Final Book
on Fundamental Theoretical Physics

by Gunn Quznetsov

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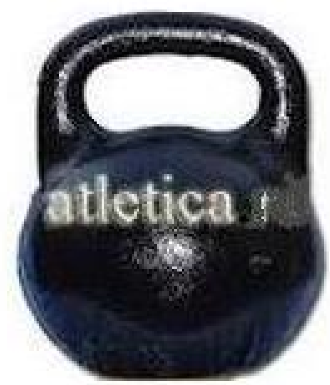

FINAL BOOK ON FUNDAMENTAL THEORETICAL PHYSICS

GUNN QUZNETSOV

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Contents

Introduction

Within last several decades many theoretical physicists investigated what isn't present in the Nature. It is the Superstrings Theory, the Higgs theory, The Dark Energy and the Dark Masses hypotheses, etc. Here some of last results from LHC and Tevatron:

WIMP-nucleon cross-section results from the second science run of ZEPLIN-III: http://arxiv.org/abs/1110.4769 : "This allows the exclusion of the scalar cross-section above 4.8E-8 pb near 50 GeV/c2 WIMP mass with 90% confidence."

A search for charged massive long-lived particles: http://arxiv.org/abs/1110.3302 : "We exclude pair-produced long-lived gaugino-like charginos below 267 GeV and higgsino-like charginos below 217 GeV at 95% C.L., as well as long-lived scalar top quarks with mass below 285 GeV."

Search for Universal Extra Dimensions with the D0 Experiment:
http://arxiv.org/pdf/1110.2991 : "No excess of data over background was observed."

Search for new physics with same-sign isolated dilepton events with jets and missing transverse energy at CMS: http://arxiv.org/PS_cache/arxiv/pdf/1110/1110.2640v1.pdf : "No evidence was seen for an excess over the background prediction."

Search for Supersymmetry with Photon at CMS: http://arxiv.org/abs/1110.2552 : "No excess of events above the standard model predictions is found."

Search for neutral Supersymmetric Higgs bosons...: http://arxiv.org/abs/1110.2421 : "In the absence of a signifcant signal, we derive upper limits for neutral Higgs boson production cross-section..."

Search for Chargino-Neutralino Associated Production in Dilepton Final States with Tau Leptons: http://arxiv.org/abs/1110.2268 : "We set limits on the production cross section as a function of SUSY particle mass for certain generic models."

Model Independent Search at the D0 experiment: http://arxiv.org/abs/1110.2266 : "No evidence of new physics is found."

Search for the Standard Model Higgs Boson in the Lepton + Missing Transverse Energy + Jets Final State in ATLAS: http://arxiv.org/abs/1110.2265 : "No significant excess of events is observed over the expected background and limits on the Higgs boson production cross section are derived for a Higgs boson mass in the range 240 GeV ¡ mH ¡ 600 GeV."...

etc.,etc. There are more than 100 similar negative results and there are no positive results absolutely (for instance: http://www.scientific.ru/dforum/scinews).

On the other hand already in 2006 - 2007 the logic analysis of these subjects described in books [1] and [2], has shown that all physical events are interpreted by well-known particles (leptons, quarks, photons, W- and Z-bozons) and forces (electroweak, strong, gravity).

"Final Book" contains development and continuation of ideas of these books. Chapter

1 gives convenient updating of the Gentzen Natural Logic [3], a logic explanation of space-time relations, and logical foundations of the Probability Theory. The reader who isn't interested in these topics, can pass this part and begin readings with Chapter 2.

Chapter 2 receives notions and statements of Quantum Theory from properties of probabilities of physical events. In Chapter 3 Electroweak Theory, Quarks-Gluon Theory and Gravity Theory are explained from these properties.

For understanding of the maintenance of this book elementary knowledge in the field of linear algebra and the mathematical analysis is sufficient.

Chapter 1

Time, Space, and Probability

1.1. Classical propositional logic

Let's consider affirmative sentences of any languages.

Def. 1.1.1: Sentence $\ll \Theta \gg$ is *true* if and only if Θ.

For example, sentence $\ll$It is raining$\gg$ is true if and only if it is raining[1] [4].

Def. 1.1.2: Sentence $\ll \Theta \gg$ is *false* if and only if there is not that Θ.

It is clear that many neither true nor false sentences exist. For example, $\ll$There is rainy 21 august 3005 year in Chelyabinsk$\gg$.

Still an example: Obviously, the following sentence isn't true and isn't false [5]:

$\ll$The sentence which has been written on this line, is false.$\gg$

Those sentences which can be either true, or false, are called as *meaningful* sentences. The previous example sentence is meaningless sentence.

Further we consider only meaningful sentences which are either true, or false.

Def. 1.1.3: Sentences A and B are *equal* (design.: $A = B$) if A is true, if and only if B is true.

Further I'm using ordinary notions of the classical propositional logic [6].

Def. 1.1.4: A sentence C is called *conjunction* of sentences A and B (design.: $C = (A\&B)$) if C is true, if and only if A and B are true.

Def. 1.1.5: A sentence C is called *negation* of sentences A (design.: $C = (\neg A)$) if C is true, if and only if A is not true.

Def. 1.1.6: A sentence C is called *disjunction* of sentences A and B (design.: $C = (A \vee B)$) if C is true, if and only if A is true or B is true or both A and B are true.

Def. 1.1.7: A sentence C is called *implication* of sentences A and B (design.: $C = (A \Rightarrow B)$) if C is true, if and only if B is true and/or B is false.

A sentence is called *a simple sentence* if it isn't neither conjunction, nor a disjunction, neither implication, nor negation.

Th. 1.1.1:

1) $(A\&A) = A$; $(A \vee A) = A$;

2) $(A\&B) = (B\&A)$; $(A \vee B) = (B \vee A)$;

3) $(A\&(B\&C)) = ((A\&B)\&C)$; $(A \vee (B \vee C)) = ((A \vee B) \vee C)$;

[1] Alfred Tarski (January 14, 1901 October 26, 1983) was a Polish logician and mathematician

4) if T is a true sentence then for every sentence A: $(A\&T) = A$ and $(A \vee T) = T$.

5) if F is a false sentence then $(A\&F) = F$ and $(A \vee F) = A$.

Proof of Th. 1.1.1: This theorem directly follows from Def. 1.1.1, 1.2, 1.3, 1.4, 1.6. □

Further I set out the version of the Gentzen Natural Propositional calculus[2] (NPC) [3]:

Expression "Sentence C is a logical consequence of a list of sentences Γ." will be wrote as the following: "$\Gamma \vdash C$". Such expressions are called *sequences*. Elements of list Γ are called *hypothesizes*.

Def. 1.1.8

1. A sequence of form $C \vdash C$ is called *NPC axiom*.

2. A sequence of form $\Gamma \vdash A$ and $\Gamma \vdash B$ is obtained from sequences of form $\Gamma \vdash (A\&B)$ by *a conjunction removing rule* (design.: R&).

3. A sequence of form $\Gamma_1, \Gamma_2 \vdash (A\&B)$ is obtained from sequence of form $\Gamma_1 \vdash A$ and a sequence of form $\Gamma_2 \vdash B$ by *a conjunction inputting rule* (design: I&).

4. A sequence of form $\Gamma \vdash (A \vee B)$ is obtained from a sequence of form $\Gamma \vdash A$ or from a sequence of form $\Gamma \vdash B$ by *a disjunction inputting rule* (design.: I∨).

5. A sequence of form $\Gamma_1[A], \Gamma_2[B], \Gamma_3 \vdash C$ is obtained from sequences of form $\Gamma_1 \vdash C$, $\Gamma_2 \vdash C$, snd $\Gamma_3 \vdash (A \vee B)$ by *a disjunction removing rule* (design.: R∨) (Here and further: $\Gamma_1[A]$ is obtained from Γ_1 by removing of sentence A, and $\Gamma_2[B]$ is obtained from Γ_2 by removing of sentence B).

6. A sequence of form $\Gamma_1, \Gamma_2 \vdash B$ is obtained from a sequence of form $\Gamma_1 \vdash A$ and from a sequence of form $\Gamma_2 \vdash (A \Rightarrow B)$ by *a implication removing rule* (design.: R⇒).

7. A sequence of form $\Gamma[A] \vdash (A \Rightarrow B)$ is obtained from a sequence of form $\Gamma \vdash B$ by *an implication inputting rule* (design.: I⇒).

8. A sequence of form $\Gamma \vdash C$ is obtained from a sequence of form $\Gamma \vdash (\neg(\neg C))$ by *a negation removing* rule (design.: R¬).

9. A sequence of form $\Gamma_1[C], \Gamma_2[C] \vdash (\neg C)$ is obtained from a sequence of form $\Gamma_1 \vdash A$ and from a sequence of form $\Gamma_2 \vdash (\neg A)$ by *negation inputting rule* (design.: I¬).

10. A finite string of sequences is called *a propositional natural deduction* if every element of this string either is a NPC axioms or is received from preceding sequences by one of the deduction rules (R&, I&, I∨, R∨, R⇒, I⇒, R¬, I¬).

Actually, these logical rules look naturally in light of the previous definitions.

Example 1: Let us consider the following string of sequences:

$$
\begin{aligned}
&1. ((R\&S)\&(R \Rightarrow G)) \vdash ((R\&S)\&(R \Rightarrow G)) \text{ - NPC axiom.}\\
&2. ((R\&S)\&(R \Rightarrow G)) \vdash (R\&S) \text{ - R\& from 1.}\\
&3. ((R\&S)\&(R \Rightarrow G)) \vdash (R \Rightarrow G) \text{ - R\& from 1.}\\
&4. ((R\&S)\&(R \Rightarrow G)) \vdash R \text{ - R\& from 2.}\\
&5. ((R\&S)\&(R \Rightarrow G)) \vdash G \text{ - R} \Rightarrow \text{ from 3. and 4.}\\
&6. ((R\&S)\&(R \Rightarrow G)) \vdash S \text{ - R\& from 2.}\\
&7. ((R\&S)\&(R \Rightarrow G)) \vdash (G\&S) \text{ - I\& from 5. and 6.}
\end{aligned}
\tag{1.1}
$$

[2]Gerhard Karl Erich Gentzen (November 24, 1909, Greifswald, Germany August 4, 1945, Prague, Czechoslovakia) was a German mathematician and logician.

This string is a propositional natural deduction of sequence

$$((R\&S)\,\&\,(R\Rightarrow G))\vdash(G\&S)\,.$$

since it fulfills to all conditions of Def. 1.1.8.

Hence sentence $(G\&S)$ is logical consequence from sentence $((R\&S)\,\&\,(R\Rightarrow G))$.

Th. 1.1.2:

$$(A\vee B)=(\neg((\neg A)\,\&\,(\neg B)))\,, \tag{1.2}$$

$$(A\Rightarrow B)=(\neg(A\&(\neg B)))\,. \tag{1.3}$$

Proof of Th. 1.1.2:

The following string is a deduction of sequence $(A\vee B)\vdash(\neg((\neg A)\,\&\,(\neg B)))$:

1. $((\neg A)\,\&\,(\neg B))\vdash((\neg A)\,\&\,(\neg B))$, NPC axiom.
2. $((\neg A)\,\&\,(\neg B))\vdash(\neg A)$, R& from 1.
3. $A\vdash A$, NPC axiom.
4. $A\vdash(\neg((\neg A)\,\&\,(\neg B)))$, I¬ from 2. and 3.
5. $((\neg A)\,\&\,(\neg B))\vdash(\neg B)$, R& from 1.
6. $B\vdash B$, NPC axiom.
7. $B\vdash(\neg((\neg A)\,\&\,(\neg B)))$, I¬ from 5. and 6.
8. $(A\vee B)\vdash(A\vee B)$, NPC axiom.
9. $(A\vee B)\vdash(\neg((\neg A)\,\&\,(\neg B)))$, R∨ from 4., 7. and 8.

A deduction of sequence $(\neg((\neg A)\,\&\,(\neg B)))\vdash(A\vee B)$ is the following:

1. $(\neg A)\vdash(\neg A)$, NPC axiom.
2. $(\neg B)\vdash(\neg B)$, NPC axiom.
3. $(\neg A),(\neg B)\vdash((\neg A)\,\&\,(\neg B))$, I& from 1. and 2.
4. $(\neg((\neg A)\,\&\,(\neg B)))\vdash(\neg((\neg A)\,\&\,(\neg B)))$, NPC axiom.
5. $(\neg((\neg A)\,\&\,(\neg B))),(\neg B)\vdash(\neg(\neg A))$, I¬ from 3. and 4.
6. $(\neg((\neg A)\,\&\,(\neg B))),(\neg B)\vdash A$, R¬ from 5.
7. $(\neg((\neg A)\,\&\,(\neg B))),(\neg B)\vdash(A\vee B)$, i∨ from 6.
8. $(\neg(A\vee B))\vdash(\neg(A\vee B))$, NPC axiom.
9. $(\neg((\neg A)\,\&\,(\neg B))),(\neg(A\vee B))\vdash(\neg(\neg B))$, I¬ from 7. and 8.
10. $(\neg((\neg A)\,\&\,(\neg B))),(\neg(A\vee B))\vdash B$, R¬ from 9.
11. $(\neg((\neg A)\,\&\,(\neg B))),(\neg(A\vee B))\vdash(A\vee B)$, I∨ from 10.
12. $(\neg((\neg A)\,\&\,(\neg B)))\vdash(\neg(\neg(A\vee B)))$, I¬ from 8. and 11.
13. $(\neg((\neg A)\,\&\,(\neg B)))\vdash(A\vee B)$, R¬ from 12.

Therefore,

$$(\neg((\neg A)\,\&\,(\neg B)))=(A\vee B)\,.$$

A deduction of sequence $(A\Rightarrow B)\vdash(\neg(A\&(\neg B)))$ is the following:

1. $(A\&(\neg B))\vdash(A\&(\neg B))$, NPC axiom.
2. $(A\&(\neg B))\vdash A$, R& from 1.
3. $(A\&(\neg B))\vdash(\neg B)$, R& from 1.

4. $(A \Rightarrow B) \vdash (A \Rightarrow B)$, NPC axiom.
5. $(A\&(\neg B)), (A \Rightarrow B) \vdash B$, R$\Rightarrow$ from 2. and 4.
6. $(A \Rightarrow B) \vdash (\neg(A\&(\neg B)))$, I$\neg$ from 3. and 5.

A deduction of sequence $(\neg(A\&(\neg B))) \vdash (A \Rightarrow B)$ is the following:

1. $A \vdash A$, NPC axiom.
2. $(\neg B) \vdash (\neg B)$, NPC axiom.
3. $A, (\neg B) \vdash (A\&(\neg B))$, I& from 1. and 2.
4. $(\neg(A\&(\neg B))) \vdash (\neg(A\&(\neg B)))$, NPC axiom.
5. $A, (\neg(A\&(\neg B))) \vdash (\neg(\neg B))$, I$\neg$ from 3. and 4.
6. $A, (\neg(A\&(\neg B))) \vdash B$, R$\neg$ from 5.
7. $(\neg(A\&(\neg B))) \vdash (A \Rightarrow B)$, I$\Rightarrow$ from 6.

Therefore,

$$(\neg(A\&(\neg B))) = (A \Rightarrow B) \square$$

Example 2:

1. $A \vdash A$ - NPC axiom.
2. $(A \Rightarrow B) \vdash (A \Rightarrow B)$ - NPC axiom.
3. $A, (A \Rightarrow B) \vdash B$ - R$\Rightarrow$ from 1. and 2.
4. $(\neg B) \vdash (\neg B)$ - NPC axiom.
5. $(\neg B), (A \Rightarrow B) \vdash (\neg A)$ - I$\neg$ from 3. and 4.
6. $(A \Rightarrow B) \vdash ((\neg B) \Rightarrow (\neg A))$ - I$\Rightarrow$ from 5.
7. $\vdash ((A \Rightarrow B) \Rightarrow ((\neg B) \Rightarrow (\neg A)))$ - I$\Rightarrow$ from 6.

This string is a deduction of sentence of form

$$((A \Rightarrow B) \Rightarrow ((\neg B) \Rightarrow (\neg A)))$$

from the empty list of sentences. I.e. sentences of such form are *logicaly provable*.

Th. 1.1.3: If sequence $\Gamma \to C$ is deduced and C is false then some false sentence is contained in Γ.

Proof of Th. 1.1.3: is received by induction of number of sequences in the deduction of sequence $\Gamma \to C$.

The recursion Basis: Let the deduction of sequence $\Gamma \to C$ contains single sentence. In accordance the definition of propositional natural deduction this deduction must be of the following type: $C \to C$. Obviously, in this case the lemma holds true.

The recursion Step:The recursion assumption: Let's admit that the lemma is carried out for any deduction which contains no more than n sequences.

Let deduction of $\Gamma \to C$ contains $n+1$ sequence. In accordance with the propositional natural deduction definition sequence $\Gamma \to C$ can be axiom NPC or can be received by the deduction rules from previous sequence.

a) If $\Gamma \to C$ is the NPC axiom then see the recursion basis.

b) Let $\Gamma \to C$ be received by R&. In this case sequence of type $\Gamma \to (C\&B)$ or sequence of type $\Gamma \to (B\&C)$ is contained among the previous sequences of this deduction. Hence, deductions of sequences $\Gamma \to (C\&B)$ and $\Gamma \to (B\&C)$ contains no more than n sequences. In accordance with the recursion assumption, these deductions submit to the lemma. Because C is false then $(C\&B)$ is false and $(B\&C)$ is false in accordance with the conjunction definition. Therefore, Γ contains some false sentence by the lemma. And in this case the lemma holds true.

c) Let $\Gamma \to C$ be received by I&. In this case sequence of type $\Gamma_1 \to A$ and sequence of type $\Gamma_2 \to B$ is contained among the previous sequences of this deduction, and $C = (A \& B)$ and $\Gamma = \Gamma_1, \Gamma_2$. Deductions of sequences $\Gamma_1 \to A$ and $\Gamma_2 \to B$ contains no more than n sequences. In accordance with the recursion assumption, these deductions submit to the lemma. Because C is false then A is false or B is false in accordance with the conjunction definition. Therefore, Γ contains some false sentence by the lemma. And in this case the lemma holds true.

d) Let $\Gamma \to C$ be received by R$\vee$. In this case sequences of type $\Gamma_1 \to (A \vee B)$, $\Gamma_2 [A] \to C$, and $\Gamma_3 [B] \to C$ are contained among the previous sequences of this deduction, and $\Gamma = \Gamma_1, \Gamma_2, \Gamma_3$. Because these previous deductions contain no more than n sequences then in accordance with the recursion assumption, these deductions submit to the lemma. Because C is false then $\Gamma_2 [A]$ contains some false sentence, and $\Gamma_3 [B]$ contains some false sentence. If A is true then the false sentence is contained in Γ_2. If B is true then the false sentence is contained in Γ_3. I.e. in these case some false sentence is contained in Γ. If A is false and B is false then $(A \vee B)$ is false in accordance with the disjunction definition. In this case Γ_1 contains some false sentence. And in all these cases the lemma holds true.

e) Let $\Gamma \to C$ be received by I$\vee$. In this case sequence of type $\Gamma \to A$ or sequence of type $\Gamma \to B$ is contained among the previous sequences of this deduction, and $C = (A \vee B)$. Deductions of sequences $\Gamma \to A$ and $\Gamma \to B$ contains no more than n sequences. In accordance with the recursion assumption, these deductions submit to the lemma. Because C is false then A is false and B is false in accordance with the disjunction definition. Therefore, Γ contains some false sentence by the lemma. And in this case the lemma holds true.

f) Let $\Gamma \to C$ be received by R$\Rightarrow$. In this case sequences of type $\Gamma_1 \to (A \Rightarrow C)$, $\Gamma_2 \to A$ are contained among the previous sequences of this deduction, and $\Gamma = \Gamma_1, \Gamma_2$. Because these previous deductions contain no more than n sequences then in accordance with the recursion assumption, these deductions submit to the lemma. If A is false then Γ_2 contains some false sentence. If A is true then $(A \Rightarrow C)$ is false in accordance with the implication defination since C is false. And in all these cases the lemma holds true.

g) Let $\Gamma \to C$ be received by I$\Rightarrow$. In this case sequences of type $\Gamma [A] \to B$ is contained among the previous sequences of this deduction, and $C = (A \Rightarrow B)$. Because deduction of $\Gamma [A] \to B$ contains no more than n sequences then in accordance with the recursion assumption, this deduction submit to the lemma. Because C is false then A is true in accordance with the implication definition. Hence, some false sentence is contained in Γ. Therefore, in this case the lemma holds true.

i) Let $\Gamma \to C$ be received by R$\neg$. In this case sequence of type $\Gamma \to (\neg (\neg C))$ is contained among the previous sequences of this deduction. This previous deduction contains no more than n sequences then in accordance with the recursion assumption, this deduction submit to the lemma Because C is false then $(\neg (\neg C))$ is false in accordance with the negation definition. Therefore, Γ contains some false sentence by the lemma. And in this case the lemma holds true.

j) Let $\Gamma \to C$ be received by I$\neg$. In this case sequences of type $\Gamma_1 [A] \to B$, and $\Gamma_2 [A] \to (\neg B)$ are contained among the previous sequences of this deduction, and $\Gamma = \Gamma_1, \Gamma_2$ and $C = (\neg A)$. Because these previous deductions contain no more than n sequences then in accordance with the recursion assumption, these deductions submit to the lemma. Because C is false then A is true. Hence, some false sentence is contained in Γ because B is false or $(\neg B)$ is false in accordance with the negation definition. Therefore, in all these cases the lemma holds true.

The recursion step conclusion: If the lemma holds true for deductions containing n sequences then the lemma holds true for deduction containing $n+1$ sequences.

The recursion conclusion: Lemma holds true for all deductions $\Box$.

Def. 1.1.9 A sentence is *naturally propositionally provable* if there exists a prpositional natural deduction of this sentence from the empty list.

In accordance with Th. 1.1.3 all naturally propositionally provable sentences are true because otherwise the list would appear not empty.

But some true sentences are not naturally propositionally provable.

