Condensation versus independence in weakly interacting CMLs

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Abstract

We propose a simple model unifying two major approaches to the analysis of large multicomponent systems: interacting particle systems (IPS) and couple map lattices (CML) and show that in the weak interaction limit depending on fine properties of the interaction potential this model may demonstrate both condensation/synchronization and independent motions. Note that one of the main paradigms of the CML theory is that the latter behavior is supposed to be generic. The model under consideration is related to dynamical networks and sheds a new light to the problem of synchronization under weak interactions.

1   Introduction

At present there are two major approaches to the analysis of large multicomponent systems: Interacting Particle Systems (IPS) and Coupled Map Lattices (CML). Ideas and especially methods used in these approaches are strikingly different and it is hard to find a single article which discuss their connections on a reasonably serious level. Nevertheless in this paper we present a simple mathematical model enjoying interpretations in terms of both of these approaches which helps a lot in its study.

Let X be a compact subset of $\mathbb{R}^d$ equipped with the Euclidian metric $\rho(\cdot, \cdot)$ and let $(\overline{T}, \overline{X}) := \bigotimes_i^N (T, X)$ be a direct product of N copies (in general $0 < N \le \infty$) of the measurable nonsingular dynamical system $(T, X, \mathcal{B}, \mu_T)$. Consider also a family of measurable maps $\overline{Q}_\varepsilon : \overline{X} \to \overline{X}$ to which we shall refer as *interactions*. The parameter $\varepsilon \ge 0$ here measures how far the map $\overline{Q}_\varepsilon$ is from the identical map, namely $\varepsilon := \sup_{\overline{x} \in \overline{X}} \overline{\rho}(\overline{Q}_\varepsilon \overline{x}, \overline{x})$, where $\overline{\rho}(\overline{x}, \overline{y}) := \max_i \rho(x_i, y_i)$.

Now a CML is defined as a composition of the direct product map and the interaction: $\overline{T}_\varepsilon := \overline{T} \circ \overline{Q}_\varepsilon$ and the system without interactions can be formally written as $\overline{T}_0 \equiv \overline{T}$.

A typical example of interactions considered in the literature is the so called "diffusive coupling":

$$(\overline{Q}_\varepsilon \overline{x})_i := \varepsilon x_i + \frac{1-\varepsilon}{|J_i|} \sum_{j \in J_i} x_j, \tag{1.1}$$

where the summation is taken over "neighboring elements" J_i of the local unit i. The formula (1.1) indicates that speaking about a CML one usually has in mind a certain structure or topology of connections, say a fixed ordered graph of interactions (see e.g. [1, 2, 5, 7, 8, 9, 10,

11, 17]). Another possibility is to consider a dynamic setup allowing a dynamical switching of connections. Some results in this direction were ob6ained in [16] (in the case of time-varying couplings) and in [12]. In what follows we propose a simple model of a dynamical switching and show that in the weak interaction limit depending on fine properties of the interaction potential this model may demonstrate both condensation/synchronization and independent motions.

Motivated by the idea of the interaction between the local systems induced by (somewhat artificial)[1] "collisions" introduced in [12] we interpret the CML as follows. A *configuration* $\overline{T}_\varepsilon^t \overline{x} := \overline{x}^t = \{x_i^t\}$ at time $t \geq 0$ is associated to the collection of identical particles with unit masses located at points $x_i^t \in X$ which wander in the common domain X independently until they come close enough to each other. As an example one can think about a billiard type system. The interaction occurs only when the particles are coming ε-close to each other and consists in the attraction to the local common center of gravity.

Assume that $(X, \mathcal{B}, m, \rho)$ is a compact convex Lebesgue metric space $X \in \mathbb{R}^d$. For a configuration $\overline{x} \in \overline{X}$ and each index i denote by $J_i = J_i(\overline{x}) := \{j : \quad \rho(x_i, x_j) \leq \varepsilon\}$ the set of indices of particles "interacting" with the i-th one.[2] Then we define the i-th coordinate of the interaction map as follows:

$$(\overline{Q}_\varepsilon \overline{x})_i := \gamma x_i + \frac{1-\gamma}{|J_i|} \sum_{j \in J_i} x_j. \tag{1.2}$$

The parameter $\gamma \in [0, 1]$ and if $\gamma = 0$ only the second summand, corresponding the "center of gravity" of particles belonging to the ε-neighborhood of the point x_i, survives in the expression above. Therefore the physical meaning of the interaction under study is that the i-th particle is moved in the direction of the center of gravity of particles belonging to the ε-neighborhood of the i-th particle.

If $\gamma \in (0, 1)$ we say that the interaction is *soft*, $\gamma = 0$ corresponds to the situation when a particle jumps directly to the center of gravity, which we call *rigid* by contrast. Finally $\gamma = 1$ nullifies the interaction.

Our main results may be formulated as follows. Let the $(T, X, \mathcal{B}, \mu_T)$ be a weakly mixing measurable dynamical system and let μ_T be its only Sinai-Bowen-Ruelle (SBR) measure (see definitions and discussion in Section 2). These assumptions are enough for the case of rigid interactions, but to study soft interactions one inevitably needs some additional smoothness type assumptions for the local map (see Section 4).

Theorem 1.1 *For each $N \in \mathbb{Z}_+$ there are constants $\varepsilon_T > 0$, $\gamma_T > 0$ such that $\forall\ 0 \leq \varepsilon < \varepsilon_T$, $0 \leq \gamma < \gamma_T$ the CML $(\overline{T}_\varepsilon, \overline{X})$ has a SBR measure $\overline{\mu}$ (which does not depend on ε), is supported by the "diagonal" of $\overline{X}$, and whose projections to each "coordinate" coincide with μ_T. If $N = 2$ this SBR measure is unique.*

This result demonstrates that arbitrary weak interactions of a CML with a "generic" local map may demonstrate discontinuity at 0 of the SBR measure $\overline{\mu}_\varepsilon$ considered as a function on ε. Previously such results were known only for "wild" examples of local maps having periodic turning points type singularities, see e.g. [2, 1]. From the point of view of interacting particle systems this theorem can be interpreted as a kind of a condensation phenomenon when particles are gathering together under dynamics.

In the opposite case when γ is close to 1 we observe a more "classical" effect, namely under some reasonably general technical assumptions specified in Section 5 the only SBR measure of

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[1]The interaction discussed in [12] resembles more closely the exchange of velocities between colliding particles rather than the collision itself. The latter is modelled by a prior arrangement of non-overlapping "traps" for pairs of particles.

[2]Later we shall consider some other choices of the set J_i.

the CML converges weakly to the direct product measure as $\varepsilon \to 0$. Bassicaly we need that the action of the transfer operator corresponding to the local map in a suitable Banach space of signed measures be quasi-compact, i.e. can be decomposed into a sum of a θ-contracting operator (with $\theta < 1$) and a finite dimensional projector.

Theorem 1.2 *For each $N \in \mathbb{Z}_+$ there is $\varepsilon'_T > 0$ such that $\forall \ 0 \le \varepsilon < \varepsilon'_T$ and each $\gamma \in (2^N \theta, 1]$ the CML $(\overline{T}_\varepsilon, \overline{X})$ has a unique SBR measure $\overline{\mu}_{\varepsilon,\gamma} \xrightarrow{\varepsilon \to 0} \overline{\mu}_T := \underbrace{\mu_T \otimes \ldots \otimes \mu_T}_{N}$.*

From the interacting particles interpretation this result tells that in this case weak interactions lead to independent particles motions. Thus soft interactions demonstrate very different and more "classical" effects in comparison to the rigid interactions. Note that here one may consider not only particles of different masses but even the local maps need not be identical.

To understand better the nature of the phenomena under study let us consider the behavior of the CML in more detail and under a bit different point of view which is more closely related to the IPS approach.

