

# HILBERT SERIES OF PI RELATIVE FREE $G$ -GRADED ALGEBRAS ARE RATIONAL FUNCTIONS

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ABSTRACT. Let  $G$  be a finite group,  $(g_1, \dots, g_r)$  an (unordered)  $r$ -tuple of  $G^{(r)}$  and  $x_{i, g_i}$ 's variables that correspond to the  $g_i$ 's,  $i = 1, \dots, r$ . Let  $F\langle x_{1, g_1}, \dots, x_{r, g_r} \rangle$  be the corresponding free  $G$ -graded algebra where  $F$  is a field of zero characteristic. Here the degree of a monomial is determined by the product of the indices in  $G$ . Let  $I$  be a  $G$ -graded  $T$ -ideal of  $F\langle x_{1, g_1}, \dots, x_{r, g_r} \rangle$  which is PI (e.g. any ideal of identities of a  $G$ -graded finite dimensional algebra is of this type). We prove that the Hilbert series of  $F\langle x_{1, g_1}, \dots, x_{r, g_r} \rangle/I$  is a rational function. More generally, we show that the Hilbert series which corresponds to any  $g$ -homogeneous component of  $F\langle x_{1, g_1}, \dots, x_{r, g_r} \rangle/I$  is a rational function.

## 1. INTRODUCTION

The Hilbert series of an affine (i.e. finitely generated) algebra and its computation, is a topic which attracted a lot of attention in the last century, classically in commutative algebra (see e.g. [29], [30]), but also (and in fact more importantly for the purpose of this paper) in non commutative algebra (see e.g. [5]). In particular the question of when the Hilbert series  $H_W$  of an algebra  $W$  is the Taylor expansion of a rational function is fundamental in the theory, and when it does, it serves as a “fine” structural invariant of  $W$  with applications to growth invariants (see [13], [14], [15], [25], [26], [28]).

In case the algebra  $W$  is a relatively free algebra, i.e. isomorphic to the quotient of an affine free algebra  $F\langle x_1, \dots, x_n \rangle$  by a  $T$ -ideal of identities  $I$  it is known that  $H_{F\langle x_1, \dots, x_n \rangle/I}$  is a rational function (see [8], [20]) and this fact has been successfully used in the estimation of the asymptotic behavior of the co-character sequence of a PI algebra.

Specifically in [11] (based on explicit formulas for the co-character sequences which appear in [10]) the authors show that if  $A$  is PI-algebra with 1 which satisfies a Capelli identity, then the asymptotic behavior of the codimension sequence is of the form

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$$c_n(A) = an^g l^n$$

where  $a$  is a scalar,  $2g$  is an integer and  $l$  is a non negative integer (we refer the reader to [17], [18] and [19] for a comprehensive account on the codimension sequence of a PI algebra).

The rationality of the Hilbert series of an affine relatively free algebra has been established also in case  $W$  is a super algebra (i.e.  $Z_2$ -graded) and our goal in this paper is to extend these results to  $G$ -graded relatively free affine PI algebras where  $G$  is an arbitrary finite group.

Let  $\Omega = (g_1, \dots, g_r)$  be an unordered  $r$ -tuple of elements of  $G$  (in particular we allow repetitions). Let  $X_G = \{x_{(1,g_1)}, \dots, x_{(r,g_r)}\}$  be a set of variables which correspond to the set  $\Omega$  and let  $F\langle X_G \rangle$  be the free algebra generated by  $X_G$  over  $F$ . In order to keep the notation as light as possible we may omit one index and write  $X_G = \{x_{g_1}, \dots, x_{g_r}\}$  with the convention that the variables  $x_{g_i}$  and  $x_{g_j}$  are different even if  $g_i = g_j$  in  $G$ . We equip  $F\langle X_G \rangle$  with a (natural)  $G$ -grading, namely a monomial of the form  $x_{g_{s_1}} x_{g_{s_2}} \cdots x_{g_{s_m}}$  is homogeneous of degree  $g_{s_1} g_{s_2} \cdots g_{s_m} \in G$ . We also consider its homogeneous degree  $m \in \mathbb{N}$ , namely the number of variables in the monomial. Let  $I$  be a  $G$ -graded  $T$ -ideal, that is,  $I$  is closed under  $G$ -graded endomorphisms of  $F\langle X_G \rangle$ . This implies (by a standard argument) that  $I$  is generated by strongly  $G$ -graded polynomials, i.e.  $G$ -graded polynomials whose different monomials are permutations of each other and moreover they all have the same  $G$ -degree (see [2]).

We will assume in addition that the  $T$ -ideal  $I$  contains an ordinary polynomial. By this we mean that  $I$  contains all  $G$ -graded polynomials obtained by assigning all possible degrees in  $G$  to the variables  $x_i$  of an ordinary non zero polynomial  $p(x_1, \dots, x_n)$ . We refer to a  $G$ -graded  $T$ -ideal which contains an ordinary polynomial as a PI  $G$ -graded  $T$ -ideal.

**Remark 1.1.** Note for instance, that the  $T$ -ideal of  $G$ -graded identities of any finite dimensional  $G$ -graded algebra is PI. On the other hand if  $G \neq \{e\}$ , then the  $G$ -graded algebra  $W$  where  $W_e$  is a free noncommutative algebra and  $W_g = 0$  for  $g \neq e$  is  $G$ -graded PI but of course not PI.

Let  $F\langle x_{g_1}, \dots, x_{g_r} \rangle$  be the free  $G$ -graded algebra generated by homogeneous variables  $\{x_{g_i}\}_{i=1}^r$ . As above we let  $I$  be a  $G$ -graded  $T$ -ideal which is PI and let  $\Phi = F\langle x_{g_1}, \dots, x_{g_r} \rangle / I$  be the corresponding relatively free  $G$ -graded algebra. Let  $\Omega_n$  be the (finite) set of monomials of degree  $n$  on the  $x_g$ 's and let  $c_n$  be the dimension of the  $F$ -subspace of  $\Phi$  spanned by the monomials of  $\Omega_n$ . We denote by

$$H_{F\langle x_{g_1}, \dots, x_{g_r} \rangle / I}(t) = \sum_n c_n t^n$$

the Hilbert series of  $\Phi$  with respect to the  $x_g$ 's.