Alphabet of Propositional Calculations:

1. symbols p_k with natural k are called *PC-letters*;
2. symbols $\cap$, $\cup$, $\supset$, $\hat{}$ are called *PC-symbols*;
3. (,) are called *brackets*.

Formula of Propositional Calculations:

1. any PC-letter is *PC-formula*.
2. if q and r are PC-formulas then $(q\cap r)$, $(q\cup r)$,$(q\supset r)$,$(\hat{}q)$ are *PC-formula*s;
3. except listed by the two first points of this definition no *PC-formulas* are exist.

Def. 1.1.10 Let function $\mathfrak{g}$ has values on the double-elements set $\{0;1\}$ and has the set of PC-formulas as a domain. And let

1) $\mathfrak{g}(\hat{}q) = 1-\mathfrak{g}(q)$ for every sentence q;
2) $\mathfrak{g}(q\cap r) = \mathfrak{g}(q)\cdot\mathfrak{g}(r)$ for all sentences q and r;
3) $\mathfrak{g}(q\cup r) = \mathfrak{g}(q)+\mathfrak{g}(r)-\mathfrak{g}(q)\cdot\mathfrak{g}(r)$ for all sentences q and r;
4) $\mathfrak{g}(q\supset r) = 1-\mathfrak{g}(q)+\mathfrak{g}(q)\cdot\mathfrak{g}(r)$ for all sentences q and r.

In this case a function $\mathfrak{g}$ is called *a Boolean function* [3].

Hence if $\mathfrak{g}$ is a Boolean function then for every sentence q:

$$(\mathfrak{g}(q))^2 = \mathfrak{g}(q).$$

A Boolean function can be defined by a table:

$\mathfrak{g}(q)$	$\mathfrak{g}(r)$	$\mathfrak{g}(q\cap r)$	$\mathfrak{g}(q\cup r)$	$\mathfrak{g}(q\supset r)$	$\mathfrak{g}(\hat{}q)$
0	0	0	0	1	1
0	1	0	1	1	1
1	0	0	1	0	0
1	1	1	1	1	0

.

Such tables can be constructed for any sentence. For example:

[3] George Boole (2 November 1815 8 December 1864) was an English mathematician and philosopher.

$\mathfrak{g}(q)$	$\mathfrak{g}(r)$	$\mathfrak{g}(s)$	$\mathfrak{g}(\hat{}((r\cap(\hat{}s))\cap(\hat{}q)))$
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

,

or:

$\mathfrak{g}(r)$	$\mathfrak{g}(s)$	$\mathfrak{g}(q)$	$\mathfrak{g}(((r\cap s)\cap(r\supset q))\supset(q\cap s))$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

. (1.4)

Def. 1.1.11 A PC-formula q is called *a t-formula* if for any Boolean function $\mathfrak{g}$: $\mathfrak{g}(q) = 1$.

For example, formula $(((r\cap s)\cap(r\supset q))\supset(q\cap s))$ is a t-formula by the table (1.4).

Def. 1.1.12 Function $\varphi(x)$ which is defined on the PC-formulas set and which has the sentences set as a range of values, is called *an interpretation function* if the following conditions are carried out:

1. if p_k is a PC-letter then $\varphi(p_k) = A$ and here A is a simple sentence and if $\varphi(p_s) = B$ then if $s \neq k$ then $A \neq B$;

2. $\varphi(r\cap s) = (\varphi(r)\&\varphi(s))$, $\varphi(r\cup s) = (\varphi(r)\vee\varphi(s))$, $\varphi(r\supset s) = (\varphi(r)\Rightarrow\varphi(s))$, $\varphi(\hat{}r) = (\neg\varphi(r))$.

Def. 1.1.13 A sencence C is called *tautology* if the following condition is carried out:

if $\varphi(q) = C$ then q is a t-formula.

Lm. 1.1.1: If $\mathfrak{g}$ is a Boolean function then every natural propositional deduction of sequence $\Gamma\vdash A$ satisfy the following condition: if $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then there exists a sentence C such that $C\in\Gamma$ and $\mathfrak{g}\left(\varphi^{-1}(C)\right) = 0$.

Proof of Lm. 1.1.1: is maked by a recursion on a number of sequences in the deduction of $\Gamma\vdash A$:

1. Basis of recursion: Let the deduction of $\Gamma\vdash A$ contains 1 sequence.

In that case a form of this sequence is $A\vdash A$ in accordance with the propositional natural deduction definition (Def. 1.1.8). Hence in this case the lemma holds true.

2. Step of recursion: The recursion assumption: Let the lemma holds true for every deduction, containing no more than n sequences.

Let the deduction of $\Gamma\vdash A$ contains $n+1$ sequences.

In that case either this sequence is a NPC-axiom or $\Gamma \vdash A$ is obtained from previous sequences by one of deduction rules.

If $\Gamma \vdash A$ is a NPC-axiom then the proof is the same as for the recursion basis.

a) Let $\Gamma \vdash A$ be obtained from a previous sequence by $R\&$.

In that case a form of this previous sequence is either the following $\Gamma \vdash (A\&B)$ or is the following $\Gamma \vdash (B\&A)$ in accordance with the definition of deduction. The deduction of this sequence contains no more than n elements. Hence the lemma holds true for this deduction in accordance with the recursion assumption.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then $\mathfrak{g}\left(\varphi^{-1}(A\&B)\right) = 0$ and $\mathfrak{g}\left(\varphi^{-1}(B\&A)\right) = 0$ in accordance with the Boolean function definition (Def. 1.1.10). Hence there exists sentence C such that $C \in \Gamma$ and $\mathfrak{g}\left(\varphi^{-1}(C)\right) = 0$ in accordance with the lemma.

Hence in that case the lemma holds true for the deduction of sequence $\Gamma \vdash A$.

b) Let $\Gamma \vdash A$ be obtained from previous sequences by I&.

In that case forms of these previous sequences are $\Gamma_1 \vdash B$ and $\Gamma_2 \vdash G$ with $\Gamma = \Gamma_1, \Gamma_2$ and $A = (B\&G)$ in accordance with the definition of deduction.

The lemma holds true for deductions of sequences $\Gamma_1 \vdash B$ and $\Gamma_2 \vdash G$ in accordance with the recursion assumption because these deductions contain no more than n elements.

In that case if $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then $\mathfrak{g}\left(\varphi^{-1}(B)\right) = 0$ or $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 0$ in accordance with the Boolean function definition. Hence there exists sentence C such that $\mathfrak{g}\left(\varphi^{-1}(C)\right) = 0$ and $C \in \Gamma_1$ or $C \in \Gamma_2$.

Hence in that case the lemma holds true for the deduction of sequence $\Gamma \vdash A$.

c) Let $\Gamma \vdash A$ be obtained from a previous sequence by R¬.

In that case a form of this previous sequence is the following: $\Gamma \vdash (\neg(\neg A))$ in accordance with the definition of deduction. The lemma holds true for the deduction of this sequence in accordance with the recursion assumption because this deduction contains no more than n elements.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then $\mathfrak{g}\left(\varphi^{-1}(\neg(\neg A))\right) = 0$ in accordance with the Boolean function definition. Hence there exists sentence C such that $C \in \Gamma$ and $\mathfrak{g}\left(\varphi^{-1}(C)\right) = 0$.

Hence the lemma holds true for the deduction of sequence $\Gamma \vdash A$.

d) Let $\Gamma \vdash A$ be obtained from previous sequences by I¬.

In that case forms of these previous sequences are $\Gamma_1 \vdash B$ and $\Gamma_2 \vdash (\neg B)$ with $\Gamma = \Gamma_1[G], \Gamma_2[G]$ and $A = (\neg G)$ in accordance with the definition of deduction.

The lemma holds true for the deductions of sequences $\Gamma_1 \vdash B$ and $\Gamma_2 \vdash (\neg B)$ in accordance with the recursion assumption because these deductions contain no more than n elements.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 1$ in accordance with the Boolean function definition.

Either $\mathfrak{g}\left(\varphi^{-1}(B)\right) = 0$ or $\mathfrak{g}\left(\varphi^{-1}(\neg B)\right) = 0$ by the same definition. Hence there exists sentence C such that either $C \in \Gamma_1[G]$ or $C \in \Gamma_2[G]$ and$\mathfrak{g}\left(\varphi^{-1}(C)\right) = 0$ in accordance with the recursion assumption.

Hence in that case the lemma holds true for the deduction of sequence $\Gamma \vdash A$.

e) Let $\Gamma \vdash A$ be obtained from a previous sequence by I∨.

In that case a form of A is $(B \vee G)$ and a form of this previous sequence is either $\Gamma \vdash B$ or $\Gamma \vdash G$ in accordance with the definition of deduction. The lemma holds true for this previous sequence deduction in accordance with the recursion assumption because this deduction contains no more than n elements.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then $\mathfrak{g}\left(\varphi^{-1}(B)\right) = 0$ and $\mathfrak{g}\left(\varphi^{-1}(G\right) = 0$ in accordance with the Boolean function definition. Hence there exists sentence C such that $C \in \Gamma$ and $\mathfrak{g}\left(\varphi^{-1}(C)\right) = 0$.

Hence in that case the lemma holds true for the deduction of sequence $\Gamma \vdash A$.

f) Let $\Gamma \vdash A$ be obtained from previous sequences by R$\vee$.

Forms of these previous sequences are $\Gamma_1 \vdash A$, $\Gamma_2 \vdash A$, and $\Gamma_3 \vdash (B \vee G)$ with $\Gamma = \Gamma_1[B], \Gamma_2[G], \Gamma_3$ in accordance with the definition of deduction. The lemma holds true for the deductions of these sequences in accordance with the recursion assumption because these deductions contain no more than n elements.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then there exists sentence C_1 such that $C_1 \in \Gamma_1$ and $\mathfrak{g}\left(\varphi^{-1}(C_1)\right) = 0$, and there exists sentence C_2 such that $C_2 \in \Gamma_2$ and $\mathfrak{g}\left(\varphi^{-1}(C_2)\right) = 0$ in accordance with the lemma.

If $\mathfrak{g}\left(\varphi^{-1}(B \vee G)\right) = 0$ then there exists sentence C such that $C \in \Gamma_3$ and $\mathfrak{g}\left(\varphi^{-1}(C)\right) = 0$ in accordance with the lemma. Hence in that case the lemma holds true for the deduction of sequence $\Gamma \vdash A$.

If $\mathfrak{g}\left(\varphi^{-1}(B \vee G)\right) = 1$ then either $\mathfrak{g}\left(\varphi^{-1}(B)\right) = 1$ or $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 1$ in accordance with the Boolean function definition.

If $\mathfrak{g}\left(\varphi^{-1}(B)\right) = 1$ then $C_1 \in \Gamma_1[B]$. Hence in that case the lemma holds true for the deduction of sequence $\Gamma \vdash A$.

If $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 1$ then a result is the same.

Hence the lemma holds true for the deduction of sequence $\Gamma \vdash A$ in all these cases.

g) Let $\Gamma \vdash A$ be obtained from previous sequences by R$\Rightarrow$.

Forms of these previous sequences are $\Gamma_1 \vdash (B \Rightarrow A)$ and $\Gamma_2 \vdash (B)$ with $\Gamma = \Gamma_1, \Gamma_2$ in accordance with the definitions of deduction. Hence the lemma holds true for these deduction in accordance with the recursion assumption because these deductions contain no more than n elements.

If $\mathfrak{g}\left(\varphi^{-1}(B \Rightarrow A)\right) = 0$ then there exists sentence C such that $C \in \Gamma_1$ and $\mathfrak{g}\left(\varphi^{-1}(C)\right) = 0$ in accordance with the lemma. Hence in that case the lemma holds true for the deduction of sequence $\Gamma \vdash A$.

If $\mathfrak{g}\left(\varphi^{-1}(B \Rightarrow A)\right) = 1$ then $\mathfrak{g}\left(\varphi^{-1}(B)\right) = 0$ in accordance with the Boolean function definition. Hence there exists sentence C such that $C \in \Gamma_2$ and $\mathfrak{g}\left(\varphi^{-1}(C)\right) = 0$.

Hence the lemma holds true for sequence $\Gamma \vdash A$ in all these cases.

h) Let $\Gamma \vdash A$ be obtained from a previous sequence by I$\Rightarrow$.

In that case a form of sentence A is $(B \Rightarrow G)$ and a form of this previous sequence is $\Gamma_1 \vdash G$ with $\Gamma = \Gamma_1[B]$ in accordance with the definition of deduction. The lemma holds true for the deduction of this sequence in accordance the recursion assumption because this deduction contain no more than n elements.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 0$ and $\mathfrak{g}\left(\varphi^{-1}(B)\right) = 1$ in accordance with the Boolean function definition. Hence there exists sentence C such that $C \in \Gamma_1[B]$ and $\mathfrak{g}\left(\varphi^{-1}(C)\right) = 0$.

The recursion step conclusion: Therefore, in each possible case, if the lemma holds true for a deduction, containing no more than n elements, then the lemma holds true for a deduction contained $n+1$ elements.

The recursion conclusion: Therefore the lemma holds true for a deduction of any length □

Th. 1.1.4: Each naturally propositionally proven sentence is a tautology.

Proof of Th. 1.1.4: If a sentence A is naturally propositionally proven then there exists a natural propositional deduction of form $\vdash A$ in accordance with Def. 1.1.9. Hence for every Boolean function g: $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 1$ in accordance with Lm. 1.1.1. Hence sentence A is a tautology in accordance with the tautology definition (Def. 1.1.13) □

Designation 1: Let $\mathfrak{g}$ be a Boolean function. In that case for every sentence A:

$$A^{\mathfrak{g}} := \left\{ \begin{array}{c} A \text{ if } \mathfrak{g}\left(\varphi^{-1}(A)\right) = 1, \\ (\neg A) \text{ if } \mathfrak{g}\left(\varphi^{-1}(A)\right) = 0. \end{array} \right|$$

Lm. 1.1.2: Let $B_1, B_2, .., B_k$ be the simple sentences making sentence A by PC-symbols ($\neg$, &, $\vee$, $\Rightarrow$).

Let $\mathfrak{g}$ be any Boolean function.

In that case there exist a propositional natural deduction of sequence

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A^{\mathfrak{g}}.$$

Proof of Lm. 1.1.2: is received by a recursion on a number of PC-symbols in sentence A.

Basis of recursion Let A does not contain PC-symbols . In this case the string of one sequence:

1. $A^{\mathfrak{g}} \vdash A^{\mathfrak{g}}$, NPC-axiom.

is a fit deduction.

Step of recursion: The recursion assumption: Let the lemma holds true for every sentence, containing no more than n PC-symbols.

Let sentence A contains $n+1$ PC-symbol. Let us consider all possible cases.

a) Let $A = (\neg G)$. In that case the lemma holds true for G in accordance with the recursion assumption because G contains no more than n PC-symbols. Hence there exists a deduction of sequence

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash G^{\mathfrak{g}}, \tag{1.5}$$

here $B_1, B_2, .., B_k$ are the simple sentences, making up sentence G. Hence $B_1, B_2, .., B_k$ make up sentence A.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 1$ then

$$A^{\mathfrak{g}} = A = (\neg G)$$

in accordance with Designation 1.

In that case $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 0$ in accordance with the Boolean function definition.

Hence

$$G^{\mathfrak{g}} = (\neg G) = A$$

in accordance with Designation 1.

Hence in that case a form of sequence (1.5) is the following:

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A^{\mathfrak{g}}.$$

Hence in that case the lemma holds true.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then

$$A^{\mathfrak{g}} = (\neg A) = (\neg(\neg G)).$$

in accordance with Designation 1.

In that case $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 1$ in accordance with the Boolean function definition.

Hence

$$G^{\mathfrak{g}} = G$$

in accordance with Designation 1.

Hence in that case a form of sequence (1.5) is

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash G.$$

Let us continue the deduction of this sequence in the following way:

1. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash G$.
2. $(\neg G) \vdash (\neg G)$, NPC-axiom.
3. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg(\neg G))$, I$\neg$ from 1. and 2.

It is a deduction of sequence

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A^{\mathfrak{g}}.$$

Hence in that case the lemma holds true.

b) Let $A = (G \& R)$.

In that case the lemma holds true both for G and for R in accordance with the recursion assumption because G and R contain no more than n PC-symbols. Hence there exist deductions of sequences

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash G^{\mathfrak{g}} \tag{1.6}$$

and

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash R^{\mathfrak{g}}, \tag{1.7}$$

here $B_1, B_2, .., B_k$ are the simple sentences, making up sentences G and R. Hence $B_1, B_2, .., B_k$ make up sentence A.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 1$ then

$$A^{\mathfrak{g}} = A = (G \& R)$$

in accordance with Designation 1.

In that case $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 1$ and $\mathfrak{g}\left(\varphi^{-1}(R)\right) = 1$ in accordance with the Boolean function definition.

Hence $G^{\mathfrak{g}} = G$ and $R^{\mathfrak{g}} = R$ in accordance with Designation 1.

Let us continue deductions of sequences (1.6) and (1.7) in the following way:

1. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash G$, (1.6).
2. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash R$, (1.7).
3. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (G\&R)$, I& from 1. and 2.

It is deduction of sequence $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A^{\mathfrak{g}}$.

Hence in that case the lemma holds true.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then

$$A^{\mathfrak{g}} = (\neg A) = (\neg (G\&R))$$

in accordance with Designation 1.

In that case $\mathfrak{g}(G) = 0$ or $\mathfrak{g}(R) = 0$ in accordance with the Boolean function definition.

Hence $G^{\mathfrak{g}} = (\neg G)$ or $R^{\mathfrak{g}} = (\neg R)$ in accordance with Designation 1.

Let $G^{\mathfrak{g}} = (\neg G)$.

In that case let us continue a deduction of sequence (1.6) in the following way:

1. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg G)$, (1.6).
2. $(G\&R) \vdash (G\&R)$, NPC-axiom.
3. $(G\&R) \vdash G$, R& from 2.
4. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg (G\&R))$, I¬ from 1. and 3.

It is a deduction of sequence $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A^{\mathfrak{g}}$.

Hence in that case the lemma holds true.

The same result is received if $R^{\mathfrak{g}} = (\neg R)$.

c) Let $A = (G \vee R)$.

In that case the lemma holds true both for G and for R in accordance with the recursion assumption because G and R contain no more than n PC-symbols. Hence there exist a deductions of sequences

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash G^{\mathfrak{g}} \tag{1.8}$$

and

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash R^{\mathfrak{g}}, \tag{1.9}$$

here $B_1, B_2, .., B_k$ are the simple sentences, making up sentences G and R. Hence $B_1, B_2, .., B_k$ make up sentence A.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then

$$A^{\mathfrak{g}} = (\neg A) = (\neg (G \vee R))$$

in accordance with Designation 1.

In that case $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 0$ and $\mathfrak{g}\left(\varphi^{-1}(R)\right) = 0$ in accordance with the Boolean function definition.

Hence $G^{\mathfrak{g}} = (\neg G)$ and $R^{\mathfrak{g}} = (\neg R)$ in accordance with Designation 1.

Let us continue deductions of sequences (1.8) and (1.9) in the following way:
1. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg G)$, (1.8).
2. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg R)$, (1.9).
3. $G \vdash G$, NPC-axiom.
4. $R \vdash R$, NPC-axiom.
5. $(G \vee R) \vdash (G \vee R)$, NPC-axiom.
6. $G, B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg (G \vee R))$, I$\neg$ from 1. and 3.
7. $R, B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg (G \vee R))$, I$\neg$ from 2. and 4.
8. $(G \vee R), B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg (G \vee R))$, R$\vee$ from 5., 6., and 7.
9. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg (G \vee R))$, I$\neg$ from 7. and 8.

It is a deduction of sequence $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A^{\mathfrak{g}}$.

Hence in that case the lemma holds true.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 1$ then

$$A^{\mathfrak{g}} = A = (G \vee R)$$

in accordance with Designation 1.

In that case $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 1$ or $\mathfrak{g}\left(\varphi^{-1}(R)\right) = 1$ in accordance with the Boolean function definition.

Hence $G^{\mathfrak{g}} = G$ or $R^{\mathfrak{g}} = R$ in accordance with Designation 1.

If $G^{\mathfrak{g}} = G$ then let us continue deduction of sequence (1.8) in the following way:
1. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash G$, (1.8).
2. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (G \vee R)$, I$\vee$ from 1.

It is deduction of sequence $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A^{\mathfrak{g}}$.

Hence in that case the lemma holds true.

The same result is received if $R^{\mathfrak{g}} = R$.

d) Let $A = (G \Rightarrow R)$.

In that case the lemma holds true both for G and for R in accordance with the recursion assumption because G and R contain no more than n PC-symbols. Hence there exist deductions of sequences

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash G^{\mathfrak{g}} \tag{1.10}$$

and

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash R^{\mathfrak{g}}, \tag{1.11}$$

here $B_1, B_2, .., B_k$ are the simple sentence, making up sentences G and R. Hence $B_1, B_2, .., B_k$ make up sentence A.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 0$ then

$$A^{\mathfrak{g}} = (\neg A) = (\neg (G \Rightarrow R))$$

in accordance with Designation 1.

In that case $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 1$ and $\mathfrak{g}\left(\varphi^{-1}(R)\right) = 0$ in accordance with the Boolean function deduction.

Hence $G^{\mathfrak{g}} = G$ and $R^{\mathfrak{g}} = (\neg R)$ in accordance with Designation 1.

Let us continue deduction of sequences (1.10) and (1.11) in the following way:

1. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash G$, (1.10).
2. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg R)$, (1.11).
3. $(G \Rightarrow R) \vdash (G \Rightarrow R)$, NPC-axiom.
4. $(G \Rightarrow R), B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash R$, R$\Rightarrow$ from 1. and 3.
5. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg (G \Rightarrow R))$, I$\neg$ from 2. and 4.

It is deduction of sequence $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A^{\mathfrak{g}}$.

Hence in that case the lemma holds true.

If $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 1$ then

$$A^{\mathfrak{g}} = A = (G \Rightarrow R)$$

in accordance with Designation 1.

