

Third-order interference and a principle of ‘quantumness’

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Abstract. Are there physical, probabilistic or information-theoretic principles which characterize the quantum probabilities and distinguish them from the classical case as well as from other probability theories, or which reveal why quantum mechanics requires its very special mathematical formalism? The paper identifies the fundamental absence of third-order interference as such a principle of ‘quantumness’. Considering three-slit experiments, the concept of third-order interference was originally introduced by Sorkin in 1994.

1 Introduction

Are there physical, probabilistic or information-theoretic principles which specify the quantum probabilities and distinguish them from the classical case as well as from other probability theories, or which reveal why quantum mechanics requires its very special mathematical formalism? The relativity principle from which Einstein derived his theory is here considered a prototype by many physicists. The more it is tried to exploit the quantum peculiarities in modern quantum information theory, quantum computing and quantum cryptography, the more vital becomes the search for such principles of ‘quantumness’. A non-exhaustive selection of recent research papers with varying approaches are Refs. [1], [2], [4], [5], [7], and [12]; references to further work in this direction can be found particularly in Ref. [4]. The present paper introduces a new approach based on Sorkin’s concept [10] of higher-order interference.

Of course, the formalism of quantum theory is well-defined by a large system of mathematical axioms. It perfectly describes all the quantum peculiarities. However, it does not reveal their origin because these axioms have only a mathematical meaning, but no obvious physical or probabilistic interpretation. The statistical interpretation is a later add-on motivated empirically, but not evident from the mathematical structure of the theory.

A most typical quantum phenomenon is the wave-like interference exhibited in two-slit experiments with small physical particles. Considering experiments with three slits, Sorkin [10] introduced the concept of third-order interference and detected that standard quantum mechanics rules out third-order interference. In a recent paper [8], the author could show that a probability theory where third-order interference does not occur must necessarily be very close to standard quantum mechanics.

This results in the following classification. A probability theory, where there is no interference at all, is classical. A theory including second-order interference, but ruling out third-order interference is quantum mechanics. The principle of ‘quantumness’ thus becomes the presence of second-order interference combined with the absence of third-order interference. A theory where third-order interference is possible would go beyond quantum mechanics.

The mathematics behind these findings has been elaborated in detail in Ref. [8]. The present paper shall outline the major result in a less mathematical way and focus on its role in a principle of ‘quantumness’. After a brief overview of classical probabilities, quantum theory and a more general probabilistic framework in the next section, second- and third-order interference will be considered in the third section, before their role in a principle of ‘quantumness’ will be studied in the fourth one.

2 Probability theories

In quantum mechanics, the measurable quantities of a physical system are represented by observables. Most simple are those observables where only the two discrete values 0 and 1 are possible as measurement result; they are called *events*. Mathematical probability theory usually starts with the identification of a structure for the events. Classically, this is a Boolean algebra. However, it is well-known that quantum mechanics requires a more general, not necessarily Boolean structure called *quantum logic*. This was pointed out by von Neumann and Birkhoff [3] already in the early days of quantum mechanics.

An *orthogonality* relation and a partial *sum* operation $+$ defined only for orthogonal events are available on a quantum logic. Orthogonality means that the events exclude each other and, in the classical case, it is the same as disjointness. Standard quantum mechanics uses a very special type of quantum logic; it consists of the self-adjoint projections on a Hilbert space or, more generally, of the self-adjoint projections in a von Neumann algebra.

The *states* on a quantum logic are the analogue of the probability measures in classical probability theory, and *conditional probabilities* can be defined similar to their classical prototype [8]. However, there are many quantum logics where no states or no conditional probabilities exist, or where the conditional probabilities are ambiguous. Therefore, only those quantum logics where sufficiently

many states and unique conditional probabilities exist can be considered a satisfying framework for general probabilistic theories.

A state μ allocates a probability $\mu(e)$ to each event e in such a way that $\mu(e_1 + e_2) = \mu(e_1) + \mu(e_2)$ holds for any two orthogonal events e_1 and e_2 . The conditional probability of an event f under another event e in the state μ is denoted by $\mu(f | e)$; this is the updated probability of the event f after the event e has been the outcome of a first measurement.