We say that a *synchronization* or *condensation* with the *basin* $\overline{Y}$ takes place in the CML $(\overline{T}_\varepsilon, \overline{X})$ if $\limsup_{t \to \infty} \max_{i,j} \rho(x_i^t, x_j^t) = 0$ for each configuration $\overline{x} \in \overline{Y}$. Similarly a *desynchronization* with the *basin* $\overline{Y}$ means that $\liminf_{t \to \infty} \max_{i,j} \rho(x_i^t, x_j^t) > 0$ for each $\overline{x} \in \overline{Y}$. We say also that a certain property with the domain A and a probabilistic measure ν is ν-*global* if $\nu(A) = 1$.

In words, the synchronization means that the dynamics converges to the *diagonal* $\overline{D} := \{\overline{x} \in \overline{X} : \ x_i = x_j \ \forall i, j\}$, while the desynchronization corresponds to the absence of such convergence. We refer the reader to [15] for the discussion of various physical aspects of the synchronization.

Let us start with the rigid case. The following statement formulated in terms of the de/synchronization not only clarifies the situation but also gives some additional insight about the dynamics.

Theorem 1.3 *Under the assumptions of Theorem 1.1 $\forall \varepsilon > 0$ the synchronization with the open basin $\overline{Y}$ of positive product measure $\overline{\mu}_T$ takes place. Moreover for $N = 2$ this synchronization is $\overline{\mu}_T$-global. On the other hand, for $N \ge 3$ there is an analytic local map T for which the desynchronization with an open positive product measure $\overline{\mu}_T$ basin takes place.*

It is worth note that our numerical simulations show that for all 1D and 2D mixing maps that we tried for *all* initial particle configurations the global synchronization was observed. This is especially striking because first one might expect the synchronization only for almost all initial particle configurations, and second even for the cases where we are able to prove the desynchronization we do not see it in the numerical experiment. The explanation is that arbitrary small round-off errors may change the behavior of a chaotic dynamical system drastically (see e.g. [1]) and they are responsible in this case as well.

The proof of Theorem 1.3 is based on the following observations. If two particles are coming ε-close to each other, then after the interaction their positions coincide. Therefore if $N = 2$ we only need to show that for almost all 2-particle configurations their trajectories will hit simultaneously the ε-neighborhood of the diagonal $\overline{D}$. To prove the latter statement one uses that outside of this ε-neighborhood there are no interactions and hence it is enough to check the same statement for the direct product system $(\overline{T}, \overline{X}, \overline{\mathcal{B}}, \overline{\mu}_T)$, which in turn follows from the weak mixing of the local map.

One is tempted to extend this construction directly for the case $N > 2$ along the following lines. First, if the trajectory of an initial configuration hits an ε-neighborhood of the diagonal $\overline{D}$

(as in the case $N = 2$) then after the interaction the synchronization takes place: all particles will share the same position. Second, additionally to the previous argument one considers multiple (triple, etc.) collisions between particles (which occur near secondary diagonals of $\overline{X}$) expecting that each collision reduces the number of unmatched particles.

Unfortunately, there are two fundamental obstacles to this naive approach. First, a single collision of $n > 2$ particles does not necessarily imply that they will share the same location after the collision. Indeed, assume that we have $n > 2$ particles uniformly distributed along a circle of radius $r_n = \frac{\varepsilon}{2} / \sin(\pi/n)$. Then in the ε-neighborhood of each particle there are two neighbors located at distance ε. Therefore after the interaction instead of coming to a single common center of gravity, the particles will be again uniformly distributed along a circle of only a slightly smaller (provided $n \gg 1$) radius $r'_n := r(1 + 2\cos(2\pi/n))/3$, e.g. $r'_4 = r_4/3$ and $r'_5 \approx r_5 \times 0.539$. Observe that we do not assume any smoothness of the local map T, and thus after the next application of T neighboring points might become arbitrary far from each other.

The second and even more important obstacle is that even a single collision changes the trajectory of a particle and one cannot apply (at least directly) arguments related to ergodic properties of the original local system $(T, X, \mathcal{B}, \mu)$. Moreover after each decrease of the number of unmatched particles one needs to check that the system is still in a "general position" in order to use again the mixing property.

The result about the desynchronization shows that the smoothness of the map T does not cure these pathologies even in the simplest setting. In a sense the doubling map turns out to be the "worst" local one-dimensional map for our problem.

Now we turn to the analysis of soft interactions. The most striking difference to the rigid case is that results of Theorem 1.1 or 1.3 are no longer available without some assumptions on the smoothness of the local system. Nevertheless Theorem 4.1 (Section 4) shows that if the local map is Lipschits continuous then we recover results of condensation type when $0 < \gamma \ll 1$.

To study the opposite case when $0 < 1 - \gamma \ll 1$ one needs to apply the transfer operators technics for suitable Banach spaces of signed measures. These matters will be discussed in Section 5 using a multidimensional version of the Lasota-Yorke inequality. The idea here is to consider the action of interactions as weak perturbations to the dynamics of each of local systems. This perturbative approach is well known but in our case there is an additional problem related to the fact that in distinction to the already studied situations the transfer operator related to the interaction might make a large contribution to the strong norm dpending on the total number of particles in the configuration.

One might argue that if $\varepsilon = 1$ and $\gamma \ll 1$ then we recover the usual weakly intearcting CML setup. The assumption that $\varepsilon \ll 1$ makes the already weak interactions to occur very rare and thus should preserve the "almost" direct structure of the invariant measure. Unfortunaly this "soft" argument does not work because the local structure of interactions leads to the creation of small regions where the "variation" of a measure might become arbitrary large under the action of interactions (see Section 5).

Additionally to the closeness in space the interaction might depend on the closeness in the "lattice position". To be precise one considers the graph of interactions and assumes that it is locally finite, i.e. the degree of each vertex is finite. The degrees need not to be uniformly bounded. This allows to consider infinite systems, discussed briefly in Section 6 (see also a mean field approach in this Section).

A few words about notation. We use a convention that for a Borel set A and a (signed/complex) measure μ the restriction of the measure μ to the set A is denoted by $\mu_{|A}$ and $\mu(\varphi) := \int \varphi \, d\mu$. The bar notation $\overline{x}$ is used to mark variables describing the CML. Note also that $|\cdot|$ is used for very different objects throughout the paper and we follow the convention that in the case of a

subset of integers $|J|$ means its cardinality, in the case of an interval - its length, in the case of a function $|\varphi| := \text{ess sup}_x |\varphi(x)|$, and in the case of a (signed/complex) measure $|\mu|$ stands for its weak norm, while $||\mu||$ stands for the strong norm (see Section 5.1).

2   Direct product systems: basic ergodic constructions

Here we give a short description of standard definitions an constructions from ergodic theory which are necessary for the proof of our results.

Recall that a measure μ is *T-invariant* if and only if $\mu(\varphi \circ T) = \mu(\varphi)$ for any μ-integrable function $\varphi : X \to \mathbb{R}^1$.

A measurable function $\varphi : X \to \mathbb{R}^1$ is called *invariant* with respect to a dynamical system $(T, X, \mathcal{B}, \mu)$ (or simply T-invariant), if $\varphi = \varphi \circ T$ almost everywhere with respect to the measure μ.

A dynamical system $(T, X, \mathcal{B}, \mu)$ is *ergodic* if each T-invariant function is a constant μ-a.e.

A dynamical system $(T, X, \mathcal{B}, \mu)$ is *weak mixing* if

$$\frac{1}{n} \sum_{k=0}^{n-1} |\mu(T^{-k}A \cap B) - \mu(A)\mu(B)| \stackrel{n \to \infty}{\longrightarrow} 0 \qquad \forall A, B \in \mathcal{B}.$$

A *direct product* of a pair of dynamical systems $(T', X', \mathcal{B}', \mu')$ and $(T'', X'', \mathcal{B}'', \mu'')$ is a new dynamical system $(T' \otimes T'', X' \otimes X'', \mathcal{B}' \otimes \mathcal{B}', \mu' \otimes \mu'')$, where the map $T' \otimes T'' : X' \otimes X'' \to X' \otimes X''$ is defined by the relation $T' \otimes T''(x', x'') := (T'x', T''x'')$, while all other objects are standard direct product of spaces, σ-algebras and measures respectively.