In that case $\mathfrak{g}\left(\varphi^{-1}(G)\right) = 0$ or $\mathfrak{g}\left(\varphi^{-1}(R)\right) = 1$ in accordance with the Boolean function definition.

Hence $G^{\mathfrak{g}} = (\neg G)$ or $R^{\mathfrak{g}} = R$ in accordance with Designation 1.

If $G^{\mathfrak{g}} = (\neg G)$ then let us continue a deduction of sequence (1.10) in the following way:

1. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg G)$, (1.10).
2. $G \vdash G$, NPC-axiom.
3. $G, B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (\neg (\neg R))$, I$\neg$ from 1. and 2.
4. $G, B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash R$, R$\neg$ from 3.
5. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (G \Rightarrow R)$, I$\Rightarrow$ from 4.

It is deduction of sequence $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A^{\mathfrak{g}}$.

Hence in that case the lemma holds true.

If $R^{\mathfrak{g}} = R$ then let us continue a deduction of sequence (1.11) in the following way:

1. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash R$, (1.11).
2. $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash (G \Rightarrow R)$, I$\Rightarrow$ from 1.

It is deduction of sequence $B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A^{\mathfrak{g}}$.

Hence in that case the lemma holds true.

The recursion step conclusion: If the lemma holds true for sentences, containing no more than n PC-symbols, then the lemma holds true for sentences, containing $n+1$ PC-symbols.

The recursion conclusion: The lemma holds true for sentences, containing any number PC-symbols □

Th. 1.1.5 (Laszlo Kalmar)[4]: Each tautology is a naturally propositionally proven sentence.

Proof of Th. 1.1.5: Let sentence A be a tautology. That is for every Boolean function $\mathfrak{g}$: $\mathfrak{g}\left(\varphi^{-1}(A)\right) = 1$ in accordance with Def. 1.1.13.

Hence there exists a deduction for sequence

$$B_1^{\mathfrak{g}}, B_2^{\mathfrak{g}}, .., B_k^{\mathfrak{g}} \vdash A \tag{1.12}$$

[4] Laszlo Kalmar (March 27, 1905 August 2, 1976) was a Hungarian mathematician and Professor at the University of Szeged. Kalmar is considered the founder of mathematical logic and theoretical Computer Science in Hungary.

for every Boolean function $\mathfrak{g}$ in accordance with Lm. 1.1.2.

There exist Boolean functions $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that

$$\mathfrak{g}_1\left(\varphi^{-1}(B_1)\right)=0,\ \mathfrak{g}_2\left(\varphi^{-1}(B_1)\right)=1,$$
$$\mathfrak{g}_1\left(\varphi^{-1}(B_s)\right)=\mathfrak{g}_2\left(\varphi^{-1}(B_s)\right)\ \text{for } s\in\{2,..,k\}.$$

Forms of sequences (1.12) for these Boolean functions are the following:

$$(\neg B_1), B_2^{\mathfrak{g}_1}, .., B_k^{\mathfrak{g}_1} \vdash A, \tag{1.13}$$

$$B_1, B_2^{\mathfrak{g}_2}, .., B_k^{\mathfrak{g}_2} \vdash A. \tag{1.14}$$

Let us continue deductions these sequence in the following way:

1. $(\neg B_1), B_2^{g_1}, .., B_k^{g_1} \vdash A$, (1.13).
2. $B_1, B_2^{g_1}, .., B_k^{g_1} \vdash A$, (1.14).
3. $(\neg A) \vdash (\neg A)$, NPC-axiom.
4. $(\neg A), B_2^{g_1}, .., B_k^{g_1} \vdash (\neg(\neg B_1))$, I$\neg$ from 1. and 3.
5. $(\neg A), B_2^{g_1}, .., B_k^{g_1} \vdash (\neg B_1)$, I$\neg$ from 2. and 3.
6. $B_2^{g_1}, .., B_k^{g_1} \vdash (\neg(\neg A))$, I$\neg$ from 4. and 5.
7. $B_2^{g_1}, .., B_k^{g_1} \vdash A$, R$\neg$ from 6.

It is deduction of sequence $B_2^{g_1}, .., B_k^{g_1} \vdash A$. This sequence is obtained from sequence (1.12) by deletion of first sentence from the hypothesizes list.

All rest hypothesizes are deleted from this list in the similar way.

Final sentence is the following:

$$\vdash A.$$

□

Therefore, in accordance with Th. 1.1.3, all tautologies are true sentences.

Therefore the natural propositional logic presents by Boolean functions.

1.2. Recorders

Any information, received from physical devices, can be expressed by a text, made of sentences.

Let $\widehat{\mathbf{a}}$ be some object which is able to receive, save, and/or transmit an information [10]. A set $\mathbf{a}$ of sentences, expressing an information of an object $\widehat{\mathbf{a}}$, is called *a recorder* of this object. Thus, statement: "Sentence ≪A≫ is an element of the set $\mathbf{a}$" denotes : "$\widehat{\mathbf{a}}$ has information that the event, expressed by sentence $\ll A \gg$, took place." In short: "$\widehat{\mathbf{a}}$ knows that A." Or by designation: "$\mathbf{a}^\bullet \ll A \gg$".

Obviously, the following conditions are satisfied:

I. For any $\mathbf{a}$ and for every A: false is that $\mathbf{a}^\bullet(A\&(\neg A))$, thus, any recorder doesn't contain a logical contradiction.

II. For every $\mathbf{a}$, every B, and all A: if B is a logical consequence from A, and $\mathbf{a}^\bullet A$, then $\mathbf{a}^\bullet B$.

*III. For all $\mathbf{a}$, $\mathbf{b}$ and for every A: if $\mathbf{a}^\bullet \ll \mathbf{b}^\bullet A \gg$ then $\mathbf{a}^\bullet A$.

For example, if device $\widehat{\mathbf{a}}$ has information that device $\widehat{\mathbf{b}}$ has information that mass of particle $\overleftarrow{\chi}$ equals to 7 then device $\widehat{\mathbf{a}}$ has information that mass of particle $\overleftarrow{\chi}$ equal to 7.

1.3. Time

Let's consider finite (probably empty) path of symbols of form $\mathbf{q}^{\bullet}$.

Def. 1.3.1 A path α is *a subpath* of a path β (design.: $\alpha \prec \beta$) if α can be got from β by deletion of some (probably all) elements.

Designation: $(\alpha)^1$ is α, and $(\alpha)^{k+1}$ is $\alpha(\alpha)^k$.

Therefore, if $k \leq l$ then $(\alpha)^k \prec (\alpha)^l$.

Def. 1.3.2 A path α is *equivalent* to a path β (design.: $\alpha \sim \beta$) if α can be got from β by substitution of a subpath of form $(\mathbf{a}^{\bullet})^k$ by a path of the same form $((\mathbf{a}^{\bullet})^s)$.

In this case:

III. If $\beta \prec \alpha$ or $\beta \sim \alpha$ then for any K:

if $\mathbf{a}^{\bullet}K$ then $\mathbf{a}^{\bullet}(K\&(\alpha A \Rightarrow \beta A))$.

Obviously, III is a refinement of condition *III.

Def. 1.3.3 A natural number q is *instant*, at which a registrates B according to κ-*clock* $\{\mathbf{g}_0, A, \mathbf{b}_0\}$ (design.: q is $[\mathbf{a}^{\bullet}B \uparrow \mathbf{a}, \{\mathbf{g}_0, A, \mathbf{b}_0\}]$) if:

1. for any K: if $\mathbf{a}^{\bullet}K$ then

$$\mathbf{a}^{\bullet}(K\&(\mathbf{a}^{\bullet}B \Rightarrow \mathbf{a}^{\bullet}(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet})^q\,\mathbf{g}_0^{\bullet}A))$$

and

$$\mathbf{a}^{\bullet}\left(K\&\left(\mathbf{a}^{\bullet}(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet})^{q+1}\,\mathbf{g}_0^{\bullet}A \Rightarrow \mathbf{a}^{\bullet}B\right)\right).$$

2. $\mathbf{a}^{\bullet}\left(\mathbf{a}^{\bullet}B\&\left(\neg\mathbf{a}^{\bullet}(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet})^{q+1}\,\mathbf{g}_0^{\bullet}A\right)\right)$.

Lm. 1.3.1 If

$$q \text{ is } [\mathbf{a}^{\bullet}\alpha B \uparrow \mathbf{a}, \{\mathbf{g}_0, A, \mathbf{b}_0\}], \tag{1.15}$$

$$p \text{ is } [\mathbf{a}^{\bullet}\beta B \uparrow \mathbf{a}, \{\mathbf{g}_0, A, \mathbf{b}_0\}], \tag{1.16}$$

$$\alpha \prec \beta, \tag{1.17}$$

then

$$q \leq p.$$

Proof of Lm. 1.3.1: From (1.16):

$$\mathbf{a}^{\bullet}\left((\mathbf{a}^{\bullet}\beta B)\&\left(\neg\mathbf{a}^{\bullet}(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet})^{(p+1)}\,\mathbf{g}_0^{\bullet}A\right)\right). \tag{1.18}$$

From (1.17) according to III:

$$\mathbf{a}^{\bullet}\left(\left(\mathbf{a}^{\bullet}\beta B\&\left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{(p+1)}\mathbf{g}_0^{\bullet}A\right)\right)\&\left(\mathbf{a}^{\bullet}\beta B\Rightarrow\mathbf{a}^{\bullet}\alpha B\right)\right). \qquad (1.19)$$

Let us designate:
$R := \mathbf{a}^{\bullet}\beta B$,
$S := \left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{(p+1)}\mathbf{g}_0^{\bullet}A\right)$,
$G := \mathbf{a}^{\bullet}\alpha B$.
In that case a shape of formula (1.18) is

$$\mathbf{a}^{\bullet}(R\&S),$$

and a shape of formula (1.19) is

$$\mathbf{a}^{\bullet}((R\&S)\&(R\Rightarrow G)).$$

Sentence $(G\&S)$ is a logical consequence from sentence $((R\&S)\&(R\Rightarrow G))$ (1.1). Hence

$$\mathbf{a}^{\bullet}(G\&S),$$

in accordance with II.
Hence

$$\mathbf{a}^{\bullet}\left(\mathbf{a}^{\bullet}\alpha B\&\left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{(p+1)}\mathbf{g}_0^{\bullet}A\right)\right)$$

in accordance with the designation.
Hence from (1.15):

$$\mathbf{a}^{\bullet}\left(\left(\mathbf{a}^{\bullet}\alpha B\&\left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{(p+1)}\mathbf{g}_0^{\bullet}A\right)\right)\&\left(\mathbf{a}^{\bullet}\alpha B\Rightarrow\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{q}\mathbf{g}_0^{\bullet}A\right)\right).$$

According to II:

$$\mathbf{a}^{\bullet}\left(\left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{(p+1)}\mathbf{g}_0^{\bullet}A\right)\&\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{q}\mathbf{g}_0^{\bullet}A\right) \qquad (1.20)$$

If $q > p$, i.e. $q \geq p+1$, then from (1.20) according to III

$$\mathbf{a}^{\bullet}\left(\begin{array}{c}\left(\left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{(p+1)}\mathbf{g}_0^{\bullet}A\right)\&\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{q}\mathbf{g}_0^{\bullet}A\right)\&\\\left(\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{q}\mathbf{g}_0^{\bullet}A\Rightarrow\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{(p+1)}\mathbf{g}_0^{\bullet}A\right)\end{array}\right).$$

According to II:

$$\mathbf{a}^{\bullet}\left(\left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{(p+1)}\mathbf{g}_0^{\bullet}A\right)\&\mathbf{a}^{\bullet}\left(\mathbf{g}_0^{\bullet}\mathbf{b}_0^{\bullet}\right)^{(p+1)}\mathbf{g}_0^{\bullet}A\right).$$

It contradicts to condition I. Therefore, $q \leq p$ □.
Lemma 1.3.1 proves that if

$$q \text{ is } \left[\mathbf{a}^{\bullet}B\uparrow\mathbf{a},\{\mathbf{g}_0,A,\mathbf{b}_0\}\right],$$

and

$$p \text{ is } [\mathbf{a}^{\bullet} B \uparrow \mathbf{a}, \{\mathbf{g}_0, A, \mathbf{b}_0\}]$$

then

$$q = p.$$

That's why, expression "q is $[\mathbf{a}^{\bullet} B \uparrow \mathbf{a}, \{\mathbf{g}_0, A, \mathbf{b}_0\}]$" is equivalent to expression "$q = [\mathbf{a}^{\bullet} B \uparrow \mathbf{a}, \{\mathbf{g}_0, A, \mathbf{b}_0\}]$."

Def. 1.3.4 κ-clocks $\{\mathbf{g}_1, B, \mathbf{b}_1\}$ and $\{\mathbf{g}_2, B, \mathbf{b}_2\}$ have *the same direction* for **a** if the following condition is satisfied:

if

$$r = [\mathbf{a}^{\bullet} (\mathbf{g}_1^{\bullet} \mathbf{b}_1^{\bullet})^q \mathbf{g}_1^{\bullet} B \uparrow \mathbf{a}, \{\mathbf{g}_2, B, \mathbf{b}_2\}],$$
$$s = [\mathbf{a}^{\bullet} (\mathbf{g}_1^{\bullet} \mathbf{b}_1^{\bullet})^p \mathbf{g}_1^{\bullet} B \uparrow \mathbf{a}, \{\mathbf{g}_2, B, \mathbf{b}_2\}],$$
$$q < p,$$

then

$$r \leq s.$$

Th. 1.3.1 All κ-clocks have the same direction.

Proof of Th. 1.3.1:

Let

$$r := [\mathbf{a}^{\bullet} (\mathbf{g}_1^{\bullet} \mathbf{b}_1^{\bullet})^q \mathbf{g}_1^{\bullet} B \uparrow \mathbf{a}, \{\mathbf{g}_2, B, \mathbf{b}_2\}],$$
$$s := [\mathbf{a}^{\bullet} (\mathbf{g}_1^{\bullet} \mathbf{b}_1^{\bullet})^p \mathbf{g}_1^{\bullet} B \uparrow \mathbf{a}, \{\mathbf{g}_2, B, \mathbf{b}_2\}],$$

$$q < p.$$

In this case

$$(\mathbf{g}_1^{\bullet} \mathbf{b}_1^{\bullet})^q \prec (\mathbf{g}_1^{\bullet} \mathbf{b}_1^{\bullet})^p.$$

Consequently, according to Lm. 1.3.1

$$r \leq s$$

□

Consequently, a recorder orders its sentences with respect to instants. Moreover, this order is linear and it doesn't matter according to which κ-clock it is set.

Def. 1.3.5 κ-clock $\{\mathbf{g}_2, B, \mathbf{b}_2\}$ is k times more precise than κ-clock $\{\mathbf{g}_1, B, \mathbf{b}_1\}$ for recorder **a** if for every C the following condition is satisfied: if

$$q_1 = [\mathbf{a}^{\bullet} C \uparrow \mathbf{a}, \{\mathbf{g}_1, B, \mathbf{b}_1\}],$$
$$q_2 = [\mathbf{a}^{\bullet} C \uparrow \mathbf{a}, \{\mathbf{g}_2, B, \mathbf{b}_2\}],$$

then

$$q_1 < \tfrac{q_2}{k} < q_1 + 1.$$

Lm. 1.3.2 If for every n:

$$q_{n-1} < \frac{q_n}{k_n} < q_{n-1} + 1, \tag{1.21}$$

then the series

$$q_0 + \sum_{n=1}^{\infty} \frac{q_n - q_{n-1}k_n}{k_1 \ldots k_n} \tag{1.22}$$

converges.
Proof of Lm. 1.3.2: According to (1.21):

$$0 \leq q_n - q_{n-1}k_n < k_n.$$

Consequently, series (1.22) is positive and majorizes next to

$$q_0 + 1 + \sum_{n=1}^{\infty} \frac{1}{k_1 \ldots k_n},$$

convergence of which is checked by d'Alambert's criterion □
Def. 1.3.6 A sequence $\widetilde{H}$ of κ-clocks:

$$\left\langle \{\mathbf{g}_0, A, \mathbf{b}_0\},\ \{\mathbf{g}_1, A, \mathbf{b}_2\}, .., \{\mathbf{g}_j, A, \mathbf{b}_j\},\ ..\ \right\rangle$$

is called *an absolutely precise* κ-clock of a recorder **a** if for every j exists a natural number k_j so that κ-clock$\{\mathbf{g}_j, A, \mathbf{b}_j\}$ is k_j times more precise than κ-clock$\{\mathbf{g}_{j-1}, A, \mathbf{b}_{j-1}\}$.
In this case if

$$q_j = \left[\mathbf{a}^{\bullet}C \uparrow \mathbf{a}, \{\mathbf{g}_j, A, \mathbf{b}_j\}\right]$$

and

$$t = q_0 + \sum_{j=1}^{\infty} \frac{q_j - q_{j-1} \cdot k_j}{k_1 \cdot k_2 \cdot .. \cdot k_j},$$

then

$$t \text{ is } \left[\mathbf{a}^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right].$$

Lm. 1.3.3: If

$$q := q_0 + \sum_{j=1}^{\infty} \frac{q_j - q_{j-1} \cdot k_j}{k_1 \cdot k_2 \cdot .. \cdot k_j} \tag{1.23}$$

with

$$q_{n-1} \leq \frac{q_n}{k_n} < q_{n-1} + 1,$$

and

$$d := d_0 + \sum_{j=1}^{\infty} \frac{d_j - d_{j-1} \cdot k_j}{k_1 \cdot k_2 \cdot .. \cdot k_j} \tag{1.24}$$

with

$$d_{n-1} \leq \frac{d_n}{k_n} < d_{n-1} + 1$$

then if $q_n \leq d_n$ then $q \leq d$.
Proof of Lm. 1.3.3: A partial sum of series (1.23) is the following:

$$Q_u := q_0 + \frac{q_1 - q_0 k_1}{k_1} + \frac{q_2 - q_1 k_2}{k_1 k_2} + \cdots + \frac{q_u - q_{u-1} k_u}{k_1 k_2 \cdots k_u},$$
$$Q_u = q_0 + \frac{q_1}{k_1} - q_0 + \frac{q_2}{k_1 k_2} - \frac{q_1}{k_1} + \cdots + \frac{q_u}{k_1 k_2 \cdots k_u} - \frac{q_{u-1}}{k_1 k_2 \cdots k_{u-1}},$$

$$Q_u = \frac{q_u}{k_1 k_2 \cdots k_u}.$$

A partial sum of series (1.24) is the following:

$$D_u = \frac{d_u}{k_1 k_2 \cdots k_u}.$$

Consequently, according to the condition of Lemma: $Q_n \leq D_n$ □
Lm. 1.3.4 If

$$q \text{ is } \left[\mathbf{a}^{\bullet} \alpha C \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$d \text{ is } \left[\mathbf{a}^{\bullet} \beta C \uparrow \mathbf{a}, \widetilde{H}\right],$$

and

$$\alpha \prec \beta$$

then

$$q \leq d.$$

Proof of Lm. 1.3.4 comes out of Lemmas 1.3.1 and 1.3.3 immediately □
Therefore, if $\alpha \sim \beta$ then $q = d$.

1.4. Space

Def. 1.4.1 A number t is called *a time, measured by a recorder* **a** *according to a* κ*-clock* $\widetilde{H}$, *during which a signal C did a path* $\mathbf{a}^{\bullet}\alpha\mathbf{a}^{\bullet}$ (design.:

$$t := \mathfrak{m}\left(\mathbf{a}\widetilde{H}\right)(\mathbf{a}^{\bullet}\alpha\mathbf{a}^{\bullet}C)),$$

if

$$t = \left[\mathbf{a}^{\bullet}\alpha\mathbf{a}^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right] - \left[\mathbf{a}^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right].$$

Th. 1.4.1

$$\mathfrak{m}\left(\mathbf{a}\widetilde{H}\right)(\mathbf{a}^{\bullet}\alpha\mathbf{a}^{\bullet}C) \geq 0.$$

Proof comes out straight of Lemma 1.3.4 □

Thus, any ”signal”, ”sent” by the recorder, ”will come back” to it not earlier than it was ”sent”.

Def. 1.4.2

1) for every recorder **a**: $(\mathbf{a}^{\bullet})^{\dagger} = (\mathbf{a}^{\bullet})$;

2) for all paths α and β: $(\alpha\beta)^{\dagger} = (\beta)^{\dagger}(\alpha)^{\dagger}$.

Def. 1.4.3 A set $\mathfrak{R}$ of recorders is *an internally stationary system* for a recorder **a** with a κ-clock $\widetilde{H}$ (design.: $\mathfrak{R}$ is $ISS\left(\mathbf{a}, \widetilde{H}\right)$) if for all sentences B and C, for all elements $\mathbf{a}_1$ and $\mathbf{a}_2$ of set $\mathfrak{R}$, and for all paths α, made of elements of set $\mathfrak{R}$, the following conditions are satisfied:

1) $\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right] - \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right] =$
$= \left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right] - \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right]$;

2) $\mathfrak{m}\left(\mathbf{a}\widetilde{H}\right)(\mathbf{a}^{\bullet}\alpha\mathbf{a}^{\bullet}C) = \mathfrak{m}\left(\mathbf{a}\widetilde{H}\right)\left(\mathbf{a}^{\bullet}\alpha^{\dagger}\mathbf{a}^{\bullet}C\right)$.

Th. 1.4.2

$$\{\mathbf{a}\} - ISS\left(\mathbf{a}, \widetilde{H}\right).$$

Proof:

1)As $\mathbf{a}^{\bullet} \sim \mathbf{a}^{\bullet}\mathbf{a}^{\bullet}$ then, according to Lemma 1.3.4 : if we symbolize

$$p := \left[\mathbf{a}^{\bullet}\mathbf{a}^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$q := \left[\mathbf{a}^{\bullet}\mathbf{a}^{\bullet}\mathbf{a}^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$r := \left[\mathbf{a}^{\bullet}\mathbf{a}^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$s := \left[\mathbf{a}^{\bullet}\mathbf{a}^{\bullet}\mathbf{a}^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right],$$

then $q = p$ and $s = r$.

