An Optimal Execution Problem with Market Impact

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Abstract

We study an optimal execution problem in a market model which considers market impact. First we study a discrete-time model and describe a value function. Then, by shortening the intervals of the execution times, we derive the value function of a continuous-time model and study some of its properties (continuity, semi-group property and viscosity property). We show that these vary with the strength of the market impact. We introduce some examples which show that the forms of the optimal strategies change completely, depending on the amount of the trader's security holdings.

Keywords : Optimal execution, Market impact, Liquidity problems, Hamilton-Jacobi-Bellman equation (HJB), Viscosity solutions

1 Introduction

An optimal portfolio management problem has been developed in [22], [23] and in other papers. These classical financial theories assumed that assets in the market are perfectly liquid. But in the real market we face various liquidity risks. For instance, the problem of transaction costs and the uncertainty of trading.

Another important problem of liquidity is market impact (MI), that is, the effect of the investment behaviour of traders on security prices. Such problems are often discussed in the framework of optimal execution problems, where a trader has a certain amount of a security holdings (shares of a security held) and tries to execute until the time horizon. The optimal execution problem considering MI was first studied in [7] as a minimization problem of an expected execution cost in a discrete-time model, and the model of [7] was generalized as a mean-variance model in [4] and [15]. A continuous-time model of the execution problem was studied in [13], [28], and [29] as a singular/impulse stochastic control problem. In [10], the author also studied the continuous-time model in the framework of mean-variance analyses

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and gave a viscosity characterization of the corresponding value functions. An infinite time horizon case is treated in [27]. The optimal execution problem in the limit-order-book (LOB) model is also studied in [1], [2], [3], [11], [12], [26], etc.

Recently there have been various studies about the optimization problem with MI, but the standard framework has not been fixed yet. In this paper, we try to construct such a framework. We formulate the optimal execution problem in discrete-time, and then derive a continuous-time model by taking the limit.

We mainly consider the case when the MI function is convex with respect to the execution volume of a trader. Although some empirical studies tell us that the MI function is concave (see [5] etc.), considering the effect of a convex MI is interesting and important in a theoretical viewpoint. From the examples given later, we can observe the way MI affects a trader's execution policy.

This paper is organized as follows. In Section 2 we introduce our model. We formulate mathematically a trader's optimization problem in a discrete-time model, and give some assumptions to derive the continuous-time model. In Section 3 we give our main results. We show that the value functions in the discrete-time model converge to the one in the continuous-time model. Then we study some properties of the continuous-time value function: continuity, the semi-group property, and a characterization of it as a viscosity solution of a certain Hamilton-Jacobi-Bellman (HJB) equation. Moreover we have the uniqueness result of the viscosity solution of HJB when MI is strong (in a meaning to be discussed later). In Section 4 we also consider a case where the trader needs to sell up their entire holdings of the security. We show that such a sell-out condition does not influence the form of the continuous-time value function in our model. In Section 5 we treat some examples of our model. We conclude this paper in Section 6. In Section 7, we give the proofs of our results.

2 The Model

In this section we present the details of the model. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be a filtered space which satisfies the usual condition (that is, $(\mathcal{F}_t)_t$ is right-continuous and $\mathcal{F}_0$ contains all P-null sets) and let $(B_t)_{0\leq t\leq T}$ be a standard one-dimensional $(\mathcal{F}_t)_t$-Brownian motion. Here $T > 0$ means a time horizon. For simplicity we assume $T = 1$.

We suppose that the market consists of one risk-free asset (namely cash) and one risky asset (namely security). The price of cash is always equal to 1, which means that a risk-free rate is equal to zero. The price of a security fluctuates according to a certain stochastic flow, and is influenced by the sales of a trader.

First we consider a discrete-time model with time interval $1/n$. We consider a single trader who has an endowment $\Phi_0 > 0$ shares of a security. This trader executes the shares Φ_0 over a time interval $[0, 1]$, but his/her sales affect the price of a security. We assume that the trader executes only at time $0, 1/n, \ldots, (n-1)/n$ for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

Now we describe the effect of the trader's execution. For $l = 0, \ldots, n$, we denote by S^n_l the price of the security at time l/n and $X^n_l = \log S^n_l$. Let $s_0 > 0$ be an initial price (i.e., $S^n_0 = s_0$) and $X^n_0 = \log s_0$. If the trader sells the amount ψ^n_l at time l/n, the log-price changes to $X^n_l - g_n(\psi^n_l)$, where $g_n : [0, \infty) \longrightarrow [0, \infty)$ is a non-decreasing and continuously differentiable function which satisfies $g_n(0) = 0$, and he/she gets the amount of cash $\psi^n_l S^n_l \exp(-g_n(\psi^n_l))$ as proceeds of the execution.

After trading at time l/n, X^n_{l+1} and S^n_{l+1} are given by

$$X^n_{l+1} = Y\Big(\frac{l+1}{n}; \frac{l}{n}, X^n_l - g_n(\psi^n_l)\Big), \quad S^n_{l+1} = \exp(X^n_{l+1}), \tag{2.1}$$

where $Y(t; r, x)$ is the solution of the following stochastic differential equation (SDE)

$$\begin{cases} dY(t; r, x) = \sigma(Y(t; r, x))dB_t + b(Y(t; r, x))dt, \quad t \geq r, \\ Y(r; r, x) = x \end{cases} \tag{2.2}$$

and $b, \sigma : \mathbb{R} \longrightarrow \mathbb{R}$ are Borel functions. We assume that b and σ are bounded and Lipschitz continuous. Then for each $r \geq 0$ and $x \in \mathbb{R}$ there exists a unique solution of (2.2).

At the end of the time interval $[0, 1]$, the trader has the amount of cash W^n_n and the amount of the security φ^n_n, where

$$W^n_{l+1} = W^n_l + \psi^n_l S^n_l \exp(-g_n(\psi^n_l)), \quad \varphi^n_{l+1} = \varphi^n_l - \psi^n_l \tag{2.3}$$

for $l = 0, \dots, n-1$ and $W^n_0 = 0$, $\varphi^n_0 = \Phi_0$. We say that an execution strategy $(\psi^n_l)^{n-1}_{l=0}$ is admissible if $(\psi^n_l)_l \in \mathcal{A}^n_n(\Phi_0)$, where $\mathcal{A}^n_k(\varphi)$ is the set of strategies $(\psi^n_l)^{k-1}_{l=0}$ such that ψ^n_l is $\mathcal{F}_{l/n}$-measurable, $\psi^n_l \geq 0$ for each $l = 0, \dots, k-1$, and $\sum_{l=0}^{k-1} \psi^n_l \leq \varphi$.

A trader whose execution strategy is in $\mathcal{A}^n_n(\Phi_0)$ is permitted to leave the unsold shares of the security, and there will be no penalty if he/she cannot finish the liquidation until the time horizon. In Section 4, we consider a case when the trader must finish the liquidation.

The trader's problem is to choose an admissible strategy to maximize the expected utility $\mathrm{E}[u(W^n_n, \varphi^n_n, S^n_n)]$, where $u \in \mathcal{C}$ is his/her utility function and $\mathcal{C}$ is the set of non-decreasing continuous functions on $D = \mathbb{R} \times [0, \Phi_0] \times [0, \infty)$ such that

$$u(w, \varphi, s) \leq C_u(1 + w^2 + s^2)^{m_u}, \quad (w, \varphi, s) \in D \tag{2.4}$$

for some constants $C_u > 0$ and $m_u \in \mathbb{N}$ (i.e., u has polynomial growth rate).

For $k = 1, \dots, n$, $(w, \varphi, s) \in D$ and $u \in \mathcal{C}$, we define the (discrete-time) value function $V^n_k(w, \varphi, s; u)$ by

$$V^n_k(w, \varphi, s; u) = \sup_{(\psi^n_l)^{k-1}_{l=0} \in \mathcal{A}^n_k(\varphi)} \mathrm{E}[u(W^n_k, \varphi^n_k, S^n_k)]$$

subject to (2.1) and (2.3) for $l = 0, \dots, k-1$ and $(W^n_0, \varphi^n_0, S^n_0) = (w, \varphi, s)$. (For $s = 0$, we set $S^n_l \equiv 0$). We denote such a triplet $(W^n_l, \varphi^n_l, S^n_l)^k_{l=0}$ by $\Xi^n_k(w, \varphi, s; (\psi^n_l)_l)$. For $k = 0$, we denote $V^n_0(w, \varphi, s; u) = u(w, \varphi, s)$. Then our problem is the same as $V^n_n(0, \Phi_0, s_0; u)$. We consider the limit of the value function $V^n_k(w, \varphi, s; u)$ as $n \to \infty$.

Let $h : [0, \infty) \longrightarrow [0, \infty)$ be a non-decreasing continuous function. We introduce the following condition.

$[A]$ $\displaystyle\lim_{n\to\infty} \sup_{\psi \in [0, \Phi_0]} \Big|\frac{d}{d\psi} g_n(\psi) - h(n\psi)\Big| = 0.$

Throughout this paper we always assume the above condition. Let $g(\zeta) = \int_0^\zeta h(\zeta')d\zeta'$ for $\zeta \in [0, \infty)$. Under condition $[A]$, we see that $\varepsilon_n \longrightarrow 0$, where

$$\varepsilon_n = \sup_{\psi \in (0, \Phi_0]} \Big|\frac{g_n(\psi)}{\psi} - \frac{g(n\psi)}{n\psi}\Big|. \tag{2.5}$$

Now we define the function which gives the limit of the discrete-time value functions. For $t \in [0,1]$ and $\varphi \in [0, \Phi_0]$ we denote by $\mathcal{A}_t(\varphi)$ the set of $(\mathcal{F}_r)_{0\le r\le t}$-progressively measurable process $(\zeta_r)_{0\le r\le t}$ such that $\zeta_r \ge 0$ for each $r \in [0,t]$, $\int_0^t \zeta_r dr \le \varphi$ almost surely and $\sup_{r,\omega} \zeta_r(\omega) < \infty$. For $t \in [0,1], (w,\varphi,s) \in D$ and $u \in \mathcal{C}$, we define $V_t(w,\varphi,s;u)$ by

$$V_t(w,\varphi,s;u) = \sup_{(\zeta_r)_r \in \mathcal{A}_t(\varphi)} \mathrm{E}[u(W_t, \varphi_t, S_t)]$$

subject to

$$dW_r = \zeta_r S_r dr, \quad d\varphi_r = -\zeta_r dr, \quad dS_r = \hat{\sigma}(S_r)dB_r + \hat{b}(S_r)dr - g(\zeta_r)S_r dr \tag{2.6}$$

and $(W_0, \varphi_0, S_0) = (w, \varphi, s)$, where $\hat{\sigma}(s) = s\sigma(\log s), \hat{b}(s) = s\{b(\log s) + \sigma(\log s)^2/2\}$. When $s > 0$, we obviously see that the process of the log-price of the security $X_r = \log S_r$ satisfies

$$dX_r = \sigma(X_r)dB_r + b(X_r)dr - g(\zeta_r)dr. \tag{2.7}$$

We denote such a triplet $(W_r, \varphi_r, S_r)_{0\le r\le t}$ by $\Xi_t(w,\varphi,s;(\zeta_r)_r)$, and $(W_r,\varphi_r,X_r)_{0\le r\le t}$ by $\Xi_t^X(w, \varphi, s;(\zeta_r)_r)$, respectively. We remark that $V_0(w,\varphi,s;u) = u(w,\varphi,s)$. We notice that $V_t(w,\varphi,s;u) < \infty$ for each $t \in [0,1]$ and $(w,\varphi,s) \in D$ (see Lemma 6 in Section 7.1).

3 Main Results

In this section we present the main results of this paper. First we give the convergence theorem for value functions.

Theorem 1. *For each $(w,\varphi,s) \in D$, $t \in [0,1]$ and $u \in \mathcal{C}$,*

$$\lim_{n\to\infty} V^n_{[nt]}(w,\varphi,s;u) = V_t(w,\varphi,s;u), \tag{3.1}$$

where $[nt]$ is the greatest integer less than or equal to nt.

The proof is given in Section 7.2. Theorem 1 implies that an optimal execution problem in a continuous-time model is derived as the limit of the ones in the discrete-time model. We call $V_t(w,\varphi,s;u)$ a continuous-time value function. We regard the stochastic processes $(\zeta_r)_r$ as the trader's execution strategies. The value of ζ_r is the instantaneous sales (in other words, execution speed) at time r.

As for the continuity of $V_t(w,\varphi,s;u)$, we have the following theorem.

Theorem 2. *Let $u \in \mathcal{C}$.*
(i) *If $h(\infty) = \infty$, then $V_t(w,\varphi,s;u)$ is continuous in $(t,w,\varphi,s) \in [0,1] \times D$.*
(ii) *If $h(\infty) < \infty$, then $V_t(w,\varphi,s;u)$ is continuous in $(t,w,\varphi,s) \in (0,1] \times D$ and $V_t(w,\varphi,s;u)$ converges to $Ju(w,\varphi,s)$ uniformly on any compact subset of D as $t \downarrow 0$, where*

$$Ju(w,\varphi,s) = \begin{cases} \sup\limits_{\psi\in[0,\varphi]} u\Big(w + \dfrac{1-e^{-h(\infty)\psi}}{h(\infty)}s, \varphi - \psi, se^{-h(\infty)\psi}\Big) & (h(\infty) > 0) \\ \sup\limits_{\psi\in[0,\varphi]} u(w + \psi s, \varphi - \psi, s) & (h(\infty) = 0). \end{cases}$$

We will show Theorem 2 in Section 7.5. As we can see, continuity in t at the origin depends on the state of the function h at infinity. When $h(\zeta) = \infty$, MI is strong (g diverges rapidly) enough to make a trader avoid to liquidate instantaneously: An optimal policy is "no-trading" in an infinitesimal time, thus V_t converges to u as $t \downarrow 0$. When $h(\infty) < \infty$, the value function is not always continuous at $t = 0$ and has the right limit $Ju(w, \varphi, s)$. In this case, MI is not so strong and there is room for sucseeding liquidation in the infinitesimal time. The function $Ju(w, \varphi, s)$ corresponds the utility of the execution of the trader who sells a part of the shares of a security ψ by dividing infinitely within an infinitely short time (enough to neglect the fluctuation of the price of a security) and makes the amount $\varphi - \psi$ remain i.e.

$$\zeta^\delta_r = \frac{\psi}{\delta} 1_{[0,\delta]}(r), \quad r \in [0, t] \quad (\delta \downarrow 0). \tag{3.2}$$

Such a strategy is also discussed in [21]. We remark that the form of Ju is strongly related to Theorem 3 in [21] (see Theorem 8 Section 4 for more details). Also note that the condition $h(\infty) = 0$ corresponds to the classical case i.e. no MI model.