**Theorem 1.2.** *The series  $H_{F\langle x_{g_1}, \dots, x_{g_r} \rangle / I}(t)$  is the Taylor series of a rational function.*

In our proof we strongly use key ingredients which appear in the proof of representability of the  $G$ -graded relatively free algebras (see [3]). These ingredients include the existence of certain polynomials, called Kemer polynomials, which are “extremal non identities” (see [3]) and their existence relies on the fundamental fact that the Jacobson radical of a PI algebra is nilpotent (see [4], [12], [22] and

[27]) (this parallels the Hilbert's Nullstellensatz in the commutative theory), and the solution of the Specht problem (see [3], [23], [24] and [31]) (which parallels the Hilbert basis theorem in the commutative theory). We emphasize however that the rationality of the Hilbert series is not a corollary of representability since there are examples of representable algebras which have a transcendental Hilbert series. Indeed, recall from [20] that the monomial algebra supported by monomials of the form  $h = x_1 x_2^m x_3 x_4^n x_5$  with the extra relation that  $h = 0$  if  $m^2 - 2n^2 = 1$  is representable but with a transcendental Hilbert series. For more on this we refer the reader to [9] and [21].

In Theorems 1.5 and 1.8 below we present different generalizations of Theorem 1.2. Then, in Theorem 1.9, these are combined into one general statement. We start with multivariate Hilbert series.

Let  $F\langle x_{g_1}, \dots, x_{g_r} \rangle$  be the free  $G$ -graded algebra generated by homogeneous variables  $\{x_{g_i}\}_{i=1}^r$ . As above we let  $I$  be a  $G$ -graded  $T$ -ideal which is PI and let  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  be the corresponding relatively free  $G$ -graded algebra. For any  $r$ -tuple of non-negative integers  $(d_1, \dots, d_r)$  we consider the (finite) set of monomials  $\Omega_{(d_1, \dots, d_r)}$  on  $x_g$ 's where the variable  $x_{g_i}$  appears exactly  $d_i$  times,  $i = 1, \dots, r$ . We denote by  $c_{(d_1, \dots, d_r)}$  the dimension of the  $F$ -subspace of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  spanned by the monomials in  $\Omega_{(d_1, \dots, d_r)}$ .

**Remark 1.3.** As mentioned above the  $T$  ideal  $I$  is generated by multilinear  $G$ -graded polynomials which are strongly homogeneous i.e. its monomials have the same homogeneous degree in  $G$ . This fact will play an important role in the sequel.

**Definition 1.4.** Notation as above. The multivariate Hilbert series of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  is given by

$$H_{F\langle x_{g_1}, \dots, x_{g_r} \rangle/I}(t_1, \dots, t_r) = \sum_{(d_1, \dots, d_r)} c_{(d_1, \dots, d_r)} t_1^{d_1} \cdots t_r^{d_r}.$$

The following result generalizes Theorem 1.2.

**Theorem 1.5.** *Notation as above. The Hilbert series  $H_{F\langle x_{g_1}, \dots, x_{g_r} \rangle/I}(t_1, \dots, t_r)$  is a rational function.*

**Remark 1.6.** Clearly, Theorem 1.2 follows from Theorem 1.5 simply by replacing all variables  $t_i$  by a single variable  $t$ .

The next generalization of Theorem 1.2 is in a different direction. We consider the Hilbert series (in one variable  $t$ ) of a unique homogeneous component. More precisely we fix  $g \in G$  and we consider the set of monomials  $\Omega_{g,n}$  of degree  $n$  whose homogeneous degree is  $g$ . We let  $c_{g,n}$  be the dimension of the subspace in  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  spanned by the monomials in  $\Omega_{g,n}$  (or rather, by the elements in  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  they represent).

**Definition 1.7.** With the above notation, the Hilbert series of the  $g$ -component of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  is given by

$$H_{g, F\langle x_{g_1}, \dots, x_{g_r} \rangle/I}(t) = \sum c_{g,n} t^n$$

**Theorem 1.8.** *With the above notation. The Hilbert series  $H_{g, F\langle x_{g_1}, \dots, x_{g_r} \rangle/I}(t)$  is a rational function.*

The case where  $g = e$  is of particular interest. Moreover we could consider the Hilbert series of any collection of  $g$ -homogeneous components and in particular the Hilbert series which corresponds to a subgroup  $H$  of  $G$ .

Finally we combine the generalizations which appeared in Theorems 1.5 and 1.8. Fix an element  $g \in G$ . For any  $r$ -tuple  $(d_1, \dots, d_r)$  of non-negative integers we consider the monomials with  $x_{g_1}$  appearing  $d_1$  times,  $x_{g_2}$  appearing  $d_2$  times,  $\dots$ , whose homogeneous degree is  $g \in G$ . We denote this set of monomials by  $\Omega_{g,(d_1,\dots,d_r)}$  and we let  $c_{g,(d_1,\dots,d_r)}$  be the dimension of the space in  $F\langle x_{g_1}, \dots, x_{g_r} \rangle / I$  spanned by elements whose representatives are the monomials in  $\Omega_{g,(d_1,\dots,d_r)}$ .

**Theorem 1.9.** *The Hilbert series*

$$H_{g,F\langle x_{g_1}, \dots, x_{g_r} \rangle / I}(t_1, \dots, t_r) = \sum c_{g,(d_1,\dots,d_r)} t_1^{d_1} \cdots t_r^{d_r}$$

which corresponds to the  $g$ -component of the  $G$ -graded algebra  $F\langle x_{g_1}, \dots, x_{g_r} \rangle / I$  is a rational function.