By A^N we denote the direct product of N identical sets $A \in \mathcal{B}$, and by $(T^{\otimes N}, X^N, \mathcal{B}^N, \mu^N)$ – the direct product of N identical copies of a dynamical system $(T, X, \mathcal{B}, \mu)$.

Theorem 2.1 *Let a dynamical system $(T, X, \mathcal{B}, \mu)$ satisfy the weak mixing property. Then for any positive integer $N < \infty$, measurable set $\overline{A} \in \mathcal{B}^N$ with $\mu^N(\overline{A}) > 0$ and for almost any (with respect to the measure μ^N) collection $\overline{x} := \{x_1, x_2, \ldots, x_N\} \in X^N$ there exists a moment of time $t \geq 0$ such that $\overline{T}^t \overline{x} \in \overline{A}$.*

Proof.   Let a dynamical system $(\tau, Y, \mathcal{B}_Y, \nu)$ be ergodic. Then for any pair of measurable sets $A, B \in \mathcal{B}_Y$ with $\nu(A)\nu(B) > 0$ there exists a positive integer $\kappa = \kappa(A, B) < \infty$ such that $\tau^\kappa A \cap B \neq \emptyset$. Indeed, assume that this is not true, i.e. $\tau^n A \cap B = \emptyset$ for any positive integer n. Consider a measurable set $A_\infty := \bigcup_{n \in \mathbb{Z}_+} \tau^n A$. Then $\nu(A_\infty) \geq \nu(A) > 0$ and $A_\infty \cap B = \emptyset$. Therefore the indicator function of a measurable set A_∞ of positive ν-measure is τ-invariant but is not a constant a.e. which contradicts to the ergodicity.

Therefore it is enough to show that the dynamical system $(T^{\otimes N}, X^N, \mathcal{B}^N, \mu^N)$ is ergodic. For that we shall take advantage of the fact that the weak mixing property is preserved under the direct product of weak mixing dynamical systems (see e.g. [13]). To complete the proof its remains to note that the weak mixing implies ergodicity. $\square$

It is of interest that one cannot weaken the conditions of Theorem 2.1 replacing the weak mixing of the original dynamical system by the ergodicity. The problem is that the direct product of ergodic dynamical systems needs not to be ergodic as well: consider a direct product of two identical irrational unit circle rotations. On the other hand, the weak mixing condition is not necessary as well (see [4] for an example of a nonergodic dynamical system for which every open neighborhood $\overline{A}$ of the diagonal of $X \times X$ satisfies the claim of Theorem 2.1).

In what follows we often deal with the ε-neighborhood of the diagonal $\overline{D}$ of $\overline{X} \equiv X^N$ which we denote by $\overline{D}_\varepsilon$. It is not difficult to calculate the Lebesgue measure of this set but the product measure $\overline{\mu}(\overline{D}_\varepsilon)$ may vary sensitively with μ. To estimate it we use the techniques of measurable partitions.

Recall that a measurable partition of $(X, \mathcal{B})$ is a collection $\Delta := \{\Delta_i\}$, $\Delta_i \in \mathcal{B}$ such that $\Delta_i \cap \Delta_j = \emptyset$, $\cap_i \Delta_i = X$. The diameter of a partition is the largest diameter of its elements.

Lemma 2.1 *For each $0 < \varepsilon$ and a probabilistic measure μ we have $0 < \overline{\mu}(\overline{D}_\varepsilon) \le 1$, moreover $\sup_\mu \overline{\mu}(\overline{D}_\varepsilon) = 1$. Assume now that $\forall \varepsilon > 0$ there exists an partition of X of diameter ε with cardinality $n_\varepsilon \le C\varepsilon^{-d}$. Then $\inf_\mu \overline{\mu}^N(\overline{D}_\varepsilon) \ge (C\varepsilon^{-d})^{-N+1}$.*

Proof. Let Δ be a finite partition of X of diameter $\varepsilon > 0$. Since X is compact such partitions always exist. Denote $a_i := \mu(\Delta_i)$ then $\sum_i a_i = \mu(X) = 1$. Observe now that $\Delta_i^N := \underbrace{\Delta_i \times \ldots \times \Delta_i}_{N} \subset \overline{D}_\varepsilon$. Therefore the lower estimate of $\overline{\mu}(\overline{D}_\varepsilon)$ follows from the trivial inequality $\overline{\mu}(\overline{D}_\varepsilon) \ge \sum_i (\mu(\Delta_i))^N = \sum_i a_i^N > 0$.

Now let μ be concentrated at a single point $x \in X$, i.e. $\mu(\{x\}) = 1$. Then obviously $\overline{\mu}(\overline{D}_\varepsilon) \equiv 1 \ \forall \varepsilon > 0$.

It remains to prove the lower estimate under the additional assumption about the cardinality of the partition Δ. Consider a power average $S_N(\{a_i\}) := \left(\frac{1}{n_\varepsilon} \sum_{i=1}^{n_\varepsilon} a_i^N \right)^{1/N}$ of n_ε nonnegative entries a_i. It is known that $S_N(\cdot)$ is monotonous on the nonnegative parameter N. Thus $S_N(\{a_i\}) \ge S_1(\{a_i\}) \equiv \frac{1}{n_\varepsilon}$. Therefore $\overline{\mu}(\overline{D}_\varepsilon) \ge \sum_i (\mu(\Delta_i))^N = \sum_i a_i^N = n_\varepsilon (S_N(\{a_i\}))^N \ge n_\varepsilon (S_1(\{a_i\}))^N = n_\varepsilon^{-N+1}$. $\square$

Lemma 2.2 *Let $\overline{A} \subset \overline{X}$ with $\overline{\mu}(\overline{A}) = 0$. Then $\overline{\mu}((\overline{T})^{-1}\overline{A}) + \overline{\mu}((\overline{Q}_\varepsilon)^{-1}\overline{A}) = 0 \ \ \forall \varepsilon, \gamma \ge 0$.*

Proof. The fact that $\overline{\mu}((\overline{T})^{-1}\overline{A}) = 0$ follows from the nonsingularity of the local map T. If $\varepsilon = 0$ the interaction does not occur and hence $\overline{Q}_0$ is an identical map. Therefore it is enough to consider $\varepsilon > 0$. $\forall N \in \mathbb{Z}_+$ and $\forall \overline{x} \in \overline{X}^N$ the interaction map $\overline{Q}_\varepsilon$ may be written as a nonsingular linear map. However the matrix defining this map depends on the "grouping" $J_i(\overline{x})$. Nevertheless for the finite number N of particles the total number of various "grouping" is finite and hence $\overline{Q}_\varepsilon$ may have only a finite number of nonsingular representations. $\square$

A map $T: X \to X$ induces the *transfer operator* T^* acting in the space of signed measures (generalized functions) $\mathcal{M}$ on X by the formula $T^*\mu(A) := \mu(T^{-1}A)$ for each $A \in \mathcal{B}$ and $\mu \in \mathcal{M}$. From this point of view a measure μ is T-invariant if and only if $T^*\mu = \mu$.

A probabilistic measure $\mu_T \in \mathcal{M}$ is called the *Sinai-Bowen-Ruelle* (SBR) measure for the dynamical system $(T, X, \mathcal{B})$ and a reference measure m (say Lebesgue measure on X) if there exists an open subset $Y \subseteq X$ such that for each probabilistic measure $\mu \in \mathcal{M}$ absolutely continuous with respect to m and such that $\mu(Y) = 1$ we have weak convergence $\frac{1}{n} \sum_{t=0}^{n-1} T^{*t}\mu \overset{n \to \infty}{\longrightarrow} \mu_T$. The set Y is called the *basin of attraction* for the measure μ_T. Obviously an SBR measure is T-invariant.

There are also different approaches for the definition of the SBR measure and we refer the reader to [5] for their discussion and conditions under which those approaches agree with each other.