Next we study the semi-group property (Bellman principle) of the family of non-linear operators corresponding with the continuous-time value function. We define an operator $Q_t : \mathcal{C} \longrightarrow \mathcal{C}$ by $Q_t u(w, \varphi, s) = V_t(w, \varphi, s; u)$. Using Theorem 2 and Lemma 6 in Section 7.1, we easily see that Q_t is well-defined. Then we have the following.

Theorem 3. *For each $r, t \in [0, 1]$ with $t + r \leq 1$, $(w, \varphi, s) \in D$ and $u \in \mathcal{C}$ it holds that $Q_{t+r} u(w, \varphi, s) = Q_t Q_r u(w, \varphi, s)$.*

The proof is in Section 7.4. Using Theorem 3, we can characterize the continuous-time value function as a viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation (HJB). Since the value functions are defined in a way that does not depend on Φ_0, we can take it that they are defined on an extended domain $\hat{D} = \mathbb{R} \times [0, \infty) \times [0, \infty)$. Let $u(w, \varphi, s) : \hat{D} \longrightarrow \mathbb{R}$ be such that u is a non-decreasing continuous function which has polynomial growth rate with respect to w, φ and s. We define a function $F : \mathscr{S} \longrightarrow [-\infty, \infty)$ by

$$F(z, p, X) = -\sup_{\zeta \geq 0} \left\{ \frac{1}{2} \hat{\sigma}(z_s)^2 X_{ss} + \hat{b}(z_s) p_s + \zeta \left(z_s p_w - p_\varphi \right) - g(\zeta) z_s p_s \right\},$$

where $\mathscr{S} = U \times \mathbb{R}^3 \times S^3$, $U = \hat{D} \setminus \partial \hat{D}$, S^3 is the space of symmetric matrices in $\mathbb{R}^3 \otimes \mathbb{R}^3$ and

$$z = \begin{pmatrix} z_w \\ z_\varphi \\ z_s \end{pmatrix} \in D, \ p = \begin{pmatrix} p_w \\ p_\varphi \\ p_s \end{pmatrix} \in \mathbb{R}^3, \ X = \begin{pmatrix} X_{ww} & X_{w\varphi} & X_{ws} \\ X_{\varphi w} & X_{\varphi\varphi} & X_{\varphi s} \\ X_{sw} & X_{s\varphi} & X_{ss} \end{pmatrix} \in S^3.$$

Although the function F may take the value $-\infty$, we can define a viscosity solution of the following Hamilton-Jacobi-Bellman equation (HJB) as usual (see [9], [19] and [24] for instance):

$$\frac{\partial}{\partial t} v + F(z, \mathcal{D} v, \mathcal{D}^2 v) = 0 \ \text{ on } (0, 1] \times U, \tag{3.3}$$

where $\mathcal{D}$ denotes the differential operator with respect to $z = (w, \varphi, s)$. Here we remark that (3.3) can be rewritten as

$$\frac{\partial}{\partial t} v(t, w, \varphi, s) - \sup_{\zeta \geq 0} \mathscr{L}^\zeta v(t, w, \varphi, s) = 0, \quad (t, w, \varphi, s) \in (0, 1] \times U, \tag{3.4}$$

where

$$\mathscr{L}^{\zeta}v(t,w,\varphi,s) = \frac{1}{2}\hat{\sigma}(s)^2\frac{\partial^2}{\partial s^2}v(t,w,\varphi,s) + \hat{b}(s)\frac{\partial}{\partial s}v(t,w,\varphi,s)$$
$$+\zeta\Big(s\frac{\partial}{\partial w}v(t,w,\varphi,s) - \frac{\partial}{\partial \varphi}v(t,w,\varphi,s)\Big) - g(\zeta)s\frac{\partial}{\partial s}v(t,w,\varphi,s).$$

Now we state the following theorem which will be proved in Section 7.6.

Theorem 4. *Assume that h is strictly increasing and $h(\infty) = \infty$. Moreover we assume*

$$\liminf_{\varepsilon\downarrow 0}\frac{V_t(w,\varphi,s+\varepsilon;u) - V_t(w,\varphi,s;u)}{\varepsilon} > 0 \tag{3.5}$$

for any $t \in (0,1]$ and $(w,\varphi,s) \in U$. Then $V_t(w,\varphi,s;u)$ is a viscosity solution of (3.3).

Finally we give the uniqueness result of viscosity solutions of (3.4).

Theorem 5. *Assume that $\hat{\sigma}$ and $\hat{b}$ are both Lipschitz continuous. Moreover we assume the conditions in Theorem 4 and the growth condition $\liminf_{\zeta\to\infty}(h(\zeta)/\zeta) = 0$. If a polynomial growth function $v : [0,1]\times\hat{D} \longrightarrow \mathbb{R}$ is a viscosity solution of* (3.4) *and satisfies the following boundary conditions*

$$\begin{aligned} v(0,w,\varphi,s) &= u(w,\varphi,s), \quad (w,\varphi,s)\in\hat{D}, \\ v(t,w,0,s) &= \mathrm{E}\left[u\left(w,0,Z\left(t;0,s\right)\right)\right], \quad (t,w,s)\in[0,1]\times\mathbb{R}\times[0,\infty), \\ v(t,w,\varphi,0) &= u(w,\varphi,0), \quad (t,w,\varphi)\in[0,1]\times\mathbb{R}\times[0,\infty), \end{aligned} \tag{3.6}$$

then $V_t(w,\varphi,s;u) = v(t,w,\varphi,s)$, where

$$Z\left(t;r,s\right) = \exp\left(Y\left(t;r,\log s\right)\right) \ (s>0), \quad 0 \ (s=0). \tag{3.7}$$

The proof is in Section 7.7. In Section 5.2, we will present an example where the assumptions in Theorem 4 and Theorem 5 are fulfilled.

4 Sell-Out Condition

In this section we consider the optimal execution problem under the “sell-out condition.” A trader has certain shares of a security at the initial time, and he/she must liquidate all of them until the time horizon. Then the spaces of admissible strategies are reduced to the following:

$$\mathcal{A}_k^{n,\mathrm{SO}}(\varphi) = \left\{(\psi_l^n)_l \in \mathcal{A}_k^n(\varphi) \ ; \ \sum_{l=0}^{k-1}\psi_l^n = \varphi\right\},$$
$$\mathcal{A}_t^{\mathrm{SO}}(\varphi) = \left\{(\zeta_r)_r \in \mathcal{A}_t(\varphi) \ ; \ \int_0^t \zeta_r dr = \varphi\right\}.$$

Now we define value functions with the sell-out condition by

$$V_k^{n,\mathrm{SO}}(w,\varphi,s;U) = \sup_{(\psi_l^n)_l \in \mathcal{A}_k^{n,\mathrm{SO}}(\varphi)} \mathrm{E}[U(W_k^n)],$$
$$V_t^{\mathrm{SO}}(w,\varphi,s;U) = \sup_{(\zeta_r)_r \in \mathcal{A}_t^{\mathrm{SO}}(\varphi)} \mathrm{E}[U(W_t)]$$

for a continuous, non-decreasing and polynomial growth function $U : \mathbb{R} \longrightarrow \mathbb{R}$. Then we have the following theorem.

Theorem 6. *$V_t^{\mathrm{SO}}(w,\varphi,s;U) = V_t(w,\varphi,s;u)$, where $u(w,\varphi,s) = U(w)$.*

Proof. The relation $V_t^{\mathrm{SO}}(w,\varphi,s;U) \le V_t(w,\varphi,s;u)$ is trivial, so we will show only the assertion $V_t^{\mathrm{SO}}(w,\varphi,s;U) \ge V_t(w,\varphi,s;u)$. Take any $(\zeta_r)_r \in \mathcal{A}_t(\varphi)$ and let $(W_r,\varphi_r,S_r)_r = \Xi_1(w,\varphi,s;(\zeta_r)_r)$. Moreover take any $\delta \in (0,t)$. We define an execution strategy $(\zeta_r^\delta)_r \in \mathcal{A}_t^{\mathrm{SO}}(\varphi)$ by $\zeta_r^\delta = \zeta_r$ $(r \in [0,t-\delta])$, $\varphi_{t-\delta}/\delta$ $(r \in (t-\delta,t])$. Let $(W_r^\delta,\varphi_r^\delta,S_r^\delta)_r = \Xi_1(w,\varphi,s;(\zeta_r^\delta)_r)$. Then we have $W_{t-\delta} = W_{t-\delta}^\delta \le W_t^\delta$. Thus we get $\mathrm{E}[U(W_{t-\delta})] \le \mathrm{E}[U(W_t^\delta)] \le V_t^{\mathrm{SO}}(w,\varphi,s;U)$. Letting $\delta \downarrow 0$, we have $\mathrm{E}[U(W_t)] \le V_t^{\mathrm{SO}}(w,\varphi,s;U)$ by using the monotone convergence theorem. Since $(\zeta_r)_r \in \mathcal{A}_t(\varphi)$ is arbitrary, we obtain the assertion. ∎

By Theorem 6, we see that the sell-out condition $\int_0^t \zeta_r dr = \varphi$ makes no change in the (value of the) value function in a continuous-time model. Thus, although the value function in a discrete-time model may depend on whether the sell-out condition is imposed or not, in the continuous-time model we need not worry about such a condition.

Moreover we obtain the following theorem which is a similar result to Theorem 1.

Theorem 7. *For each $(w,\varphi,s) \in D$*

$$\lim_{n\to\infty} V_{[nt]}^{n,\mathrm{SO}}(w,\varphi,s;U) = V_t^{\mathrm{SO}}(w,\varphi,s;U) \quad (= V_t(w,\varphi,s;U).)$$

Proof. We may assume $t > 0$. Wee see that for large n

$$V_{[nt]-1}^n(w,\varphi,s;u) \le V_{[nt]}^{n,\mathrm{SO}}(w,\varphi,s;U) \le V_{[nt]}^n(w,\varphi,s;u). \tag{4.1}$$

We notice that the first inequality of (4.1) is obtained since any strategy in $\mathcal{A}_{[nt]-1}^n(\varphi)$ can always be extended to the one in $\mathcal{A}_{[nt]}^{\mathrm{SO}}(\varphi)$ by liquidating all remaining inventory at the last period. By similar arguments as in the proof of Theorem 1, we get

$$\lim_{n\to\infty} V_{[nt]-1}^n(w,\varphi,s;u) = V_{t-}(w,\varphi,s;u), \quad \lim_{n\to\infty} V_{[nt]}^n(w,\varphi,s;u) = V_t(w,\varphi,s;u). \tag{4.2}$$

By (4.1), (4.2) and Theorem 2, we get the assertion. ∎

When $g(\zeta)$ is linear, we can apply the variable reduction method (9')–(12') in [21] (The author thanks to Professor N.Touzi to point out this reference) to get the following.

Theorem 8. *Assume $g(\zeta) = \alpha\zeta$ for $\alpha > 0$.*
(i) $V_t^{\mathrm{SO}}(w, \varphi, s; U) = \overline{V}_t^{\varphi}\left(w + \dfrac{1 - e^{-\alpha\varphi}}{\alpha} s, e^{-\alpha\varphi} s; U\right)$, *where*

$$
\begin{aligned}
\overline{V}_t^{\varphi}(\bar{w}, \bar{s}; U) &= \sup_{(\bar{\varphi}_r)_r \in \overline{\mathcal{A}}_t(\varphi)} \mathrm{E}[U(\bar{W}_t)] \\
&\qquad \text{s.t.} \quad d\bar{S}_r = \bar{S}_r \bar{b}(\bar{S}_r + \alpha\bar{\varphi}_r)dr + \bar{S}_r \sigma(\bar{S}_r + \alpha\bar{\varphi}_r)dB_r, \\
&\qquad \qquad d\bar{W}_r = \frac{e^{\alpha\bar{\varphi}_r} - 1}{\alpha} d\bar{S}_r, \\
&\qquad \qquad \bar{S}_0 = s, \quad \bar{W}_0 = w, \\
\overline{\mathcal{A}}_t(\varphi) &= \left\{ \left(\varphi - \int_0^r \zeta_v dv\right)_{0 \le r \le t} \; ; \; (\zeta_r)_{0 \le r \le t} \in \mathcal{A}_t^{\mathrm{SO}}(\varphi) \right\}
\end{aligned}
$$

and $\bar{b}(x) = b(x) + \sigma(x)^2/2$.
(ii) *If U is concave and $\bar{b} \le 0$, then*

$$
V_t^{\mathrm{SO}}(w, \varphi, s; U) = U\left(w + \frac{1 - e^{-\alpha\varphi}}{\alpha} s\right). \tag{4.3}
$$

The proof is in Section 7.9. We notice that the assertion (ii) is the same as Theorem 3 in [21] and in this case we can get the explicit form of the value function. The right-hand side of (4.3) is equal to $Ju(w, \varphi, s)$ for $u(w, \varphi, s) = U(w)$ and the nearly optimal strategy for $V_t^{\mathrm{SO}}(w, \varphi, s; U) = V_t(w, \varphi, s; u)$ is given by (3.2).