That’s why $q - p = s - r$.

2) Since any series α, made of elements of set $\{\mathbf{a}\}$ coincides with $\alpha^{\dagger}$ then

$$\mathfrak{m}\left(\mathbf{a}\widetilde{H}\right)\left(\mathbf{a}^{\bullet}\alpha\mathbf{a}^{\bullet}C\right)=\mathfrak{m}\left(\mathbf{a}\widetilde{H}\right)\left(\mathbf{a}^{\bullet}\alpha^{\dagger}\mathbf{a}^{\bullet}C\right).\ \square$$

Thus every singleton is an internally stationary systeminternally stationary system.
Lm. 1.4.1: If $\{\mathbf{a},\mathbf{a}_1,\mathbf{a}_2\}$ is$ISS\left(\mathbf{a},\widetilde{H}\right)$ then

$$\begin{aligned}&\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right]-\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right]=\\&=\left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right]-\left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right]\end{aligned}$$

Proof: Let's symbolize

$$\begin{aligned}p&:=\left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right],\\q&:=\left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right],\\r&:=\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right],\\s&:=\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right],\\u&:=\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right],\\w&:=\left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right].\end{aligned}$$

Thus, according to statement 1.4.3

$$u-p=s-w,w-r=q-u.$$

Thus,

$$s-r=q-p$$

$\square$

Def. 1.4.4 A number l is called *an* $\mathbf{a}\widetilde{H}(B)$*-measure* of recorders $\mathbf{a}_1$ and $\mathbf{a}_2$ (design.:

$$l=\ell\left(\mathbf{a},\widetilde{H},B\right)(\mathbf{a}_1,\mathbf{a}_2)$$

if

$$l=0.5\cdot\left(\left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right]-\left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right]\right).$$

Lm. 1.4.2 If $\{\mathbf{a},\mathbf{a}_1,\mathbf{a}_2\}$ is $ISS\left(\mathbf{a},\widetilde{H}\right)$ then for all B and C:

$$\ell\left(\mathbf{a},\widetilde{H},B\right)(\mathbf{a}_1,\mathbf{a}_2)=\ell\left(\mathbf{a},\widetilde{H},C\right)(\mathbf{a}_1,\mathbf{a}_2).$$

Proof: Let us designate: Let us design:

$$p := \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$q := \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$r := \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$s := \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$u := \left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$w := \left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right].$$

Thus, according to Def. 1.4.3:

$$u - p = w - r, q - u = s - w.$$

Thus,

$$q - p = s - r$$

□

Therefore, one can write expression of form "$\ell\left(\mathbf{a}, \widetilde{H}, B\right)(\mathbf{a}_1, \mathbf{a}_2)$" as the following: "$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_2)$".

Th. 1.4.3: If $\{\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is $ISS\left(\mathbf{a}, \widetilde{H}\right)$ then

1) $\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_2) \geq 0$;

2) $\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_1) = 0$;

3) $\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_2) = \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_2, \mathbf{a}_1)$;

4) $\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_2) + \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_2, \mathbf{a}_3) \geq \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_3)$.

Proof: 1) and 2) come out straight from Lemma 1.3.4 and 3) from Lemma 1.4.2.

Let's symbolize

$$p := \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$q := \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$r := \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_3^{\bullet}\mathbf{a}_1^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$s := \left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$u := \left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_3^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right],$$
$$w = \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_3^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}C \uparrow \mathbf{a}, \widetilde{H}\right].$$

Thus, according to statement 1.4.3

$$w-u=q-s.$$

Therefore,

$$w-p=(q-p)+(u-s).$$

According to Lemma 1.3.4

$$w\geq r.$$

Consequently,

$$(q-p)+(u-s)\geq r-p$$

□

Thus, all four axioms of the metrical space [7] are accomplished for $\ell\left(\mathbf{a},\widetilde{H}\right)$ in an internally stationary systeminternally stationary system of recorders.

Consequently, $\ell\left(\mathbf{a},\widetilde{H}\right)$ is a distance length similitude in this space.

Def. 1.4.5 A set $\mathfrak{R}$ of recorders is *degenerated into a beam* $\mathbf{ab}_1$ *and point* $\mathbf{a}_1$ if there exists C such that the following conditions are satisfied:

1) For any sequence α, made of elements of set $\mathfrak{R}$, and for any K: if $\mathbf{a}^\bullet K$ then

$$\mathbf{a}^\bullet\left(K\&\left(\alpha\mathbf{a}_1^\bullet C\Rightarrow\alpha\mathbf{b}_1^\bullet\mathbf{a}_1^\bullet C\right)\right).$$

2) There is sequence β, made of elements of the set$\mathfrak{R}$, and there exist sentence S such that $\mathbf{a}^\bullet\left(\beta\mathbf{b}_1^\bullet C\&S\right)$ and it's false that $\mathbf{a}^\bullet\left(\beta\mathbf{a}_1^\bullet\mathbf{b}_1^\bullet C\&S\right)$

Further we'll consider only not degenerated sets of recorders.

Def. 1.4.6: *B* took place *in the same place as* $\mathbf{a}_1$ *for* $\mathbf{a}$ (design.: $\natural\left(\mathbf{a}\right)\left(\mathbf{a}_1,B\right)$) if for every sequence α and for any sentence K the following condition is satisfied:

if $\mathbf{a}^\bullet K$ then $\mathbf{a}^\bullet\left(K\&\left(\alpha B\Rightarrow\alpha\mathbf{a}_1^\bullet B\right)\right)$.

Th. 1.4.4:

$$\natural\left(\mathbf{a}\right)\left(\mathbf{a}_1,\mathbf{a}_1^\bullet B\right).$$

Proof: Since $\alpha\mathbf{a}_1^\bullet\sim\alpha\mathbf{a}_1^\bullet\mathbf{a}_1^\bullet$ then according to III: if $\mathbf{a}_1^\bullet K$ then

$$\mathbf{a}_1^\bullet\left(K\&\left(\alpha\mathbf{a}_1^\bullet B\Rightarrow\alpha\mathbf{a}_1^\bullet\mathbf{a}_1^\bullet B\right)\right)$$

□

Th. 1.4.5: If

$$\natural\left(\mathbf{a}\right)\left(\mathbf{a}_1,B\right),\tag{1.25}$$

$$\natural\left(\mathbf{a}\right)\left(\mathbf{a}_2,B\right),\tag{1.26}$$

then

$$\natural(\mathbf{a})(\mathbf{a}_2, \mathbf{a}_1^{\bullet} B).$$

Proof: Let $\mathbf{a}^{\bullet} K$.
In this case from (1.26):

$$\mathbf{a}^{\bullet}(K\&(\alpha\mathbf{a}_1^{\bullet} B \Rightarrow \alpha\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet} B)).$$

From (1.25):

$$\mathbf{a}^{\bullet}((K\&(\alpha\mathbf{a}_1^{\bullet} B \Rightarrow \alpha\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet} B))\&(\alpha\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet} B \Rightarrow \alpha\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet} B)).$$

According to II:

$$\mathbf{a}^{\bullet}(K\&(\alpha\mathbf{a}_1^{\bullet} B \Rightarrow \alpha\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet} B)).$$

According to III:

$$\mathbf{a}^{\bullet}((K\&(\alpha\mathbf{a}_1^{\bullet} B \Rightarrow \alpha\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet} B))\&(\alpha\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet} B \Rightarrow \alpha\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet} B)).$$

According to II:

$$\mathbf{a}^{\bullet}(K\&(\alpha\mathbf{a}_1^{\bullet} B \Rightarrow \alpha\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet} B))$$

□

Lm. 1.4.3: If

$$\natural(\mathbf{a})(\mathbf{a}_1, B), \tag{1.27}$$

$$t = \left[\mathbf{a}^{\bullet}\alpha B \uparrow \mathbf{a}, \widetilde{H}\right], \tag{1.28}$$

then

$$t = \left[\mathbf{a}^{\bullet}\alpha\mathbf{a}_1^{\bullet} B \uparrow \mathbf{a}, \widetilde{H}\right].$$

Proof: Let's symbolize:

$$t_j := \left[\mathbf{a}^{\bullet}\alpha B \uparrow \mathbf{a}, \{\mathbf{g}_j, A, \mathbf{b}_j\}\right].$$

Therefore,

$$\mathbf{a}^{\bullet}\left(\mathbf{a}^{\bullet}\alpha B\&\left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_j^{\bullet}\mathbf{b}_j^{\bullet}\right)^{t_j+1}\mathbf{g}_j^{\bullet} A\right)\right),$$

from (1.27):

$$\mathbf{a}^{\bullet}\left(\left(\mathbf{a}^{\bullet}\alpha B\&\left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_j^{\bullet}\mathbf{b}_j^{\bullet}\right)^{t_j+1}\mathbf{g}_j^{\bullet} A\right)\right)\&(\mathbf{a}^{\bullet}\alpha B \Rightarrow \mathbf{a}^{\bullet}\alpha\mathbf{a}_1^{\bullet} B)\right).$$

According to II:

$$\mathbf{a}^{\bullet}\left(\mathbf{a}^{\bullet}\alpha\mathbf{a}_{1}^{\bullet}B\&\left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_{j}^{\bullet}\mathbf{b}_{j}^{\bullet}\right)^{t_{j}+1}\mathbf{g}_{j}^{\bullet}A\right)\right), \tag{1.29}$$

Let $\mathbf{a}^{\bullet}K$. In this case from (1.28):

$$\mathbf{a}^{\bullet}\left(K\&\left(\mathbf{a}^{\bullet}\alpha B\Rightarrow\mathbf{a}^{\bullet}\left(\mathbf{g}_{j}^{\bullet}\mathbf{b}_{j}^{\bullet}\right)^{t_{j}}\mathbf{g}_{j}^{\bullet}A\right)\right).$$

Therefore, according to III:

$$\mathbf{a}^{\bullet}\left(\left(K\&\left(\mathbf{a}^{\bullet}\alpha B\Rightarrow\mathbf{a}^{\bullet}\left(\mathbf{g}_{j}^{\bullet}\mathbf{b}_{j}^{\bullet}\right)^{t_{j}}\mathbf{g}_{j}^{\bullet}A\right)\right)\&\left(\mathbf{a}^{\bullet}\alpha\mathbf{a}_{1}^{\bullet}B\Rightarrow\mathbf{a}^{\bullet}\alpha B\right)\right).$$

According to II:

$$\mathbf{a}^{\bullet}\left(K\&\left(\mathbf{a}^{\bullet}\alpha\mathbf{a}_{1}^{\bullet}B\Rightarrow\mathbf{a}^{\bullet}\left(\mathbf{g}_{j}^{\bullet}\mathbf{b}_{j}^{\bullet}\right)^{t_{j}}\mathbf{g}_{j}^{\bullet}A\right)\right). \tag{1.30}$$

From (1.27):

$$\mathbf{a}^{\bullet}\left(\left(K\&\left(\mathbf{a}^{\bullet}\left(\mathbf{g}_{j}^{\bullet}\mathbf{b}_{j}^{\bullet}\right)^{t_{j}+1}\mathbf{g}_{j}^{\bullet}A\Rightarrow\mathbf{a}^{\bullet}\alpha B\right)\right)\&\left(\mathbf{a}^{\bullet}\alpha B\Rightarrow\mathbf{a}^{\bullet}\alpha\mathbf{a}_{1}^{\bullet}B\right)\right).$$

according to II:

$$\mathbf{a}^{\bullet}\left(K\&\left(\mathbf{a}^{\bullet}\left(\mathbf{g}_{j}^{\bullet}\mathbf{b}_{j}^{\bullet}\right)^{t_{j}+1}\mathbf{g}_{j}^{\bullet}A\Rightarrow\mathbf{a}^{\bullet}\alpha\mathbf{a}_{1}^{\bullet}B\right)\right).$$

From (1.29), (1.30) for all j:

$$t_{j}=\left[\mathbf{a}^{\bullet}\alpha\mathbf{a}_{1}^{\bullet}B\uparrow\mathbf{a},\left\{\mathbf{g}_{j},A,\mathbf{b}_{j}\right\}\right].$$

Consequently,

$$t\ =\ \left[\mathbf{a}^{\bullet}\alpha\mathbf{a}_{1}^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right]$$

□

Th. 1.4.6: If $\{\mathbf{a},\mathbf{a}_{1},\mathbf{a}_{2}\}$ is $ISS\left(\mathbf{a},\widetilde{H}\right)$,

$$\natural\left(\mathbf{a}\right)\left(\mathbf{a}_{1},B\right), \tag{1.31}$$

$$\natural\left(\mathbf{a}\right)\left(\mathbf{a}_{2},B\right), \tag{1.32}$$

then

$$\ell\left(\mathbf{a},\widetilde{H}\right)\left(\mathbf{a}_{1},\mathbf{a}_{2}\right)=0.$$

Proof: Let's symbolize:

$$t:=\left[\mathbf{a}^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right].$$

According to Lemma 1.4.3:

from (1.31):

$$t = \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right],$$

from (1.32):

$$t = \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right],$$

again from (1.31):

$$t = \left[\mathbf{a}^{\bullet}\mathbf{a}_1^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{a}_1^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right].$$

Consequently,

$$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_2) = 0.5 \cdot (t - t) = 0$$

□

Th. 1.4.7: If $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is $ISS\left(\mathbf{a}, \widetilde{H}\right)$ and there exists sentence B such that

$$\natural(\mathbf{a})(\mathbf{a}_1, B), \quad (1.33)$$

$$\natural(\mathbf{a})(\mathbf{a}_2, B), \quad (1.34)$$

then

$$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_3, \mathbf{a}_2) = \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_3, \mathbf{a}_1).$$

Proof: According to Theorem 1.4.6 from (1.33) and (1.34):

$$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_2) = 0; \quad (1.35)$$

according to Theorem 1.4.3:

$$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_2) + \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_2, \mathbf{a}_3) \geq \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_3),$$

therefore, from (1.35):

$$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_2, \mathbf{a}_3) \geq \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_3),$$

i.e. according to Theorem 1.4.3:

$$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_3, \mathbf{a}_2) \geq \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_3). \quad (1.36)$$

From

$$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_3, \mathbf{a}_1) + \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_2) \geq \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_3, \mathbf{a}_2):$$

and from (1.35):

$$\ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a}_3,\mathbf{a}_1) \geq \ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a}_3,\mathbf{a}_2).$$

From (1.36):

$$\ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a}_3,\mathbf{a}_1) = \ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a}_3,\mathbf{a}_2)$$

□

Def. 1.4.7 A real number t is an instant of a sentence B in *frame of reference* $\left(\Re\mathbf{a}\widetilde{H}\right)$ (design.: $t = \left[B \mid \Re\mathbf{a}\widetilde{H}\right]$) if

1) $\Re$ is $ISS\left(\mathbf{a},\widetilde{H}\right)$,

2) there exists a recorder $\mathbf{b}$ so that $\mathbf{b} \in \Re$ and $\natural(\mathbf{a})(\mathbf{b},B)$,

3) $t = \left[\mathbf{a}^{\bullet}B \uparrow \mathbf{a},\widetilde{H}\right] - \ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{b})$.

Lm. 1.4.4:

$$\left[\mathbf{a}^{\bullet}B \uparrow \mathbf{a},\widetilde{H}\right] = \left[\mathbf{a}^{\bullet}B \mid \Re\mathbf{a}\widetilde{H}\right].$$

Proof: Let $\Re$ is $ISS\left(\mathbf{a},\widetilde{H}\right)$, $\mathbf{a}_1 \in \Re$ and

$$\natural(\mathbf{a})(\mathbf{a}_1,\mathbf{a}^{\bullet}B). \tag{1.37}$$

According to Theorem 1.4.4:

$$\natural(\mathbf{a})(\mathbf{a},\mathbf{a}^{\bullet}B).$$

From (1.37) according to Theorem 1.4.6:

$$\ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{a}_1) = 0,$$

therefore

$$\left[\mathbf{a}^{\bullet}B \mid \Re\mathbf{a}\widetilde{H}\right] = \left[\mathbf{a}^{\bullet}B \uparrow \mathbf{a},\widetilde{H}\right] - \ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{a}_1) = \left[\mathbf{a}^{\bullet}B \uparrow \mathbf{a},\widetilde{H}\right]$$

□

Def. 1.4.8 A real number z is *a distance length* between B and C in a frame of reference $\left(\Re\mathbf{a}\widetilde{H}\right)$ (design.: $z = \ell\left(\Re\mathbf{a}\widetilde{H}\right)(B,C)$) if

1) $\Re$ is $ISS\left(\mathbf{a},\widetilde{H}\right)$,

2) there exist recorders $\mathbf{a}_1$ and $\mathbf{a}_2$ so that $\mathbf{a}_1 \in \Re$, $\mathbf{a}_2 \in \Re$, $\natural(\mathbf{a})(\mathbf{a}_1,B))$ and $\natural(\mathbf{a})(\mathbf{a}_2,C))$,

3) $z = \ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a}_2,\mathbf{a}_1)$.

According to Theorem 1.4.3 such distance length satisfies conditions of all axioms of a metric space.

1.5. Relativity

Def. 1.5.1: Recorders $\mathbf{a}_1$ and $\mathbf{a}_2$ *equally receive a signal* about B for a recorder $\mathbf{a}$ if

$$\ll \natural(\mathbf{a})(\mathbf{a}_2, \mathbf{a}_1^{\bullet} B) \gg = \ll \natural(\mathbf{a})(\mathbf{a}_1, \mathbf{a}_2^{\bullet} B) \gg.$$

Def. 1.5.2: Set of recorders are called *a homogeneous space of recorders*, if all its elements equally receive all signals.

Def. 1.5.3: A real number c is *an information velocity about B to the recorder* $\mathbf{a}_1$ in a frame of reference $\left(\Re\mathbf{a}\widetilde{H}\right)$ if

$$c = \frac{\ell\left(\Re\mathbf{a}\widetilde{H}\right)(B, \mathbf{a}_1^{\bullet} B)}{\left[\mathbf{a}_1^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right] - \left[B \mid \Re\mathbf{a}\widetilde{H}\right]}.$$

Th. 1.5.1: In all homogeneous spaces:

$$c = 1.$$

Proof: Let c represents information velocity about B to a recorder $\mathbf{a}_1$ in a frame of reference $\left(\Re\mathbf{a}\widetilde{H}\right)$.

Thus, if

$$\Re \text{ is } ISS\left(\mathbf{a}, \widetilde{H}\right),$$

$$z := \ell\left(\Re\mathbf{a}\widetilde{H}\right)(B, \mathbf{a}_1^{\bullet} B), \tag{1.38}$$

$$t_1 := \left[B \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.39}$$

$$t_2 := \left[\mathbf{a}_1^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.40}$$

then

$$c = \frac{z}{t_2 - t_1}. \tag{1.41}$$

According to (1.38) there exist elements $\mathbf{b}_1$ and $\mathbf{b}_2$ of set $\Re$ such that:

$$\natural(\mathbf{a})(\mathbf{b}_1, B), \tag{1.42}$$

$$\natural(\mathbf{a})(\mathbf{b}_2, \mathbf{a}_2^{\bullet} B), \tag{1.43}$$

$$z = \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{b}_1, \mathbf{b}_2). \tag{1.44}$$

According to (1.39) and (1.40) there exist elements $\mathbf{b}'_1$ and $\mathbf{b}'_2$ of set $\Re$ such that:

$$\natural(\mathbf{a})\left(\mathbf{b}'_1, B\right), \tag{1.45}$$

$$\natural(\mathbf{a})\left(\mathbf{b}'_2, \mathbf{a}_2^{\bullet}B\right), \tag{1.46}$$

$$t_1 = \left[\mathbf{a}^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right] - \ell\left(\mathbf{a}, \widetilde{H}\right)\left(\mathbf{a}, \mathbf{b}'_1\right), \tag{1.47}$$

$$t_2 = \left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right] - \ell\left(\mathbf{a}, \widetilde{H}\right)\left(\mathbf{a}, \mathbf{b}'_2\right). \tag{1.48}$$

From (1.38), (1.42), (1.45) according to Theorem 1.4.7:

$$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}, \mathbf{b}_1) = \ell\left(\mathbf{a}, \widetilde{H}\right)\left(\mathbf{a}, \mathbf{b}'_1\right). \tag{1.49}$$

Analogously from (1.38), (1.43), (1.46):

$$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}, \mathbf{b}_2) = \ell\left(\mathbf{a}, \widetilde{H}\right)\left(\mathbf{a}, \mathbf{b}'_2\right). \tag{1.50}$$

Analogously from (1.47), (1.42), (1.49)according to Lemma 1.4.3:

$$t_1 = \left[\mathbf{a}^{\bullet}\mathbf{b}_1^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right] - \ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}, \mathbf{b}_1). \tag{1.51}$$

From (1.43) according to Lemma 1.4.3:

$$\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right] = \left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}\mathbf{a}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right]. \tag{1.52}$$

According to Lemma 1.3.4:

$$\left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}\mathbf{a}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right] \geq \left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right]. \tag{1.53}$$

From (1.43):

$$\natural(\mathbf{a})(\mathbf{a}_2, \mathbf{b}_2^{\bullet}B).$$

According to Lemma 1.4.3

$$\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{b}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right] = \left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right]. \tag{1.54}$$

Again according to Lemma 1.3.4:

$$\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}\mathbf{b}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right] \geq \left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right].$$

From (1.54), (1.52), (1.53):

$$\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right] \geq \left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right] \geq \left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}B \uparrow \mathbf{a}, \widetilde{H}\right],$$

therefore,

$$\left[\mathbf{a}^{\bullet}\mathbf{a}_2^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right]=\left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right].$$

From (1.48), (1.50):

$$t_2=\left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right]-\ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{b}_2).$$

From (1.42) according to Lemma 1.4.3

$$t_2=\left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}\mathbf{b}_1^{\bullet}B\uparrow\mathbf{a},\widetilde{H}\right]-\ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{b}_2). \tag{1.55}$$

Let's symbolize

$$u:=\left[\mathbf{a}^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right], \tag{1.56}$$

$$d:=\left[\mathbf{a}^{\bullet}\mathbf{b}_1^{\bullet}\mathbf{a}^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right], \tag{1.57}$$

$$w:=\left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}\mathbf{a}^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right], \tag{1.58}$$

$$j:=\left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}\mathbf{b}_1^{\bullet}\mathbf{a}^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right], \tag{1.59}$$

$$q:=\left[\mathbf{a}^{\bullet}\mathbf{b}_1^{\bullet}\mathbf{b}_2^{\bullet}\mathbf{a}^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right],$$

$$p:=\left[\mathbf{a}^{\bullet}\mathbf{b}_1^{\bullet}\mathbf{b}_2^{\bullet}\mathbf{b}_1^{\bullet}\mathbf{a}^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right], \tag{1.60}$$

$$r:=\left[\mathbf{a}^{\bullet}\mathbf{b}_2^{\bullet}\mathbf{b}_1^{\bullet}\mathbf{b}_2^{\bullet}\mathbf{a}^{\bullet}C\uparrow\mathbf{a},\widetilde{H}\right].$$

Since $\mathfrak{R}$ is $ISS\left(\mathbf{a},\widetilde{H}\right)$ then

$$q-w=p-j, \tag{1.61}$$

$$j=q. \tag{1.62}$$

From (1.55), (1.51), (1.57), (1.59):

$$\left(t_2+\ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{b}_2)\right)-\left(t_1+\ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{b}_1)\right)=j-d,$$

therefore

$$t_2-t_1=j-d-\ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{b}_2)+\ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{b}_1). \tag{1.63}$$

From (1.56), (1.57), (1.58) according to Lemma 1.3.4:

$$\ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{b}_2)=0.5\cdot(w-u),\ \ell\left(\mathbf{a},\widetilde{H}\right)(\mathbf{a},\mathbf{b}_1)=0.5\cdot(d-u).$$

From (1.61), (1.62), (1.63):

$$t_2 - t_1 = 0.5 \cdot ((j-d) + (j-w)) = 0.5 \cdot (j - d + p - j) = 0.5 \cdot (p-d).$$

From (1.60), (1.57), (1.44):

$$z = 0.5 \cdot (p-d).$$

Consequently

$$z = t_2 - t_1$$

□

That is in every homogenous space a propagation velocity of every information to every recorder for every frame reference equals to 1.