With two successive measurements, the probability that the first one provides the result e and the second one then the result f is the product $\mu(f | e)\mu(e)$. In the classical case, the system of events forms a Boolean algebra and this probability becomes $\mu(f | e)\mu(e) = \mu(f \cap e)$, which is additive in f as well as in e . In the general case, it remains additive only in f , but not in e . This is the origin of quantum interference which shall be considered in the following section.

3 Interference

For a pair of orthogonal events e_1 and e_2 , a further event f and a state μ , the following mathematical term I_2 shall be studied:

$$I_2 := \mu(f | e_1 + e_2)\mu(e_1 + e_2) - \mu(f | e_1)\mu(e_1) - \mu(f | e_2)\mu(e_2)$$

For classical probabilities, the identity $I_2 = 0$ holds, but not for quantum mechanics. Many quantum peculiarities can directly be traced back to the fact that $I_2 = 0$ is not valid. This is the essence of quantum interference which is exhibited e.g., in the two-slit experiments with small physical particle like electrons, photons and others, or in experiments measuring the spin of electrons and photons twice along differently oriented spatial axes.

Sorkin [10] introduced the concept of third-order interference. For a triple of orthogonal events e_1 , e_2 and e_3 , a further event f and a state μ , he defined the following mathematical term I_3 :

$$\begin{aligned} I_3 := & \mu(f | e_1 + e_2 + e_3)\mu(e_1 + e_2 + e_3) \\ & - \mu(f | e_1 + e_2)\mu(e_1 + e_2) \\ & - \mu(f | e_1 + e_3)\mu(e_1 + e_3) \\ & - \mu(f | e_2 + e_3)\mu(e_2 + e_3) \\ & + \mu(f | e_1)\mu(e_1) + \mu(f | e_2)\mu(e_2) + \mu(f | e_3)\mu(e_3) \end{aligned}$$

Sorkin's original definition refers to probability measures on 'sets of histories'. With the use of conditional probabilities, it gets the above shape, which was seen by Ududec, Barnum and Emerson [11] who adapted it into an operational probabilistic framework.

Classical probabilities satisfy the identity $I_3 = 0$; this follows from $I_2 = 0$. Sorkin detected that $I_3 = 0$ also holds in quantum mechanics. The identity $I_3 = 0$ can be rewritten in the following form:

$$\begin{aligned}
& \mu(f \mid e_1 + e_2 + e_3) \mu(e_1 + e_2 + e_3) - \mu(f \mid e_1) \mu(e_1) - \mu(f \mid e_2) \mu(e_2) - \mu(f \mid e_3) \mu(e_3) \\
&= \mu(f \mid e_1 + e_2) \mu(e_1 + e_2) - \mu(f \mid e_1) \mu(e_1) + \mu(f \mid e_2) \mu(e_2) \\
&+ \mu(f \mid e_1 + e_3) \mu(e_1 + e_3) - \mu(f \mid e_1) \mu(e_1) + \mu(f \mid e_3) \mu(e_3) \\
&+ \mu(f \mid e_2 + e_3) \mu(e_2 + e_3) - \mu(f \mid e_2) \mu(e_2) + \mu(f \mid e_3) \mu(e_3)
\end{aligned}$$

The left-hand side of this equation is a measure for the interference involved in the case with three mutually exclusive events e_1 , e_2 , and e_3 . It is identical with the sum of the measures for the pair interferences when the three different pairs are considered which can be built from the three events. Therefore, $I_3 = 0$ means that interference with three mutually exclusive events does not provide anything new compared to the cases with only two exclusive events. However, $I_3 \neq 0$ would mean that a fundamentally new form of interference exists in addition to the pair interferences.

While the familiar two slit-experiment involves only two possible paths, three different paths are available for the particle in a three-slit configuration. In this case, the identity $I_3 = 0$ means that quantum interference is limited to pairs of paths and that quantum mechanics does not exhibit a new form of interference involving path triples. The interference pattern observed with three open slits is a simple combination of the patterns observed in the six different cases with only one or two among the three available slits open, which could be confirmed in a recent experimental test with photons [9].

As well as I_2 , the term I_3 defines a very general concept and is not restricted to multiple-slit experiments. The new type of interference which is present whenever $I_3 \neq 0$ holds is called *third-order interference*, and the one present whenever $I_2 \neq 0$ holds is called *second-order interference*.