5 Examples

In this section we consider two examples of our model. Let $b(x) \equiv -\mu$ and $\sigma(x) \equiv \sigma$ for some constants $\mu, \sigma \ge 0$ and suppose $\tilde{\mu} = \mu - \sigma^2/2 > 0$. We assume that a trader has a risk-neutral utility function $u(w, \varphi, s) = u_{\mathrm{RN}}(w, \varphi, s) = w$. We remark that we can replace the stochastic control problem $V_t(w, \varphi, s; u_{\mathrm{RN}})$ with the deterministic control problem $f(t, \varphi)$, where

$$
\begin{aligned}
f(t, \varphi) &= \sup_{(\zeta_r)_r \in \mathcal{A}_t^{\det}(\varphi)} \tilde{f}(t, \varphi; (\zeta_r)_r), \\
\tilde{f}(t, \varphi; (\zeta_r)_r) &= \int_0^t \zeta_r \exp\left(-\tilde{\mu} r - \int_0^r g(\zeta_v) dv\right) dr, \\
\mathcal{A}_t^{\det}(\varphi) &= \{(\zeta_r)_r \in \mathcal{A}_t(\varphi) \; ; \; (\zeta_r)_r \text{ is deterministic}\}.
\end{aligned}
$$

Indeed we have the following.

Proposition 1. $V_t(w, \varphi, s; u_{\mathrm{RN}}) = w + s f(t, \varphi)$.

This is proved in Section 7.8. By Proposition 1, we see that

$$
\frac{\partial}{\partial s} V_t(w, \varphi, s; u_{\mathrm{RN}}) = f(t, \varphi) > 0, \quad t, \varphi > 0.
$$

5.1 Log-Linear Impact

Set $g(\zeta) = \alpha\zeta$ for $\alpha > 0$. The following theorem is a direct consequence of Theorem 8(ii).

Theorem 9. *It holds that*

$$V_t(w, \varphi, s; u_{\mathrm{RN}}) = w + \frac{1 - e^{-\alpha\varphi}}{\alpha} s \tag{5.1}$$

for each $t \in (0, 1]$ *and* $(w, \varphi, s) \in D$.

The right-hand side of (9) converges to $w + \varphi s$ as $\alpha \downarrow 0$, which is the profit gained by choosing the execution strategy of so-called block liquidation such that the trader sells all shares φ at $t = 0$ when there is no market impact. Theorem 9 implies that the optimal strategy in this case is to execute all shares dividing infinitely within an infinitely short time at $t = 0$. This is almost the same as a block liquidation at the initial time, and the trader does not delay the execution time (although MI lowers the profit of the execution). Therefore we cannot see the essential influence of the MI in this example.

5.2 Log-Quadratic Impact

In this subsection we consider the case of a strictly convex MI function. Set $g(\zeta) = \alpha\zeta^2$ for $\alpha > 0$. We remark that the continuous-time value function in this example is the unique viscosity solution of (3.3) with boundary conditions (3.6).

As we will see, we can derive the explicit form of an optimal strategy. However, when φ is not so small, such a strategy has in fact unbounded execution speed and is not subject to $\mathcal{A}_t(\varphi)$. Thus we extend the set of admissible strategies such that

$$\begin{aligned} \tilde{\mathcal{A}}_t(\varphi) &= \Big\{(\zeta_r)_{0\le r\le t}\ ;\ (\mathcal{F}_r)_r\text{-adapted},\ \zeta_r \ge 0,\ \int_0^t \zeta_r dr \le \varphi \\ &\qquad \text{and} \sup_{(r,\omega)\in[0,t-\varepsilon]\times\Omega} \zeta_r(\omega) < \infty \text{ for all } \varepsilon \in (0,t)\Big\}, \\ \tilde{\mathcal{A}}_t^{\mathrm{det}}(\varphi) &= \{(\zeta_r)_r \in \tilde{\mathcal{A}}_t(\varphi)\ ;\ (\zeta_r)_r \text{ is deterministic}\}. \end{aligned}$$

to allow unbounded execution speed at t. We see that the values of $V_t(w, \varphi, s; u_{\mathrm{R}N})$ and $f(t, \varphi)$ do not change by replacing $\mathcal{A}_t(\varphi)$ with $\tilde{\mathcal{A}}_t^{\mathrm{det}}(\varphi)$. Indeed, for each $(\zeta_r)_r \in \tilde{\mathcal{A}}_t^{\mathrm{det}}(\varphi)$, we have

$$\begin{aligned} \tilde{f}(t, \varphi; (\zeta_r)_r) &= \lim_{\varepsilon\to 0} \tilde{f}(t-\varepsilon, \varphi; (\zeta_r)_r) \\ &\le \lim_{\varepsilon\to 0} f(t-\varepsilon, \varphi) = f(t, \varphi) \end{aligned} \tag{5.2}$$

by virtue of the dominated convergence theorem, so we get

$$f(t, \varphi) = \sup_{(\zeta_r)_r \in \tilde{\mathcal{A}}_t^{det}(\varphi)} \tilde{f}(t, \varphi; (\zeta_r)_r).$$

We define functions $\hat{v}^i(t, w, \varphi, s)$ and $\hat{\zeta}_t^i$, $i = 1, 2$, by

$$\hat{v}^1(t, w, \varphi, s) = w + \frac{s\sqrt{1 - e^{-2\tilde{\mu}t}}}{2\sqrt{\alpha\tilde{\mu}}}, \quad \hat{\zeta}_t^1 = \sqrt{\frac{\tilde{\mu}}{\alpha(1 - e^{-2\tilde{\mu}(t-r)})}}$$

and

$$\hat{v}^2(t,w,\varphi,s) = w + \frac{s}{2\sqrt{\alpha\tilde{\mu}}}(1 - e^{-2\sqrt{\alpha\tilde{\mu}}\varphi}), \quad \hat{\zeta}^2_t = \sqrt{\frac{\tilde{\mu}}{\alpha}} 1_{[0,\varphi\sqrt{\alpha/\tilde{\mu}}]}(r).$$

Moreover we set

$$\hat{\Phi}^1(t) = \frac{\mathrm{arctanh}\sqrt{1 - e^{-2\tilde{\mu}t}}}{\sqrt{\alpha\tilde{\mu}}}, \quad \hat{\Phi}^2(t) = \sqrt{\frac{\tilde{\mu}}{\alpha}} t.$$

Then we have the following.

Theorem 10.
(i) *If* $\varphi \geq \hat{\Phi}^1(t)$*, then* $V_t(w,\varphi,s;u_{\mathrm{RN}}) = \hat{v}^1(t,w,\varphi,s)$ *and* $(\hat{\zeta}^1_r)_r$ *is an optimal strategy.*
(ii) *If* $\varphi \leq \hat{\Phi}^2(t)$*, then* $V_t(w,\varphi,s;u_{\mathrm{RN}}) = \hat{v}^2(t,w,\varphi,s)$ *and* $(\hat{\zeta}^2_r)_r$ *is an optimal strategy.*

Proof. Let $(\hat{W}^i_r, \hat{\varphi}^i_r, \hat{S}^i_r)_r = \Xi_t(w,\varphi,s;(\hat{\zeta}^i_r)_r)$ for $i = 1,2$. A straight calculation shows that $\mathrm{E}[\hat{W}^i_t] = \hat{v}^i(t,w,\varphi,s)$. Then we have $\hat{v}^i(t,w,\varphi,s) \leq V_t(w,\varphi,s;u_{\mathrm{RN}})$. Since $\hat{v}^i$ satisfies (3.4) at (t,w,φ,s), we see that $\hat{v}^i(t,w,\varphi,s) \geq V_t(w,\varphi,s;u_{\mathrm{RN}})$ by Theorem 5.2.1 in [24]. Then we have the assertions. ∎

This theorem implies that the form of optimal strategies and value functions vary depending on the amount of the security holdings φ. If a trader has a small amount of securities, then we have case (ii) and the optimal strategy is to sell the entire holdings of the security until the time $\varphi\sqrt{\alpha/\tilde{\mu}}$. If he/she has a large amount, then we have case (i) and the trader cannot finish the selling.

We have not had an explicit form for $V_t(w,\varphi,s;u_{\mathrm{RN}})$ on a whole space. So we try to solve this example numerically. $V_1(w,\varphi,s;u_{\mathrm{RN}})$ is approximated by $V^n_n(w,\varphi,s;u_{\mathrm{RN}})$ for enough large n, and we can assume that the optimal strategy is deterministic. We can get the value of $V^n_n(w,\varphi,s;u_{\mathrm{RN}})$ numerically for finite n. Figure 1 describes the form of the execution strategies and Figure 2 describes the form of the corresponding processes of the amount of a security when we set $n = 500$, $w = 0, s = 1, \alpha = 0.01, \tilde{\mu} = 0.05, \sigma = 0$ and $\varphi = 1, 10$ and 100. We also get the form of the function $f(t,\varphi)$ of Proposition 1 numerically, which is described in Figure 3. If a pair (t,φ) is in the range (a) of Figure 4, then we have $f(t,\varphi) = \sqrt{1 - e^{-2\tilde{\mu}t}}/(2\sqrt{\alpha\tilde{\mu}})$, and if (t,φ) is in the range (c), we have $f(t,\varphi) = (1 - e^{-2\sqrt{\alpha\tilde{\mu}}\varphi})/(2\sqrt{\alpha\tilde{\mu}})$. We have not had the form of $f(t,\varphi)$ analytically when (t,φ) is in the range (b).

We remark that in case (i) we can also construct a nearly optimal strategy with the sell-out condition. Let $\hat{\zeta}^{1,\delta}_r = \hat{\zeta}^1_r$ $(r \leq t - \delta)$, $(\varphi - \hat{\varphi}_{t-\delta})/\delta$ $(t - \delta < r \leq t)$, where

$$\hat{\varphi}_{t-\delta} = \frac{\mathrm{arctanh}\sqrt{1 - e^{-2\tilde{\mu}t}} - \mathrm{arctanh}\sqrt{1 - e^{-2\tilde{\mu}\delta}}}{\sqrt{\alpha\tilde{\mu}}}.$$

Then $(\zeta^{\delta}_r)_r \in \mathcal{A}^{\mathrm{SO}}_t(\varphi)$ and the corresponding expected profit $\mathrm{E}[\hat{W}^{\delta}_t]$ converges to $V_t(w,\varphi,s;u_{\mathrm{RN}})$ as $\delta \to 0$.

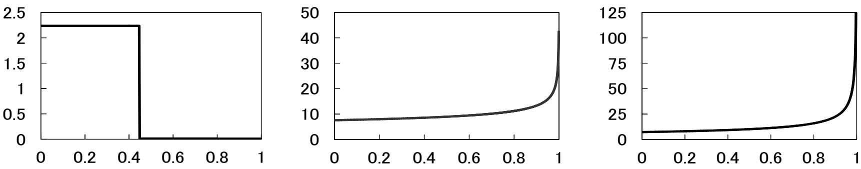

Figure 1: The forms of optimal execution strategies $(\zeta_r)_r$. Horizontal axis is time r. The left graph: $\varphi = 1$. The middle graph: $\varphi = 10$. The right graph: $\varphi = 100$. In the middle graph we calculate $(\zeta_r)_r$ numerically.

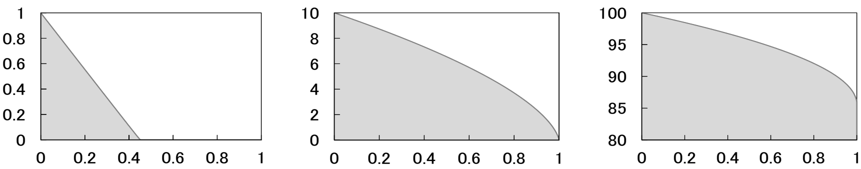

Figure 2: The forms of the amount of security holdings $(\varphi_r)_r$ corresponding with optimal strategies. Horizontal axis is time r. The left graph: $\varphi = 1$. The middle graph: $\varphi = 10$. The right graph: $\varphi = 100$. In the middle graph we calculate $(\varphi_r)_r$ numerically.

6 Concluding Remarks

In this paper we studied the optimal execution problem when MI is considered. First we formulated the discrete-time model and then took the limit. We showed that the discrete-time value functions converge to the continuous-time value function.

We mainly treated the case when the MI function is convex. This is not only for mathematical reasons, but also from a financial viewpoint. In a Black-Scholes type market, an optimal execution strategy of a risk-neutral trader is a block liquidation when there is no MI. As we saw in Section 5, the form of the optimal strategy entirely changes when MI is (log-)quadratic. In contrast, when MI is not convex, especially (log-)linear, then a trader's optimal strategy is almost block liquidation.

However, in the real market, many traders execute the selling in taking time in spite of recognizing that the MI is concave. One of the reasons is that the trader may have a risk-averse utility function. As another reason, we surmise that MI can be divided into two parts: permanent impact and temporary impact (see [4] and [14]). As time passes, the temporary impact disappears and the price once pushed down transitorily, recovers. Our examples treat permanent impact only, but we can also consider temporary impact and price recovery effects. If the process of security prices follows some mean-reverting process, such as an Ornstein-Uhlenbeck process, then we may deal with the optimization problem with MI and price recovery. We study such a case in [18].

It is also meaningful to characterize the continuous-time value function as the solution of the corresponding HJB. We have shown that the value function is a viscosity solution under some strong assumptions. Such assumptions would not be necessary if we considered only bounded strategies, but the control region of our model is unbounded. We avoid this difficulty

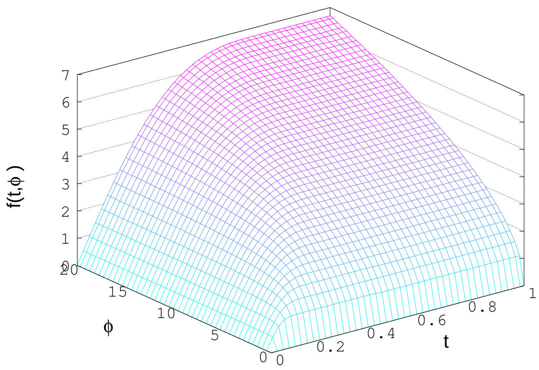

Figure 3: The form of the function $f(t, \varphi)$ in Section 5.2.