Th. 1.5.2: If $\Re$ is a homogeneous space, then

$$\left[\mathbf{a}_1^\bullet B \mid \Re\mathbf{a}\widetilde{H}\right] \geq \left[B \mid \Re\mathbf{a}\widetilde{H}\right].$$

Proof comes out straight from Theorem 1.5.1.

Consequently, in any homogeneous space any recorder finds out that B ”took place” not earlier than B ”actually take place”. ”Time” is irreversible.

Th. 1.5.3 If $\mathbf{a}_1$ and $\mathbf{a}_2$ are elements of $\Re$,

$$\Re isISS\left(\mathbf{a}, \widetilde{H}\right), \tag{1.64}$$

$$p := \left[\mathbf{a}_1^\bullet B \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.65}$$

$$q := \left[\mathbf{a}_2^\bullet \mathbf{a}_1^\bullet B \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.66}$$

$$z := \ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{a}_1, \mathbf{a}_2),$$

then

$$z = q - p.$$

Proof: In accordance with Theorem 1.5.1
from (1.64), (1.65), (1.66):

$$q - p = \ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{a}_1^\bullet B, \mathbf{a}_2^\bullet \mathbf{a}_1^\bullet B),$$

thus in accordance with Definition 1.4.8 there exist elements $\mathbf{b}_1$ and $\mathbf{b}_2$ of $\Re$ such that

$$\natural(\mathbf{a})(\mathbf{b}_1, \mathbf{a}_1^\bullet B), \tag{1.67}$$

$$\natural(\mathbf{a})(\mathbf{b}_2, \mathbf{a}_2^\bullet \mathbf{a}_1^\bullet B), \tag{1.68}$$

$$q - p = \ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{b}_1, \mathbf{b}_2).$$

Moreover, in accordance with Theorem 1.4.4

$$\begin{aligned}&\natural(\mathbf{a})(\mathbf{a}_1^\bullet, \mathbf{a}_1^\bullet B),\\&\natural(\mathbf{a})(\mathbf{a}_2^\bullet, \mathbf{a}_2^\bullet\mathbf{a}_1^\bullet B).\end{aligned} \tag{1.69}$$

From (1.68) in accordance with Theorem 1.4.7:

$$\ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{b}_1, \mathbf{b}_2) = \ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{b}_1, \mathbf{a}_2). \tag{1.70}$$

In accordance with Theorem 1.4.3:

$$\ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{b}_1, \mathbf{a}_2) = \ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{a}_2, \mathbf{b}_1). \tag{1.71}$$

Again in accordance with Theorem 1.4.7 from (1.69), (1.67):

$$\ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{a}_2, \mathbf{b}_1) = \ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{a}_2, \mathbf{a}_1). \tag{1.72}$$

Again in accordance with Theorem 1.4.3:

$$\ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{a}_2, \mathbf{a}_1) = \ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{a}_1\mathbf{a}_2).$$

From (1.72), (1.71), (1.70):

$$\ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{b}_1, \mathbf{b}_2) = \ell\left(\Re\mathbf{a}\widetilde{H}\right)(\mathbf{a}_1\mathbf{a}_2)$$

□

According to Urysohn‘s theorem[5] [8]: any homogeneous space is homeomorphic to some set of points of real Hilbert space. If this homeomorphism is not Identical transformation, then $\Re$ will represent a non- Euclidean space. In this case in this ”space-time” corresponding variant of General Relativity Theory can be constructed. Otherwise, $\Re$ is Euclidean space. In this case there exists *coordinates system* R^μ such that the following condition is satisfied: for all elements $\mathbf{a}_1$ and $\mathbf{a}_2$ of set $\Re$ there exist points $\mathbf{x}_1$ and $\mathbf{x}_2$ of system R^μ such that

$$\ell\left(\mathbf{a}, \widetilde{H}\right)(\mathbf{a}_k, \mathbf{a}_s) = \left(\sum_{j=1}^{\mu}\left(x_{s,j} - x_{k,j}\right)^2\right)^{0.5}.$$

[5]Pavel Samuilovich Urysohn, Pavel Uryson (February 3, 1898, Odessa - August 17, 1924, Batz-sur-Mer) was a Jewish mathematician who is best known for his contributions in the theory of dimension, and for developing Urysohn’s Metrization Theorem and Urysohn’s Lemma, both of which are fundamental results in topology.

In this case R^{μ} is called *a coordinates system of frame of reference* $\left(\Re\mathbf{a}\widetilde{H}\right)$ and numbers $\left\langle x_{k,1}, x_{k,2}, \ldots, x_{k,\mu}\right\rangle$ are called *coordinates of recorder* $\mathbf{a}_k$ in R^{μ}.

A coordinates system of a frame of reference is specified accurate to transformations of shear, turn, and inversion.

Def. 1.5.4: Numbers $\left\langle x_1, x_2, \ldots, x_{\mu}\right\rangle$ are called *coordinates* of B in *a coordinate system* R^{μ} of *a frame of reference* $\left(\Re\mathbf{a}\widetilde{H}\right)$ if there exists a recorder $\mathbf{b}$ such that $\mathbf{b} \in \Re$, $\natural(\mathbf{a})(\mathbf{b}, B)$ and these numbers are the coordinates in R^{μ} of this recorder.

Th. 1.5.4: In a coordinate system R^{μ} of a frame of reference $\left(\Re\mathbf{a}\widetilde{H}\right)$: if z is a distance length between B and C, coordinates of B are $(b_1, b_2, .., b_n)$, coordinates of C are $(c_1, c_2, .., c_3)$, then

$$z = \left(\sum_{j=1}^{\mu} (c_j - b_j)^2\right)^{0.5}.$$

Proof came out straight from Definition 1.5.4 □

Def. 1.5.5: Numbers $\left\langle x_1, x_2, \ldots, x_{\mu}\right\rangle$ are called *coordinates of the recor-der* $\mathbf{b}$ *in the coordinate system* R^{μ} *at the instant* t *of the frame of reference* $\left(\Re\mathbf{a}\widetilde{H}\right)$ if for every B the condition is satisfied: if

$$t = \left[\mathbf{b}^{\bullet}B \mid \Re\mathbf{a}\widetilde{H}\right]$$

then coordinates of $\ll \mathbf{b}^{\bullet}B \gg$ in coordinate system R^{μ} of frame of reference $\left(\Re\mathbf{a}\widetilde{H}\right)$ are the following:

$$\left\langle x_1, x_2, \ldots, x_{\mu}\right\rangle.$$

Lm. 1.5.1 If

$$\tau := \left[\mathbf{b}^{\bullet}C \uparrow \mathbf{b}, \{\mathbf{g}_0, B, \mathbf{b}_0\}\right], \tag{1.73}$$

$$p := \left[\mathbf{a}^{\bullet}b^{\bullet}\left(\mathbf{g}_0^{\bullet}b_0^{\bullet}\right)^{\tau}\mathbf{g}_0^{\bullet}B \uparrow \mathbf{a}, \{\mathbf{g}_1, A, \mathbf{b}_1\}\right], \tag{1.74}$$

$$q := \left[\mathbf{a}^{\bullet}b^{\bullet}\left(\mathbf{g}_0^{\bullet}b_0^{\bullet}\right)^{\tau+1}\mathbf{g}_0^{\bullet}B \uparrow \mathbf{a}, \{\mathbf{g}_1, A, \mathbf{b}_1\}\right], \tag{1.75}$$

$$t := \left[\mathbf{a}^{\bullet}b^{\bullet}C \uparrow \mathbf{a}, \{\mathbf{g}_1, A, \mathbf{b}_1\}\right] \tag{1.76}$$

then

$$p \leq t \leq q.$$

Proof

1) From (1.75):

$$\mathbf{a}^{\bullet}\left(\mathbf{a}^{\bullet}b^{\bullet}\left(\mathbf{g}_0^{\bullet}b_0^{\bullet}\right)^{\tau+1}\mathbf{g}_0^{\bullet}B \& \left(\neg\mathbf{a}^{\bullet}\left(\mathbf{g}_1^{\bullet}\mathbf{b}_1^{\bullet}\right)^{q+1}\mathbf{g}_1^{\bullet}A\right)\right). \tag{1.77}$$

Hence from (1.73):

$$\left(\mathbf{b}^\bullet \left(\mathbf{g}_0^\bullet \mathbf{b}_0^\bullet\right)^{\tau+1} \mathbf{g}_0^\bullet B \Rightarrow \mathbf{b}^\bullet C\right)$$

then from (1.77) according to II:

$$\mathbf{a}^\bullet \left(\mathbf{a}^\bullet b^\bullet C \& \left(\neg \mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{q+1} \mathbf{g}_1^\bullet A\right)\right).$$

According to II, since from (1.76):

$$\left(\mathbf{a}^\bullet b^\bullet C \Rightarrow \mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{t} \mathbf{g}_1^\bullet A\right)$$

then

$$\mathbf{a}^\bullet \left(\mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{t} \mathbf{g}_1^\bullet A \& \left(\neg \mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{q+1} \mathbf{g}_1^\bullet A\right)\right). \tag{1.78}$$

If $t > q$ then $t \geq q+1$. Hence according to III from (1.78):

$$\mathbf{a}^\bullet \left(\mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{q+1} \mathbf{g}_1^\bullet A \& \left(\neg \mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{q+1} \mathbf{g}_1^\bullet A\right)\right),$$

it contradicts to I. So $t \leq q$.
2) From (1.76):

$$\mathbf{a}^\bullet \left(\mathbf{a}^\bullet b^\bullet C \& \left(\neg \mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{t+1} \mathbf{g}_1^\bullet A\right)\right). \tag{1.79}$$

Since from (1.73):

$$\left(\mathbf{b}^\bullet C \Rightarrow \mathbf{b}^\bullet \left(\mathbf{g}_0^\bullet \mathbf{b}_0^\bullet\right)^{\tau} \mathbf{g}_0^\bullet B\right)$$

then from (1.79) according to II:

$$\mathbf{a}^\bullet \left(\mathbf{a}^\bullet b^\bullet \left(\mathbf{g}_0^\bullet \mathbf{b}_0^\bullet\right)^{\tau} \mathbf{g}_0^\bullet B \& \left(\neg \mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{t+1} \mathbf{g}_1^\bullet A\right)\right). \tag{1.80}$$

Since from (1.74):

$$\left(\mathbf{a}^\bullet b^\bullet \left(\mathbf{g}_0^\bullet b_0^\bullet\right)^{\tau} \mathbf{g}_0^\bullet B \Rightarrow \mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{p} \mathbf{g}_1^\bullet A\right)$$

then according to II from (1.80):

$$\mathbf{a}^\bullet \left(\mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{p} \mathbf{g}_1^\bullet A \& \left(\neg \mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{t+1} \mathbf{g}_1^\bullet A\right)\right). \tag{1.81}$$

If $p > t$ then $p \geq t+1$. In that case from (1.81) according to III:

$$\mathbf{a}^\bullet \left(\mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{t+1} \mathbf{g}_1^\bullet A \& \left(\neg \mathbf{a}^\bullet \left(\mathbf{g}_1^\bullet \mathbf{b}_1^\bullet\right)^{t+1} \mathbf{g}_1^\bullet A\right)\right),$$

it contradicts to I. So $p \leq t$ □

Th. 1.5.5 In a coordinates system R^{μ} of a frame of reference $\left(\Re\mathbf{a}\widetilde{H}\right)$: if in every instant t: coordinates of[6]:

$\mathbf{b}$: $\left\langle x_{\mathbf{b},1} + v \cdot t, x_{\mathbf{b},2}, x_{\mathbf{b},3}, \ldots, x_{\mathbf{b},\mu} \right\rangle$;
$\mathbf{g}_0$: $\left\langle x_{0,1} + v \cdot t, x_{0,2}, x_{0,3}, \ldots, x_{0,\mu} \right\rangle$;
$\mathbf{b}_0$: $\left\langle x_{0,1} + v \cdot t, x_{0,2} + l, x_{0,3}, \ldots, x_{0,\mu} \right\rangle$; and
$t_C = \left[\mathbf{b}^{\bullet} C \mid \Re\mathbf{a}\widetilde{H}\right]$;
$t_D = \left[\mathbf{b}^{\bullet} D \mid \Re\mathbf{a}\widetilde{H}\right]$;
$q_C = [\mathbf{b}^{\bullet} C \uparrow \mathbf{b}, \{\mathbf{g}_0, A, \mathbf{b}_0\}]$;
$q_D = [\mathbf{b}^{\bullet} D \uparrow \mathbf{b}, \{\mathbf{g}_0, A, \mathbf{b}_0\}]$,
then

$$\lim_{l \to 0} 2 \cdot \frac{l}{\sqrt{(1 - v^2)}} \cdot \frac{q_D - q_C}{t_D - t_C} = 1.$$

Proof: Let us designate:

$$t_1 := \left[\mathbf{b}^{\bullet} \left(\mathbf{g_0}^{\bullet} b_0^{\bullet}\right)^{q_C} \mathbf{g_0}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.82}$$

$$t_2 := \left[\mathbf{b}^{\bullet} \left(\mathbf{g_0}^{\bullet} b_0^{\bullet}\right)^{q_C+1} \mathbf{g_0}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.83}$$

$$t_3 := \left[\left(\mathbf{g_0}^{\bullet} b_0^{\bullet}\right)^{q_C} \mathbf{g_0}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.84}$$

$$t_4 := \left[\left(\mathbf{g_0}^{\bullet} b_0^{\bullet}\right)^{q_C+1} \mathbf{g_0}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right]. \tag{1.85}$$

In that case coordinates of:

$$\ll \mathbf{b}^{\bullet} \left(\mathbf{g_0}^{\bullet} b_0^{\bullet}\right)^{q_C} \mathbf{g_0}^{\bullet} B \gg: \left\langle x_{\mathbf{b},1} + v \cdot t_1, x_{\mathbf{b},2}, x_{\mathbf{b},3}, \ldots, x_{\mathbf{b},\mu} \right\rangle, \tag{1.86}$$

$$\ll \mathbf{b}^{\bullet} \left(\mathbf{g_0}^{\bullet} b_0^{\bullet}\right)^{q_C+1} \mathbf{g_0}^{\bullet} B \gg: \left\langle x_{\mathbf{b},1} + v \cdot t_2, x_{\mathbf{b},2}, x_{\mathbf{b},3}, \ldots, x_{\mathbf{b},\mu} \right\rangle, \tag{1.87}$$

$$\ll \left(\mathbf{g_0}^{\bullet} b_0^{\bullet}\right)^{q_C} \mathbf{g_0}^{\bullet} B \gg: \left\langle x_{0,1} + v \cdot t_3, x_{0,2}, x_{0,3}, \ldots, x_{0,\mu} \right\rangle, \tag{1.88}$$

$$\ll \left(\mathbf{g_0}^{\bullet} b_0^{\bullet}\right)^{q_C+1} \mathbf{g_0}^{\bullet} B \gg: \left\langle x_{0,1} + v \cdot t_4, x_{0,2}, x_{0,3}, \ldots, x_{0,\mu} \right\rangle, \tag{1.89}$$

$$\ll \mathbf{b}^{\bullet} C \gg: \left\langle x_{\mathbf{b},1} + v \cdot t_C, x_{\mathbf{b},2}, x_{\mathbf{b},3}, \ldots, x_{\mathbf{b},\mu} \right\rangle. \tag{1.90}$$

According to Theorem 1.5.1 and Lemma 1.4.4 from (1.82), (1.86), (1.83), (1.87), (1.90):

$$\left[\mathbf{a}^{\bullet} b^{\bullet} \left(\mathbf{g_0}^{\bullet} b_0^{\bullet}\right)^{q_C} \mathbf{g_0}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right] = \left[\mathbf{a}^{\bullet} b^{\bullet} \left(\mathbf{g_0}^{\bullet} b_0^{\bullet}\right)^{q_C} \mathbf{g_0}^{\bullet} B \uparrow \mathbf{a}, \widetilde{H}\right] = t_1 + \left(\left(x_{b,1} + v t_1\right)^2 + \sum_{j+2}^{\mu} x_{b,j}^2\right)^{0.5},$$

[6] below v is a real positive number such that $|v| < 1$)

$$\left[\mathbf{a}^{\bullet} b^{\bullet} \left(\mathbf{g_0}^{\bullet} b_0{}^{\bullet}\right)^{qc+1} B \mid \Re \mathbf{a} \widetilde{H}\right] =$$
$$\left[\mathbf{a}^{\bullet} b^{\bullet} \left(\mathbf{g_0}^{\bullet} b_0{}^{\bullet}\right)^{qc+1} B \uparrow \mathbf{a}, \widetilde{H}\right] =$$
$$t_2 + \left(\left(x_{b,1} + vt_2\right)^2 + \sum_{j=2}^{\mu} x_{b,j}^2\right)^{0.5}.$$

According to Lemma 1.5.1:

$$\begin{aligned} & t_1 + \left(\left(x_{b,1} + vt_1\right)^2 + \sum_{j=2}^{\mu} x_{b,j}^2\right)^{0.5} \\ \leq \quad & t_C + \left(\left(x_{b,1} + vt_C\right)^2 + \sum_{j=2}^{\mu} x_{b,j}^2\right)^{0.5} \\ \leq \quad & t_2 + \left(\left(x_{b,1} + vt_2\right)^2 + \sum_{j=2}^{\mu} x_{b,j}^2\right)^{0.5}. \end{aligned} \tag{1.91}$$

According to Theorem 1.5.1 from (1.82), (1.84), (1.86), (1.88):

$$t_1 = t_3 + \left(\left(x_{0,1} + vt_3 - x_{b,1} - vt_1\right)^2 + \sum_{j=2}^{\mu} \left(x_{0,j} - x_{b,j}\right)^2\right)^{0.5}.$$

From (1.83), (1.85), (1.87), (1.89):

$$t_2 = t_4 + \left(\left(x_{0,1} + vt_4 - x_{b,1} - vt_2\right)^2 + \sum_{j=2}^{\mu} \left(x_{0,j} - x_{b,j}\right)^2\right)^{0.5}.$$

Hence:

$$(t_1 - t_3)^2 = v^2 (t_1 - t_3)^2 - 2v (t_1 - t_3)(x_{0,1} - x_{b,1}) + \sum_{j=2}^{\mu} \left(x_{0,j} - x_{b,j}\right)^2,$$
$$(t_2 - t_4)^2 = v^2 (t_2 - t_4)^2 - 2v (t_2 - t_4)(x_{0,1} - x_{b,1}) + \sum_{j=2}^{\mu} \left(x_{0,j} - x_{b,j}\right)^2.$$

Therefore,

$$t_2 - t_4 = t_1 - t_3. \tag{1.92}$$

Let us designate:

$$t_5 := \left[\mathbf{b_0}^{\bullet} \left(\mathbf{g_0}^{\bullet} b_0{}^{\bullet}\right)^{qc} \mathbf{g_0}^{\bullet} B \mid \Re \mathbf{a} \widetilde{H}\right]. \tag{1.93}$$

In that case coordinates of:

$$\ll \mathbf{b_0}^{\bullet} \left(\mathbf{g_0}^{\bullet} b_0{}^{\bullet}\right)^{qc} \mathbf{g_0}^{\bullet} B \gg: \left\langle x_{0,1} + v \cdot t_5, x_{0,2} + l, x_{0,3}, \ldots, x_{0,\mu}\right\rangle.$$

hence from (1.84), (1.88) according to Theorem 1.5.1:

$$t_5 - t_3 = \left(\left(x_{0,1} + vt_5 - x_{0,1} - vt_3\right)^2 + \left(x_{0,2} + l - x_{0,2}\right)^2 + \sum_{j=3}^{\mu} \left(x_{0,j} - x_{0,j}\right)^2\right)^{0.5},$$

hence:

$$t_5 - t_3 = \frac{l}{\sqrt{1-v^2}}. \tag{1.94}$$

Analogously from (1.93), (1.85), (1.89):

$$t_4 - t_5 = \frac{l}{\sqrt{1-v^2}}.$$

From (1.94):

$$t_4 - t_3 = \frac{2l}{\sqrt{1-v^2}}.$$

From (1.92):

$$t_2 - t_1 = \frac{2l}{\sqrt{1-v^2}}.$$

Hence from (1.91):

$$\begin{aligned} & t_1 + \left((x_{b,1} + vt_1)^2 + \sum_{j=2}^{\mu} x_{b,j}^2 \right)^{0.5} \\ \leq \quad & t_C + \left((x_{b,1} + vt_C)^2 + \sum_{j=2}^{\mu} x_{b,j}^2 \right)^{0.5} \\ \leq \quad & t_1 + \frac{2l}{\sqrt{1-v^2}} + \left(\left(x_{b,1} + v \left(t_1 + \frac{2l}{\sqrt{1-v^2}} \right) \right)^2 + \sum_{j=2}^{\mu} x_{b,j}^2 \right)^{0.5}. \end{aligned}$$