4 A principle of ‘quantumness’

The absence of second-order interference ($I_2 = 0$) characterizes the classical probabilities. The presence of second-order interference ($I_2 \neq 0$) is typical of the quantum probabilities, but third order-interference is not possible in quantum theory. To what extent does now the absence of third-order interference ($I_3 = 0$) characterize the quantum probabilities? Can there be other probability theories with $I_3 = 0$, or is quantum mechanics the only one?

In Ref. [8], it could be shown that that a probability theory where third-order interference does not occur must necessarily be very close to standard quantum mechanics. The events must then be projections in a Jordan algebra. This is the major result (Theorem 11.1) of Ref. [8]. Only the exceptional Jordan algebras provide examples with $I_3 = 0$ which are not covered by standard quantum mechanics (i.e., do not have a representation as operators on a Hilbert space). This results in the classification of the different probability theories which is depicted in Figure 1.

The combination of the presence of second-order interference ($I_2 \neq 0$) with the absence of third-order interference ($I_3 = 0$) can therefore be identified as an essential principle of ‘quantumness’. The presence of second-order interference ($I_2 \neq 0$) distinguishes quantum theory from the classical case. It is the reason why classical probability theory cannot cover the quantum probabilities. The fundamental absence of third-order interference ($I_3 = 0$) entails the very special mathematical formalism of quantum mechanics. It defines what quantum theory is, leaving only little room beyond standard quantum mechanics. A further still unknown principle might do the rest and rule out the exceptional Jordan algebras.

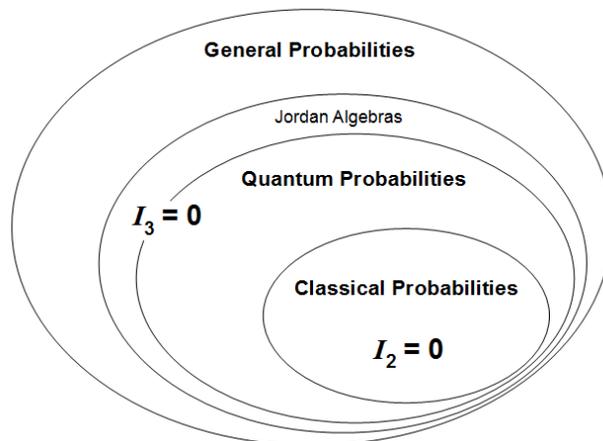


Figure 1: Classification of probability theories

5 Remarks

The above results and their mathematical proofs have been elaborated in detail in Ref. [8]. Without going too far into the mathematical details, some assumptions which they are based upon shall be presented here. The general probabilistic framework is given by those quantum logics where sufficiently many states and unique conditional probabilities exist. Beyond that, three further assumptions are needed.

The first assumption is that real-valued (i.e., not necessarily positive) generalized states satisfy a Hahn-Jordan decomposition property similar to the classical real-valued measures.

A quantum logic with sufficiently many states can always be embedded in an ordered linear space. In Ref. [8], it was shown that the absence of third-order interference ($I_3 = 0$) is equivalent to the existence of a product in this ordered linear space. Generally, this product is neither commutative nor associative, and the second assumption is that each element of the ordered linear space generates an associative subalgebra. The third one is that the square of an element is positive.

The elements of the ordered linear space are candidates for observables, for which the last two assumptions are quite natural postulates. The first assumption is a mathematical technical requirement.

Only the exceptional Jordan algebras satisfy these assumptions and the identity $I_3 = 0$ without being included in standard quantum mechanics. There are not too many of them and, in a certain sense, there is only one; it is formed by the self-adjoint 3×3 -matrices the entries of which are octonions (see Ref. [6]).

6 Conclusions

The concept of third-order interference is a quite natural extension of the second-order interference which is so typical of quantum mechanics, but surprisingly third-order interference is ruled out by quantum theory. Quantum interference involves only pairs, but no triples of mutually exclusive alternatives.

In the present paper, the absence of third-order interference (i.e., $I_3 = 0$) has been identified as a fundamental principle of ‘quantumness’ which entails the very special structure of quantum theory. It defines the theory, leaving only little room beyond standard quantum mechanics.

A violation of the identity $I_3 = 0$ has never been detected, and a recent experimental test [9] has confirmed it. One can therefore still assume that quantum mechanics is universally valid in nature. However, its universal validity is sometimes questioned because a successful unification with relativity theory and a satisfying quantum gravity theory are still missing. This might require a new theory going beyond standard quantum mechanics and possibly exhibiting third-order interference.

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