**Remark 1.10.** Clearly, Theorems 1.2, 1.5, 1.8 are direct corollaries of Theorem 1.9.

Theorem 1.9 is proved in Section 2. As mentioned above the proof uses ingredients from the proof of the representability of relatively free  $G$ -graded, affine, PI algebras. For the reader convenience we recall in the first part of the section the required results from [3] which are used in the proof.

In Section 3 we consider the  $T$ -ideal of  $G$ -graded identities  $I$  of the group algebra  $FG$  and show, in a rather direct way, the rationality of the Hilbert series of the corresponding relatively free algebra  $F\langle x_{g_1}, \dots, x_{g_r} \rangle / I$ . We also give in this case (see Corollary 3.4) the asymptotic behavior of the corresponding codimension sequence. Furthermore, these results can be easily extended to twisted group algebras  $F^\alpha G$ , where  $\alpha \in Z^2(G, F^*)$  (see Remark 3.5). The  $G$ -grading determined by twisted group algebras is called “fine” and it plays an important role in the classification of  $G$ -graded simple algebras (see [6]).

We close the introduction by mentioning that one may consider the Hilbert series of an affine  $G$ -graded free algebra. It is easy to see that the corresponding Hilbert series is a rational function. Of course one may ask whether the different generalizations (Theorems 1.5 and 1.8) apply also in this case (i.e. when  $I = 0$ ). It is easy to see that this is indeed the case when extending to multivariate series. As for the second generalization, we refer the reader to [16]. The authors prove the rationality of the Hilbert series which corresponds to the sub algebra of coinvariant elements, that is, in case  $g = e$ .

## 2. PRELIMINARIES AND PROOFS

We start this section by recalling some facts on  $G$ -graded algebras  $W$  over a field of characteristic zero  $F$  and their corresponding  $G$ -graded identities. We refer the reader to [3] for a detailed account on this topic.

Let  $W$  be an affine  $G$ -graded PI algebra over  $F$ . We denote by  $I = \text{id}_G(W)$  the ideal of  $G$ -graded identities of  $W$ . These are polynomials in the free  $G$ -graded algebra over  $F$  which are generated by  $X_G$  and that vanish upon any admissible evaluation on  $W$ . Here  $X_G = \bigcup X_g$  and  $X_g$  is a set of countably many variables of degree  $g$ . An evaluation is admissible if the variables from  $X_g$  are replaced only by elements from  $W_g$ . It is known that  $I$  is a  $G$ -graded  $T$ -ideal, i.e. closed under  $G$ -graded endomorphisms of  $F\langle X_G \rangle$ .

We recall from [3] that the  $T$ -ideal  $I$  is generated by multilinear polynomials. Consequently, the  $T$ -ideal of identities does not change when passing to  $\bar{F}$ , the

algebraic closure of  $F$ , in the sense that the ideal of identities of  $W_{\overline{F}}$  over  $\overline{F}$  is the span (over  $\overline{F}$ ) of the  $T$ -ideal of identities of  $W$  over  $F$ . It is easily checked that the Hilbert series remains the same when passing to the algebraic closure of  $F$ . Thus, from now on we assume  $F = \overline{F}$ .

Next, we recall some terminology and some facts from Kemer theory extended to the context of  $G$ -graded algebras as they appear in [3]. We start with the concept of alternating polynomial on a set of variables.

Let  $f(x_{1,g}, \dots, x_{r,g}; y_1, \dots, y_n)$  be a multilinear polynomial with variables  $(x_{1,g}, \dots, x_{r,g})$ , homogeneous of degree  $g$ , and some other variables  $y$ 's, homogeneous of unspecified degrees. We say that  $f$  is alternating on the set  $x_{1,g}, \dots, x_{r,g}$  if there is a multilinear polynomial  $h(x_{1,g}, \dots, x_{r,g}; y_1, \dots, y_n)$  such that

$$f(x_{1,g}, \dots, x_{r,g}; y_1, \dots, y_n) = \sum_{\sigma \in \text{Sym}(r)} (-1)^\sigma h(x_{\sigma(1),g}, \dots, x_{\sigma(r),g}; y_1, \dots, y_n).$$

We say that a polynomial  $f$  alternates on a collection of disjoint sets of homogeneous variables (each set constitute of variables of the same degree), if it is alternating on each set.

In the sequel we will need to consider multilinear polynomials  $f$  which alternates on  $d$  disjoint sets of  $g$ -elements, each of cardinality  $r$ . More generally, we will consider multilinear polynomials such that for any  $g \in G$ ,  $f$  contains  $n_g$  disjoint sets of variables of homogeneous degree  $g$  and each set of cardinality  $d_g$ .

We recall from [3] that a  $G$ -graded polynomial with an alternating sets of  $g$ -homogeneous variables which is “large enough” is necessarily an identity. More precisely, for any affine PI  $G$ -graded algebra  $W$  and for any  $g \in G$  there exists an integer  $d_g$  such that any  $G$ -graded polynomial which has an alternating set of  $g$  variables of cardinality exceeding  $d_g$  is necessarily a  $G$ -graded identity of  $W$ .

In particular this holds for a finite dimensional  $G$ -graded algebra  $A$ . Note that in that case, if a polynomial  $f$  has an alternating set of  $g$ -homogeneous elements whose cardinality exceeds the dimension of  $A_g$ , it is clearly an identity of  $A$ .

Next we recall that by Wedderburn-Malcev decomposition theorem, a  $G$ -graded finite dimensional algebra  $A$  over  $F$ , may be decomposed into the direct sum of  $\overline{A} \oplus J$  (decomposition as vector spaces) where  $J$  is the Jacobson radical ( $G$ -graded) and  $\overline{A}$  is a (semisimple) subalgebra of  $A$  isomorphic to  $A/J$  as  $G$ -graded algebras. As a consequence we have decompositions of  $\overline{A}$  and  $J$  to the corresponding  $g$ -homogeneous components.