Or if $l \to 0$ then $t_2 \to t_1$, and

$$\begin{aligned} & \lim_{l \to 0} \left(t_1 + \left((x_{b,1} + vt_1)^2 + \sum_{j=2}^{\mu} x_{b,j}^2 \right)^{0.5} \right) \\ = \quad & t_C + \left((x_{b,1} + vt_C)^2 + \sum_{j=2}^{\mu} x_{b,j}^2 \right)^{0.5}. \end{aligned}$$

Since, if $v^2 < 1$ then function

$$f(t) = t + \left((x_{b,1} + vt)^2 + \sum_{j=2}^{\mu} x_{b,j}^2 \right)^{0.5}$$

is a monotonic one, then

$$\lim_{l \to 0} t_1 = t_C,$$

hence

$$\lim_{l\to 0}\left[\mathbf{b}^{\bullet}\left(\mathbf{g_0}^{\bullet}b_0{}^{\bullet}\right)^{q_C}\mathbf{g_0}^{\bullet}B\mid \Re\mathbf{a}\widetilde{H}\right]=t_C. \tag{1.95}$$

Analogously,

$$\lim_{l\to 0}\left[\mathbf{b}^{\bullet}\left(\mathbf{g_0}^{\bullet}b_0{}^{\bullet}\right)^{q_D}\mathbf{g_0}^{\bullet}B\mid \Re\mathbf{a}\widetilde{H}\right]=t_D. \tag{1.96}$$

According to Theorem 1.5.1 from (1.82) and (1.83):

$$\begin{aligned}
&\left[\mathbf{b}^{\bullet}\left(\mathbf{g_0}^{\bullet}b_0{}^{\bullet}\right)^{q_D}\mathbf{g_0}^{\bullet}B\mid \Re\mathbf{a}\widetilde{H}\right]-\left[\mathbf{b}^{\bullet}\left(\mathbf{g_0}^{\bullet}b_0{}^{\bullet}\right)^{q_C}\mathbf{g_0}^{\bullet}B\mid \Re\mathbf{a}\widetilde{H}\right]\\
=&\left(t_1+\frac{2l}{\sqrt{1-v^2}}\left(q_D-q_C\right)\right)-t_1\\
=&\frac{2l\left(q_D-q_C\right)}{\sqrt{1-v^2}}.
\end{aligned}$$

From (1.95) and (1.96):

$$\lim_{l\to 0}\frac{2l\left(q_D-q_C\right)}{t_D-t_C}=\sqrt{1-v^2}$$

□

Corollary of Theorem 1.5.5: If designate: $q_D^{st}:=q_D$ and $q_C^{st}:=q_C$ for $v=0$, then

$$\lim_{l\to 0}2l\frac{q_D^{st}-q_C^{st}}{t_D-t_C}=1,$$

hence:

$$\lim_{l\to 0}\frac{q_D-q_C}{q_D^{st}-q_C^{st}}=\sqrt{1-v^2}.$$

For an absolutely precise κ-clock:

$$q_D^{st}-q_C^{st}=\frac{q_D-q_C}{\sqrt{1-v^2}}\square$$

Consequently, moving at speed v κ-clock are times slower than the one at rest.

Th. 1.5.6 Let: v ($|v|<1$) and l be real numbers and k_i be natural ones.

Let in a coordinates system R^{μ} of a frame of reference $\left(\Re\mathbf{a}\widetilde{H}\right)$: in each instant t coordinates of:

$\mathbf{b}$: $\left\langle x_{b,1}+v\cdot t, x_{b,2}, x_{b,3},\ldots,x_{b,\mu}\right\rangle$,
$\mathbf{g}_j$: $\left\langle y_{j,1}+v\cdot t, y_{j,2}, y_{j,3},\ldots,y_{j,\mu}\right\rangle$,
$\mathbf{u}_j$: $\left\langle y_{j,1}+v\cdot t, y_{j,2}+l/\left(k_1\cdot\ldots\cdot k_j\right), y_{j,3},\ldots,y_{j,\mu}\right\rangle$,
for all $\mathbf{b}_i$: if $\mathbf{b}_i\in\Im$, then coordinates of
$\mathbf{b}_i$: $\left\langle x_{i,1}+v\cdot t, x_{i,2}, x_{i,3},\ldots,x_{i,\mu}\right\rangle$,
$\widetilde{T}$ is $\left\langle\left\{\mathbf{g}_1,A,\mathbf{u}_1\right\},\ \left\{\mathbf{g}_2,A,\mathbf{u}_2\right\},..,\left\{\mathbf{g}_j,A,\mathbf{u}_j\right\},\ ..\ \right\rangle$.

In that case: $\mathfrak{I}$ is $ISS\left(\mathbf{b},\widetilde{T}\right)$.

Proof

1) Let us designate:

$$p := \left[\mathbf{b}^{\bullet}b_1^{\bullet}B \uparrow \mathbf{b},\widetilde{T}\right],$$
$$q := \left[\mathbf{b}^{\bullet}b_2^{\bullet}b_1^{\bullet}B \uparrow \mathbf{b},\widetilde{T}\right],$$
$$r := \left[\mathbf{b}^{\bullet}b_1^{\bullet}C \uparrow \mathbf{b},\widetilde{T}\right],$$
$$s := \left[\mathbf{b}^{\bullet}b_2^{\bullet}b_1^{\bullet}C \uparrow \mathbf{b},\widetilde{T}\right],$$

$$t_p := \left[\mathbf{b}^{\bullet}b_1^{\bullet}B \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.97}$$
$$t_q := \left[\mathbf{b}^{\bullet}b_2^{\bullet}b_1^{\bullet}B \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.98}$$
$$t_r := \left[\mathbf{b}^{\bullet}b_1^{\bullet}C \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.99}$$
$$t_s := \left[\mathbf{b}^{\bullet}b_2^{\bullet}b_1^{\bullet}B \mid \Re\mathbf{a}\widetilde{H}\right]. \tag{1.100}$$

According to Corollary of Theorem 1.5.5:

$$t_q - t_p = \frac{q-p}{\sqrt{1-v^2}}, \tag{1.101}$$
$$t_s - t_r = \frac{s-r}{\sqrt{1-v^2}}. \tag{1.102}$$

From (1.97-1.100) coordinates of:

$$\ll \mathbf{b}^{\bullet}b_1^{\bullet}B \gg: \left\langle x_{b,1}+vt_p, x_{b,2}, x_{b,3}, \ldots, x_{b,\mu}\right\rangle, \tag{1.103}$$
$$\ll \mathbf{b}^{\bullet}b_2^{\bullet}b_1^{\bullet}B \gg: \left\langle x_{b,1}+vt_q, x_{b,2}, x_{b,3}, \ldots, x_{b,\mu}\right\rangle,$$

$$\ll \mathbf{b}^{\bullet}b_1^{\bullet}C \gg: \left\langle x_{b,1}+vt_r, x_{b,2}, x_{b,3}, \ldots, x_{b,\mu}\right\rangle, \tag{1.104}$$
$$\ll \mathbf{b}^{\bullet}b_2^{\bullet}b_1^{\bullet}C \gg: \left\langle x_{b,1}+vt_s, x_{b,2}, x_{b,3}, \ldots, x_{b,\mu}\right\rangle.$$

Let us designate:

$$t_1 := \left[\mathbf{b}_1^{\bullet}B \mid \Re\mathbf{a}\widetilde{H}\right], \tag{1.105}$$
$$t_2 := \left[\mathbf{b}_1^{\bullet}C \mid \Re\mathbf{a}\widetilde{H}\right]. \tag{1.106}$$

Consequently, coordinates of:

$$\ll \mathbf{b}_1^{\bullet} B \gg: \left\langle x_{1,1}+vt_1, x_{1,2}, x_{1,3}, \ldots, x_{1,\mu}\right\rangle,$$
$$\ll \mathbf{b}_1^{\bullet} C \gg: \left\langle x_{1,1}+vt_2, x_{1,2}, x_{1,3}, \ldots, x_{1,\mu}\right\rangle.$$

According to Theorem 1.5.1 from (1.104), (1.106), (1.99):

$$t_r - t_2 = \left((x_{b,1}+vt_r - x_{1,1} - vt_2)^2 + \sum_{j=2}^{\mu} \left(x_{b,j} - x_{1,j}\right)^2 \right)^{0.5}.$$

Analogously from (1.103), (1.105), (1.97):

$$t_p - t_1 = \left((x_{b,1}+vt_p - x_{1,1} - vt_1)^2 + \sum_{j=2}^{\mu} \left(x_{b,j} - x_{1,j}\right)^2 \right)^{0.5}.$$

Hence,

$$t_r - t_2 = t_p - t_1. \tag{1.107}$$

Let us denote:

$$t_3 := \left[\mathbf{b_2}^{\bullet} b_1^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right],$$
$$t_4 := \left[\mathbf{b_2}^{\bullet} b_1^{\bullet} C \mid \Re\mathbf{a}\widetilde{H}\right].$$

Hence, coordinates of:

$$\ll \mathbf{b_2}^{\bullet} b_1^{\bullet} B \gg: \left\langle x_{2,1}+vt_3, x_{2,2}, x_{2,3}, \ldots, x_{2,\mu}\right\rangle,$$
$$\ll \mathbf{b_2}^{\bullet} b_1^{\bullet} C \gg: \left\langle x_{2,1}+vt_4, x_{2,2}, x_{2,3}, \ldots, x_{2,\mu}\right\rangle.$$

According to Theorem 1.5.1:

$$t_3 - t_1 = \left((x_{2,1}+vt_3 - x_{1,1} - vt_1)^2 + \sum_{j=2}^{\mu} (x_{2,j} - x_{1,j})^2 \right)^{0.5}.$$
$$t_4 - t_2 = \left((x_{2,1}+vt_4 - x_{1,1} - vt_2)^2 + \sum_{j=2}^{\mu} (x_{2,j} - x_{1,j})^2 \right)^{0.5}.$$

Hence:

$$t_3 - t_4 = t_1 - t_2. \tag{1.108}$$

And analogously:

$$t_q - t_3 = t_s - t_4. \tag{1.109}$$

From (1.108), (1.109), (1.107):

$$t_q - t_p = t_s - t_r.$$

From (1.102), (1.101):

$$q - p = s - r. \tag{1.110}$$

2) Let us designate:

$$p' := \left[\mathbf{b}^{\bullet}C \uparrow \mathbf{b}, \widetilde{T}\right],$$
$$q' := \left[\mathbf{b}^{\bullet}\alpha\mathbf{b}^{\bullet}C \uparrow \mathbf{b}, \widetilde{T}\right],$$
$$r' := \left[\mathbf{b}^{\bullet}\alpha^{\dagger}\mathbf{b}^{\bullet}C \uparrow \mathbf{b}, \widetilde{T}\right];$$

here α is $\mathbf{b}_1^{\bullet}b_2^{\bullet}\ldots b_k^{\bullet}b_{k+1}^{\bullet}\ldots b_N^{\bullet}$.
Hence according Definition 1.4.1:

$$\mathfrak{m}\left(\mathbf{b}\widetilde{T}\right)\left(\mathbf{b}^{\bullet}\alpha\mathbf{b}^{\bullet}C\right) = q' - p', \tag{1.111}$$
$$\mathfrak{m}\left(\mathbf{b}\widetilde{T}\right)\left(\mathbf{b}^{\bullet}\alpha^{\dagger}\mathbf{b}^{\bullet}C\right) = r' - p'. \tag{1.112}$$

Let us designate:

$$\begin{aligned}
&t_0 := \left[\mathbf{b}^{\bullet}C \mid \Re\mathbf{a}\widetilde{H}\right],\\
&t_1 = \left[\mathbf{b}_1^{\bullet}b^{\bullet}C \mid \Re\mathbf{a}\widetilde{H}\right],\\
&t_2 := \left[\mathbf{b}_2^{\bullet}b_1^{\bullet}b^{\bullet}C \mid \Re\mathbf{a}\widetilde{H}\right],\\
&\cdots,\\
&t_k := \left[\mathbf{b}_k^{\bullet}\ldots\mathbf{b}_2^{\bullet}b_1^{\bullet}b^{\bullet}C \mid \Re\mathbf{a}\widetilde{H}\right],\\
&t_{k+1} := \left[\mathbf{b}_{k+1}^{\bullet}b_k^{\bullet}\ldots\mathbf{b}_2^{\bullet}b_1^{\bullet}b^{\bullet}C \mid \Re\mathbf{a}\widetilde{H}\right],\\
&\cdots,\\
&t_N := \left[\mathbf{b}_N^{\bullet}\ldots\mathbf{b}_{k+1}^{\bullet}b_k^{\bullet}\ldots\mathbf{b}_2^{\bullet}b_1^{\bullet}b^{\bullet}C \mid \Re\mathbf{a}\widetilde{H}\right],\\
&t_{N+1} := \left[\mathbf{b}^{\bullet}\alpha^{\dagger}\mathbf{b}^{\bullet}C \mid \Re\mathbf{a}\widetilde{H}\right].
\end{aligned} \tag{1.113}$$

Hence in accordance with this theorem condition coordinates of:

$$\begin{aligned}
\ll\ &\mathbf{b}^{\bullet}C \gg:\\
&\left\langle x_{b,1} + vt_0, x_{b,2}, x_{b,3}, \ldots, x_{b,\mu}\right\rangle,\\
\ll\ &\mathbf{b}_1^{\bullet}b^{\bullet}C \gg:\\
&\left\langle x_{1,1} + vt_1, x_{1,2}, x_{1,3}, \ldots, x_{1,\mu}\right\rangle,
\end{aligned}$$

$$\begin{aligned}
\ll\ & \mathbf{b}_2^{\bullet} b_1^{\bullet} b^{\bullet} C \gg: \\
& \left\langle x_{2,1} + vt_2, x_{2,2}, x_{2,3}, \ldots, x_{2,\mu} \right\rangle, \\
& \ldots, \\
\ll\ & \mathbf{b}_k^{\bullet} \cdots \mathbf{b}_2^{\bullet} b_1^{\bullet} b^{\bullet} C \gg: \\
& \left\langle x_{k,1} + vt_k, x_{k,2}, x_{k,3}, \ldots, x_{k,\mu} \right\rangle, \\
\ll\ & \mathbf{b}_{k+1}^{\bullet} b_k^{\bullet} \cdots \mathbf{b}_2^{\bullet} b_1^{\bullet} b^{\bullet} C \gg: \\
& \left\langle x_{k+1,1} + vt_{k+1}, x_{k+1,2}, x_{k+1,3}, \ldots, x_{k+1,\mu} \right\rangle, \\
& \ldots, \\
\ll\ & \mathbf{b}_N^{\bullet} \cdots \mathbf{b}_{k+1}^{\bullet} b_k^{\bullet} \cdots \mathbf{b}_2^{\bullet} b_1^{\bullet} b^{\bullet} C \gg: \\
& \left\langle x_{N,1} + vt_N, x_{N,2}, x_{N,3}, \ldots, x_{N,\mu} \right\rangle, \\
\ll\ & \mathbf{b}^{\bullet} \alpha^{\dagger} \mathbf{b}^{\bullet} C \gg: \\
& \left\langle x_{N+1,1} + vt_{N+1}, x_{N+1,2}, x_{N+1,3}, \ldots, x_{N+1,\mu} \right\rangle.
\end{aligned}$$

Hence from (1.113) according Theorem 1.5.1:

$$\begin{aligned}
& t_1 - t_0 \\
= & \left((x_{1,1} + vt_1 - x_{b,1} - vt_0)^2 + \sum_{j=2}^{\mu} \left(x_{1,j} - x_{b,j} \right)^2 \right)^{0.5}, \\
& t_2 - t_1 \\
= & \left((x_{2,1} + vt_2 - x_{1,1} - vt_1)^2 + \sum_{j=2}^{\mu} \left(x_{2,j} - x_{1,j} \right)^2 \right)^{0.5}, \\
& \ldots, \\
& t_{k+1} - t_k \\
= & \left((x_{k+1,1} + vt_{k+1} - x_{k,1} - vt_k)^2 + \sum_{j=2}^{\mu} \left(x_{k+1,j} - x_{k,j} \right)^2 \right)^{0.5}, \\
& \ldots, \\
& t_{N+1} - t_N \\
= & \left((x_{b,1} + vt_{N+1} - x_{N,1} - vt_N)^2 + \sum_{j=2}^{\mu} \left(x_{b,j} - x_{N,j} \right)^2 \right)^{0.5}.
\end{aligned}$$

If designate:

$$\rho_{a,b}^2 := \sum_{j=1}^{\mu} (x_{b,1} - x_{a,1})^2,$$

then for every k:

$$\begin{aligned}
t_{k+1} - t_k = & \frac{v}{1 - v^2} (x_{k+1,1} - x_{k,1}) \\
& + \frac{1}{1 - v^2} \left(\rho_{k,k+1}^2 - v^2 \sum_{j=2}^{\mu} \left(x_{k+1,j} - x_{k,j} \right)^2 \right)^{0.5}.
\end{aligned}$$

Hence:

$$t_{N+1}-t_0=$$
$$=\frac{1}{1-v^2}\left(\begin{array}{c}\left(\rho_{b,1}^2-v^2\sum_{j=2}^{\mu}\left(x_{1,j}-x_{b,j}\right)^2\right)^{0.5}\\+\left(\rho_{N,b}^2-v^2\sum_{j=2}^{\mu}\left(x_{b,j}-x_{N,j}\right)^2\right)^{0.5}\\+\sum_{k=1}^{N-1}\left(\rho_{k,k+1}^2-v^2\sum_{j=2}^{\mu}\left(x_{k+1,j}-x_{k,j}\right)^2\right)^{0.5}\end{array}\right).$$

Analogously, if designate:

$$\tau_{N+1}:=\left[\mathbf{b}^{\bullet}\alpha\mathbf{b}^{\bullet}C\mid\Re\mathbf{a}\widetilde{H}\right]$$

then

$$\tau_{N+1}-t_0=$$
$$=\frac{1}{1-v^2}\left(\begin{array}{c}\left(\rho_{1,b}^2-v^2\sum_{j=2}^{\mu}\left(x_{b,j}-x_{1,j}\right)^2\right)^{0.5}\\+\left(\rho_{b,N}^2-v^2\sum_{j=2}^{\mu}\left(x_{N,j}-x_{b,j}\right)^2\right)^{0.5}\\+\sum_{k=1}^{N-1}\left(\rho_{k+1,k}^2-v^2\sum_{j=2}^{\mu}\left(x_{k,j}-x_{k+1,j}\right)^2\right)^{0.5}\end{array}\right),$$

hence

$$t_{N+1}-t_0=\tau_{N+1}-t_0. \tag{1.114}$$

According to Theorem 1.5.5:

$$\tau_{N+1}-t_0=\frac{q'-p'}{\sqrt{1-v^2}}\text{ and }t_{N+1}-t_0=\frac{r'-p'}{\sqrt{1-v^2}}.$$

From (1.114), (1.111), (1.112):

$$\mathfrak{m}\left(\mathbf{b}\widetilde{T}\right)\left(\mathbf{b}^{\bullet}\alpha\mathbf{b}^{\bullet}C\right)=\mathfrak{m}\left(\mathbf{b}\widetilde{T}\right)\left(\mathbf{b}^{\bullet}\alpha^{\dagger}\mathbf{b}^{\bullet}C\right).$$

From (1.110) according to Definition 1.4.3: $\Im$ is $ISS\left(\mathbf{b},\widetilde{T}\right)$ □

Therefore, a inner stability survives on a uniform straight line motion.