3 Rigid interactions

3.1 Proof of Theorem 1.3

Let $N = 2$. Consider the ε-neighborhood of the diagonal $\overline{D}_\varepsilon$ in $\overline{X}$. By Lemma 2.1 $\overline{\mu}(\overline{D}_\varepsilon) > 0$. Therefore by Theorem 2.1 for $\overline{\mu}$-a.a. $\overline{x} \in \overline{X}$ there exists the first moment of time $0 \le t(\overline{x}) < \infty$ such that $\overline{T}^{t(\overline{x})}\overline{x} \in \overline{D}_\varepsilon$. Denote the set of full $\overline{\mu}$-measure for which this holds by $\overline{X}_\varepsilon$.

By the definition of the rigid interaction, $\overline{Q}_\varepsilon \overline{D}_\varepsilon \subseteq \overline{D}$ while $\overline{Q}_\varepsilon \overline{x} \equiv \overline{x}$ if $\overline{x} \notin \overline{D}_\varepsilon$. Therefore $\forall \overline{x} \in \overline{X}_\varepsilon, \quad t \in \{0, \ldots, t(\overline{x}) - 1\}$ we have $\overline{T}^t_\varepsilon \overline{x} \equiv \overline{T}^t \overline{x}$ and $\overline{T}^{t(\overline{x})}_\varepsilon \overline{x} \in \overline{D}$ which proves the global synchronization if $N = 2$.

Local synchronization for an arbitrary $N \ge 2$ follows from the invariance of the ε-neighborhood of the diagonal with respect to the dynamics. However, if $N \ge 3$ the observation that $\overline{T}^t_\varepsilon \overline{x} \equiv \overline{T}^t \overline{x} \quad \forall t \in \{0, \ldots, t(\overline{x}) - 1\}$ does not hold.

It remains to show that when $N \ge 3$ even the analyticity of the map T does not guarantee the global synchronization.

Lemma 3.1 *Let $X := S^1$ (unit circle), the local system be governed by the doubling map $Tx := \{2x\}$ and let $N = 3$. Then $\forall\ 0 < \varepsilon \ll 1$ the desynchronization with the domain of positive Lebesgue measure occurs.*

Proof. Let $x_i \in X$, $i \in \{1, 2, 3\}$ and denote $a := x_2 - x_1$, $b := x_3 - x_2$. Denote by A the subset of $\overline{X}$ for which $0 < a, b \le \varepsilon$ and $a + b > \varepsilon$. Under the interaction the coordinates x_i will be changed to x_i' such that the distances between them will become equal to $a' := (a + 2b)/6$ and $b' := (2a + b)/6$. Applying the doubling map we get the new pair of distances $a'' := (a + 2b)/3$ and $b'' := (2a + b)/3$. Since $0 < a'', b'' \le \varepsilon$ and $a'' + b'' = a + b > \varepsilon$ the new configuration again belongs to the set A. Now the observation that the product Lebesgue measure $\overline{m}(A) = \varepsilon^2/2 > 0$ finishes the proof. $\square$

The extension of Lemma 3.1 for the case when $\dim(X) > 1$ is straightforward. Moreover, the discussion after the formulation of Theorem 1.3 demonstrates that for large N the local map demonstrating the desynchronization may be chosen $o(1/N)$-close to identical for each $\varepsilon > 0$.

3.2 Proof of Theorem 1.1

One might think that Theorem 1.1 is a direct consequence of Theorem 1.3. Indeed by Theorem 1.3 a.a. trajectories of our CML $\forall \varepsilon > 0$ after a finite number of iterations hit the diagonal. Hence any forward invariant set of $\overline{T}_\varepsilon$ belongs to the diagonal $\overline{D}$ and obviously the analysis of the invariant measures may be restricted to the "one-dimensional" dynamics on the diagonal. On the other hand, we do not assume that the T-invariant measure μ is unique and hence typically there is a subset $Y \subset X$ on untypical points leading to statistics different from μ. Still the μ-measure of this exceptional set is zero. Therefore using that number of iterations before to hit the diagonal is finite (but not uniformly bounded) and the result of Lemma 2.2 we deduce that the set of $\overline{\mu}$-"typical" points is of full $\overline{\mu}$-measure.

3.3 The closest rigid interactions

As usual only the particles from the ε-neighborhood of x_i will be included to $J_i(\overline{x})$, but now we consider a special (simplified) choice of

$$J_i(\overline{x}) := \{j: \quad \rho(x_i, x_j) \le \varepsilon, \quad \rho(x_i, x_j) = \min_{x_k \ne x_i} \rho(x_i, x_k)\},$$

i.e. this collection contains only the *closest* particles to the i-th one. We shall refer to this choice of $J_i(\overline{x})$ as the *closest interaction*. Note that the set of configurations $\overline{x}$ for which $\max_i |J_i(\overline{x})| > 2$ has Lebesgue measure 0.

Theorem 3.1 *Let the $(T, X, \mathcal{B}, \mu)$ be a weakly mixing measurable DS and μ be its only SBR measure. Assume also that the interaction is rigid with the above choice of $J_i(\overline{x})$. Then for each $N \in \mathbb{Z}_+$, $\varepsilon > 0$ the $\overline{\mu}$-global synchronization takes place.*

Proof. For a configuration $\overline{x} \in \overline{X}$ denote by $\ell(\overline{x})$ the minimum distance between particles in $\overline{x}$. Observe that for $\overline{\mu}$-a.a. configurations each $J_i(\overline{x})$ consists of at most one particle and the minimal distance is achieved at a single pair of particles. By the definition of the interaction if $\ell(\overline{x}) \le \varepsilon$ these two particles may interact only between themselves and hence their positions after the interaction will coincide with the common center of gravity.

On the other hand, if $\ell(\overline{x}) > \varepsilon$ no interactions occur and one may use the same argument as in the proof of Theorem 1.3 to show that $\overline{\mu}$-a.s. this effect takes place.

To finalize the inductive construction, observe that for each particle the matching with some other particles takes place after a finite number of time steps and hence by Lemma 2.2 the $\overline{\mu}$ measure of "non generic" initial configurations leading to the non-uniqueness of $\ell(\overline{x})$ is zero. $\quad\square$

4 Soft interactions. Case $0 \le \gamma \ll 1$

In this Section we study the intermediate case when $0 < \gamma < 1$. It is easy to see that if γ is close enough to 1 then we are basically in the same situation as in the case of rigid interactions.

Theorem 4.1 *Let the $(T, X, \mathcal{B}, \mu)$ be a weakly mixing measurable DS and μ be its only SBR measure. Assume also that there is a constant $0 < \Lambda < 1/\gamma$ such that $\rho(Tx, Ty) \le \Lambda\rho(x, y) \ \forall x, y \in X$ Then the claims of Theorems 1.1 and 1.3 hold true.*

Proof. Let $\overline{x} \in \overline{D}_\varepsilon$. Then $\rho(x_i, x_j) \le \varepsilon \ \forall i, j$ and hence all "particles" do interact with each other. Denote by z their common center of gravity. Then $(\overline{Q}_\varepsilon \overline{x})_i := \gamma x_i + (1-\gamma)z$ which implies that

$$\rho((\overline{Q}_\varepsilon \overline{x})_i, (\overline{Q}_\varepsilon \overline{x})_j) \le \gamma\rho(x_i, x_j) \ \forall i, j.$$

Thus $\overline{Q}_\varepsilon \overline{D}_\varepsilon \subseteq \overline{D}_{\gamma\varepsilon}$.