Now, we know that in order to test whether a multilinear polynomial is an identity of an algebra, it is sufficient to evaluate its variables on a base and hence, applying the above decomposition, we may consider semisimple and radical  $G$ -graded evaluations. From these considerations we conclude that if a polynomial has sufficiently many alternating sets of  $g$ -homogeneous elements of cardinality that exceeds the dimension of the  $g$ -homogeneous component of  $\overline{A}$ , the polynomial is necessary an identity of  $A$ . Indeed, in any evaluation we either have a semisimple basis element which appears twice or at least one of the evaluations is radical. In the first case we get zero as a result of the alternation of two elements which are equal, while in the second, we get zero if the number of alternating sets is at least the nilpotency index of  $J$ . We therefore see that if a  $G$ -graded polynomial  $f$  has a number of alternating sets (same cardinality) of  $g$ -variables which is at least the nilpotency index of  $J$  (and in particular if it has “sufficiently many”) then the

cardinality of the sets must be bounded by the dimension of  $\overline{A}_g$  if we know that  $f$  is a non ( $G$ -graded) identity of  $A$ . Thus, if  $f$  is a non ( $G$ -graded) identity of  $A$  with  $\alpha_g$  disjoint alternating sets of  $g$ -homogeneous elements of cardinality  $d_g + 1$ ,  $g \in G$  where  $d_g = \dim(\overline{A}_g)$  then,  $\sum_g \alpha_g \leq n - 1$ , where  $n$  is the nilpotency index of  $J$ .

In [3] the notion of a finite dimensional  $G$ -graded *basic* algebra was introduced. For our exposition here, it is not necessary to recall its precise definition but only say (as a result of Kemer's Lemmas 1 and 2 for  $G$ -graded algebras (see sections 5 and 6 in [3])) that any basic algebra  $A$  admits non-identities  $G$ -graded polynomials which have "arbitrary many" (say at least  $n$ , the nilpotency index of  $J$ ) alternating sets of  $g$ -homogeneous variables of cardinality  $d_g$  and precisely (a total of)  $n - 1$  alternating sets of  $g$ -homogeneous variables of cardinality  $d_g + 1$  for some  $g \in G$ . These extremal non identities are the so called " $G$ -graded Kemer polynomials" of the basic algebra  $A$ .

Thus to each basic algebra  $A$  corresponds an  $r + 1$ -tuple of non-negative integers  $(d_{g_1}, \dots, d_{g_r}; n - 1)$  where  $d_g$  is the dimension of the  $g$ -component of the semisimple part of  $A$  and  $n$  is the nilpotency index of  $J$ , the radical of  $A$ . We refer to such a tuple as the Kemer point of the basic algebra  $A$ .

The representability theorem for affine  $G$ -graded algebras can be stated as follows.

Given an affine PI  $G$ -graded algebra  $W$  over  $F$ , where  $F$  is a field of zero characteristic, there exists a finite number of  $G$ -graded basic algebras  $A_1, \dots, A_m$  over a field extension  $K$  of  $F$  such that  $W$  satisfies the same  $G$ -graded identities as  $A_1 \oplus \dots \oplus A_m$ . Note that since  $\text{id}_G(A_1 \oplus \dots \oplus A_m) = \bigcap \text{id}_G(A_i)$  we may assume  $\text{id}(A_i) \not\subseteq \text{id}(A_j)$  for every  $1 \leq i, j \leq m$ .

**Remark 2.1.** In fact, by passing to the algebraic closure of  $K$ , we may assume that the algebras  $A_i$  above are finite dimensional over the same field  $F$ .

**Definition 2.2.** With the above notation, we say that a finite dimensional  $G$ -graded algebra  $A$  is subdirectly irreducible if it has no non trivial, two sided  $G$ -graded ideals  $I$  and  $J$  such that  $I \cap J = (0)$ .

**Remark 2.3.** Note that if the algebra  $A_1 \oplus \dots \oplus A_m$  is subdirectly irreducible then  $m = 1$ .

A key ingredient in the proof of Theorem 1.9 is the existence of an essential Shirshov base for the relatively free algebra  $F\langle x_{g_1}, \dots, x_{g_r} \rangle / I$ .

For the reader convenience we recall the necessary definitions and statements from [3], starting from the ordinary case (i.e. ungraded).

**Definition 2.4.** Let  $W$  be an affine PI-algebra over  $F$ . Let  $\{a_1, \dots, a_s\}$  be a set of generators of  $W$ . Let  $m$  be a positive integer and let  $Y$  be the set of all words in  $\{a_1, \dots, a_s\}$  of length  $\leq m$ . We say that  $W$  has *Shirshov* base of length  $m$  and of height  $h$  if elements of the form  $y_{i_1}^{k_1} \dots y_{i_l}^{k_l}$  where  $y_{i_i} \in Y$  and  $l \leq h$ , span  $W$  as a vector space over  $F$ .

**Theorem 2.5.** *If  $W$  is an affine PI-algebra, then it has a Shirshov base for some  $m$  and  $h$ . More precisely, suppose  $W$  is generated by a set of elements of cardinality  $s$  and suppose it has PI-degree  $m$  (i.e. there exists an identity of degree  $m$  and  $m$  is minimal) then  $W$  has a Shirshov base of length  $m$  and of height  $h$  where  $h = h(m, s)$ .*

In fact we will need a weaker condition (see [3])

**Definition 2.6.** Let  $W$  be an affine PI-algebra. We say that a set  $Y$  as above is an *essential Shirshov base* of  $W$  (of length  $m$  and of height  $h$ ) if there exists a finite set  $D(W)$  such that the elements of the form  $d_{i_1} y_{i_1}^{k_1} d_{i_2} \cdots d_{i_l} y_{i_l}^{k_l} d_{i_{l+1}}$  where  $d_{i_j} \in D(W)$ ,  $y_{i_j} \in Y$  and  $l \leq h$  span  $W$ .

An essential Shirshov's base gives the following.

**Theorem 2.7.** *Let  $C$  be a commutative ring and let  $W = C\langle\{a_1, \dots, a_s\}\rangle$  be an affine algebra over  $C$ . If  $W$  has an essential Shirshov base (in particular, if  $W$  has a Shirshov base) whose elements are integral over  $C$ , then it is a finite module over  $C$ .*

Returning to  $G$  graded algebras we have the following.