Th. 1.5.7

Let:

1) in a coordinates system R^{μ} of a frame of reference $\left(\Re\mathbf{a}\widetilde{H}\right)$ in every instant t:

$\mathbf{b}:\left\langle x_{\mathbf{b},1}+v\cdot t,x_{\mathbf{b},2},x_{\mathbf{b},3},\ldots,x_{\mathbf{b},\mu}\right\rangle$,

$\mathbf{g}_j$: $\left\langle y_{j,1}+v\cdot t,y_{j,2},y_{j,3},\ldots,y_{j,\mu}\right\rangle$,

$\mathbf{u}_j$: $\left\langle y_{j,1}+v\cdot t,y_{j,2}+l/\left(k_1\cdot\ldots\cdot k_j\right),y_{j,3},\ldots,y_{j,\mu}\right\rangle$,

for every recorder $\mathbf{q}_i$: if $\mathbf{q}_i\in\Im$ then coordinates of

$\mathbf{q}_i:\left\langle x_{i,1}+v\cdot t,x_{i,2},x_{i,3},\ldots,x_{i,\mu}\right\rangle$,

$\widetilde{T}$ is $\left\langle \{\mathbf{g}_1, A, \mathbf{u}_1\}, \{\mathbf{g}_2, A, \mathbf{u}_2\}, .., \{\mathbf{g}_j, A, \mathbf{u}_j\}, .. \right\rangle$.
$C : \langle C_1, C_2, C_3, \ldots, C_\mu \rangle$,
$D : \langle D_1, D_2, D_3, \ldots, D_\mu \rangle$,
$t_C = \left[C \mid \Re\mathbf{a}\widetilde{H}\right]$,
$t_D = \left[D \mid \Re\mathbf{a}\widetilde{H}\right]$;
2) in a coordinates system $R^{\mu\prime}$ of a frame of reference $\left(\Im\mathbf{b}\widetilde{T}\right)$:
$C : \langle C'_1, C'_2, C'_3, \ldots, C'_\mu \rangle$,
$D : \langle D'_1, D'_2, D'_3, \ldots, D'_\mu \rangle$,
$t'_C = \left[C \mid \Im\mathbf{b}\widetilde{T}\right]$,
$t'_D = \left[D \mid \Im\mathbf{b}\widetilde{T}\right]$.
In that case:

$$\begin{aligned} t'_D - t'_C &= \frac{(t_D - t_C) - v(D_1 - C_1)}{\sqrt{1 - v^2}}, \\ D'_1 - C'_1 &= \frac{(D_1 - C_1) - v(t_D - t_C)}{\sqrt{1 - v^2}}. \end{aligned}$$

Proof:
Let us designate:

$$\rho_{\mathbf{a},\mathbf{b}} := \left(\sum_{j=1}^{\mu} (b_j - a_j)^2\right)^{0.5}.$$

According to Definition 1.4.8 there exist elements $\mathbf{q}_C$ and $\mathbf{q}_D$ of set $\Im$ such that

$$\natural(\mathbf{b})(\mathbf{q}_C, C)), \natural(\mathbf{b})(\mathbf{q}_D, D)$$

and

$$\ell\left(\Im\mathbf{b}\widetilde{T}\right)(C, D) = \ell\left(\mathbf{b}, \widetilde{T}\right)(\mathbf{q}_C, \mathbf{q}_D).$$

In that case:

$$\begin{aligned} t'_C = \left[C \mid \Im\mathbf{b}\widetilde{T}\right] &= \left[\mathbf{q}_C^{\bullet} C \mid \Im\mathbf{b}\widetilde{T}\right], \\ t'_D = \left[D \mid \Im\mathbf{b}\widetilde{T}\right] &= \left[\mathbf{q}_D^{\bullet} D \mid \Im\mathbf{b}\widetilde{T}\right]. \end{aligned}$$

According to Corollary of Theorem 1.5.5:

$$\begin{aligned} \left[\mathbf{q}_C^{\bullet} C \mid \Re\mathbf{a}\widetilde{H}\right] &= \left[C \mid \Re\mathbf{a}\widetilde{H}\right] = t_C, \\ \left[\mathbf{q}_D^{\bullet} D \mid \Re\mathbf{a}\widetilde{H}\right] &= \left[D \mid \Re\mathbf{a}\widetilde{H}\right] = t_D. \end{aligned}$$

Let us designate:

$$\begin{aligned}
\tau_1 &:= \left[\mathbf{b}^{\bullet} C \uparrow \mathbf{b}, \widetilde{T}\right],\\
\tau_2 &:= \left[\mathbf{b}^{\bullet} D \uparrow \mathbf{b}, \widetilde{T}\right],\\
t_1 &:= \left[\mathbf{b}^{\bullet} C \mid \Re\mathbf{a}\widetilde{H}\right],\\
t_2 &:= \left[\mathbf{b}^{\bullet} D \mid \Re\mathbf{a}\widetilde{H}\right],\\
t_3 &:= \left[\mathbf{b}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right],\\
t_4 &:= \left[\mathbf{q}_C^{\bullet}\mathbf{b}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right],\\
t_5 &:= \left[\mathbf{b}^{\bullet}\mathbf{q}_C^{\bullet}\mathbf{b}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right],\\
t_6 &:= \left[\mathbf{q}_D^{\bullet}\mathbf{q}_C^{\bullet}\mathbf{b}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right],\\
t_7 &:= \left[\mathbf{q}_C^{\bullet}\mathbf{q}_D^{\bullet}\mathbf{q}_C^{\bullet}\mathbf{b}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right],\\
t_8 &:= \left[\mathbf{b}^{\bullet}\mathbf{q}_C^{\bullet}\mathbf{q}_D^{\bullet}\mathbf{q}_C^{\bullet}\mathbf{b}^{\bullet} B \mid \Re\mathbf{a}\widetilde{H}\right].
\end{aligned}$$

Under such designations:

$$\begin{aligned}
&t_8 - t_7 = t_5 - t_4 \text{ hence: } t_8 - t_5 = t_7 - t_4 \text{ and}\\
&\ell\left(\Im\mathbf{b}\widetilde{T}\right)(C,D)\\
&= 0.5\left(\left[\mathbf{b}^{\bullet}\mathbf{q}_C^{\bullet}\mathbf{q}_D^{\bullet}\mathbf{q}_C^{\bullet}\mathbf{b}^{\bullet} B \uparrow \mathbf{b}, \widetilde{T}\right] - \left[\mathbf{b}^{\bullet}\mathbf{q}_C^{\bullet}\mathbf{b}^{\bullet} B \uparrow \mathbf{b}, \widetilde{T}\right]\right),
\end{aligned}$$

hence:

$$\begin{gathered}
\ell\left(\Im\mathbf{b}\widetilde{T}\right)(C,D) = 0.5\,(t_8 - t_5)\sqrt{1-v^2} = 0.5\,(t_7 - t_4)\sqrt{1-v^2},\\
(t_7 - t_6)^2 = (x_{C,1} + vt_7 - x_{D,1} - vt_6)^2 + \sum_{j=2}^{\mu}(x_{C,j} - x_{D,j})^2,\\
(t_6 - t_4)^2 = (x_{D,1} + vt_6 - x_{C,1} - vt_4)^2 + \sum_{j=2}^{\mu}(x_{C,j} - x_{D,j})^2,
\end{gathered}$$

hence:

$$\begin{gathered}
(t_7 - t_6)^2 = v^2(t_7 - t_6)^2 + 2v(x_{C,1} - x_{D,1})(t_7 - t_6) + \rho^2_{\mathbf{q}_C,\mathbf{q}_D},\\
(t_6 - t_4)^2 = v^2(t_6 - t_4)^2 + 2v(x_{D,1} - x_{C,1})(t_6 - t_4) + \rho^2_{\mathbf{q}_D,\mathbf{q}_C}.
\end{gathered}$$

Sequencely:

$$t_7 - t_4 = \frac{2}{\sqrt{1-v^2}}\left(v^2(x_{D,1} - x_{C,1})^2 + \left(1 - v^2\right)\rho^2_{\mathbf{q}_C,\mathbf{q}_D}\right)^{0.5}.$$

Let us designate:

$$R_{\mathbf{a},\mathbf{b}} := \left(\rho^2_{\mathbf{a},\mathbf{b}} - v^2 \sum_{j=2}^{\mu} (a_j - b_j)^2\right)^{0.5}.$$

Under such designation:

$$\ell\left(\Im \mathbf{b}\widetilde{T}\right)(C,D) = \frac{R_{\mathbf{q}_C,\mathbf{q}_D}}{\sqrt{1-v^2}}.$$

Since

$$\begin{array}{c} C_1 = x_{C,1} + vt_C, D_1 = x_{D,1} + vt_D, \\ C_{j+1} = x_{C,j+1}, D_{j+1} = x_{D,j+1} \end{array}$$

then

$$R_{\mathbf{q}_C,\mathbf{q}_D} = \left(\begin{array}{c} v^2 (D_1 - vt_D - C_1 + vt_C)^2 \\ +(1-v^2)\left(\begin{array}{c} (D_1 - vt_D - C_1 + vt_C)^2 \\ +\sum_{j=2}^{\mu} (D_j - C_j)^2 \end{array} \right) \end{array} \right)^{0.5},$$

hence:

$$R_{\mathbf{q}_C,\mathbf{q}_D} = \left(\begin{array}{c} v^2 (t_D - t_C)^2 - 2v (t_D - t_C)(D_1 - C_1) \\ +\rho^2_{C,D} \\ -v^2 \sum_{j=2}^{\mu} (D_j - C_j)^2 \end{array} \right)^{0.5}. \tag{1.115}$$

Moreover, according to Definition 1.4.7:

$$\begin{array}{c} t'_D - t'_C = (\tau_2 - \tau_1) - \\ -\left(\ell\left(\mathbf{b},\widetilde{T}\right)(\mathbf{b},\mathbf{q}_D) - \ell\left(\mathbf{b},\widetilde{T}\right)(\mathbf{b},\mathbf{q}_C)\right) \end{array} \tag{1.116}$$

According to Theorem 1.5.5:

$$\tau_2 - \tau_1 = (t_2 - t_1)\sqrt{1-v^2}. \tag{1.117}$$

According to Theorem 1.5.3:

$$\begin{array}{c} (t_1 - t_C)^2 = (x_{\mathbf{b},1} + vt_1 - C_1)^2 + \sum_{j=2}^{\mu} (x_{\mathbf{b},j} - C_j)^2, \\ (t_2 - t_D)^2 = (x_{\mathbf{b},1} + vt_2 - D_1)^2 + \sum_{j=2}^{\mu} (x_{\mathbf{b},j} - D_j)^2. \end{array}$$

Therefore,

$$\begin{array}{c} (t_1 - t_C)^2 = v^2 (t_1 - t_C)^2 + 2v (x_{\mathbf{b},1} - x_{C,1})(t_1 - t_C) + \rho^2_{\mathbf{b},\mathbf{q}_C}, \\ (t_2 - t_D)^2 = v^2 (t_2 - t_D)^2 + 2v (x_{\mathbf{b},1} - x_{D,1})(t_2 - t_D) + \rho^2_{\mathbf{b},\mathbf{q}_D}. \end{array}$$

Hence,

$$\begin{array}{c} t_2 - t_1 = \\ = (t_D - t_C) + \frac{v}{1-v^2}\left(x_{C,1} - x_{D,1}\right) \\ +\frac{1}{1-v^2}\left(R_{\mathbf{b},\mathbf{q}_D} - R_{\mathbf{b},\mathbf{q}_C}\right). \end{array}$$

Because

$$\ell\left(\mathbf{b},\widetilde{T}\right)(\mathbf{b},\mathbf{q}_D) = \frac{R_{\mathbf{b},\mathbf{q}_D}}{\sqrt{1-v^2}}, \ell\left(\mathbf{b},\widetilde{T}\right)(\mathbf{b},\mathbf{q}_C) = \frac{R_{\mathbf{b},\mathbf{q}_C}}{\sqrt{1-v^2}},$$

then from (1.116), (1.117), (1.118):

$$t'_D - t'_C = (t_D - t_C)\sqrt{1-v^2} - \frac{v}{\sqrt{1-v^2}}\left(x_{D,1} - x_{C,1}\right),$$

hence:

$$\begin{array}{rl} & t'_D - t'_C \\ = & (t_D - t_C)\sqrt{1-v^2} - \frac{v}{\sqrt{1-v^2}}\left((D_1 - C_1) - v\,(t_D - t_C)\right), \end{array}$$

hence:

$$\begin{array}{rcl} t'_D - t'_C & = & \frac{(t_D - t_C) - v\,(D_1 - C_1)}{\sqrt{1-v^2}}, \\ D'_1 - C'_1 & = & \frac{(D_1 - C_1) - v\,(t_D - t_C)}{\sqrt{1-v^2}}. \end{array}$$

It is the Lorentz spatial-temporal transformations[7] $\square$.

1.6. Probability

There is the evident nigh affinity between the classical probability function and the Boolean function of the classical propositional logic [9]. These functions are differed by the range of value, only. That is if the range of values of the Boolean function shall be expanded from the two-elements set $\{0;1\}$ to the segment $[0;1]$ of the real numeric axis then the logical analog of the Bernoulli Large Number Law [13] can be deduced from the logical axioms. These topics is considered in this article.

Further we consider set of all meaningfull sentences.

[7] Hendrik Antoon Lorentz (18 July 1853 - 4 February 1928) was a Dutch physicist who shared the 1902 Nobel Prize in Physics with Pieter Zeeman for the discovery and theoretical explanation of the Zeeman effect. He also derived the transformation equations subsequently used by Albert Einstein to describe space and time.

1.6.1. Events

Def. 1.6.1.1: A set $\mathcal{B}$ of sentences is called *event, expressed by sentence* C, if the following conditions are fulfilled:

1. $C \in \mathcal{B}$;
2. if $A \in \mathcal{B}$ and $D \in \mathcal{B}$ then $A = D$;
3. if $D \in \mathcal{B}$ and $A = D$ then $A \in \mathcal{B}$.

In this case denote: $\mathcal{B} := {}^{\circ}C$.

Def. 1.6.1.2: An event $\mathcal{B}$ *occurs* if here exists a true sentence A such that $A \in \mathcal{B}$.

Def. 1.6.1.3: Events $\mathcal{A}$ and $\mathcal{B}$ *equal* (denote: $\mathcal{A} = \mathcal{B}$) if $\mathcal{A}$ occurs if and only if $\mathcal{B}$ occurs.

Def. 1.6.1.4: Event $\mathcal{C}$ is called *product* of event $\mathcal{A}$ and event $\mathcal{B}$ (denote: $\mathcal{C} = (\mathcal{A} \cdot \mathcal{B})$) if $\mathcal{C}$ occurs if and only if $\mathcal{A}$ occurs and $\mathcal{B}$ occurs.

Def. 1.6.1.5: Events $\mathcal{C}$ is called *complement* of event $\mathcal{A}$ (denote: $\mathcal{C} = (\#\mathcal{A})$) if $\mathcal{C}$ occurs if and only if $\mathcal{A}$ does not occur.

Def. 1.6.1.6: $(\mathcal{A} + \mathcal{B}) := (\#((\#\mathcal{A}) \cdot (\#\mathcal{B})))$. Event $(\mathcal{A} + \mathcal{B})$ is called *sum* of event $\mathcal{A}$ and event $\mathcal{B}$.

Therefore, the sum of event occurs if and only if there is at least one of the addends.

Def. 1.6.1.7: *The authentic event* (denote: $\mathcal{T}$) is the event which contains a tautology.

Hence, $\mathcal{T}$ occurs in accordance Def. 1.6.1.2:

The impossible event (denote: $\mathcal{F}$) is event which contains negation of a tautology.

Hence, $\mathcal{F}$ does not occur.

1.6.2. B-functions

Def. 1.6.2.1: Let $\mathfrak{b}(X)$ be a function defined on the set of events.

And let this function has values on he real numbers segment $[0;1]$.

Let there exists an event $\mathcal{C}_0$ such that $\mathfrak{b}(\mathcal{C}_0) = 1$.

Let for all events $\mathcal{A}$ and $\mathcal{B}$: $\mathfrak{b}(\mathcal{A} \cdot \mathcal{B}) + \mathfrak{b}(\mathcal{A} \cdot (\#\mathcal{B})) = \mathfrak{b}(\mathcal{A})$.

In that case function $\mathfrak{b}(X)$ is called *B-function*.

By this definition:

$$\mathfrak{b}(\mathcal{A} \cdot \mathcal{B}) \leq \mathfrak{b}(\mathcal{A}). \tag{1.118}$$

Hence, $\mathfrak{b}(\mathcal{T} \cdot \mathcal{C}_0) \leq \mathfrak{b}(\mathcal{T})$. Because $\mathcal{T} \cdot \mathcal{C}_0 = \mathcal{C}_0$ (by Def.1.6.1.4 and Def.1.6.1.7) then $\mathfrak{b}(\mathcal{C}_0) \leq \mathfrak{b}(\mathcal{T})$. Because $\mathfrak{b}(\mathcal{C}_0) = 1$then

$$\mathfrak{b}(\mathcal{T}) = 1. \tag{1.119}$$

From Def.1.6.2.1: $\mathfrak{b}(\mathcal{T} \cdot \mathcal{B}) + \mathfrak{b}(\mathcal{T} \cdot (\#\mathcal{B})) = \mathfrak{b}(\mathcal{T})$. Because $\mathcal{T}\mathcal{D} = \mathcal{D}$ for any $\mathcal{D}$ then $\mathfrak{b}(\mathcal{B}) + \mathfrak{b}(\#\mathcal{B}) = \mathfrak{b}(\mathcal{T})$. Hence, by (1.119): for any $\mathcal{B}$:

$$\mathfrak{b}(\mathcal{B}) + b(\#\mathcal{B}) = 1. \tag{1.120}$$

Therefore, $\mathfrak{b}(\mathcal{T}) + b(\#\mathcal{T}) = 1$. Hence, in accordance $(1.119): 1 + b(\mathcal{F}) = 1$. Therefore,

$$\mathfrak{b}(\mathcal{F}) = 0. \tag{1.121}$$

In accordance with Def.1.6.2.1, Def.1.6.1.6, and (1.120):
$\mathfrak{b}(\mathcal{A}\cdot(\mathcal{B}+\mathcal{C}))=\mathfrak{b}(\mathcal{A}\cdot(\#((\#\mathcal{B})\cdot(\#\mathcal{C}))))=$
$=\mathfrak{b}(\mathcal{A})-\mathfrak{b}((\mathcal{A}\cdot(\#\mathcal{B}))\cdot(\#\mathcal{C}))=\mathfrak{b}(\mathcal{A})-\mathfrak{b}(\mathcal{A}\cdot(\#\mathcal{B}))+\mathfrak{b}((\mathcal{A}\cdot(\#\mathcal{B}))\cdot\mathcal{C})=$
$=\mathfrak{b}(\mathcal{A})-\mathfrak{b}(\mathcal{A})+\mathfrak{b}(\mathcal{A}\cdot\mathcal{B}))+\mathfrak{b}((\#\mathcal{B})\cdot(\mathcal{A}\cdot\mathcal{C}))=$
$=\mathfrak{b}(\mathcal{A}\cdot\mathcal{B}))+\mathfrak{b}(\mathcal{A}\cdot\mathcal{C})-\mathfrak{b}(\mathcal{B}\cdot\mathcal{A}\cdot\mathcal{C}).$
And $\mathfrak{b}((\mathcal{A}\cdot\mathcal{B})+(\mathcal{A}\cdot\mathcal{C}))=\mathfrak{b}(\#((\#(\mathcal{A}\cdot\mathcal{B}))\cdot(\#(\mathcal{A}\cdot\mathcal{C}))))=$
$=1-\mathfrak{b}((\#(\mathcal{A}\cdot\mathcal{B}))\cdot(\#(\mathcal{A}\cdot\mathcal{C})))=$
$=1-\mathfrak{b}(\#(\mathcal{A}\cdot\mathcal{B}))+\mathfrak{b}((\#(\mathcal{A}\cdot\mathcal{B}))\cdot(\mathcal{A}\cdot\mathcal{C}))=$
$=1-1+\mathfrak{b}(\mathcal{A}\cdot\mathcal{B})+\mathfrak{b}((\#(\mathcal{A}\cdot\mathcal{B}))\cdot(\mathcal{A}\cdot\mathcal{C}))=$
$=\mathfrak{b}(\mathcal{A}\cdot\mathcal{B})+\mathfrak{b}((\mathcal{A}\cdot\mathcal{C}))-\mathfrak{b}((\mathcal{A}\cdot\mathcal{B})\cdot(\mathcal{A}\cdot\mathcal{C}))=$
$=\mathfrak{b}(\mathcal{A}\cdot\mathcal{B})+\mathfrak{b}((\mathcal{A}\cdot\mathcal{C}))-\mathfrak{b}(\mathcal{A}\cdot\mathcal{B}\cdot\mathcal{C})$ because $\mathcal{A}\cdot\mathcal{A}=\mathcal{A}$.
Therefore:

$$\mathfrak{b}(\mathcal{A}\cdot(\mathcal{B}+\mathcal{C}))=\mathfrak{b}(\mathcal{A}\cdot\mathcal{B})+(\mathcal{A}\cdot\mathcal{C})-\mathfrak{b}(\mathcal{A}\cdot\mathcal{B}\cdot\mathcal{C})) \tag{1.122}$$

and

$$\mathfrak{b}((\mathcal{A}\cdot\mathcal{B})+(\mathcal{A}\cdot\mathcal{C}))=\mathfrak{b}(\mathcal{A}\cdot\mathcal{B}))+\mathfrak{b}(\mathcal{A}\cdot\mathcal{C})-\mathfrak{b}(\mathcal{A}\cdot\mathcal{B}\cdot\mathcal{C}). \tag{1.123}$$

Hence (*distributivity*):

$$\mathfrak{b}(\mathcal{A}\cdot(\mathcal{B}+\mathcal{C}))=\mathfrak{b}((\mathcal{A}\cdot\mathcal{B})+(\mathcal{A}\cdot\mathcal{C})). \tag{1.124}$$

If $\mathcal{A}=\mathcal{T}$ then from (1.122) and (1.123) (*the addition formula of probabilities*):

$$\mathfrak{b}(\mathcal{B}+\mathcal{C})=\mathfrak{b}(\mathcal{B})+\mathfrak{b}(\mathcal{C})-\mathfrak{b}(\mathcal{B}\cdot\mathcal{C}). \tag{1.125}$$

Def. 1.6.2.2– 19: Events $\mathcal{B}$ and $\mathcal{C}$ are *antithetical events* if $(\mathcal{B}\cdot\ \mathcal{C})=\mathcal{F}$.
From (1.125) and (1.121) for antithetical events $\mathcal{B}$ and $\mathcal{C}$:

$$\mathfrak{b}(\mathcal{B}+\mathcal{C})=\mathfrak{b}(\mathcal{B})+\mathfrak{b}(\mathcal{C}). \tag{1.126}$$

Def. 1.6.2.3-20: Events $\mathcal{B}$ and $\mathcal{C}$ are *independent for $\mathcal{B}$-function* $\mathfrak{b}$ *events* if $\mathfrak{b}(\mathcal{B}\cdot\mathcal{C})=\mathfrak{b}(\mathcal{B})\cdot\mathfrak{b}(\mathcal{B})$.