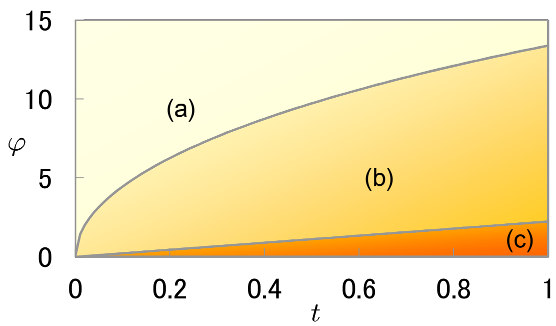

Figure 4: The region of pairs (t, φ). The region (a) (resp., (c)) corresponds to Theorem 10 (i) (resp., (ii)).

by supposing (3.5).

In trading operations, a trader should execute while considering the fluctuation of the price of other assets (e.g., rebalancing an index fund). In [16], a multi-dimensional version of this model was studied to consider such a case. However, in the case of rebalancing, it is necessary to consider not only selling but also buying securities. We should formulate such a model of an optimal execution problem carefully to avoid the opportunity of a free-lunch when MI is large.

The complete solution of our example in Section 5.2 is another remaining task. This is a representative example where a trading policy is influenced vastly by MI, and it would be interesting to solve this completely in future research.

7 Proofs

7.1 Preliminaries

We introduce some lemmas which we use to prove our main results.

Lemma 1. *For each $m \in \mathbb{N}$ there is a constant $C > 0$ depending only on b, σ and m such that $\mathrm{E}[\hat{Z}(s)^m] \le Cs^m$, where $\hat{Z}(s) = \sup_{0 \le t \le 1} Z(t; 0, s)$.*

Proof. We may assume $s > 0$. By the definition of $\hat{Z}(s)$, we have

$$\mathrm{E}[\hat{Z}(s)^m] \le s^m \, \mathrm{E}[\sup_{t \in [0,1]} \exp(\tilde{Y}_t)],$$

where $(\tilde{Y}_t)_t$ is given by

$$d\tilde{Y}_t = m\sigma(Y(t; 0, \log s))dB_t + mb(Y(t; 0, \log s))dt, \quad \tilde{Y}_0 = 0.$$

Using Corollary 2.5.10 in [20] for the process $(\exp(\tilde{Y}_t))_t$, we have the assertion. ■

Lemma 2. *Let Γ_k, $k \in \mathbb{N}$, be sets, $u \in \mathcal{C}$ and $(W^i_{k,\gamma}, \varphi^i_{k,\gamma}, S^i_{k,\gamma}) \in D$, $\gamma \in \Gamma_k$, $k \in \mathbb{N}$, $i = 1, 2$, be random variables. Let $m_u \in \mathbb{N}$ be as in (2.4). Suppose*

$$\lim_{k \to \infty} \sup_{\gamma \in \Gamma_k} \mathrm{E}[|W^1_{k,\gamma} - W^2_{k,\gamma}| + |\varphi^1_{k,\gamma} - \varphi^2_{k,\gamma}| + |S^1_{k,\gamma} - S^2_{k,\gamma}|] = 0$$

and $\sum_{i=1}^{2} \sup_{k \in \mathbb{N}} \sup_{\gamma \in \Gamma_k} \mathrm{E}[(W^i_{k,\gamma})^{4m_u} + (S^i_{k,\gamma})^{4m_u}] < \infty$. *Then*

$$\lim_{k \to \infty} \sup_{\gamma \in \Gamma_k} \left| \mathrm{E}[u(W^1_{k,\gamma}, \varphi^1_{k,\gamma}, S^1_{k,\gamma})] - \mathrm{E}[u(W^2_{k,\gamma}, \varphi^2_{k,\gamma}, S^2_{k,\gamma})] \right| = 0.$$

This lemma is obtained by standard arguments using the Chebyshev inequality and the uniform continuity of $u(w, \varphi, s)$ on D_R for any $R > 0$, where $D_R = [-R, R] \times [0, \Phi_0] \times [0, R]$.

Lemma 3. *Let $0 \le t_0 \le \cdots \le t_k \le 1$ and $f : [0,1] \longrightarrow [0, \infty)$ be a Borel measurable function. Suppose that f is continuous on $[0,1] \setminus \{t_0, \ldots t_k\}$ and there is a Borel measurable function $\gamma : [0,1] \longrightarrow [0, \infty)$ and a constant $\beta > 0$ such that*

$$f(t) \le \gamma(t) + \beta \int_0^t f(r)dr, \quad t \in [0,1].$$

Then

$$f(t) \le \gamma(t) + \beta \int_0^t \gamma(r) e^{\beta(t-r)} dr, \quad t \in [0,1].$$

Lemma 3 is obtained by applying the same arguments as in the proof of the Gronwall inequality to $f(t)$ on $[0, t_l]$, inductively in l.

Using the Burkholder-Davis-Gundy inequality and the Hölder inequality, we have the following lemma.

Lemma 4. *Let $t \in [0,1]$, $\varphi \geq 0$, $x \in \mathbb{R}$, $(\zeta_r)_{0\leq r\leq t} \in \mathcal{A}_t(\varphi)$ and let $(X_r)_{0\leq r\leq t}$ be given by* (2.7) *with $X_0 = x$. Then there is a constant $C > 0$ depending only on b and σ such that*

$$\mathrm{E}\Big[\sup_{r\in[r_0,r_1]}\Big|X_r - X_{r_0} + \int_{r_0}^{r} g(\zeta_v)dv\Big|^4\Big] \leq C(r_1 - r_0)^2$$

for each $0 \leq r_0 \leq r_1 \leq t$.

Lemma 5. *Let $t \in [0,1]$, $\varphi \geq 0$, $x \in \mathbb{R}$, $(\zeta_r)_{0\leq r\leq t}, (\zeta'_r)_{0\leq r\leq t} \in \mathcal{A}_t(\varphi)$ and let $(X_r)_{0\leq r\leq t}$ (resp., $(X'_r)_{0\leq r\leq t}$) be given by* (2.7) *with $(\zeta_r)_r$ (resp., $(\zeta'_r)_r$) and $X_0 = x \leq X'_0$. Suppose $\zeta_r \leq \zeta'_r$ for each $r \in [0,t]$ almost surely. Then $X_r \geq X'_r$ for each $r \in [0,t]$ almost surely. In particular we have* $\exp(X_r) \leq \hat{Z}(e^x)$.

This lemma is obtained by the same arguments as in the proof of Proposition 5.2.18 in [17].

Lemma 1 and Lemma 5 imply the following.

Lemma 6. *For $n \in \mathbb{R}$, $k = 0,\ldots,n$, $t \in [0,1]$ and $u \in \mathcal{C}$, $V^n_k(w,\varphi,s;u)$ and $V_t(w,\varphi,s;u)$ are non-decreasing in w,φ and s, and have polynomial growth rate with respect to w and s.*

By standard arguments, we obtain the next lemma.

Lemma 7. *Let*

$$\begin{aligned} & q_n(w,\varphi,s,\psi;u) \\ = \; & \mathrm{E}[u(w + (\psi\wedge\varphi)e^{-g_n(\psi\wedge\varphi)}, \varphi - (\psi\wedge\varphi), Z(1/n; 0, se^{-g_n(\psi\wedge\varphi)}))] \end{aligned}$$

for $u \in \mathcal{C}$, where $a \wedge b = \min\{a,b\}$. Then q_n is continuous on $D \times [0,\Phi_0]$.

We remark that $V^n_1(w,\varphi,s;u) = \sup_{\psi\in[0,\Phi_0]} q_n(w,\varphi,s,\psi;u)$. By Lemma 6, Lemma 7 and the arguments of the Bellman equation in discrete-time dynamic programming theory (see [6]), $V^n_k(\cdot;u) \in \mathcal{C}$.

7.2 Proof of Theorem 1

We divide the proof of Theorem 1 into the following two propositions.

Proposition 2. $\limsup_{n\to\infty} V^n_{[nt]}(w,\varphi,s;u) \leq V_t(w,\varphi,s;u)$.

Proposition 3. $\liminf_{n\to\infty} V^n_{[nt]}(w,\varphi,s;u) \geq V_t(w,\varphi,s;u)$.

Proof of Proposition 2. For brevity, we suppose $t = 1$. For $u' \in \mathcal{C}$ and $(w',\varphi',s') \in D$, let $\hat{\psi}_n(w',\varphi',s';u')$ be an optimal strategy for the value function $V^n_1(w',\varphi',s';u')$. By Proposition 7.33 in [6], we can take $\hat{\psi}_n(w',\varphi',s';u')$ as a measurable function with respect to (w',φ',s').

We define $(\psi^n_l)^{n-1}_{l=0} \in \mathcal{A}^n_n(\varphi)$ and $(W^n_l,\varphi^n_l,S^n_l)^n_{l=0}$ by $(W^n_0,\varphi^n_0,S^n_0) = (w,\varphi,s)$, $\psi^n_l = \hat{\psi}_n(W^n_l,\varphi^n_l,S^n_l;V^n_{n-l-1}(\cdot;u)) \wedge \varphi^n_l$, (2.1)–(2.3) inductively in l and let $X^n_l = \log S^n_l$. We also define a strategy $(\zeta_r)_{0\leq r\leq 1}$ by $\zeta_r = n\psi^n_{[nr]}$. Then $(\zeta_r)_r \in \mathcal{A}_1(\varphi)$. Let $(W_r,\varphi_r,X_r)_{0\leq r\leq 1} = \Xi^X_1(w,\varphi,s;(\zeta_r)_r)$.

Step 1. First we show that there is a constant $C^* > 0$ and a sequence $(c_n^*)_{n\in\mathbb{N}} \subset (0,\infty)$ with $c_n^*/n \longrightarrow 0$ as $n \to \infty$ such that

$$g_n(\psi_l^n) \le C^* \wedge (c_n^* \psi_l^n), \quad l = 0, \ldots, n-1.$$

If $\lim_{\zeta\to\infty} h(\zeta) < \infty$, the assertion is obvious. So we may assume $h(\infty) = \infty$. Let $f_n(\psi) = \psi \frac{d}{d\psi} g_n(\psi)$ and $p_n(\psi) = \psi e^{-g_n(\psi)}$ for $\psi \in [0, \Phi_0]$. Then we have $\frac{d}{d\psi} p_n(\psi) = e^{-g_n(\psi)}(1 - f_n(\psi))$. Put $A_n = \{\psi \in (0, \Phi_0] \ ; \ f_n(\psi) = 1\}$. By $[A]$ and the assumption $h(\infty) = \infty$, we see that A_n is not empty and the function $p_n(\psi)$ has a maximum at one of the points in A_n for large enough n. We denote by ψ_n^* a point at which $p_n(\psi)$ has a maximum.

We see that $p_n(\psi) \le p_n(\psi_n^*)$ for $\psi \in (\psi_n^*, \Phi_0]$ and that Lemma 5 implies that $Y(t; r, x - g_n(\psi))$ is non-increasing with respect to ψ. Moreover the function $u(w, \varphi, s)$ is non-decreasing in (w, φ, s). Thus $\hat{\psi}_n(w, \varphi, s; u) \le \psi_n^*$ holds for large n. Then, by the definition of ψ_l^n, we get

$$\psi_l^n \le \psi_n^*, \quad l = 0, \ldots, n-1 \ \text{ and } \ n > n_0 \tag{7.1}$$

for some $n_0 \in \mathbb{N}$. Moreover $[A]$ implies

$$n\psi_n^* \longrightarrow \infty, \quad n \to \infty. \tag{7.2}$$

Indeed, if (7.2) does not hold, there is a constant $M > 0$ and a subsequence $(n_k)_k \subset \mathbb{N}$ such that $n_k \psi_{n_k}^* \le M$. Then we have

$$n_k = n_k f_{n_k}(\psi_{n_k}^*) \le n_k \psi_{n_k}^* (h(n_k \psi_{n_k}^*) + \varepsilon'_{n_k}) \le M(h(M) + \varepsilon'_{n_k})$$

for each k, where $\varepsilon'_n = \sup_\psi \left| \frac{dg_n}{d\psi}(\psi) - h(n\psi) \right|$. This is a contradiction.

Since $h(\zeta)$ is non-decreasing, we have

$$\begin{aligned} g_n(\psi) &\le \int_0^\psi (h(n\psi') + \varepsilon'_n) d\psi' \le \int_0^\psi (h(n\psi_n^*) + \varepsilon'_n) d\psi' \\ &\le \left(\frac{d}{d\psi} g_n(\psi_n^*) + 2\varepsilon'_n \right) \psi = \left(\frac{1}{\psi_n^*} + 2\varepsilon'_n \right) \psi, \quad \psi \in [0, \psi_n^*] \end{aligned} \tag{7.3}$$

for each $n \in \mathbb{N}$. Thus

$$g_n(\psi) \le 1 + 2\Phi_0 \sup_{n'} \varepsilon'_{n'}, \quad \psi \in [0, \psi_n^*]. \tag{7.4}$$

By (7.1)–(7.4), we have the assertion by letting

$$C^* = \max_{n \le n_0} g_n(\Phi_0) + 1 + 2\Phi_0 \sup_n \varepsilon'_n, \quad c_n^* = \frac{1}{\psi_n^*} + 2\varepsilon'_n.$$

Step 2. In this step we will show that

$$\lim_{n\to\infty} \mathrm{E}[\max_{k=0,\ldots,n} |X_k^n - X_{k/n}|^2] = 0. \tag{7.5}$$

We define $\tilde{X}_r^n$, $r \in [0,1]$, by

$$\tilde{X}_r^n = Y\Big(r; \frac{k}{n}, X_k^n - g_n(\psi_k^n)\Big), \quad r \in \Big(\frac{k}{n}, \frac{k+1}{n}\Big] \tag{7.6}$$

and $\tilde{X}_0^n = \log s$. Then we see that $\tilde{X}_{k/n}^n = X_k^n$ for each $k = 0, \ldots, n$ and that $\tilde{X}_r^n$ satisfies

$$\tilde{X}_r^n = \log s + \int_0^r \sigma(\tilde{X}_v^n) dB_v + \int_0^r b(\tilde{X}_v^n) dv - \sum_{k=0}^{c_n(r)} g_n(\psi_k^n),$$

where $c_n(r) = [nr] - 1_{\mathbb{Z}_+}(nr)$ and $\sum_{k=0}^{-1} g_n(\psi_k^n) = 0$.