**Proposition 2.8.** *Let  $W$  be an affine, PI,  $G$ -graded algebra. Then it has an essential  $G$ -graded Shirshov base of elements of  $W_e$ .*

For the proof of Theorem 1.9 we will assume below there exists a  $G$ -graded  $T$ -ideal  $I$  which is non-rational (that is the Hilbert series of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle / I$  is non-rational) and get a contradiction. The next two lemmas will be used to reduce the problem to the case where  $I$  is maximal with respect to being non-rational and also that the relatively free algebra  $F\langle x_{g_1}, \dots, x_{g_r} \rangle / I$  is  $G$ -graded PI equivalent to one basic algebra (rather than to a direct sum of them).

Before stating the lemmas we simplify the terminology as follows.

**Remark 2.9.** If  $X$  is a subspace of an algebra  $U = F\langle x_{g_1}, \dots, x_{g_r} \rangle / I$  spanned by strongly homogeneous polynomials, then we may consider naturally its corresponding multivariate Hilbert series. Then when we say ‘‘Hilbert series of  $X$ ’’ we mean ‘‘multivariate  $G$ -graded Hilbert series which corresponds to the  $g$ -component of  $X$ ’’ for any given  $g \in G$ .

**Lemma 2.10.** *Let  $J$  be a  $G$ -graded  $T$ -ideal containing  $I$ . Let  $H_U$ ,  $H_{U/J}$  and  $H_{J/I}$  be the Hilbert series of  $U$ ,  $U/J$  and  $J/I$  respectively. Then*

$$H_{F\langle x_{g_1}, \dots, x_{g_r} \rangle / I} = H_{F\langle x_{g_1}, \dots, x_{g_r} \rangle / J} + H_{J/I}.$$

*Proof.* This is clear since  $J$  is spanned by strongly homogeneous polynomials. □

**Lemma 2.11.** *Let  $I'$  and  $I''$  be two  $G$ -graded  $T$ -ideals which contain  $I$ . Then the following holds:*

$$H_{F\langle x_{g_1}, \dots, x_{g_r} \rangle / (I' \cap I'')} = H_{F\langle x_{g_1}, \dots, x_{g_r} \rangle / I'} + H_{F\langle x_{g_1}, \dots, x_{g_r} \rangle / I''} - H_{F\langle x_{g_1}, \dots, x_{g_r} \rangle / (I' + I'')}$$

*Proof.* The proof is similar to the proof of Lemma 9.40 in [20] and hence is omitted. □

Let us assume now that there exist  $G$ -graded  $T$ -ideals of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle$  which are PI and such that the Hilbert series of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle / I$  is non-rational.

**Proposition 2.12.** *Under the above assumption there exists a  $G$ -graded  $T$ -ideal of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle$  which is PI and*

- (1) *is maximal with respect to the property that the Hilbert series of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  is non-rational.*
- (2) *the relatively free algebra  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  is  $G$ -graded PI equivalent to a  $G$ -graded basic algebra  $A$ .*

*Proof.* The first assertion follows from the  $G$ -graded Specht property (see section 12 in [3]). Indeed, if there is no such an ideal then we get an infinite ascending sequence of ideals which does not stabilize and this contradicts the fact that the union of the  $T$ -ideals is finitely generated. We thus may assume that the Hilbert series of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  is non-rational and the Hilbert series of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/J$  for any  $G$ -graded  $T$ -ideal  $J$  which properly contains  $I$  is a rational function.

For the proof of the second assertion let us show that the maximality of  $I$  already implies that  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  is PI equivalent to a  $G$ -graded basic algebra. Assuming the converse, we have that  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  is  $G$ -graded PI equivalent to a direct sum of basic algebras  $A_1 \oplus A_2 \oplus \dots \oplus A_m$  where  $m \geq 2$  and  $\text{id}_G(A_i) \not\subseteq \text{id}_G(A_j)$  for any  $1 \leq i, j \leq m$ . It follows that the ideal  $\text{id}_G(F\langle x_{g_1}, \dots, x_{g_r} \rangle/I) = \text{id}_G(A_1 \oplus A_2 \oplus \dots \oplus A_m) = \bigcap \text{id}_G(A_i)$  and  $\text{id}_G(F\langle x_{g_1}, \dots, x_{g_r} \rangle/I) \subsetneq \text{id}_G(A_i)$  for any  $i$ . Now, consider the evaluations  $I_i$  of the  $T$ -ideals  $\text{id}_G(A_i)$  on  $F\langle x_{g_1}, \dots, x_{g_r} \rangle$ . Clearly,  $I_i$  properly contains  $I$  and their intersection is  $I$ . By Lemma 2.11 we conclude that the Hilbert series of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  is rational. Contradiction.  $\square$

It is convenient to view the relatively free algebra  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  as an algebra of generic elements,  $G$ -graded embedded in a matrix algebra over a suitable rational function field over  $F$ . Indeed, by part (2) of Proposition 2.12 we have that  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  is  $G$ -graded PI equivalent to a  $G$ -graded basic algebra which we denote by  $A$  and which from now on will be viewed a  $G$ -graded subalgebra of the  $n \times n$  matrices over  $F$  (see [3]). If  $\{v_{g,1}, \dots, v_{g,s_g}\}$  is an  $F$ -basis of  $A_g$ , the  $g$ -homogeneous component of  $A$ , we consider different sets of central indeterminates  $\{t_{g,1}, \dots, t_{g,s_g}\}$ ,  $g \in G$  (one set for each generator  $x_g$  of the free algebra).

Let  $K = F\langle \{t_{g,i}\} \rangle$  be the field of rational functions on the  $t$ 's and  $A_K$  be the algebra over  $K$  obtained from  $A$  by extending scalars from  $F$  to  $K$ . The algebra of generic elements will be an  $F$ -subalgebra of  $A_K$ . For each variable  $x_g = x_{g_j}$  in the generating set of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle$  we form an element  $z_g = \sum_i t_{g,i} v_{g,i}$  in  $A_K$ . Following the embedding of  $A$  in  $M_n(F)$  we view the generating elements  $z_g$ 's as  $n \times n$ -matrices over the field  $K$ . Observe that these generating elements are matrices whose entries are homogeneous polynomials (on the  $t$ 's) of degree one.