On the other hand $\overline{T D}_\varepsilon \subseteq \overline{D}_{\Lambda\varepsilon}$ by the assumption on the map T. Therefore

$$\overline{T} \circ \overline{Q}_\varepsilon \overline{D}_\varepsilon \subseteq \overline{D}_{\Lambda\gamma\varepsilon} \subseteq \overline{D}_\varepsilon.$$

Moreover,

$$\rho((\overline{T}_\varepsilon^t \overline{x})_i, (\overline{T}_\varepsilon^t \overline{x})_j) \le (\Lambda\gamma)^t \rho(x_i, x_j) \stackrel{t\to\infty}{\Longrightarrow} 0 \ \forall i, j.$$

The completion of the proof follows exactly to the same arguments as in the case of rigid interactions. $\quad\square$

Corollary 4.1 *Let Λ_T be the modulus of the largest Lyapunov multiplier of the map T. Then the claims of Theorems 1.1 and 1.3 hold true if and only if $\gamma < \gamma_0 := 1/\Lambda_T$.*

Proof. The direct statement follows from the argument above applied to $\overline{D}_{\varepsilon/\Lambda_T}$ rather than to $\overline{D}_\varepsilon$. To prove the inverse statement one observes that the modulus of the largest Lyapunov multiplier of the map $\overline{T} \circ \overline{Q}_\varepsilon$ cannot be smaller than $\Lambda_T \gamma > 1$. $\qquad\qquad\square$

Remark. Despite the similarity between the case under consideration and the case of rigid interactions there is an important difference in that once $\overline{x} \in \overline{D}_\varepsilon$ all "particles" will immediately form a cluster on the next time step in the rigid case, while an infinite number of iterations is nessasary for this in the soft case.

5 Soft interactions. Case $0 < 1 - \gamma \ll 1$

Since $\gamma = 1$ correspond to the absence of interactions (and thus the SBR measure of the multicomponent system is equal to the direct product of local SBR measures) one expects to observe a kind of phase transition when the parameter γ grows from 0 to 1. In what follows we shall study what happens when the parameter γ becomes very close to 1.

In the Introduction we already mentioned that for our purpose it is enough to assume that the local transfer operator T^* be quasi-compact in a suitable Banach space of signed measures. This means that T^* may be represented as a sum of a contraction and a compact operator. Below we shall show that this property is satisfied e.g. for the so called piecewise-expanding maps.

To this end we need to introduce a proper Banach space of signed measures and to describe their properties.

5.1 Transfer-operator approach and BV measures

Recall that $\overline{X}$ is a unit N-dimensional Eucledian cube equipped with the standard Borel σ-algebra $\overline{\mathcal{B}} \equiv \mathcal{B}^N$. The map $\overline{T} : \overline{X} \to \overline{X}$ induces the *transfer operator* $\overline{T}^*$ acting in the space of signed measures (generalized functions) $\overline{\mathcal{M}}$ on $\overline{X}$ by the formula $\overline{T}^* \mu(\overline{A}) := \mu(\overline{T}^{-1}\overline{A})$ for each $\overline{A} \in \overline{\mathcal{B}}$ and $\overline{\mu} \in \overline{\mathcal{M}}$.

Starting from [6, 3] the approach to the analysis of transfer operators in terms of the so called dual norms proved to be efficient and became popular. To introduce the dual norms in the space of signed measures (generalized functions) on $\overline{X}$ we start with spaces of test-functions

$$\mathcal{F} := \{\varphi \in C^1(\overline{X}): \ |\varphi| \le 1\}, \quad \mathcal{F}_0 := \{\varphi \in C^1(\overline{X}): \ |\varphi| \le 1, \ \varphi|_{\partial \overline{X}} = 0\},$$
$$\mathcal{F}_L := \{\varphi \in \mathbf{L}(\overline{X}): \ |\varphi|_\infty \le 1\}.$$

The following are two versions of the "variation" of a signed measure:

$$\mathbf{V}(\overline{\mu}) := \max_i \sup_{\varphi \in \mathcal{F}} \overline{\mu}(\partial_i \varphi), \qquad \mathbf{V}_0(\overline{\mu}) := \max_i \sup_{\varphi \in \mathcal{F}_0} \overline{\mu}(\partial_i \varphi).$$

The latter functional gives the variation of the density of the measure μ with respect to the Lebesgue measure, while the former also gives the variation of this density but considered as a function from $\mathbb{R}^N$ taking zero value outside of X. An important advantage of the above definition of the variation is that in the case of a measure having the direct product structure properties of its variation can be easily obtained from their one-dimensional counterparts.

Therefore we shall give proofs only for one-dimensional statements and refer the reader e.g. to [11] for the multidimensional setting.

Define also the L^1-norm of the (signed) measure:

$$|\overline{\mu}| := \sup_{\varphi \in \mathcal{F}_L} \overline{\mu}(\varphi)$$

which we shall call the *weak* norm.

Lemma 5.1 *(a)* $\mathbf{V}(\overline{\mu}_{|\overline{Y}}) \leq \mathbf{V}(\overline{\mu})$;
(b) $\mathbf{V}_0(\overline{\mu}_{|\overline{Y}}) \leq \mathbf{V}(\overline{\mu}_{|\overline{Y}}) \leq 2\mathbf{V}_0(\overline{\mu}_{|\overline{Y}}) + 2|\overline{\mu}_{|Y}|/\overline{m}(\overline{Y})$;
(c) if $\overline{Y}$ is a proper rectangle[3] then $|\overline{\mu}_{|Y}| \leq \frac{1}{2}\,\overline{m}(\overline{Y})\,\mathbf{V}(\overline{\mu})$, *in particular,* $|\overline{\mu}| \leq \frac{1}{2}\mathbf{V}(\overline{\mu})$.

Proof. (a) $\mathbf{V}(\mu_{|Y}) = \sup_{\varphi \in \mathcal{F}} \mu_{|Y}(\varphi) \leq \sup_{\varphi \in \mathcal{F}} \mu(\varphi) = \mathbf{V}(\mu)$.

(b) For $\varphi \in \mathcal{F}$ set $\varphi_0 := x(\varphi(x) - \varphi(1)) + (1 - x)(\varphi(x) - \varphi(0))$. Then $|\varphi_0| \leq 2|\varphi|$ and since $\varphi_0(0) = \varphi_0(1) = 0$ we obtain that $\frac{1}{2}\varphi_0 \in \mathcal{F}_0$. Therefore $\mu(\varphi') = \mu(\varphi_0 + (\varphi(1) - \varphi(0))) \leq 2\mathbf{V}_0(\mu) + 2|\mu|$, which proves the inequality for the case $Y = X$. The general case can be proven similarly.

(c) Decompose the signed measure $\mu_{|Y} := \mu_+ - \mu_-$ into positive and negative components and set $Y_\pm := \text{supp}(\mu_\pm)$. The function $\varphi(x) := m(Y_+ \cap [0, x]) - m(Y_- \cap [0, x]) - \frac{1}{2}m(Y)$ is continuous on X and $|\varphi| \leq \frac{1}{2}m(Y)$. On the other hand, by definition $|\mu_{|Y}| = \mu(\varphi') \leq \frac{1}{2}m(Y)\mathbf{V}(\mu)$ since $2\varphi/m(Y) \in \mathcal{F}$ is a valid test-function. $\qquad\square$

Therefore the functional $\mathbf{V}(\overline{\mu})$ is actually a norm (which we denote by $||\overline{\mu}||$) and is equivalent to a more common strong norm $\mathbf{V}_0(\overline{\mu}) + |\overline{\mu}|$. Therefore we shall refer to $|| \cdot ||$ as a *strong norm*. Note also that for the Lebesgue measure $\mathbf{V}(\overline{m}) = 2$. The set of (signed) measures μ with $||\overline{\mu}|| < \infty$ we shall call measures of *bounded variation* and denote this set by $\mathbf{BV}$.

Using this terminology we may rewrite Theorem 1.2 claiming the convergence to the direct product measure as follows.

Theorem 5.1 *Let the map T have the only one SBR measure μ_T, and let there are constants $0 \leq \theta < 1 \leq \Theta < \infty$ such that*

$$||T^*\mu|| \leq \theta||\mu|| + \Theta|\mu| \tag{5.1}$$

for each $\mu \in \mathbf{BV}$. If $2^N\theta/\gamma < 1$ then for each $0 \leq \varepsilon \ll 1$ the CML $(\overline{T}_\varepsilon, \overline{X})$ has the only one SBR measure $\overline{\mu}_\varepsilon \xrightarrow{\varepsilon \to 0} \overline{\mu}_T$ – the direct product of the local SBR measures.