Let $\Delta_r^n = \mathrm{E}\left[\max\Big\{|\tilde{X}_{r'}^n - X_{r'}|^2 \ ; \ r' = 0, \frac{1}{n}, \ldots, \frac{[nr]}{n}, r\Big\}\right]$. We have

$$\Big|\sum_{k=0}^{c_n(r')} g_n(\psi_k^n) - \int_0^{r'} g(\zeta_v) dv\Big| \ \leq \ \sum_{k=0}^{n-1} \Big|g_n(\psi_k^n) - \frac{1}{n} g(n\psi_k^n)\Big| + d_n(r) g_n(\psi_{[nr]}^n)$$

for $r' = 0, \frac{1}{n}, \ldots, \frac{[nr]}{n}, r$, where $d_n(r) = c_n(r) + 1 - nr$. Then Step 1 implies

$$\begin{aligned}
|\tilde{X}_{r'}^n - X_{r'}|^2 &\leq 4\Big\{\Big|\int_0^{r'} (\sigma(X_v) - \sigma(\tilde{X}_v^n)) dB_v\Big|^2 + \Big|\int_0^{r'} (b(X_v) - b(\tilde{X}_v^n)) dv\Big|^2 \\
&\qquad + \Big(\sum_{k=0}^{n-1} \Big|g_n(\psi_k^n) - \frac{1}{n} g(n\psi_k^n)\Big|\Big)^2 + d_n(r)^2 g_n(\psi_{[nr]}^n)^2\Big\} \\
&\leq \ 4\Big\{\sup_{0\leq v\leq r} \Big|\int_0^v (\sigma(X_{v'}) - \sigma(\tilde{X}_{v'}^n)) dB_{v'}\Big|^2 + \sup_{0\leq v\leq r} \Big|\int_0^v (b(X_{v'}) - b(\tilde{X}_{v'}^n)) dv'\Big|^2 \\
&\qquad + \Phi_0^2 \varepsilon_n^2 + C^* c_n^* d_n(r)^2 \psi_{[nr]}^n\Big\}
\end{aligned}$$

for $r' = 0, \frac{1}{n}, \ldots, \frac{[nr]}{n}, r$, where ε_n is defined by (2.5). Thus, using the Burkholder-Davis-Gundy inequality and the Hölder inequality, we get

$$\Delta_r^n \leq C_0\Big\{\gamma_n(r) + \mathrm{E}\Big[\int_0^r |\tilde{X}_v - X_v|^2 dv\Big]\Big\} \leq C_0\Big\{\gamma_n(r) + \int_0^r \Delta_v^n dv\Big\}$$

for some constant $C_0 > 0$, where $\gamma_n(r) = \Phi_0^2 \varepsilon_n^2 + C^* c_n^* d_n(r)^2\, \mathrm{E}[\psi_{[nr]}^n]$. Then Lemma 3 implies

$$\Delta_r^n \ \leq \ C_0 \gamma_n(r) + C_0^2 \int_0^r \gamma_n(v) e^{C_0(r-v)} dv.$$

Since $0 \le d_n(v) \le 1$ for $v \in [0,1]$ and $d_n(1) = 0$, we have

$$\begin{aligned} \mathrm{E}[\max_{k=0,\dots,n} |X_k^n - X_{k/n}|^2] &= \Delta_1^n \le C_1 \Big\{ \Phi_0^2 \varepsilon_n^2 + \int_0^1 \gamma_n(v) dv \Big\} \\ \le \;& C_1 \Big\{ 2\Phi_0^2 \varepsilon_n^2 + \frac{C^* c_n^*}{n} \sum_{l=0}^{n-1} \mathrm{E}[\psi_l^n] \Big\} \le C_1 \Big\{ 2\Phi_0^2 \varepsilon_n^2 + \frac{\Phi_0 C^* c_n^*}{n} \Big\} \end{aligned} \tag{7.7}$$

for some $C_1 > 0$. By (2.5) and the assertion of Step 1, the right-hand side of (7.7) tends to zero as $n \to \infty$. Then we have (7.5).

Step 3. Let $\tilde{W}_n^n = w + \sum_{l=0}^{n-1} \int_{l/n}^{(l+1)/n} n\psi_l^n \exp(X_l^n - (nr - l) g_n(\psi_l^n)) dr$. From

$$|e^x - e^y| \le \int_0^1 e^{rx} e^{(1-r)y} dv |x - y| \le (e^x + 1)(e^y + 1)|x - y|, \tag{7.8}$$

it follows that

$$|\tilde{W}_n^n - W_1| \le \Phi_0 (\hat{Z}(s) + 1)^2 \max_{l=0,\dots,n-1} \sup_{r \in [l/n,(l+1)/n]} |X_l^n - (nr - l) g_n(\psi_l^n) - X_r|.$$

Since

$$\begin{aligned} & |X_l^n - (nr - l) g_n(\psi_l^n) - X_r| \\ \le \;& \Big| X_r - X_{l/n} + \int_{l/n}^r g(\zeta_v) dv \Big| + |X_l^n - X_{l/n}| + (nr - l)\varepsilon_n \psi_l^n \end{aligned}$$

for each $l = 0, \dots, n-1$ and $r \in [l/n, (l+1)/n]$, by virtue of Lemma 4, we have

$$\begin{aligned} & \mathrm{E}[\max_{l=0,\dots,n-1} \sup_{r \in [l/n,(l+1)/n]} |X_l^n - (nr - l) g_n(\psi_l^n) - X_r|^2]^{1/2} \\ \le \;& \Big\{ \sum_{l=0}^{n-1} \mathrm{E}\Big[\sup_{r \in [l/n,(l+1)/n]} \Big| X_r - X_{l/n} + \int_{l/n}^r g(\zeta_v) dv \Big|^4 \Big] \Big\}^{1/4} + \delta_n + \Phi_0 \varepsilon_n \\ \le \;& C_0 \times \frac{1}{n^{1/4}} + \delta_n + \Phi_0 \varepsilon_n \end{aligned} \tag{7.9}$$

for some $C_0 > 0$, where $\delta_n = \mathrm{E}[\max_{k=0,\dots,n} |X_k^n - X_{k/n}|^2]^{1/2}$. Thus

$$\mathrm{E}[|\tilde{W}_n^n - W_1|] \le \Phi_0 \, \mathrm{E}[(\hat{Z}(s) + 1)^4]^{1/2} \tilde{\delta}_n \le C_1 \tilde{\delta}_n \tag{7.10}$$

for some $C_1 > 0$, where $\tilde{\delta}_n$ is a right-hand side of (7.9).

On the other hand we have

$$\mathrm{E}[|S_n^n - \exp(X_1)|] \le \mathrm{E}[(\hat{Z}(s) + 1)^4]^{1/2} \, \mathrm{E}[|X_n^n - X_1|^2]^{1/2} \le C_2 \delta_n \tag{7.11}$$

for some $C_2 > 0$. Since Step 2 implies that δ_n and $\tilde{\delta}_n$ converge to zero as $n \to \infty$, by (7.10), (7.11) and Lemma 1, we can apply Lemma 2 and then we obtain

$$\lim_{n \to \infty} \big| \mathrm{E}[u(\tilde{W}_n^n, \varphi_n^n, S_n^n)] - \mathrm{E}[u(W_1, \varphi_1, \exp(X_1))] \big| = 0. \tag{7.12}$$

Since u is non-decreasing in w and $\tilde{W}^n_n \geq W^n_n$, we have

$$V^n_n(w,\varphi,s;u) - V_1(w,\varphi,s;u) \leq \mathrm{E}[u(W^n_n,\varphi^n_n,S^n_n)] - \mathrm{E}[u(W_1,\varphi_1,\exp(X_1))]$$
$$\leq \mathrm{E}[u(\tilde{W}^n_n,\varphi^n_n,S^n_n)] - \mathrm{E}[u(W_1,\varphi_1,\exp(X_1))]. \tag{7.13}$$

Now the assertion of Proposition 2 is given by (7.12) and (7.13).

Proof of Proposition 3. Again we suppose $t=1$. Take any $(\zeta_r)_{0\leq r\leq 1} \in \mathcal{A}_1(\varphi)$ and let $\psi^n_l = \int_{((l-1)/n)\vee 0}^{l/n} \zeta_r dr$, where $a \vee b = \max\{a,b\}$. Then we have $(\psi^n_l)_l \in \mathcal{A}^n_n(\varphi)$. Let $(W_r,\varphi_r,X_r)_{0\leq r\leq 1} = \Xi^X_1(w,\varphi,s;(\zeta_r)_r)$ and $(W^n_l,\varphi^n_l,S^n_l)^n_{l=0} = \Xi^n_n(w,\varphi,s;(\psi^n_l)_l)$ and $X^n_l = \log S^n_l$.

Step 1. First we will show that

$$\mathrm{E}[\max_{k=0,\ldots,n} |X^n_k - X_{k/n}|^2] \longrightarrow 0, \quad n\to\infty. \tag{7.14}$$

Define $\tilde{X}^n_r$ by (7.6) and let $\tilde{\Delta}^n_r = \mathrm{E}[\sup_{0\leq r'\leq r} |\tilde{X}^n_{r'} - X_{r'}|^2]$. By a similar calculation as in the proof of Step 2 of Proposition 2, we get

$$\mathrm{E}[\max_{k=0,\ldots,n} |X^n_k - X_{k/n}|^2] \leq \tilde{\Delta}^n_1 \leq C_0 z_n$$

for some $C_0>0$ depending only on b and σ, where

$$z_n = \Phi_0^2\varepsilon_n^2 + \frac{M^2}{n^2} + \mathrm{E}\Big[\int_0^1 |G_n(v)|^2 dv\Big], \quad M = g\big(\sup_{r,\omega}\zeta_r(\omega)\big),$$
$$G_n(v) = g\Big(n\int_{[nv]/n}^{([nv]+1)/n} \zeta_{v'}dv'\Big) - g(\zeta_v).$$

Here $|G_n(r,\omega)| \leq 2M$ and Lebesgue's differentiation theorem implies $G_n(v,\omega)\longrightarrow 0$ as $n\to\infty$ for almost all $(v,\omega)\in[0,1]\times\Omega$. Then, using the dominated convergence theorem, we have $z_n \longrightarrow 0$ as $n\to\infty$. Then we obtain (7.14).

Step 2. Let $\hat{W}^n_1 = w + \sum_{l=0}^{n-1} \psi^n_l n \int_{l/n}^{(l+1)/n} \exp(X_r)dr$. Then we have

$$\begin{aligned}
&\mathrm{E}[|\hat{W}^n_1 - W_1|] \\
\leq\ & \mathrm{E}\Big[\sum_{l=0}^{n-2}\int_{l/n}^{(l+1)/n} \Big|\exp(X_{r+1/n})n\int_{l/n}^{(l+1)/n}\zeta_v dv - \exp(X_r)\zeta_r\Big| dr\Big] \\
& + \mathrm{E}\Big[\int_{(n-1)/n}^1 \exp(X_r)\zeta_r dr\Big] \\
\leq\ & C_0\Big\{\Phi_0\,\mathrm{E}[\sup_{v\in[0,1-1/n]} |X_{v+1/n} - X_v|^2]^{1/2} + K_n + \frac{1}{n}\Big\}
\end{aligned}$$

for some $C_0 > 0$ depending only on $b, \sigma, (\zeta_r)_r$ and s, where

$$K_n = \Big(\int_0^1 \mathrm{E}[|H_n(r)|^2] dr \Big)^{1/2}, \quad H_n(r) = n \int_{[nr]/n}^{([nr]+1)/n} \zeta_v dv - \zeta_r.$$

By Lemma 4, we have

$$\mathrm{E}[\sup_{v \in [0,1-1/n]} |X_{v+1/n} - X_v|^2]^{1/2} \le C_2 \times \frac{1}{\sqrt{n}} + \frac{M}{n}$$

for some $C_2 > 0$. Lebesgue's differentiation theorem and the dominated convergence theorem imply $K_n \longrightarrow 0$. Then we obtain $\mathrm{E}[|\hat{W}_1^n - W_1|] \longrightarrow 0$. On the other hand, a similar calculation to Step 2 of the proof of Proposition 2 implies $\mathrm{E}[|W_n^n - \hat{W}_1^n|] \longrightarrow 0$. Thus $\mathrm{E}[|W_n^n - W_1^n|] \longrightarrow 0$ converges. Moreover, by (7.14), we have $\mathrm{E}[|S_n^n - \exp(X_1)|] \longrightarrow 0$. Then we can apply Lemma 2 and we get

$$\mathrm{E}[u(W_1, \varphi_1, \exp(X_1))] = \lim_{n\to\infty} \mathrm{E}[u(W_n^n, \varphi_n^n, S_n^n)] \le \liminf_{n\to\infty} V_n^n(w, \varphi, s; u).$$

Since $(\zeta_r)_r \in \mathcal{A}_1(\varphi)$ is arbitrary, we obtain the assertion. ■

By Proposition 2 and Proposition 3, we obtain Theorem 1.

7.3 Strategy-Restricted Value Functions

In this subsection we prepare strategy-restricted value functions to prove Theorem 2 and Theorem 3. For $L > 0$, we define

$$\begin{aligned}
\mathcal{A}_k^{n,L}(\varphi) &= \{(\psi_l)_{l=0}^{k-1} \in \mathcal{A}_k^n(\varphi)\ ;\ \psi_l \le L/n,\ l = 0, \dots, k-1\}, \\
\mathcal{A}_t^L(\varphi) &= \{(\zeta_r)_{0 \le r \le t} \in \mathcal{A}_t(\varphi)\ ;\ \sup_{r,\omega} |\zeta_r(\omega)| \le L\}, \\
V_k^{n,L}(w, \varphi, s; u) &= \sup_{(\psi_l)_{l=0}^{k-1} \in \mathcal{A}_k^{n,L}(\varphi)} \mathrm{E}[u(W_k^n, \varphi_k^n, S_k^n)], \\
V_t^L(w, \varphi, s; u) &= \sup_{(\zeta_r)_{r \le t} \in \mathcal{A}_t^L(\varphi)} \mathrm{E}[u(W_t, \varphi_t, S_t)].
\end{aligned}$$

We see easily that $V_k^n(w, \varphi, s; u) = \sup_{L>0} V_k^{n,L}(w, \varphi, s; u)$ and $V_t(w, \varphi, s; u) = \sup_{L>0} V_t^L(w, \varphi, s; u)$. By similar arguments as in Section 7.2, we see that for each $L > 0$, $(w, \varphi, s) \in D$, $t \in [0,1]$ and $u \in \mathcal{C}$

$$\lim_{n\to\infty} V_{[nt]}^{n,L}(w, \varphi, s; u) = V_t^L(w, \varphi, s; u). \tag{7.15}$$

Now we consider the continuity of $V_t^L(w, \varphi, s; u)$. Our purpose in this section is to prove the following proposition.