**Proposition 2.13** (see [3]). *There is an  $F$ -isomorphism of  $G$ -graded algebras of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  with  $\mathcal{A}$ , the  $F$ -sub algebra of  $A_K$  generated by the elements  $\{z_{g_1}, \dots, z_{g_r}\}$ .*

**Remark 2.14.** In the rest of the proof we use without further notice the identification of the relatively free algebra with the algebra of generic elements  $\mathcal{A}$  embedded in  $M_n(K)$ .

Now we recall from Proposition 2.8 that the relatively free algebra  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  has an essential Shirshov  $\Theta$  base which is contained in the  $e$ -component. Picking the natural set of generators of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle/I$  we note that  $\Theta$  consists of monomials on the  $x_{g_i}$ 's and hence, viewed in  $\mathcal{A}$ , they consist of matrices over  $K$  whose entries are homogeneous polynomial in the  $t_i$ 's. It then follows that the characteristic values of the elements in  $\Theta$  are homogeneous polynomials on the  $t_i$ 's.

Denote by  $C$  the algebra over  $F$  generated by these homogeneous polynomials. Note that  $C$  is an affine commutative algebra (the essential Shirshov base is finite). Moreover, if we extend the algebra of generic elements  $\mathcal{A}$  to  $C$  we have (by Theorem 2.7) that  $\mathcal{A}_C$  is a finite module over  $C$ .

For  $\mathcal{A}_C$  we know the following result.

**Lemma 2.15.** *The Hilbert series of  $\mathcal{A}_C$  is rational. More generally, the rationality of the Hilbert series is independent of the integer degree given to the generators.*

*Proof.* Indeed, the algebra is a finitely generated module over an affine domain and hence its Hilbert series is rational. (see [20], Prop. 9.33). □

More generally, we may consider  $C$ -submodules  $M$  of  $\mathcal{A}_C$  which are generated by strongly homogeneous polynomials on the  $x_{g_i}$ 's.

**Lemma 2.16.** *The Hilbert series of  $M$  is rational.*

*Proof.* Since  $\mathcal{A}_C$  is a finite module over a Noetherian domain, the module  $M$  is finitely generated as well. The result now follows from the second part of Lemma 2.15. □

We can complete now the proof of Theorem 1.9. Let  $f$  be a  $G$ -graded Kemer polynomial of the basic algebra  $A$  and let  $J$  be the  $G$ -graded  $T$ -ideal it generates together with  $I$ . Note that since  $f$  is a non identity of  $A$  (and hence a non identity of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle / I$ ) the  $T$ -ideal  $J$  strictly contains  $I$ , and hence, by the maximality of  $I$  we have that the Hilbert series of  $F\langle x_{g_1}, \dots, x_{g_r} \rangle / J$  is rational. The key property which we need here is that the ideal  $J/I$  is closed under the multiplication of the characteristic values (see [3], Prop. 8.2). Hence the ideal  $J/I$  of the relatively free algebra is in fact a  $C$ -submodule of  $\mathcal{A}_C$ . Hence its Hilbert series is rational. Applying Lemma 2.10 the result follows. This completes the proof of Theorem 1.9.

### 3. A SPECIAL CASE

In this section we show by direct computations the rationality of the Hilbert series of the affine relative free  $G$ -graded algebra in case  $I$  is the  $T$ -ideal of  $G$ -graded identities of the group algebra  $FG$ . In addition we obtain a precise estimation of the asymptotic behavior of the  $G$ -graded codimension sequence for that case.

Let  $\bar{\alpha} = (g_1, g_2, \dots, g_r)$  be an  $r$ -tuple in  $G^{(r)}$ . As in previous sections we consider the free  $G$ -graded algebra  $F\langle x_{1,g_1}, \dots, x_{r,g_r} \rangle$ , where the  $x_{i,g_i}$ 's are non-commuting variables which are in one to one correspondence with the entries of  $\bar{\alpha}$ . As above, we may abuse notation by deleting the index  $i$  and simply write  $x_{g_i}$ .

Next we consider the  $T$ -ideal of  $G$ -graded identities  $\text{id}_G(FG)$  of the group algebra  $FG$ . Recall from [2] that  $\text{id}_G(FG)$  is generated as a  $T$ -ideal by binomial identities of the form  $x_{g_{i_1}} x_{g_{i_2}} \cdots x_{g_{i_n}} - x_{g_{i_{\sigma(1)}}} x_{g_{i_{\sigma(2)}}} \cdots x_{g_{i_{\sigma(n)}}}$  where  $\sigma$  is a permutation in  $S_n$  and the products  $g_{i_1} g_{i_2} \cdots g_{i_n}$  and  $g_{i_{\sigma(1)}} g_{i_{\sigma(2)}} \cdots g_{i_{\sigma(n)}}$  coincide in  $G$ . That is two monomials are equivalent if and only if they have the same variables and they determine elements in the same  $g$ -homogeneous component. In particular if the group  $G$  is abelian, then two monomials are equivalent if and only if they have the same variables.

**Remark 3.1.** Clearly, by passing to a subgroup of  $G$  if necessary, we may assume the elements of  $\bar{\alpha} = (g_1, g_2, \dots, g_r)$  generate the group  $G$ .

Fix a natural number  $n$  and consider monomials of degree  $n$  with  $n_1$  variables  $x_{g_1}$ ,  $n_2$  variables  $x_{g_2}, \dots, n_r$  variables  $x_{g_r}$  where  $n_1 + n_2 + \dots + n_r = n$ . Now consider permutations of any monomial of that form. Clearly, any permutation determines the same element in the abelianization of  $G$ . In other words the elements in  $G$  determined by two monomials which have the same variables lie in the same coset of the commutator  $G'$ . In the next lemma we show that if the monomial is “rich enough” we may obtain all elements of a  $G'$ -coset. In order to state the lemma we need the following notation.