To give a specific model satisfying to our assumptions consider the class of piecewise expanding maps. Let $X := [0, 1]$ and $\tau : [0, 1] \to [0, 1]$ be a piecewise C^2-smooth map, i.e. there is a finite partition of X into intervals X_i on each of which the map τ is bijective, C^2-smooth, and $\inf_x |\tau'(x)| \geq \lambda > 0$, $\beta_1(\tau) := \frac{2}{\lambda \min_i |X_i|}$, $\beta_2(\tau) := \sup_x |(1/\tau'(x))'| < \infty$. Such maps are called λ-*expanding*. Set $\beta(\tau) := \beta_1(\tau) + \beta_2(\tau)$.

Lemma 5.2 *(Lasota-Yorke inequality) Let the maps $\tau_1, \tau_2, \ldots, \tau_N$ be λ-expanding, $\overline{\tau}$ stands for their direct product, and let $\beta(\overline{\tau}) := \max_i \beta(\tau_i)$. Then*

$$||\overline{\tau}^*\overline{\mu}|| \leq \frac{2}{\lambda}||\overline{\mu}|| + \beta(\overline{\tau})\,|\overline{\mu}|. \tag{5.2}$$

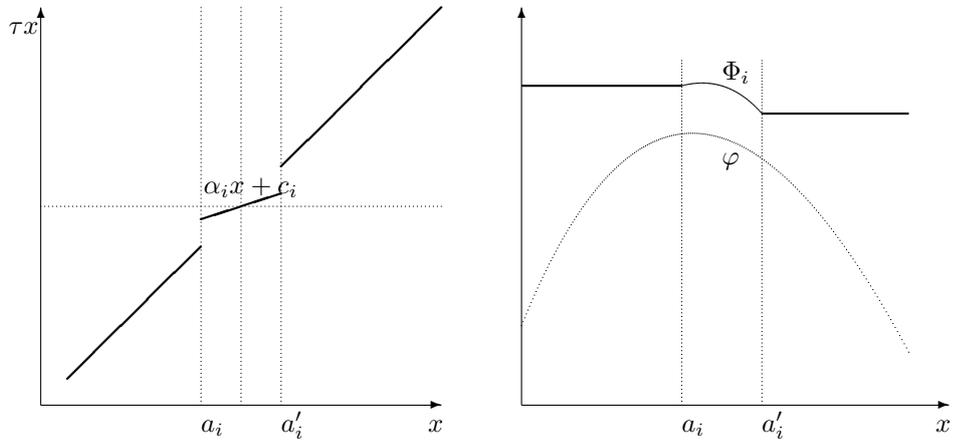[3]i.e. a direct product of n intervals.

Figure 1: Local structure of the map τ (left). Decomposition of the test-function (right).

Proof. First observe that from Lemma 5.1,(b) it follows that for each $\varphi \in \mathcal{F}$

$$\mu(\varphi') \leq 2|\varphi| \left(\mathbf{V}_0(\mu) + \frac{1}{|X_i|} |\mu| \right) \tag{5.3}$$

As $(\varphi \circ \tau)'(x) = \varphi'(\tau(x)) \cdot \tau'(x)$ for each $x \in X \setminus (\cup_i \partial X_i)$, we have

$$\tau^*\mu(\varphi') = \mu(\varphi' \circ \tau) = \mu((\varphi \circ \tau)'/\tau') = \mu(((\varphi \circ \tau)/\tau)') - \mu((\varphi \circ \tau) \cdot (1/\tau)').$$

To estimate the first term we apply (5.3), while the second term is bounded by $\beta_2(\tau) |\mu|$. □

Corollary 5.3 *Under the assumptions of Lemma 5.2 there exists a probabilistic $\overline{\tau}$-invariant measure $\mu_{\overline{\tau}}$.*

Proof. Choose $k \in \mathbb{Z}_+$ large enough such that $\lambda^k > 2$. Denote $r := 2/\lambda^k < 1$ and let $\overline{\mu} \in \mathbf{BV}$ be a probabilistic measure. Then for each $n \in \mathbb{Z}_+$ we have:

$$||\overline{\tau}^{*nk}(\overline{\mu})|| \leq r^n ||\overline{\mu}|| + \frac{\beta(\overline{\tau})}{1-r} |\overline{\mu}|.$$

Thus the sequence $\mu_n := \overline{\tau}^{*nk}\overline{\mu}$ satisfies the conditions of the embedding of $\mathbf{BV}$ to the set of measures having absolutely continuous densities with respect to Lebesgue measure, which we denote by $\mathbf{L}^1$, and, hence, there exists the limit point of this sequence, i.e. $\overline{\mu}_{n_i} \xrightarrow{i \to \infty} \overline{\mu}_\infty \in \mathbf{BV}$ with $||\mu_\infty|| \leq \frac{\beta(\overline{\tau})}{1-r}|\mu|$. On the other hand, being a limit point of this sequence the measure $\overline{\mu}_\infty$ satisfies the relation $\overline{\tau}^*\overline{\mu}_\infty = \overline{\mu}_\infty$ and thus is $\overline{\tau}$-invariant. □

Corollary 5.4 *The transfer operator corresponding to the map $\overline{\tau}$ under an additional assumption of the uniqueness of SBR measures for each map τ_i satisfies the conditions of Theorem 5.1 with $\theta := 2/\lambda$ and $\Theta := \beta(\overline{\tau})$.*

Consider now a more specific family of piecewise C^2-smooth maps $\tau : X \to X$. For a given positive integer n let $0 \leq a_1 < a'_1 \leq a_2 < a'_2 \leq \ldots \leq a_n < a'_n \leq 1$ and set $A_i := [a_i, a'_i]$, $i \in \{1, 2, \ldots, n\}$, $A := \cup_i A_i$, and $B := X \setminus A$. These intervals define a partition of X.

Then we define $\tau x := \begin{cases} \tau_i x := \alpha_i x + c_i & \text{if } x \in A_i \\ x & \text{otherwise} \end{cases}$, where $\tau_i A_i \subseteq A_i$, $\alpha_i > 0$ $\forall i$. Fig. 1 (left) demonstrates the shape of τ in a neighborhood of an interval A_i.

We shall be interested in the properties of the transfer-operator τ^* when $|A| \ll 1$. Due to this restriction the application of Theorem 5.2 gives an estimate with the second term going to infinity as the diameter of the partition goes to zero. In order to overcome this difficulty we develop a new approach to estimate the norm of the transfer operator in this case.

Lemma 5.5 $||\tau^* \mu|| \le (n + 1 + \sum_i \frac{1}{\alpha_i}) \, ||\mu||$, and $|\mu - \tau^* \mu| \le \frac{1}{2}(n + 2 + \sum_i \frac{1}{\alpha_i}) \, m(A) \, ||\mu||$.

Proof. The idea used in the proof of Lemma 5.1,(b) is to interpolate linearly between the values of the test-function at boundary points of the partition $\{X_i\}$ and to estimate the contribution of this interpolation into the integral against the weak norm, rather than the strong one. In the case under consideration the lengths of the intervals of monotonicity might be arbitrary small which does not allow to apply his trick. Instead we shall treat each interval of monotonicity separately extending the test-function by two constants equal to the values at boundary points outside of the interval (see Fig. 1 (right)).

Observe that for each i the function $\tau_i(x)$ can be extended as a linear function to the whole X. For a test-function $\varphi \in \mathcal{F}$ and a (signed) measure μ we have

$$\tau^* \mu(\varphi') = \mu(\varphi' \circ \tau) = \mu_{|B}(\varphi') + \sum_i \mu_{|A_i}(\varphi' \circ \tau)$$

$$= \mu_{|B}(\varphi') + \mu_{|A}(\varphi') - \mu_{|A}(\varphi') + \sum_i \mu_{|A_i}(\varphi' \circ \tau)$$

$$= \mu(\varphi') + \sum_i^n \left(\mu_{|A_i}(\varphi' \circ \tau) - \mu_{|A_i}(\varphi') \right).$$

Let $x \in A_i$ then $\varphi' \circ \tau = (\varphi \circ \tau)' \cdot (\tau_i')^{-1} = \frac{1}{\alpha_i}(\varphi \circ \tau)'$ and $\varphi \circ \tau \in \mathcal{F}$ is a valid test-function. Therefore $\mu_{|A_i}(\varphi' \circ \tau) \le \frac{1}{\alpha_i} \mathbf{V}(\mu)$. Summing up all contributions from the integrations over A_i we get

$$\tau^* \mu(\varphi') \le \left(1 + n + \sum_i \frac{1}{\alpha_i} \right) \, ||\mu||$$

since $\mu_{|A_i}(\varphi') \le ||\mu||$ $\forall i$ by Lemma 5.1.