Proposition 4. *$V_t^L(w, \varphi, s; u)$ is continuous with respect to $(t, w, \varphi, s) \in [0,1] \times D$.*

To prove Proposition 4, we will prove the following lemmas.

Lemma 8. *For each $(w, \varphi, s) \in D$ and $t \in [0,1]$*

$$\lim_{(w',\varphi',s')\to(w,\varphi,s)} \sup_{L>0} |V_t^L(w',\varphi',s';u) - V_t^L(w,\varphi,s;u)| = 0.$$

Proof. Let $R > 0$ and $(w, \varphi, s), (w', \varphi', s') \in D_R$. We may assume $s' > 0$. Take any $(\zeta_r)_{r\le t} \in \mathcal{A}_t^L(\varphi)$. Let $\rho = \inf\{r > 0 \ ; \ \int_0^r \zeta_v dv > \varphi \wedge \varphi'\} \wedge t$ and $\zeta'_r = \zeta_r 1_{\{r\le\rho\}}$. Then $(\zeta'_r)_{r\le t} \in \mathcal{A}_t^L(\varphi')$. Let $(W_r, \varphi_r, S_r)_{r\le t} = \Xi_t(w, \varphi, s; (\zeta_r)_r)$ and $(W'_r, \varphi'_r, S'_r)_{r\le t} = \Xi_t(w', \varphi', s'; (\zeta'_r)_r)$. Moreover let us define $(\tilde{S}'_r)_{r\le t}$ by

$$d\tilde{S}'_r = \hat{\sigma}(\tilde{S}'_r)dB_r + \hat{b}(\tilde{S}'_r)dr - g(\zeta_r)\tilde{S}'_r dr, \quad \tilde{S}'_0 = s'.$$

Then Lemma 5 implies $S'_r \ge \tilde{S}'_r$ for each $r \in [0,t]$ almost surely. Thus

$$\mathrm{E}[u(W_t, \varphi_t, S_t)] - V_t(w', \varphi', s'; u) \le \mathrm{E}[|u(W_t, \varphi_t, S_t) - u(W'_t, \varphi'_t, \tilde{S}'_t)|]. \tag{7.16}$$

By a simple calculation we get

$$|W_t - W'_t| \le |w - w'| + \hat{Z}(s)|\varphi - \varphi'| + \Phi_0 \sup_{r\in[0,t]} |S_r - \tilde{S}'_r|$$

and $|\varphi_t - \varphi'_t| \le |\varphi - \varphi'|$. Moreover Theorem 3.2.7 in [24] and Lemma 1 imply

$$\mathrm{E}[\sup_{r\in[0,t]} |S_r - \tilde{S}'_r|] \le \begin{cases} C_0 s' & (s = 0) \\ C_0|\log s - \log s'| & (s > 0) \end{cases}$$

for some $C_0 > 0$ depending only on b, σ and R. Then we obtain

$$\sup_{L>0} \sup_{(\zeta_r)_r\in\mathcal{A}_t^L(\varphi)} \mathrm{E}[|u(W_t, \varphi_t, S_t) - u(W'_t, \varphi'_t, \tilde{S}'_t)|] \longrightarrow 0 \tag{7.17}$$

as $(w', \varphi', s') \to (w, \varphi, s)$ by using Lemma 2. Then (7.16) and (7.17) imply

$$\lim_{(w',\varphi',s')\to(w,\varphi,s)} \sup_{L>0} (V_t^L(w,\varphi,s;u) - V_t^L(w',\varphi',s';u)) \le 0.$$

A similar argument gives us

$$\lim_{(w',\varphi',s')\to(w,\varphi,s)} \sup_{L>0} (V_t^L(w',\varphi',s';u) - V_t^L(w,\varphi,s;u)) \le 0.$$

So we get the assertions. ■

By Proposition 4, it follows that the convergence of (7.15) is uniform on any compact subset of D for each fixed t (we remark that $V_k^{n,L}(w, \varphi, s; u)$ and $V_t^L(w, \varphi, s; u)$ are non-decreasing in w, φ and s).

Lemma 9. *For each compact set $E \subset D$,*

$$\limsup_{r\uparrow t} \sup_{L>0} \sup_{(w,\varphi,s)\in E} (V_r^L(w,\varphi,s;u) - V_t^L(w,\varphi,s;u)) \le 0, \quad t \in (0,1],$$

$$\limsup_{t\downarrow r} \sup_{L>0} \sup_{(w,\varphi,s)\in E} (V_r^L(w,\varphi,s;u) - V_t^L(w,\varphi,s;u)) \le 0, \quad r \in [0,1).$$

Proof. Let $r, t \in [0,1]$ with $r < t$. Lemma 2 and Lemma 4 imply

$$\sup_{L>0} \sup_{(w,\varphi,s)\in E} \sup_{(\zeta_v)_v \in \mathcal{A}^L_r(\varphi)} \mathrm{E}[|u(W_r, \varphi_r, \exp(X_r))] - u(\tilde{W}_t, \tilde{\varphi}_t, \exp(\tilde{X}_t))|] \longrightarrow 0$$

as $r \uparrow t$ and $t \downarrow r$, where $(W_v, \varphi_v, X_v)_{0\le v\le r} = \Xi^X_t(w, \varphi, s; (\zeta_r)_r)$, $(\tilde{W}_v, \tilde{\varphi}_v, \tilde{X}_v)_{0\le v\le t} = \Xi^X_t(w, \varphi, s; (\tilde{\zeta}_r)_r)$ and $\tilde{\zeta}_v = \zeta_v 1_{[0,r]}(v)$. This implies the assertions. ■

Similar arguments give us the following lemma.

Lemma 10. *For each $L > 0$ and compact set $E \subset D$,*

$$\limsup_{r\uparrow t} \sup_{(w,\varphi,s)\in E} (V^L_t(w, \varphi, s; u) - V^L_r(w, \varphi, s; u)) \le 0, \quad t \in (0, 1],$$

$$\limsup_{t\downarrow r} \sup_{(w,\varphi,s)\in E} (V^L_t(w, \varphi, s; u) - V^L_r(w, \varphi, s; u)) \le 0, \quad r \in [0, 1).$$

By Lemmas 8–10, we obtain Proposition 4. We remark that Lemma 6 and Proposition 4 imply $V^L_t(\cdot; u), V_t(\cdot; u) \in \mathcal{C}$.

7.4 Proof of Theorem 3

In order to show Theorem 3, we define the operators $Q^L_t : \mathcal{C} \longrightarrow \mathcal{C}$ and $Q^{n,L}_t : \mathcal{C} \longrightarrow \mathcal{C}$ by $Q^L_t u(w, \varphi, s) = V^L_t(w, \varphi, s; u)$ and $Q^{n,L}_t u(w, \varphi, s) = V^{2^n,L}_{[2^n t]} u(w, \varphi, s)$. We see that Q^L_t and $Q^{n,L}_t$ are also well-defined. First we will show

$$Q^L_{t+r} u(w, \varphi, s) = Q^L_t Q^L_r u(w, \varphi, s) \tag{7.18}$$

for each $t, r \in I$ with $t + r \le 1$, where $I = \{k/2^l \ ; \ k, l \in \mathbb{Z}_+\} \cap [0, 1]$. Let $n \in \mathbb{N}$ be large enough so that $2^n t, 2^n r \in \mathbb{Z}_+$. By the Bellman equation of the discrete-time case ([6]), we have

$$Q^{n,L}_{t+r} u(w, \varphi, s) = Q^{n,L}_t Q^{n,L}_r u(w, \varphi, s). \tag{7.19}$$

By (7.15), we see that the left-hand side of (7.19) converges to that of (7.18) as $n \to \infty$ for each $t, r \in I$. To see the convergence of the right-hand side, we prove the following proposition.

Proposition 5. *Let $u_n, u \in \mathcal{C}$ be utility functions satisfying (2.4) for some C_u and m_u. Assume that u_n is converges to u uniformly on any compact subset of D as $n \to \infty$. Then*

$$\lim_{n\to\infty} \sup_{k=0,\dots,n} |V^{n,L}_k(w, \varphi, s; u_n) - V^{n,L}_k(w, \varphi, s; u)| = 0, \quad (w, \varphi, s) \in D.$$

Proof. Take any $R > 0$. Then we have

$$\begin{aligned} & |V^{n,L}_k(w, \varphi, s; u_n) - V^{n,L}_k(w, \varphi, s; u)| \\ \le & \sup_{(\psi^n_l)_l \in \mathcal{A}^{n,L}_k(\varphi)} \mathrm{E}\left[|u_n(W^n_k, \varphi^n_k, S^n_k) - u(W^n_k, \varphi^n_k, S^n_k)|\right] \\ \le & \sup_{(w',\varphi',s')\in D_R} |u_n(w', \varphi', s') - u(w', \varphi', s')| + \frac{C_0}{R} \end{aligned}$$

by virtue of Lemma 1 and the Chebyshev inequality, where $C_0 > 0$ depends only on b, σ, C_u, m_u and (w, φ, s). Now we easily see the assertion. ■

Using Proposition 5 and the uniform convergence of (7.15) on any compact set, we see that the right-hand side of (7.19) converges to that of (7.18). Moreover Proposition 4 implies that (7.18) also holds for each $t, r \in [0,1]$. Theorem 3 is obtained by (7.18), the relation $Q_t u(w,\varphi,s) = \sup_{L>0} Q_t^L u(w,\varphi,s)$ and similar calculation to the proof of Proposition 4 in [25].

7.5 Proof of Theorem 2

In this section we give the proof of Theorem 2. First we consider the right-continuity at $t=0$ when $h(\infty)=\infty$.

Lemma 11. *Assume $h(\infty)=\infty$. Then for each $t \in [0,1]$ and $(\zeta_r)_{0\le r\le t} \in \mathcal{A}_t(\varphi)$,*

$$\int_0^r \exp\Big(-\int_0^v g(\zeta_{v'})dv'\Big)\zeta_v dv \le \phi(r), \quad r \in [0,t],$$

where $\phi(r)$, $r \in (0,1]$, is a continuous function depending only on the function $h(\zeta)$ and Φ_0 such that $\lim_{r\to 0}\phi(r) = 0$.

Proof. Let $\pi_r = \displaystyle\int_0^r g(\zeta_v)dv$ and $\tau_R = \inf\{v \in [r/2, r] \ ;\ \pi_v > R\} \wedge r$ for $r \in (0,t]$ and $R>0$. Then we have

$$\int_0^r \exp(-\pi_v)\zeta_v dv \le \int_0^{\tau_R}\zeta_v dv + \int_{\tau_R}^r e^{-R}\zeta_v dv \le \int_0^{\tau_R}\zeta_v dv + \Phi_0 e^{-R}$$

for $r \in (0,t]$ and $R>0$. Since $g(\zeta)$ is convex, the Jensen inequality implies

$$\int_0^{\tau_R}\zeta_v dv \le \tau_R g^{-1}\Big(\int_0^1 g(\zeta_{\tau_R v})dv\Big) \le r g^{-1}\Big(\frac{2}{r}\int_0^{\tau_R} g(\zeta_v)dv\Big) \le r g^{-1}(2R/r),$$

where $g^{-1}(y) = \inf\{\zeta \in [0,\infty) \ ;\ g(\zeta)=y\}$, $y \ge 0$. The function $g^{-1}(y)$ is well-defined at any $y \ge 0$ and continuous for large y.

Let us define a function $f(\zeta)$, $\zeta \ge 0$, by $f(\zeta) = \zeta\sqrt{h(\zeta/2)}$. Then $f(\zeta)$ is continuous, strictly increasing for large y and satisfies $f(0)=0$ and $\lim_{\zeta\to\infty} f(\zeta)=\infty$. Thus $f(\zeta)$ has an inverse function $f^{-1}(y)$ on $[0,\infty)$ such that $f^{-1}(0)=0$, $\lim_{y\to\infty} f^{-1}(y)=\infty$ and $f^{-1}(y)$ is continuous for large y. So we can define $M(r) = f^{-1}(1/r)$ and $R(r) = rg(M(r))/2$ for $r \in (0,1]$. Then we see that $M(r), R(r) \longrightarrow \infty$ as $r \to 0$ and that

$$R(r) \ge \frac{r}{2}\int_{M(r)/2}^{M(r)} h(\zeta)d\zeta \ge \frac{rM(r)h(M(r)/2)}{4} = \frac{\sqrt{h(M(r)/2)}}{4} \longrightarrow \infty$$

as $r \to 0$. Moreover we have

$$r g^{-1}(2R(r)/r) = rM(r) = \frac{1}{\sqrt{h(M(r)/2)}} \longrightarrow 0, \quad r \to 0.$$

Then the assertion holds by letting $\phi(r) = rg^{-1}(2R(r)/r) + \Phi_0\exp(-R(r))$. ■

Proposition 6. *Assume $h(\infty) = \infty$. Then for each compact set $E \subset D$*

$$\lim_{t \downarrow 0} \sup_{(w,\varphi,s) \in E} |V_t(w,\varphi,s;u) - u(w,\varphi,s)| = 0.$$

Proof. Take any $t \in (0,1)$. Let $\hat{S}_t = s \exp\Big(-\int_0^t g(\zeta_v)dv\Big)$ and $(W_r, \varphi_r, S_r)_{0 \le r \le t} = \Xi_t(w,\varphi,s;(\zeta_r)_r)$. Then we have

$$V_t(w,\varphi,s;u) - u(w,\varphi,s) \le \sup_{(\zeta_r)_r \in \mathcal{A}_t(\varphi)} \Big| \mathrm{E}[u(W_t, \varphi_t, S_t)] - \mathrm{E}[u(w, \varphi_t, \hat{S}_t)] \Big| \tag{7.20}$$

by virtue of the relations $\varphi_t \le \varphi$ and $\hat{S}_t \le s$. Using Lemma 11, the Burkholder-Davis-Gundy inequality and the Hölder inequality, we have $\mathrm{E}[|S_t - \hat{S}_t|] \le C_0 s t^{1/2}$ and

$$\begin{aligned} \mathrm{E}[|W_t - w|] &\le s\,\mathrm{E}\Big[\int_0^t \exp\Big(-\int_0^r g(\zeta_v)dv\Big)\zeta_r dr\Big] + \mathrm{E}\Big[\int_0^t |S_r - \hat{S}_r|\zeta_r dr\Big] \\ &\le s\phi(t) + C_0 \Phi_0 s t^{1/2} \end{aligned}$$

for some $C_0 > 0$ which is independent of t, w, φ, s and $(\zeta_r)_r$. Then, by (7.20) and Lemma 2, we get $\limsup_{t \downarrow 0} \sup_{(w,\varphi,s) \in E} (V_t(w,\varphi,s;u) - u(w,\varphi,s)) \le 0$. The inequality $\limsup_{t \downarrow 0} \sup_{(w,\varphi,s) \in E} (u(w,\varphi,s) - V_t(w,\varphi,s;u)) \le 0$ is obtained by Lemma 9. Then we have the assertion. ■

Next we consider the case of $h(\infty) < \infty$.

