$, in this case in continuous space-time gravity can be considered as classical, that is equivalent to the *semiclassical approximation*–"quantized material fields in the classical space-time geometry".

Since the energies are low, using **Remark 2.5**, in this case we can take a *discrete* QFT as an (*almost-continuous*) theory with a cut-off at a certain upper level of the momenta which are significantly lower than the Planck's momentum and with substitution of formula (44) in the corresponding formulae of a quantum theory in curved space-time [16],[27], considering substitution of the **measurable** metric $g_{\alpha\beta}(x, \{N\})$ for the metric $g_{\alpha\beta}(x)$.

Nevertheless, the differences, as compared to the continuous theory, really exist and are associated with selection of $N_{x_\chi} \in \{N\}$. The selection should be determined by the energies for which the theory is considered.

In continuous consideration, with the abstract infinitesimal quantities $dx_\chi, dp_i, dE, \chi = 0, ...3; i = 1, ...3$, the theory fails to "sense" specific energies. In the **measurable** form this is not the case due to the theory construction per se. Further studies are needed to find the corresponding inferences for different problems in curved space-time (for example, properties of pure and mixed states, entanglement depending on dynamics of the elements $\{N\}$), specifically for solution of the Information Paradox Problem (IPP)[28].

5.3.3. High energies $E \approx E_p$ and curved space-time.

This is a pure quantum-gravitational phase. When the material field Lagrangian is studied in this phase, in the **measurable** form, in accordance with the above formulae, we resolve a pure discrete theory. The geometry in such a "space" arises from the metrics satisfying the equation $\mathcal{EEM}[N_q]$ (61). In this case all "minimal" variations for gauge fields and material fields in the coordinate and momentum representations should be taken from formulae for the corresponding **GMQ**,i.e. from the expressions for $p_{\{N\}}, l_H(p_{\{N\}}), |\{N\}| \approx 1$ with regard to formulae (71),(72).

Then in the low-energy limit we have the case **5.3.2.** And, if the geometry determined by the metric $g_{\alpha\beta}(x, \{N\})$ is asymptotically flat, for very great $|\{N\}|$ we have the case **5.3.1.**

6 Conclusion

6.1. in the proposed approach the mathematical apparatus of the well-known theories in continuous space-time based on the use of the **abstract** infinitesimal quantities dx_μ, dp_i, dE is replaced by the apparatus based on the **measurability** notion and involving the ordered small quantities dependent on the existent energies. All small space-time variations in the indicated theories are generated by the momenta, (**primarily measurable** at low energies and **generalized measurable** at high energies). Considering the involvement of the minimal length $\ell \propto l_p$, in this case the initial theory becomes *discrete* but at low energies, far from the Planck energy $E \ll E_p$, it is **very close to** the initial theory in continuous space-time. Real *discreteness* is revealed at high energies $E \ll E_p$. Such an approach enables one to study the theories (specifically, QFT and gravity) in the same terms at all the energy scales.

6.2. In terms of the **measurability** notion the author has conducted a comparative analysis of passage to high energies for gravity and gauge theories. It has been shown that **measurability** in gravity is closely associated with *quantum fluctuations* of the space-time geometry (or at high energies of the **"space-time foam"**) introduced by J.A.Wheeler.

6.3. Of course, the words **"very close** given in bold type are not meaning **coincident**. In the last paragraph of **5.3.2** it is noted that **measurability** offers additional possibilities for solution of the known problems in curved space-time.
Because of this, it should be noted that in [5] the author first analyzed the potentialities of using **measurability** to avoid pathological solutions in GR – Closed Timelike Curves (CTC) [29]–[32].

6.4. In the proposed approach, within the scope of the **measurability** notion, the terms *classical* and *quantum* considerations common for the continuous space-time paradigm, generally speaking, lose their initial meaning. Indeed, the use of these terms is justified only at low energies $E \ll E_p$ but at these energies all minimal variations in the coordinate space take the

form $\ell/\{N\}, |\{N\}| \gg 1$ and ℓ in its definition has all the three fundamental constants including $\hbar$, because $\ell \propto l_p$. On the other hand, due to the condition $|\{N\}| \gg 1$, a quantum nature of the variations $\ell/\{N\}$ is not felt. The same is true for the momentum representation.
In fact, in the proposed approach the *classical* consideration is associated with the limiting transition $|\{N\}| \to \infty$. However, as shown in [5], for real physical systems at low energies $E \ll E_p$ is always $|\{N\}| < \infty$ and we have

$$N^* \geq |\{N\}| \geq N_* \gg 1, \tag{79}$$

where N_*, N^* – some lower and upper bounds.
As noted in **5.3.1**, in this case passage to higher or to lower energies means going to consideration of a theory with higher or lower absolute values of the numbers $\{N\}$, respectively, compared to the initial ones.

6.5. From formula (68) it follows that

$$\Lambda(x,\{N_q\}),(|\{N_q\}| \approx 1) \overset{|\{N_q\}|\approx 1 \to |\{N\}| \gg 1}{\Rightarrow} \Lambda(x,\{N\}),(|\{N\}| \gg 1), \tag{80}$$

where the right side of (80) is a dynamic cosmological term in the **measurable** form at low energies $E \ll E_p$. According to the results of Subsection 5.2, $\Lambda(x,\{N\})$ has little differences from the cosmological constant Λ in continuous consideration.
In his earlier works [33],[34] the author uses other methods, within the holographic principle validity, to show that

$$\frac{\Lambda(x,\{N\})}{\Lambda(x,\{N_q\})} \approx 10^{-123}. \tag{81}$$

It should be noted that $\Lambda(x,\{N\})$, to a high accuracy, agrees with the experimental cosmological constant.

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