To estimate $|\mu - \tau^* \mu|$ observe that the measures differ only on the intervals A_i. Set $\nu := \mu - \tau^* \mu$ and $A := \cup_i A_i$. Then

$$|\nu| = |\nu_{|A}| \le \frac{1}{2} m(A) \mathbf{V}(\nu) \le \frac{1}{2} m(A)(||\mu|| + ||\tau^* \mu||)$$

$$\le \frac{1}{2} m(A) \left(n + 2 + \sum_i \frac{1}{\alpha_i} \right) ||\mu||.$$

$\square$

It might seem that the multiplier $(n + 1 + \sum_i \frac{1}{\alpha_i})$ is overpessimistic, but the trivial example of the Lebesgue measure m on X immediately shows that $||\tau^* m|| = 2(n + 1 + \sum_i \frac{1}{\alpha_i})$ while $||m|| = 2$.

As we shall see the argument used in this proof is the key point in the proof of Theorem 5.1. Apart from this Lemma 5.5 allows to apply the operator approach to a new class of small but discontinuous perturbations.

Theorem 5.2 *Let $T : X \to X$ be a λ-expanding map with $\lambda > 1$ having a unique SBR measure μ_T and let τ be a piecewise linear map described above with $|A| = \delta$ and a given collection of slopes $\{\alpha_i\}$ such that $2(n + 1 + \sum_i \frac{1}{\alpha_i}) < \lambda$. Then for each $0 < \delta \ll 1$ the dynamical system $(\tau \circ T, X)$ has a unique smooth invariant measure $\mu_\delta \xrightarrow{\delta \to 0} \mu_T$.*

Proof. Combining results of Lemmas 5.2, 5.5 we get

$$||(\tau \circ T)^* \mu|| \leq (n + 1 + \sum_i \frac{1}{\alpha_i}) \, ||T\mu||$$

$$\leq \left(n + 1 + \sum_i \frac{1}{\alpha_i} \right) \frac{2}{\lambda} ||\mu|| + \left(n + 1 + \sum_i \frac{1}{\alpha_i} \right) \beta(\tau) |\mu|.$$

Therefore we are in a position to apply Corollary 5.3 to the map $\tau \circ T$ which yields the existence of the probabilistic invariant measure $\mu_\delta \in \mathbf{BV}$.

On the other hand, by Lemma 5.5

$$|\mu - \tau^* \mu| \leq C\delta ||\mu||$$

which by the now standard perturbation argument (see e.g. [1, 6]) implies the convergence $\mu_\delta \xrightarrow{\delta \to 0} \mu_T$. $\square$

To finish this preparatory part let us formulate an estimate of the action of the transfer operator of a contracting affine map which can be proven by a direct inspection.

Lemma 5.6 *Let $\overline{X}$ be the N-dimensional unit cube and let $\overline{\tau}(\overline{x}) := G\overline{x} + H$ be an affine map from $\overline{X}$ into itself such that $\ell(G\xi) \geq \alpha\ell(\xi)$ for each $\xi \in \mathbb{R}^N$ and any norm $\ell(\cdot)$. Then $||\overline{\tau}^* \overline{\mu}|| \leq 1/\alpha ||\overline{\mu}||$.*

5.2 Proof of Theorem 5.1

One is tempted to argue as follows. Assume that the parameter ε is of order of the diameter of the set X. Then each pair of particles is interacting between themselves and we are coming to the well known mean field interaction model. One can show that when γ goes to 1 the only SBR measure of the mean field model converges to the direct product measure, i.e. the subsystems behave independently. Now the decrease of ε leads only to the decrease of the frequency of interactions between particles. Thus already "almost" independent particle are becoming even more independent. In fact the situation is much more complicated. The point is that the existence of a large number of small islands in the phase space where different combinatorial types of interactions actually take place leads to a severe amplification of the variation of a measure under the action of the operator $\overline{Q}_\varepsilon^*$. Consider this in detail.

The definition of the map $\overline{Q}_\varepsilon$ implies that configurations $\overline{x} \in \overline{X}$ whose coordinates have pair distances larger than ε are fixed points of the map $\overline{Q}_\varepsilon$. On the remaining part of the phase space $\overline{X}$ consisting of a large (of order 2^N) number of disjoint components of small Lebesgue measure the map $\overline{Q}_\varepsilon$ is linear and contracting in each of them. Thus the structure of the multi-dimensional map $\overline{Q}_\varepsilon$ is very similar to the one-dimensional map τ considered in Lemma 5.5. Therefore we shall use basically the same strategy for the proof.

For a given test-function $\varphi \in \mathcal{F}$ we need to evaluate the functional $\max_i \overline{Q}_\varepsilon^* \overline{\mu}(\partial_i \varphi)$.

Fix some index i and for each subset J of different integers belonging to the set $\{1, 2, \ldots, N\}$ and containing the index i define a set

$$A_J := \{\overline{x} \in \overline{X} : \quad |x_i - x_j| \le \varepsilon \ \forall j \in J, \quad |x_i - x_k| > \varepsilon \ \forall k \notin J\},$$

and let $A := \cup_{|J|>1} A_J$, $B := \overline{X} \setminus A$. Then the interaction with the i-th particle occurs only for $\overline{x} \in A$ and the sets A, B define a finite partition of X.

We have

$$\overline{Q}_\varepsilon^* \overline{\mu}(\partial_i \varphi) = \overline{\mu}(\partial_i \varphi \circ \overline{Q}_\varepsilon) = \overline{\mu}_{|B}(\partial_i \varphi \circ \overline{Q}_\varepsilon) + \sum_{|J|>1} \overline{\mu}_{|A_J}(\partial_i \varphi \circ \overline{Q}_\varepsilon)$$

$$= \overline{\mu}_{|B}(\partial_i \varphi) + \overline{\mu}_{|A}(\partial_i \varphi) - \overline{\mu}_{|A}(\partial_i \varphi) + \sum_{|J|>1} \overline{\mu}_{|A_J}(\partial_i \varphi \circ \overline{Q}_\varepsilon)$$

$$= \overline{\mu}(\partial_i \varphi) + \sum_{|J|>1} \left(\overline{\mu}_{|A_J}(\partial_i \varphi \circ \overline{Q}_\varepsilon) - \overline{\mu}_{|A_J}(\partial_i \varphi) \right)$$

Denote by φ' the vector of partial derivatives of φ, by $\overline{Q}'_\varepsilon$ the matrix of partial derivatives of the map $\overline{Q}_\varepsilon$, and by (q_{ij}) the matrix inverse to the matrix $\overline{Q}'_\varepsilon$ (i.e. $(q_{ij}) = (\overline{Q}'_\varepsilon)^{-1}$). Then

$$(\partial_i \varphi) \circ \overline{Q}_\varepsilon = ((\varphi \circ \overline{Q}_\varepsilon) \cdot (\overline{Q}'_\varepsilon)^{-1})_i = \sum_j \partial_j (\varphi \circ \overline{Q}_\varepsilon) \cdot q_{ji}$$

$$= \sum_j \partial_j (\varphi \circ \overline{Q}_\varepsilon \cdot q_{ji}) - \sum_j \varphi \circ \overline{Q}_\varepsilon \cdot \partial_j q_{ji}.$$

Denoting $\varphi_{ij} := \varphi \circ \overline{Q} \cdot q_{ij} \in \mathbf{C}^1$ we rewrite the expression for the action of the transfer-operator on a measure restricted to A_J as follows:

$$\overline{\mu}_{|A_J}(\partial_i \varphi \circ \overline{Q}_\varepsilon) \le \sum_j \sup_x |\varphi_{ji}(x)| \cdot \mathbf{V}(\overline{\mu}) + \sum_j \sup_x |\partial_j q_{ji}(x)| \cdot |\overline{\mu}|.$$

The interaction inside of each region A_J is described by a linear function and thus the last term is equal to zero for $\overline{x} \in A_J$.