Proposition 7. *Assume $h(\infty) < \infty$. Then for each compact set $E \subset D$*

$$\limsup_{t \downarrow 0} \sup_{(w,\varphi,s) \in E} (V_t(w,\varphi,s;u) - Ju(w,\varphi,s)) \le 0.$$

Proof. Take any $t \in (0,1)$ and $(\zeta_r)_{0 \le r \le t} \in \mathcal{A}_t(\varphi)$. Let $(W_r, \varphi_r, X_r)_{0 \le r \le t} = \Xi_t^X(w,\varphi,s;(\zeta_r)_r)$. Easily we have

$$\begin{aligned} \lim_{t \downarrow 0} \sup_{(w,\varphi,s) \in E} \sup_{(\zeta_r)_r \in \mathcal{A}_t(\varphi)} \Big| &\mathrm{E}[u(W_t, \varphi_t, S_t)] \\ &- \mathrm{E}\Big[u\Big(w + s\int_0^t e^{-\tilde{\eta}_r}\zeta_r dr, \varphi - \eta_t, s e^{-\tilde{\eta}_t}\Big)\Big]\Big| = 0 \end{aligned} \tag{7.21}$$

by virtue of Lemma 2, where $\eta_r = \int_0^r \zeta_v dv$ and $\tilde{\eta}_r = \int_0^r g(\zeta_v)dv$.

Now we define

$$\hat{\eta}_r = 1_{(0,t]}(r)\int_0^{\eta_r} h(\zeta'/r)d\zeta', \quad \hat{w}_t = \int_0^{\eta_t} \exp\Big(-\int_0^p h(\zeta'/t)d\zeta'\Big)\,dp.$$

Since $g(\zeta)$ is convex, the Jensen inequality implies $\tilde{\eta}_r \ge r g(\eta_r/r) = \hat{\eta}_r$ and

$$\hat{w}_t \ge \int_0^t \exp\Big(-\int_0^{\eta_r} h(\zeta'/r)d\zeta'\Big)\zeta_r dr \ge \int_0^t e^{-\tilde{\eta}_r}\zeta_r dr$$

for $r \in (0,t]$. Moreover $h(\zeta)$ is non-decreasing in ζ and so is $u(w,\varphi,s)$ in w. Thus we get

$$\mathrm{E}\left[u\Big(w + s\int_0^t e^{-\tilde{\eta}_r}\zeta_r dr, \varphi - \eta_t, se^{-\tilde{\eta}_t}\Big)\right] \le \mathrm{E}[u(w + s\hat{w}_t, \varphi - \eta_t, se^{-\hat{\eta}_t})]$$

for each $(\zeta_r)_r \in \mathcal{A}_t(\varphi)$. By this inequality and (7.21), we get

$$\limsup_{t\downarrow 0} \sup_{(w,\varphi,s)\in E} \big(V_t(w,\varphi,s;u) - \sup_{(\zeta_r)_r\in\mathcal{A}_t(\varphi)} \mathrm{E}[u(w + s\hat{w}_t, \varphi - \eta_t, se^{-\hat{\eta}_t})]\big) \le 0. \tag{7.22}$$

Next let us define

$$\tilde{\varepsilon}_t = \int_0^{\Phi_0} (h(\infty) - h(\zeta/t))d\zeta, \quad F(\psi) = \int_0^{\psi} e^{-h(\infty)p}dp. \tag{7.23}$$

Then we have $|e^{-\hat{\eta}_t} - e^{-h(\infty)\eta_t}| \le 4\tilde{\varepsilon}_t$ and $|\hat{w}_t - F(\eta_t)| \le 4\Phi_0\tilde{\varepsilon}_t$. Since the dominated convergence theorem implies $\tilde{\varepsilon}_t \longrightarrow 0$ as $t \downarrow 0$, using Lemma 2, we get

$$\lim_{t\downarrow 0} \sup_{(w,\varphi,s)\in E} \sup_{(\zeta_r)_r\in\mathcal{A}_t(\varphi)} \Big| \mathrm{E}[u(w + s\hat{w}_t, \varphi - \eta_t, s\exp(-\hat{\eta}_t))] - \mathrm{E}[u(w + F(\eta_t)s, \varphi - \eta_t, se^{-h(\infty)\eta_t})]\Big| = 0.$$

By this and (7.22), we get the assertion. ∎

Proposition 8. *Assume $h(\infty) < \infty$. Then for each compact set $E \subset D$,*

$$\limsup_{t\downarrow 0} \sup_{(w,\varphi,s)\in E} (Ju(w,\varphi,s) - V_t(w,\varphi,s;u)) \le 0.$$

Proof. Let $t \in (0,1)$. For each $(w,\varphi,s) \in E$, take any $\psi \in [0,\varphi]$ and define $(\zeta_r)_{0\le r\le t} \in \mathcal{A}_t(\varphi)$ by $\zeta_r = \psi/t$ and $(W_r,\varphi_r,S_r)_{0\le r\le t} = \Xi_t(w,\varphi,s;(\zeta_r)_r)$. Similarly to the proof of Proposition 7, we get

$$\lim_{t\downarrow 0} \sup_{(w,\varphi,s)\in E} \sup_{\psi\in[0,\varphi]} \big|u(w + F(\psi)s, \varphi - \psi, se^{-h(\infty)\psi}) - \mathrm{E}[u(W_t,\varphi_t,S_t)]\big| = 0,$$

which implies our assertion. ∎

Finally we consider the continuity with respect to $t \in (0,1]$.

Proposition 9. *For each compact set $E \subset D$ we have*
(i) $\lim_{t'\uparrow t} \sup_{(w,\varphi,s)\in E} |V_{t'}(w,\varphi,s;u) - V_t(w,\varphi,s;u)| = 0, \quad t \in (0,1]$,
(ii) $\lim_{t'\downarrow t} \sup_{(w,\varphi,s)\in E} |V_{t'}(w,\varphi,s;u) - V_t(w,\varphi,s;u)| = 0, \quad t \in (0,1)$.

Proof. Lemma 9 implies

$$\limsup_{t'\uparrow t} \sup_{(w,\varphi,s)\in E} (V_{t'}(w,\varphi,s;u) - V_t(w,\varphi,s;u)) \le 0.$$

By the following uniform convergence (which is given by Dini's theorem)

$$\lim_{L\to\infty}\sup_{(w,\varphi,s)\in E}|V_t^L(w,\varphi,s;u)-V_t(w,\varphi,s;u)|=0$$

and Lemma 10, we have

$$\limsup_{t'\uparrow t}\sup_{(w,\varphi,s)\in E}(V_t(w,\varphi,s;u)-V_{t'}(w,\varphi,s;u))\le 0.$$

Then we get the assertion (i).

Next we will check (ii). If $h(\infty)=\infty$, this assertion holds by Proposition 6 and Theorem 3. So we may assume $h(\infty)<\infty$.

By Propositions 7–8 and Theorem 3, we get

$$\lim_{t'\downarrow t}\sup_{(w,\varphi,s)\in E}|V_{t'}(w,\varphi,s;u)-JV_t(w,\varphi,s;u)|=0,$$

and obviously $V_t(w,\varphi,s;u)\le JV_t(w,\varphi,s;u)$. So it suffices to show

$$JV_t(w,\varphi,s;u)\le V_t(w,\varphi,s;u),\quad t>0. \tag{7.24}$$

Take any $\psi\in[0,\varphi]$ and $(\zeta_r)_{0\le r\le t}\in\mathcal{A}_t(\varphi-\psi)$. Let $\delta\in(0,t)$ and define $(\tilde\zeta_r)_{0\le r\le t}\in\mathcal{A}_t(\varphi)$ by $\tilde\zeta_r=(\psi/\delta)1_{[0,\delta]}(r)+\zeta_r$. Let $(W_r,\varphi_r,X_r)_{0\le r\le t}=\Xi_t^X(w+F(\psi)s,\varphi-\psi,se^{-h(\infty)\psi};(\zeta_r)_r)$ and $(\tilde W_r,\tilde\varphi_r,\tilde X_r)_{0\le r\le t}=\Xi_t^X(w,\varphi,s;(\tilde\zeta_r)_r)$, where $F(x)$ is given by (7.23). Then we have for $r\in[\delta,t]$

$$\tilde X_r-X_r=\int_0^r(\sigma(\tilde X_v)-\sigma(X_v))dB_v+\int_0^r(b(\tilde X_v)-b(X_v))dv+e_\delta,$$

where

$$e_\delta=h(\infty)\psi-\int_0^\delta(g(\tilde\zeta_v)-g(\zeta_v))dv=\frac{1}{\delta}\int_0^\delta\int_0^\psi\left(h(\infty)-h\left(\frac{\zeta'}{\delta}+\zeta_v\right)\right)d\zeta'dv.$$

Using the Burkholder-Davis-Gundy inequality and the Hölder inequality, we get

$$\begin{aligned}
&\mathrm{E}[\sup_{v\in[\delta,r]}|\tilde X_v-X_v|^2]\le C_0\Big\{\int_0^r\mathrm{E}[|\tilde X_v-X_v|^2]dv+\mathrm{E}[e_\delta]\Big\}\\
\le\ & C_0\Big\{\int_\delta^r\mathrm{E}[|\tilde X_v-X_v|^2]dv+2\int_0^\delta\mathrm{E}[|\tilde X_v|^2+|X_v|^2]dv+\mathrm{E}[e_\delta]\Big\}\\
\le\ & C_1\Big\{\int_\delta^r\mathrm{E}[\sup_{v'\in[\delta,v]}|\tilde X_{v'}-X_{v'}|^2]dv+\delta+\mathrm{E}[e_\delta]\Big\},\quad r\in[\delta,t]
\end{aligned}$$

for some $C_0,C_1>0$ depending only on b, σ and E. So the Gronwall inequality implies

$$\begin{aligned}
\mathrm{E}[\sup_{r\in[\delta,t]}|\tilde X_r-X_r|^2]\ &\le\ C_1\left\{\delta+\mathrm{E}[e_\delta]+(\delta+\mathrm{E}[e_\delta])\int_\delta^t e^{C_1(t-r)}dr\right\}\\
&\le\ C_2(\delta+\mathrm{E}[e_\delta])
\end{aligned}$$

for some $C_2 > 0$. Since $\mathrm{E}[e_\delta] \le \tilde{\varepsilon}_\delta \longrightarrow 0$ as $\delta \to 0$, where $\tilde{\varepsilon}_\delta$ is given by (7.23), we get $\mathrm{E}[\sup_{r\in[\delta,t]} |\tilde{X}_r - X_r|^2]$, $\mathrm{E}[\sup_{r\in[0,t]} |\exp(\tilde{X}_r) - \exp(X_r)|] \longrightarrow 0, \quad \delta \to 0$. Moreover we have

$$
\begin{aligned}
& \mathrm{E}[|\tilde{W}_t - W_t|] \\
\le \ & \mathrm{E}[|\frac{\psi}{\delta}\int_0^\delta \exp(\tilde{X}_r)dr - F(\psi)s|] + \mathrm{E}[\int_0^t |\exp(\tilde{X}_r) - \exp(X_r)|\zeta_r dr] \\
\le \ & \mathrm{E}[\frac{\psi}{\delta}\int_0^\delta \left|\exp(\tilde{X}_r) - se^{-h(\infty)\psi r/\delta}\right| dr] + \Phi_0\, \mathrm{E}[\sup_{r\in[0,t]} |\exp(\tilde{X}_r) - \exp(X_r)|] \\
\le \ & \Phi_0(s+1)\, \mathrm{E}[(\hat{Z}(s)+1)^2]^{1/2} \Big\{ \mathrm{E}[\sup_{r\in[0,\delta]} |\tilde{X}_r - \log s + \int_0^r g(\zeta_v)dv|^2]^{1/2} \\
& + \int_0^1 \tilde{\varepsilon}_{\delta r} dr + \delta g\left(\sup_{r,\omega} \zeta_r(\omega)\right) \Big\} + \Phi_0\, \mathrm{E}[\sup_{r\in[0,t]} |\exp(\tilde{X}_r) - \exp(X_r)|],
\end{aligned}
$$

thus $\mathrm{E}[|\tilde{W}_t - W_t|] \longrightarrow 0, \quad \delta \to 0$ by virtue of Lemma 4. Then Lemma 2 implies

$$\lim_{\delta\to 0} \left| \mathrm{E}[u(W_t, \varphi_t, \exp(X_t))] - \mathrm{E}[u(\tilde{W}_t, \tilde{\varphi}_t, \exp(\tilde{X}_t))] \right| = 0. \tag{7.25}$$

By (7.25), we easily get $\mathrm{E}[(W_t, \varphi_t, \exp(X_t))] \le V_t(w, \varphi, s; u)$. Since $(\zeta_r)_r \in \mathcal{A}_t(\varphi - \psi)$ is arbitrary, we have

$$V_t(w + F(\psi)s, \varphi - \psi, se^{-h(\infty)\psi}; u) \le V_t(w, \varphi, s; u).$$

Moreover, since $\psi \in [0, \varphi]$ is arbitrary, we get (7.24). ■

By Propositions 6–9 and the relation $V_t(\cdot\, ; u) \in \mathcal{C}$, we complete the proof of Theorem 2.