We say that the word  $\Sigma = g_{i_1}g_{i_2} \cdots g_{i_n}$  is a presentation of  $g \in G$  (in terms of the entries of  $\bar{\alpha} = (g_1, g_2, \dots, g_r)$ ) if  $g = g_{i_1}g_{i_2} \cdots g_{i_n}$  in  $G$ . For any word  $\Sigma$  we may consider the corresponding *monomial* (in the free algebra)  $X_\Sigma = x_{g_{i_1}}x_{g_{i_2}} \cdots x_{g_{i_n}}$ . We say that the monomial  $X_\Sigma$  represents  $g$  in  $G$ .

**Lemma 3.2.** *For every  $z \in G$  there exists an integer  $n$  and a word  $g_{i_1}g_{i_2} \cdots g_{i_n}$  in  $G$  such that the set of words in*

$$\Omega_{g_{i_1}g_{i_2} \cdots g_{i_n}} = \{g_\sigma = g_{i_{\sigma(1)}}g_{i_{\sigma(2)}} \cdots g_{i_{\sigma(n)}}, \sigma \in \text{Sym}(n)\}$$

*represent all elements of the form  $zg$  where  $g \in G'$  (i.e. the full coset of  $G'$  in  $G$  represented by  $z$ ).*

*Proof.* Clearly it is sufficient to find a word whose permutations yield all elements of  $G'$ . It is well known that the commutator subgroup  $G'$  of  $G$  is generated by commutators  $[g, h] = ghg^{-1}h^{-1}$  where  $g, h \in G$ . For any commutator  $[g, h]$  we write the elements  $g$  and  $h$  as words in the entries of  $\bar{\alpha}$  and then we write  $g^{-1}$  and  $h^{-1}$  by inverting the corresponding words. We denote by  $\Sigma_{[g, h]}$  the corresponding word in the entries of  $\bar{\alpha}$  and their inverses (which is equal to  $[g, h]$  in  $G$ ). Clearly, the total degree (in  $\Sigma_{[g, h]}$ ) of each entry  $g_i$  is zero and hence permuting the elements of  $\Sigma_{[g, h]}$  we obtain the identity element  $e$ . It follows easily that taking products of such words we obtain a word  $\Sigma = \Sigma_z$  whose different permutations yield all of  $G'$ . This would complete the proof of the lemma if we assume that whenever  $g$  is an entry in  $\bar{\alpha}$  then also  $g^{-1}$  is. But clearly, if this is not the case, we can replace  $g^{-1}$  by  $g^{\text{ord}(g)-1}$  and the result follows.  $\square$

Consider the ( $r$ -dimensional) lattice  $\Gamma_r = (\mathbb{Z}_+)^{(r)}$  of non negative integer points, where  $r$  is the cardinality of  $\bar{\alpha}$ . We refer to  $\Gamma_r$  as the  $r$ -dimensional non-negative Euclidean lattice. Similarly we will consider  $\Gamma_k$ ,  $k \leq r$ , the  $k$ -dimensional non-negative Euclidean lattices and their translations,  $\vec{x} + \Gamma_k$  where  $\vec{x} \in (\mathbb{Z}_+)^{(k)}$ . We view the lattice  $\Gamma_r$ , as a partial ordered set where  $A = (n_1, \dots, n_r) \prec B = (m_1, \dots, m_r)$  if and only if  $n_i \leq m_i$  for  $1 \leq i \leq r$ . Clearly, this partial order inherits a partial order on  $\Gamma_k$ ,  $k \leq r$ , and their translations. To any point  $A = (n_1, \dots, n_r)$ ,  $n_i \geq 0$  in  $\Gamma_r$  we attach all monomials  $X_\Sigma$  with number of variables as prescribed by the point  $A$ , that is, the variable  $x_{1, g_1}$  appears  $n_1$  times,  $x_{2, g_2}$  appears  $n_2$  and so on. Clearly any word  $\Sigma$  determines a unique lattice point  $A$  in which case we write  $A = A_\Sigma$  or  $\Sigma \in A$ . Clearly the elements in  $G$  represented by all monomials that correspond to a point  $A \in \Gamma_r$  lie in the same coset of  $G'$  and hence

denoting by  $N_A = \{g \in G : g \text{ is represented by monomials in } A\}$ , we have that  $1 \leq \text{ord}(N_A) \leq \text{ord}(G')$ .

**Lemma 3.3.** *The function  $\text{ord}(N(A)) : \Gamma_r \rightarrow \{1, \dots, \text{ord}(G')\}$  is monotonic (increasing) with respect to partial ordering on  $\Gamma_r$ .*

*In particular if  $A_\Sigma \in \Gamma_r$ , where  $\Sigma$  is a word in the entries of  $\bar{\alpha}$  whose different permutations represent all elements of  $G'$  (as constructed in Lemma 3.2), then for any word  $\Pi$  such that  $A_\Pi \succeq A_\Sigma$  we have  $\text{ord}(N(A_\Pi)) = \text{ord}(G')$ .*

*Proof.* This is clear. □

We can now complete the proof that the Hilbert series of the corresponding relatively free algebra is rational.

Let  $A = (n_1, \dots, n_r) \in \Gamma_r$ , where  $n = n_1 + \dots + n_r$ , and  $P_A$  be the space spanned by all monomials  $X_\Pi$  where  $\Pi \in A$ . Since the  $T$ -ideal of identities is spanned by strongly homogeneous polynomials we have that the subspace of the relatively free algebra spanned by monomials of degree  $n$  is decomposed into the direct sum of spaces which are spanned by monomials in lattice points  $A$  of degree  $n$ .