In general the prefactor $\sum_j \sup_x |\varphi_{ji}(x)|$ can be estimated as

$$\sum_j \sup_x |\varphi_{ji}(x)| \le \sum_j \sup_x |((\overline{Q}'_\varepsilon)^{-1})_{ji}|.$$

In our case the upper estimate can be done explicitly. Observe that for a given set A_J the restriction of the map $\overline{Q}_\varepsilon$ to A_J is a linear map (which we denote by L) has a very simple structure. Namely, considering only "interacting coordinates" one can rewrite this map as an affine map $L\overline{y} := G\overline{y} + H$ with $G := \gamma + \frac{1-\gamma}{n} E$. Here n stands for the number of the "interacting coordinates" and all entries of the $n \times n$ matrix E are ones. It is easy to check that $\ell(G\xi) \ge \gamma \ell(\xi)$ for any norm $\ell(\cdot)$ and $\xi \in \mathbb{R}^n$ and the equality is achieved on a vector ξ having the only one nontrivial coordinate. Therefore using Lemma 5.6 and setting $\alpha := \gamma$ we get $||L^* \overline{\mu}|| \le 1/\gamma$. Thus the prefactor can be estimated from above as $1/\gamma$ uniformly on J.

Now using that the number of different collections J cannot exceed 2^N we get

$$||\overline{Q}_\varepsilon^* \overline{\mu}|| \le \frac{2^N \theta}{\gamma} ||\overline{\mu}||.$$

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Combining this result with the Lasota-Yorke type inequality (5.1) for the direct product map we estimate the strong norm of the action of the transfer-operator of the CML as

$$||\overline{T}_\varepsilon^* \overline{\mu}|| \leq \theta ||\overline{Q}_\varepsilon^* \overline{\mu}|| + \Theta |\overline{Q}_\varepsilon^* \overline{\mu}| \leq \frac{2^N \theta}{\gamma} ||\overline{\mu}|| + \Theta |\overline{\mu}|.$$

Now using again the same trick as in the proof of Lemma 5.5 we show that the measures $\overline{T}_\varepsilon^* \overline{\mu}$ and $\overline{T}^* \overline{\mu}$ are close in the weak ($\mathbf{L}^1$) norm for small enough $\varepsilon > 0$.

For a collection of indices I introduce a set

$$B_I := \{\overline{x} \in \overline{X} : \quad \forall i_1 \in I \; \exists i_2, \ldots, i_k \in I : \quad |x_{i_j} - x_{i_{j+1}}| \leq \varepsilon, \quad \min_{i \in I, j \notin I} |x_i - x_j| > \varepsilon\}.$$

In words, the configurations from the set B_I satisfy the condition that all I-particles (i.e. those with indices from I) are connected by ε-chains, while all others are far enough.

The sets $\{B_I\}$ define a finite partition of $\overline{X}$ and the map $\overline{Q}_\varepsilon$ differs from the identical map only on the sets B_I with $|I| > 1$. Thus the signed measure $\overline{\nu} := \overline{Q}_\varepsilon \overline{\mu} - \overline{\mu}$ is supported only on the sets B_I with $|I| > 1$. Denote $\tilde{B}_I := \{\overline{x} \in \overline{X} : \quad |x_i - x_j| \leq \varepsilon |I| \; \forall i, j \in I\}$. Obviously $\tilde{B}_I$ is a proper rectangle and $B_I \subseteq \tilde{B}_I \; \forall I$. Applying Lemma 5.1(c) we get

$$|\overline{\nu}| = |\overline{\nu}_{|\cup_{|I|>1} B_I}| = \sum_{|I|>1} |\overline{\nu}_{|B_I}| \leq \sum_{|I|>1} |\overline{\nu}_{|\tilde{B}_I}|$$

$$\leq \frac{1}{2} \sum_{|I|>1} |\tilde{B}_I|(||\overline{\mu}|| + ||\overline{Q}_\varepsilon^* \overline{\mu}||)$$

$$\leq \frac{1}{2} \sum_{|J|>1} (\varepsilon |I|)^{|J|} \left(1 + \frac{2^N \theta}{\gamma}\right) ||\overline{\mu}|| \xrightarrow{\varepsilon \to 0} 0.$$

The completion of the proof follows the same perturbation argument as in the proof of Theorem 5.2. □

6 Generalizations

1. General interaction potential. So far we have considered local interactions based on a somewhat non-physical model of attraction. Indeed, normally by the attraction one means something more close to the gravitation law. In order to include these more general local interactions (excluded by (1.2)) consider an "interaction potential" $U : X \to \mathbb{R}$ and set $U_\varepsilon(x) := U(x/\varepsilon)$. Then one defines the following generalization of the dynamically switching interactions:

$$(\overline{Q}_\varepsilon \overline{x})_i := \gamma x_i + \frac{1 - \gamma}{|J_i|} \sum_{j \in J_i} x_j \cdot U_\varepsilon(x_j - x_i). \tag{6.1}$$

The interaction in (1.2) corresponds to the potential $U(x)$ defined by the indicator function $1_{[-1,1]}$ of the interval $[-1, 1]$. Assuming that $U(x) \geq 0$ and making some regularity type assumptions on the potential one recovers all results obtained in Theorems 1.1 and 1.2.

2. Random local dynamics. It worth notice that the dynamics of the local units of the multicomponent system under consideration needs not to be deterministic. Indeed, it might be defined by a stochastic Markov chain satisfying the weak mixing condition (in Theorem 1.1)

and some additional assumptions about the induced operator acting in the space of signed measures (in Theorem 1.2).

3. Infinite particle systems. Strictly speaking our definition of the interaction does not allow to consider infinite particle systems since the notion of the center of gravity is not well defined in this case. To overcome this difficulty one may assume that there exists a certain locally finite[4] graph of interactions and that the dynamical switching occurs only between neighboring elements in this graph. This means that the sets J_i satisfy the property that only elements J connected to i may belong to them. All our results can be extended to this setup.

Another and potentially more promising approach (at least in the "independent phase") is to consider a mean field approximation scheme. Let μ be a probability distribution describing the position of a particle (and assume that it is the same for all particles). If a given particle is located at a point $x \in X$ then the mean field approximation allows to calculate the center of gravity of the particles in the ε-neighborhood $B_\varepsilon(x)$ of this point as $\frac{1}{\mu(B_\varepsilon(x))} \int_{B_\varepsilon(x)} y d\mu(y)$. Therefore one rewrites the interaction as

$$Q_{\varepsilon,\gamma,\mu} x := \gamma x + \frac{1-\gamma}{\mu(B_\varepsilon(x))} \int_{B_\varepsilon(x)} y d\mu(y).$$

Denoting (for a given μ) by $Q^*_{\varepsilon,\gamma,\mu}$ the induced action of the map $Q_{\varepsilon,\gamma,\mu}$ in the space of signed measures we obtain the description of the mean field approximation in this space:

$$T^*_{\varepsilon,\gamma} \mu := Q^*_{\varepsilon,\gamma,T^*\mu} T^* \mu. \tag{6.2}$$

In distinction to the transfer operators considered in the previous Sections the operator $T^*_{\varepsilon,\gamma}$ is nonlinear which complicates its analysis a lot.

In the simplest case when $X := S^1$ and $Tx := 2x$ it is easy to show that the Lebesgue measure is $T^*_{\varepsilon,\gamma}$-invariant for all ε, γ and that any measure uniformly distributed on a periodic trajectory is invariant for small enough $\varepsilon > 0$. Nevertheless the analysis of stability of these measures (i.e. the construction of the analogue of the SBR measure) is a much more delicate task. Even in this simple example one needs to develop a special technique to study properties of the nonlinear transfer operator. Therefore this analysis will be discussed in a separate publication

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[4]i.e. the degree of each vertex is finite, but not necessarily uniformly bounded.

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