7.6 Proof of Theorem 4

In Section 7.6 and Section 7.7 we always assume that h is strictly increasing and $h(\infty) = \infty$. First we consider the characterization of $V_t^L(w, \varphi, s; u)$ as the viscosity solution of the corresponding HJB. We define a function $F^L : \mathscr{S} \longrightarrow \mathbb{R}$ by

$$F^L(z, p, X) = -\sup_{0\le\zeta\le L} \left\{ \frac{1}{2}\hat{\sigma}(z_s)^2 X_{ss} + \hat{b}(z_s)p_s + \zeta\,(z_s p_w - p_\varphi) - g(\zeta) z_s p_s \right\}.$$

Then we have the following.

Proposition 10. *Assume $h(\infty) = \infty$. Then, for each $u \in \mathcal{C}$, the function $V_t^L(w, \varphi, s; u)$ is the viscosity solution of*

$$\frac{\partial}{\partial t}v + F^L(z, \mathcal{D}v, \mathcal{D}^2 v) = 0 \quad \text{on } (0,1] \times U. \tag{7.26}$$

Since the control region $[0, L]$ is compact, we obtain Proposition 10 by using (7.18) and the standard arguments of the Bellman principle (see Theorem 5.4.1 in [24]).

Next we treat HJB (3.3). Let $\mathscr{U} = \{(z, p, X) \in \mathscr{S}\ ;\ F(z, p, X) > -\infty\}$. A direct calculation gives the following.

Proposition 11. *For* $(z,p,X) \in \mathscr{U}$*,*

$$\begin{aligned} F(z,p,X) &= -\frac{1}{2}\hat{\sigma}(z_s)^2 X_{ss} - \hat{b}(z_s)p_s \\ &\quad - \max\left\{\zeta^*(z,p)\,(z_s p_w - p_\varphi) - g(\zeta^*(z,p))z_s p_s, 0\right\}, \end{aligned}$$

where $\zeta^*(z,p) = h^{-1}\left(\frac{z_s p_w - p_\varphi}{z_s p_s} \vee h(0)\right) 1_{\{p_s>0\}}$*. In particular* F *is continuous on* $\mathscr{U}$*.*

Now we prove Theorem 4. We define an open set $\mathscr{R} = U \times (\mathbb{R}^2 \times (0,\infty)) \times S^3 \subset \mathscr{U}$. Since F is continuous on $\mathscr{R}$ and F^L converges to F monotonuously, we see that this convergence is uniform on any compact set in $\mathscr{R}$ by Dini's theorem. Similarly, using Dini's theorem again, we see that V^L converges to V uniformly on any compact set in $[0,1] \times \hat{D}$. Moreover we notice that if we take $\hat{v} \in C^{1,2}((0,1] \times U)$ such that $V - \hat{v}$ has a local maximum 0 at (t,z), then (3.5) implies $(\partial \hat{v}/\partial z_s)(t,z) > 0$ and $(z, \mathcal{D}\hat{v}(t,z), \mathcal{D}^2\hat{v}(t,z)) \in \mathscr{R}$. Then the same arguments as in the proof of Lemma 5.7.1 in [24] lead us to the assertion. ■

7.7 Proof of Theorem 5

First we remark that Lemmas 1 and 5 also imply that $V_t(w,\varphi,s;u)$ has polynomial growth in w, φ and s.

Let $\tilde{U} \subset U$ be open and bounded. Let $\mathscr{P}^{2,\pm}_{(0,1]\times\tilde{U}}$ be parabolic variants of semijets and $\overline{\mathscr{P}}^{2,\pm}_{(0,1]\times\tilde{U}}$ be their closures (see [8]). For $\lambda > 0$, we define $F_\lambda(z,r,p,X) = \lambda r + F(z,p,X)$. We see that the following (a.) and (b.) are equivalent:

(a.) A function v is a viscosity subsolution (resp., supersolution) of (3.3),

(b.) A function $v_\lambda(t,z) = e^{-\lambda t}v(t,z)$ is a viscosity subsolution (resp., supersolution) of

$$\frac{\partial}{\partial t}v_\lambda + F_\lambda(z,t,\mathcal{D}v,\mathcal{D}^2 v) = 0. \tag{7.27}$$

By Proposition 11, we can easily prove the following lemma.

Lemma 12. *Suppose* v *is a viscosity subsolution (resp., supersolution) of* (7.27)*. Then*

$$a + F_\lambda(z,t,p,X) \le 0 \ (\textit{resp.}, \ \ge 0)$$

for any $(t,a,z,p,X) \in (0,1] \times \mathbb{R} \times \tilde{U} \times \mathbb{R}^3 \times S^3$ *with* $(z,p,X) \in \overline{\mathscr{P}}^{2,+}_{(0,1]\times\tilde{U}}v(z)$ *(resp.,* $(a,p,X) \in \overline{\mathscr{P}}^{2,-}_{(0,1]\times\tilde{U}}v(z)$*).*

Especially we note that $\overline{\mathscr{P}}^{2,-}_{(0,1]\times\tilde{U}}v(z) \subset \mathscr{U}$ when v is a viscosity supersolution of (7.27). Now we consider the comparison principle on a bounded domain.

Proposition 12. *Suppose* v *(resp.,* v'*) be a viscosity subsolution (resp., supersolution) of* (7.27) *on* $(0,1] \times \tilde{U}$*. Moreover suppose* $v(0,z) \le v'(0,z)$ *for* $z \in \tilde{U}$ *and* $v \le 0 \le v'$ *on* $(0,1] \times \partial\tilde{U}$*. Then* $v \le v'$ *on* $[0,1] \times \tilde{U}$*.*

By Lemma 12 and Theorem 8.12 in [8], we see that to prove Proposition 12 it suffices to show the following Proposition 13.

Proposition 13. *The function F satisfies the following structure condition*

$$F_\lambda(z', r, \alpha(z-z'), Y) - F_\lambda(z, r, \alpha(z-z'), X) \le \rho\left(\alpha|z-z'|^2 + |z-z'|\right)$$

for $\lambda > 0$, $\alpha > 1$, $\rho \in C([0,\infty);[0,\infty))$ with $\rho(0) = 0$, $z, z' \in \tilde{U}$, $X, Y \in S^3$ with $F(z', \alpha(z-z'), Y) > -\infty$ and

$$-3\alpha\begin{pmatrix} I & O \\ O & I \end{pmatrix} \le \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \le 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{7.28}$$

where $I \in \mathbb{R}^3 \otimes \mathbb{R}^3$ denotes the unit matrix.

Proof. The condition $F_\lambda(r, z', \alpha(z-z'), Y) > -\infty$ implies $(z', \alpha(z-z'), Y) \in \mathscr{U}$, thus either (i) $z_s > z'_s$ or (ii) $z_s = z'_s$ and $z'_s(p_w - p'_w) - (p_\varphi - p'_\varphi) \le 0$. In each case we have $F(z, \alpha(z-z'), X) > -\infty$ and

$$\begin{aligned} & F_\lambda(z', r, \alpha(z-z'), Y) - F_\lambda(z, r, \alpha(z-z'), X) \\ = \;& F(z', \alpha(z-z'), Y) - F(z, \alpha(z-z'), X) \\ \le \;& \frac{1}{2}(\hat{\sigma}^2(z_s)X_{ss} - \hat{\sigma}^2(z'_s)Y_{ss}) + |\hat{b}(z_s) - \hat{b}(z'_s)|\alpha|z_s - z'_s| \\ & + \alpha \sup_{\zeta \ge 0}\left\{-(z_s - z'_s)^2 g(\zeta) + (z_s - z'_s)(z_w - z'_w)\zeta\right\}. \end{aligned} \tag{7.29}$$

Since (7.28) implies

$$\hat{\sigma}^2(z_s)X_{ss} - \hat{\sigma}^2(z'_s)Y_{ss} \le 3\alpha(\hat{\sigma}(z_s) - \hat{\sigma}(z'_s))^2$$

and, $\hat{\sigma}$ and $\hat{b}$ are both Lipschitz continuous and linear growth, we have

$$\frac{1}{2}(\hat{\sigma}^2(z_s)X_{ss} - \hat{\sigma}^2(z'_s)Y_{ss}) + |\hat{b}(z_s) - \hat{b}(z'_s)|\alpha|z_s - z'_s| \le C_0\alpha|z_s - z'_s|^2$$

for some $C_0 > 0$.

Next we estimate the last term of the right-hand side of (7.29). If $z_s = z'_s$, it is obvious that this term is equal to zero, so we consider the case $z_s > z'_s$. Since $\liminf_{\zeta\to\infty}(h(\zeta)/\zeta) > 0$, we see that there exist $\beta > 0$ and $\zeta_0 > 0$ such that $g(\zeta) \ge \beta\zeta^2$ for any $\zeta \ge \zeta_0$. Thus

$$\begin{aligned} & \sup_{\zeta \ge 0}\left\{-(z_s - z'_s)^2 g(\zeta) + (z_s - z'_s)(z_w - z'_w)\zeta\right\} \\ \le \;& -(g(\zeta_0) + \zeta_0)|z - z'|^2 + \sup_{\zeta\ge 0}\left\{-(z_s - z'_s)^2\beta\zeta^2 + (z_s - z'_s)(z_w - z'_w)\zeta\right\} \\ \le \;& (g(\zeta_0) + \zeta_0)|z - z'|^2 + |z_w - z'_w|\left(\frac{z_w - z'_w}{2\beta} \vee 0\right) \;\le\; C_2|z - z'|^2 \end{aligned}$$

for some $C_1, C_2 > 0$. Thus we obtain the assertion. ∎

Now we present the following proposition which includes the assertion of Theorem 5.

Proposition 14. *Let v (resp., v') be functions such that*

$$|v(t,z)| + |v'(t,z)| \le C(1 + z_w^2 + z_\varphi^2 + z_s^2)^m, \quad (t,z) \in [0,1] \times \hat{D}$$

for some $C, m > 0$. Suppose that v (resp., v') is a viscosity subsolution (resp., supersolution) of (3.3) on $(0,1] \times \hat{D}$. Moreover suppose that v and v' satisfy (3.6). Then $v \le v'$ on $[0,1] \times \hat{D}$.

Proof. Let $q(z) = (1+z_w^2+z_\varphi^2+z_s^2)^{m+1}$. By the similar arguments as in the proof of Proposition 13, we have

$$|F(z, q(z), \mathcal{D}q(z), \mathcal{D}^2 q(z))| \le C_0 q(z), \quad z \in \hat{D}$$

for some $C_0 > 0$. Let $\lambda > C_0$ and take any $\varepsilon > 0$. We define $\bar{v}(t,z) = e^{-\lambda t} v(t,z) - \varepsilon q(z)$ and $\bar{v}'(t,z) = e^{-\lambda t} v'(t,z) + \varepsilon q(z)$. Then there is some $R_\varepsilon > 0$ such that $\bar{v} < 0 < \bar{v}'$ holds on $[0,1] \times \{|z| \ge R_\varepsilon\}$. By a straightforward calculation, we see that $\bar{v}$ (resp., $\bar{v}'$) is a viscosity subsolution (resp., supersolution) of (7.27). Thus Proposition 12 implies $\bar{v} \le \bar{v}'$ on $[0,1] \times \hat{D}$. Since $\varepsilon > 0$ is arbitrary, we obtain the assertion. ■

7.8 Proof of Proposition 1

First we prove the following lemma.

Lemma 13. *Under the assumptions of Section 5, $V_k^n(w, \varphi, s; u)$ is equal to the discrete-time value function with $b(x) = -\tilde{\mu}$ and $\sigma(x) = 0$.*

This lemma is easily proved by mathematical induction. Then we see that the optimal strategy in the discrete-time model is deterministic. By the proof of Theorem 1, the optimal strategy in continuous-time model is also deterministic. For $(\zeta_r)_r \in \mathcal{A}_t^{\mathrm{det}}(\varphi)$, we have

$$\mathrm{E}[S_t] = s \exp\left(-\tilde{\mu} t - \int_0^t g(\zeta_r) dr \right)$$

and

$$\mathrm{E}[W_t] = w + \int_0^t \zeta_r \, \mathrm{E}[S_r] dr = w + s \int_0^t \zeta_r \exp\left(-\tilde{\mu} r - \int_0^r g(\zeta_v) dv \right) dr.$$

This implies the assertion.

7.9 Proof of Theorem 8

The assertion (i) is directly obtained by (9')–(12') in [21]. We can show the inequality

$$V_t^{\mathrm{SO}}(w, \varphi, s; U) \ge U\left(w + \frac{1 - e^{-\alpha\varphi}}{\alpha} s \right)$$

by considering the strategy (3.2) and letting $\delta \downarrow 0$. To see the opposite inequality, it suffices to show that

$$\overline{V}_t^{\varphi}(\bar{w}, \bar{s}) \le U(\bar{w}). \tag{7.30}$$

Since U is concave and non-decreasing and $\bar{b}$ is non-positive, the Jensen inequality implies

$$\mathrm{E}[U(\bar{W}_t)] \le U(\mathrm{E}[\bar{W}_t]) = U\left(\bar{w} + \int_0^t \frac{e^{\alpha\bar{\varphi}_r} - 1}{\alpha} \bar{S}_r \bar{b}(\log \bar{S}_r + \alpha\varphi_r) dr \right) \le U(\bar{w})$$

for each $(\varphi_r)_r \in \overline{\mathcal{A}}_t(\varphi)$, hence (7.30) holds. ■

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