We claim that for any integer  $\lambda$ ,  $1 \leq \lambda \leq \text{ord}(G')$ , the set of points  $A \in \Gamma_r$  such that

$$\dim(P_A/P_A \cap (\text{id}_G(FG))) = \lambda$$

is a *finite* union of *disjoint* sets which are translations of lattices  $\Gamma_k$ ,  $0 \leq k \leq r$ . We present here a proof which was shown to us by Uri Bader. Consider the one point compactification  $\widehat{\mathbb{Z}}_+$  of  $\mathbb{Z}_+$ . It is convenient to view the space  $\widehat{\mathbb{Z}}_+$  as homeomorphic to the set of points  $I_{\mathbb{N}} = \{1/n : n \in \mathbb{N}\} \cup \{0\}$  with the induced topology of the Lebesgue measure on the interval  $[0, 1]$ . The closed sets are either the finite sets or those that contain 0. Consequently the sets which are closed and open are either the finite sets without 0 or sets that contain a set of the form  $\{1/n : n \geq d\} \cup \{0\}$ . Consider the  $r$ -fold cartesian product

$$I_{\mathbb{N}}^r = I_{\mathbb{N}} \times I_{\mathbb{N}} \times \dots \times I_{\mathbb{N}}$$

Clearly  $I_{\mathbb{N}}^r$  is compact. Furthermore the function  $\text{ord}(N(A))$  with values in the finite set  $T = \{1, \dots, \text{ord}(G')\}$  (viewed as a function on  $I_{\mathbb{N}}^r$ ) is monotonic decreasing and hence continuous. It follows that the inverse image of any point in  $T$  is an open and closed subset of  $I_{\mathbb{N}}^r$  and the result now follows easily.

Returning to  $\Gamma_r = (\mathbb{Z}_+)^{(r)}$  we see that the rationality of the series will follow if we know the rationality of the Hilbert series which corresponds to the enumeration of lattice points of degree  $n$  in  $\Gamma_k$ ,  $k \leq r$ . Thus we need to check the rationality of the following power series

$$\sum_n ((n+k-1)!/(n!)(k-1)!)t^n$$

and this is clear. This gives a direct proof of Theorem 1.2 in case  $I = \text{id}_G(FG)$ .

We close the article with an estimate of the sequence of codimensions which corresponds to the  $T$ -ideal  $I = \text{id}_G(FG)$ . For the calculation we consider the  $F$ -space  $P_n$  spanned by all  $G$ -graded multilinear monomials of degree  $n$  on the variables

$\{x_{g,i}\}_{g \in G, i=1, \dots, n}$  which are permutations of monomials of the form  $x_{1,g_1} x_{2,g_2} \dots x_{n,g_n}$ , where  $(g_1, \dots, g_n) \in G^{(n)}$ . Clearly,  $\dim(P_n) = \text{ord}(G)^n \times n!$ . Let

$$c_G^n(FG) = P_n / (P_n \cap I).$$

The integer  $c_G^n(FG)$  is the  $n$ -codimension which corresponds to the  $G$ -graded  $T$ -ideal  $I$ .

**Corollary 3.4.** *Let  $A = FG$  be the group algebra over a field  $F$  and let  $\text{id}_G(FG)$  be the  $G$ -graded  $T$ -ideal of identities. Let  $c_G^n(FG)$  be the  $n$ -th coefficient of the codimension sequence of  $\text{id}_G(FG)$ . Then for any integer  $n$  we have*

(1)

$$\text{ord}(G)^n \leq c_G^n(FG) \leq \text{ord}(G') \text{ord}(G)^n.$$

(2)

$$\lim_{n \rightarrow \infty} (c_G^n(FG) / \text{ord}(G') \text{ord}(G)^n) = 1$$

In particular  $\exp_G(FG) = \lim_{n \rightarrow \infty} \sqrt[n]{c_G^n(FG)} = \text{ord}(G)$  (see [1]).

*Proof.* Let  $\mathbf{x} = x_{1,g_1} x_{2,g_2} \dots x_{n,g_n}$  be a  $G$ -graded monomial in  $P_n$ . As noted above, the set of all  $n!$  permutations of  $\mathbf{x}$  decomposes into at most  $G'$  equivalence classes where two monomials are equivalent if and only if the difference is a binomial identity of  $FG$ . It then follows easily that  $\text{ord}(G)^n \leq c_G^n(FG) \leq \text{ord}(G') \text{ord}(G)^n$ .

For the second part, recall from the second part of Lemma 3.3 that if a monomial contains  $G$ -graded variables with degrees in  $G$  as in the word  $\Sigma$  (including multiplicities), then its permutations yield precisely  $G'$  non equivalent classes. So the second part of the corollary will follow easily if we show that

$$\lim_{n \rightarrow \infty} \frac{d_n}{\text{ord}(G)^n \times n!} = 0$$

where  $d_n$  denotes the number of monomials in  $P_n$  which do not contain the set of elements of  $\Sigma$  (with repetitions). But this of course follows from an easy calculation which is omitted. □

**Remark 3.5.** One may replace the group algebra  $FG$  above by any twisted group algebra  $F^\alpha G$ , where  $\alpha$  is a 2-cocycle of  $G$  with values in  $F^*$ . As above, also here, two monomials are equivalent if and only if they have the same variables and they determine elements in the same  $g$ -homogeneous component. The only difference (comparing to the case where  $\alpha \equiv 1$ ) is that here, for two such monomials, the polynomial identity they determine has the form

$$x_{g_{i_1}} x_{g_{i_2}} \dots x_{g_{i_n}} - \gamma x_{g_{i_{\sigma(1)}}} x_{g_{i_{\sigma(2)}}} \dots x_{g_{i_{\sigma(n)}}}$$

where  $\gamma$  is a non zero element of  $F$  which is determined by the 2-cocycle  $\alpha$  and the words  $g_{i_1} g_{i_2} \dots g_{i_n}$  and  $g_{i_{\sigma(1)}} g_{i_{\sigma(2)}} \dots g_{i_{\sigma(n)}}$  (see [2]). One sees easily that the twisting by the 2-cocycle  $\alpha$  has no effect neither on the Hilbert series nor on the sequence of codimensions. Details are left to the reader.

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