If events $\mathcal{B}$ and $\mathcal{C}$ are independent for B-function $\mathfrak{b}$ events then:
$\mathfrak{b}(\mathcal{B}\cdot(\#\mathcal{C}))=\mathfrak{b}(\mathcal{B})-\mathfrak{b}(\mathcal{B}\cdot\mathcal{C})=\mathfrak{b}(\mathcal{B})-\mathfrak{b}(\mathcal{B})\cdot\mathfrak{b}(\mathcal{C})=\mathfrak{b}(\mathcal{B})\cdot(1-\mathfrak{b}(\mathcal{C}))=\mathfrak{b}(\mathcal{B})\cdot\mathfrak{b}(\#\mathcal{C})$.
Hence, if events $\mathcal{B}$ and $\mathcal{C}$ are independent for $\mathcal{B}$-function b events then:

$$\mathfrak{b}(\mathcal{B}\cdot(\#\mathcal{C}))=\mathfrak{b}(\mathcal{B})\cdot\mathfrak{b}(\#\mathcal{C}). \tag{1.127}$$

Let calculate:

$$\mathfrak{b}(\mathcal{A}\cdot(\#\mathcal{A})\cdot\mathcal{C})=\mathfrak{b}(\mathcal{A}\cdot\mathcal{C})-\mathfrak{b}(\mathcal{A}\cdot\mathcal{A}\cdot\mathcal{C})=\mathfrak{b}(\mathcal{A}\cdot\mathcal{C})-\mathfrak{b}(\mathcal{A}\cdot\mathcal{C})=0. \tag{1.128}$$

1.6.3. Independent Tests

Definition 1.6.3.1: Let $st(n)$ be a function such that $st(n)$ has domain on the set of natural numbers and has values in the events set.

In this case event $\mathcal{A}$ is a *[st]-series of range r* with *V- number k* if A, r and k fulfill to some one amongst the following conditions:

1) $r = 1$ and $k = 1$, $\mathcal{A} = st(1)$ or $k = 0$, $\mathcal{A} = (\#st(1))$;

2) $\mathcal{B}$ is [st]-series of range $r-1$ with V-number $k-1$ and

$$\mathcal{A} = (\mathcal{B} \cdot st(r)),$$

or $\mathcal{B}$ is [st]-series of range $r-1$ with V-number k and

$$\mathcal{A} = (\mathcal{B} \cdot (\#st(r))).$$

Let us denote a set of [st]-series of range r with V-number k as $[st](r,k)$.

For example, if $st(n)$ is a event $\mathcal{B}_n$ then the sentences:

$(\mathcal{B}_1 \cdot \mathcal{B}_2 \cdot (\#\mathcal{B}_3))$, $(\mathcal{B}_1 \cdot (\#\mathcal{B}_2) \cdot \mathcal{B}_3)$, $((\#\mathcal{B}_1) \cdot \mathcal{B}_2 \cdot \mathcal{B}_3)$

are the elements of $[st](3,2)$, and

$\left(\mathcal{B}_1 \cdot \mathcal{B}_2 \cdot (\#\mathcal{B}_3) \cdot \mathcal{B}_4 \cdot \overline{\mathcal{B}_5}\right) \in [st](5,3)$.

Definition 1.6.3.2: Function $st(n)$ is *independent* for B-function $\mathfrak{b}$ if for $\mathcal{A}$: if $\mathcal{A} \in [st](r,r)$ then:

$$\mathfrak{b}(\mathcal{A}) = \prod_{n=1}^{r} \mathfrak{b}(st(n)).$$

Definition 1.6.3.3: Let $st(n)$ be a function such that $st(n)$ has domain on the set of natural numbers and has values in the set of events.

In this case sentence $\mathcal{A}$ is *[st]-disjunction of range r* with *V-number k* (denote: $t[st](r,k)$) if $\mathcal{A}$ is the disjunction of all elements of $[st](r,k)$.

For example, if $st(n)$ is event $\mathcal{C}_n$ then:

$((\#\mathcal{C}_1) \cdot (\#\mathcal{C}_2) \cdot (\#\mathcal{C}_3)) = t[st](3,0)$,

$t[st](3,1) = ((\mathcal{C}_1 \cdot (\#\mathcal{C}_2) \cdot (\#\mathcal{C}_3)) + ((\#\mathcal{C}_1) \cdot \mathcal{C}_2 \cdot (\#\mathcal{C}_3)) + ((\#\mathcal{C}_1) \cdot (\#\mathcal{C}_2) \cdot \mathcal{C}_3))$,

$t[st](3,2) = ((\mathcal{C}_1 \cdot \mathcal{C}_2 \cdot (\#\mathcal{C}_3)) + ((\#\mathcal{C}_1) \cdot \mathcal{C}_2 \cdot \mathcal{C}_3) + (\mathcal{C}_1 \cdot (\#\mathcal{C}_2) \cdot \mathcal{C}_3))$,

$(\mathcal{C}_1 \cdot \mathcal{C}_2 \cdot \mathcal{C}_3) = t[st](3,3)$.

Definition 1.6.3.4: A rational number ω is called *frequency of sentence $\mathcal{A}$* in the [st]-series of r independent for B-function $\mathfrak{b}$ tests (designate: $\omega = \nu_r[st](\mathcal{A})$) if

1) $st(n)$ is independent for B-function $\mathfrak{b}$,

2) for all n: $\mathfrak{b}(st(n)) = \mathfrak{b}(\mathcal{A})$,

3) $t[st](r,k)$ is true and $\omega = k/r$.

Theorem: 1.6.3.1: (the J.Bernoulli[8] formula [13]) If $st(n)$ is independent for B-function $\mathfrak{b}$ and there exists a real number p such that for all n: $\mathfrak{b}(st(n)) = p$ then

$$\mathfrak{b}(t[st](r,k)) = \frac{r!}{k! \cdot (r-k)!} \cdot p^k \cdot (1-p)^{r-k}.$$

[8]Jacob Bernoulli (also known as James or Jacques) (27 December 1654 16 August 1705) was one of the many prominent mathematicians in the Bernoulli family.

Proof of the Theorem 1.6.3.1: By the Definition 1.6.3.2 and formula (1.127): if $\mathcal{B} \in [st](r,k)$ then:

$$\mathfrak{b}(\mathcal{B}) = p^k \cdot (1-p)^{r-k}.$$

Since $[st](r,k)$ contains $r!/(k! \cdot (r-k)!)$ elements then by the Theorems (1.127), (1.128) and (1.126) this Theorem is fulfilled.

Definition 1.6.3.5: Let function $st(n)$ has domain on the set of the natural numbers and has values in the set of the events.

Let function $f(r,k,l)$ has got the domain in the set of threes of the natural numbers and has got the range of values in the set of the events.

In this case $f(r,k,l) = T[st](r,k,l)$ if

1) $f(r,k,k) = t[st](r,k)$,

2) $f(r,k,l+1) = (f(r,k,l) + t[st](r,l+1))$.

Definition 1.6.3.6: If a and b are real numbers and $k-1 < a \leq k$ and $l \leq b < l+1$ then $T[st](r,a,b) = T[st](r,k,l)$.

Theorem: 1.6.3.2:

$$T[st](r,a,b) = ^{\circ}\ll \frac{a}{r} \leq \nu_r[st](\mathcal{A}) \leq \frac{b}{r} \gg .$$

Proof of the Theorem 1.6.3.2: By the Definition 1.6.3.6: there exist natural numbers r and k such that $k-1 < a \leq k$ and $l \leq b < l+1$.

The recursion on l:

1. Let $l = k$.

In this case by the Definition 1.6.3.4:

$$T[st](r,k,k) = t[st](r,k) = ^{\circ}\ll \nu_r[st](\mathcal{A}) = \frac{k}{r} \gg .$$

2. Let n be any natural number.

The recursive assumption: Let

$$T[st](r,k,k+n) = ^{\circ}\ll \frac{k}{r} \leq \nu_r[st](\mathcal{A}) \leq \frac{k+n}{r} \gg .$$

By the Definition 1.6.3.5:

$$T[st](r,k,k+n+1) = (T[st](r,k,k+n) + t[st](r,k+n+1)).$$

By the recursive assumption and by the Definition 1.6.3.4:

$$T[st](r,k,k+n+1) =$$

$$= (^{\circ}\ll \frac{k}{r} \leq \nu_r[st](\mathcal{A}) \leq \frac{k+n}{r} \gg +^{\circ} \ll \nu_r[st](\mathcal{A}) = \frac{k+n+1}{r} \gg).$$

Hence, by the Definition 2.10:

$$T[st](r,k,k+n+1) =^\circ \ll \frac{k}{r} \le \nu_r [st] (\mathcal{A}) \le \frac{k+n+1}{r} \gg .$$

Theorem: 1.6.3.3 If $st(n)$ is independent for B-function $\mathfrak{b}$ and there exists a real number p such that $\mathfrak{b}(st(n)) = p$ for all n then

$$\mathfrak{b}(T[st](r,a,b)) = \sum_{a \le k \le b} \frac{r!}{k! \cdot (r-k)!} \cdot p^k \cdot (1-p)^{r-k}.$$

Proof of the Theorem 1.6.3.3: This is the consequence from the Theorem 1.6.3.1 by the Theorem 3.6.

Theorem: 1.6.3.4 If $st(n)$ is independent for the B-function $\mathfrak{b}$ and there exists a real number p such that $\mathfrak{b}(st(n)) = p$ for all n then

$$\mathfrak{b}(T[st](r, r \cdot (p-\varepsilon), r \cdot (p+\varepsilon))) \ge 1 - \frac{p \cdot (1-p)}{r \cdot \varepsilon^2}$$

for every positive real number ε.

Proof of the Theorem 1.6.3.4: Because

$$\sum_{k=0}^{r} (k - r \cdot p)^2 \cdot \frac{r!}{k! \cdot (r-k)!} \cdot p^k \cdot (1-p)^{r-k} = r \cdot p \cdot (1-p)$$

then if

$$J = \{k \in \mathbf{N} | 0 \le k \le r \cdot (p-\varepsilon)\} \cap \{k \in \mathbf{N} | r \cdot (p+\varepsilon) \le k \le r\}$$

then

$$\sum_{k \in J} \frac{r!}{k! \cdot (r-k)!} \cdot p^k \cdot (1-p)^{r-k} \le \frac{p \cdot (1-p)}{r \cdot \varepsilon^2}.$$

Hence, by (1.120) this Theorem is fulfilled.

Hence

$$\lim_{r \to \infty} \mathfrak{b}(T[st](r, r \cdot (p-\varepsilon), r \cdot (p+\varepsilon))) = 1 \tag{1.129}$$

for all tiny positive numbers ε.

1.6.4. The logic probability function

Definition 1.6.4.1: B-function P is P-*function* if for every event $^\circ \ll \Theta \gg$:
If $P(^\circ \ll \Theta \gg) = 1$ then $\ll \Theta \gg$ is true sentence.

Hence from Theorem 1.6.3.2 and (1.129): if $\mathfrak{b}$ is a P-function then the sentence

$$\ll (p-\varepsilon) \le \nu_r [st] (\mathcal{A}) \le (p+\varepsilon) \gg$$

is almost true sentence for large r and for all tiny ε. Therefore, it is almost truely that

$$\nu_r [st] (\mathcal{A}) = p$$

for large r.

Therefore, it is almost true that

$$\mathfrak{b} (\mathcal{A}) = \nu_r [st] (\mathcal{A})$$

for large r.

Therefore, the function, defined by the Definition 1.6.4.1 has got the statistical meaning. That is why I'm call such function as the logic probability function.

1.6.5. Conditional probability

Definition 1.6.5.1: *Conditional probability* $\mathcal{B}$ for $\mathcal{C}$ is the following function:

$$\mathfrak{b} (\mathcal{B}/\mathcal{C}) := \frac{\mathfrak{b} (\mathcal{C} \cdot \mathcal{B})}{\mathfrak{b} (\mathcal{C})}. \tag{1.130}$$

Theorem 1.6.5.1 The conditional probability function is a B-function.

Proof of Theorem 1.6.5.1 From Definition 1.6.5.1:

$$\mathfrak{b} (\mathcal{C}/\mathcal{C}) = \frac{\mathfrak{b} (\mathcal{C} \cdot \mathcal{C})}{\mathfrak{b} (\mathcal{C})}.$$

Hence by Theorem 1.1.1:

$$\mathfrak{b} (\mathcal{C}/\mathcal{C}) = \frac{\mathfrak{b} (\mathcal{C})}{\mathfrak{b} (\mathcal{C})} = 1.$$

Form Definition 1.6.5.1:

$$\mathfrak{b} ((\mathcal{A} \cdot \mathcal{B}) / \mathcal{C}) + \mathfrak{b} ((\mathcal{A} \cdot (\#\mathcal{B})) / \mathcal{C}) = \frac{\mathfrak{b} (\mathcal{C} \cdot (\mathcal{A} \cdot \mathcal{B}))}{\mathfrak{b} (\mathcal{C})} + \frac{\mathfrak{b} (\mathcal{C} \cdot (\mathcal{A} \cdot (\#\mathcal{B})))}{\mathfrak{b} (\mathcal{C})}.$$

Hence:

$$\mathfrak{b} ((\mathcal{A} \cdot \mathcal{B}) / \mathcal{C}) + \mathfrak{b} ((\mathcal{A} \cdot (\#\mathcal{B})) / \mathcal{C}) = \frac{\mathfrak{b} (\mathcal{C} \cdot (\mathcal{A} \cdot \mathcal{B})) + \mathfrak{b} (\mathcal{C} \cdot (\mathcal{A} \cdot (\#\mathcal{B})))}{\mathfrak{b} (\mathcal{C})}.$$

By Theorem 1.1.1:

$$\mathfrak{b} ((\mathcal{A} \cdot \mathcal{B}) / \mathcal{C}) + \mathfrak{b} ((\mathcal{A} \cdot (\#\mathcal{B})) / \mathcal{C}) = \frac{\mathfrak{b} ((\mathcal{C} \cdot \mathcal{A}) \cdot \mathcal{B}) + \mathfrak{b} ((\mathcal{C} \cdot \mathcal{A}) \cdot (\#\mathcal{B}))}{\mathfrak{b} (\mathcal{C})}.$$

Hence by Definition 1.6.2.1:

$$\mathfrak{b} ((\mathcal{A} \cdot \mathcal{B}) / \mathcal{C}) + \mathfrak{b} ((\mathcal{A} \cdot (\#\mathcal{B})) / \mathcal{C}) = \frac{\mathfrak{b} (\mathcal{C} \cdot \mathcal{A})}{\mathfrak{b} (\mathcal{C})}.$$

Hence by Definition 1.6.5.1:

$$\mathfrak{b} ((\mathcal{A} \cdot \mathcal{B}) / \mathcal{C}) + \mathfrak{b} ((\mathcal{A} \cdot (\#\mathcal{B})) / \mathcal{C}) = \mathfrak{b} (\mathcal{A}/\mathcal{C}) \square$$

1.6.6. Classical probability

Let P be *P*-function.

Definition 1.6.6.1: $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\}$ is called as *complete set* if the following conditions are fulfilled:

1. if $k \neq s$ then $(\mathcal{B}_k \cdot \mathcal{B}_s)$ is a false sentence;
2. $(\mathcal{B}_1 + \mathcal{B}_2 + \ldots + \mathcal{B}_n)$ is a true sentence.

Definition 1.6.6.2: $\mathcal{B}$ is favorable for $\mathcal{A}$ if $(\mathcal{B} \cdot (\#\mathcal{A}))$ is a false sentence, and $\mathcal{B}$ is unfavorable for $\mathcal{A}$ if $(\mathcal{B} \wedge A)$ is a false sentence.

Let

1. $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\}$ be complete set;
2. for $k \in \{1, 2, \ldots, n\}$ and $s \in \{1, 2, \ldots, n\}$: $\mathrm{P}(\mathcal{B}_k) = \mathrm{P}(\mathcal{B}_s)$;
3. if $1 \leq k \leq m$ then $\mathcal{B}_k$ is favorable for $\mathcal{A}$, and if $m+1 \leq s \leq n$ then $\mathcal{B}_s$ is unfavorable for $\mathcal{A}$.

In that case from Theorem 1.1.1 and from (1.119) and (1.120):

$$\mathrm{P}((\#\mathcal{A}) \cdot \mathcal{B}_k) = 0$$

for $k \in \{1, 2, \ldots, m\}$ and

$$\mathrm{P}(\mathcal{A} \cdot \mathcal{B}_s) = 0$$

for $s \in \{m+1, m+2, \ldots, n\}$.

Hence from Definition 1.6.2.1:

$$\mathrm{P}(\mathcal{A} \cdot \mathcal{B}_k) = \mathrm{P}(\mathcal{B}_k)$$

for $k \in \{1, 2, \ldots, n\}$.

By point 4 of Theorem 1.1.1:

$$\mathcal{A} = (\mathcal{A} \cdot (\mathcal{B}_1 + \mathcal{B}_2 + \ldots + \mathcal{B}_m + \mathcal{B}_{m+1} \ldots + \mathcal{B}_n)).$$

Hence by formula (1.124):

$\mathrm{P}(\mathcal{A}) = \mathrm{P}(\mathcal{A} \cdot \mathcal{B}_1) + \mathrm{P}(\mathcal{A} \cdot \mathcal{B}_2) + \ldots +$
$+ \mathrm{P}(\mathcal{A} \cdot \mathcal{B}_m) + \mathrm{P}(\mathcal{A} \cdot \mathcal{B}_{m+1}) + \ldots + \mathrm{P}(\mathcal{A} \cdot \mathcal{B}_n) =$
$= \mathrm{P}(\mathcal{B}_1) + \mathrm{P}(\mathcal{B}_2) + \ldots + \mathrm{P}(\mathcal{B}_m)$.

Therefore

$$\mathrm{P}(\mathcal{A}) = \frac{m}{n}.$$

1.6.7. Probability and Logic

Let P be the probability function and let B be the set of events A such that either A occurs or $(\#A)$ occurs.

In this case if $\mathrm{P}(A) = 1$ then A occurs, and $(A \cdot B) = B$ in accordance with Def. 1.6.1.4. Consequently, if $\mathrm{P}(B) = 1$ then $\mathrm{P}(A \cdot B) = 1$. Hence, in this case $\mathrm{P}(A \cdot B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$.

If $\mathrm{P}(A) = 0$ then $\mathrm{P}(A \cdot B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$ because $\mathrm{P}(A \cdot B) \leq \mathrm{P}(A)$ in accordance with (1.118).

Moreover in accordance with (1.120): $\mathrm{P}(\#A) = 1 - \mathrm{P}(A)$ since the function P is a B-function.

If event A occurs then $(A \cdot B) = B$ and $(A \cdot (\#B)) = (\#B)$ Hence, $\mathrm{P}(A \cdot B) + \mathrm{P}(A \cdot (\#B)) = \mathrm{P}(A) = \mathrm{P}(B) + \mathrm{P}(\#B) = 1$.

Consequently, if an element A of B occurs then $\mathrm{P}(A) = 1$. If does not occurs then $(\#A)$ occurs. Hence, $\mathrm{P}(\#A) = 1$ and because $\mathrm{P}(A) + \mathrm{P}(\#A) = 1$ then $\mathrm{P}(A) = 0$. Therefore, on B the range of values of is the two-element set $\{0;1\}$ similar the Boolean function range of values. Hence, on set B the probability function obeys definition of a Boolean function (Def.1.1.10).

The logic probability function is the extension of the logic B-function. Therefore, **the probability is some generalization of the classic propositional logic.** That is the probability is the logic of events such that these events do not happen, yet.

Chapter 2

Quants

Quantum theory evolved as a new branch of theoretical physics during the first few decades of the 20th century in an endeavour to understand the fundamental properties of matter. It began with the study of the interactions of matter and radiation. Certain radiation effects could neither be explained by classical mechanics, nor by the theory of electromagnetism.

Quantum theory was not the work of one individual, but the collaborative effort of some of the most brilliant physicists of the 20th century, among them Niels Bohr[1], Erwin Schrodinger[2], Wolfgang Pauli[3], and Max Born[4], Max Planck[5] and Werner Heisenberg[6].

Quantum Field Theory (QFT) is the mathematical and conceptual framework for contemporary elementary particle physics (Eugene Wigner[7], Hans Bethe[8], Tomonaga[9], Schwinger[10], Feynman[11], Dyson[12], Yang[13] and Mills[14]).

2.1. Physical Events and Equation of Moving

Denote:

$$\mathbf{x} \ : \ = (x_1, x_2, x_3),$$

[1]Niels Henrik David Bohr (7 October 1885 - 18 November 1962) was a Danish physicist

[2]Erwin Rudolf Josef Alexander Schrodinger (12 August 1887 - 4 January 1961) was an Austrian physicist and theoretical biologist who was one of the fathers of quantum mechanics

[3]Wolfgang Ernst Pauli (25 April 1900 15 December 1958) was an Austrian theoretical physicist

[4]Max Born (11 December 1882 5 January 1970) was a German-born physicist and mathematician

[5]Max Karl Ernst Ludwig Planck (April 23, 1858 October 4, 1947) was a German physicist

[6]Werner Karl Heisenberg (5 December 1901 1 February 1976) was a German theoretical physicist

[7]Eugene Paul Wigner (Hungarian Wigner Jeno Pal; November 17, 1902 - January 1, 1995) was a Hungarian American physicist and mathematician.

[8]Hans Albrecht Bethe (July 2, 1906 - March 6, 2005) [1] was a German-American nuclear physicist,

[9]Sin-Itiro Tomonaga (March 31, 1906 July 8, 1979) was a Japanese physicist

[10]Julian Seymour Schwinger (February 12, 1918 - July 16, 1994) was an American theoretical physicist.

[11]Richard Phillips Feynman (May 11, 1918 - February 15, 1988)[2] was an American physicist

[12]Freeman John Dyson FRS (born December 15, 1923) is a British-born American theoretical physicist and mathematician

[13]Chen-Ning Franklin Yang (born October 1, 1922) is a Chinese-American physicist

[14]Robert L. Mills (April 15, 1927 – October 27, 1999) was an English physicist

$$\begin{aligned}
\underline{x} &: = (x_0, \mathbf{x}),\\
\int d^{3+1}\underline{x} &: = \int dx_0 \int dx_1 \int dx_2 \int dx_3,\\
\int d^3\mathbf{y} &: = \int dy_1 \int dy_2 \int dy_3,\\
t &: = \frac{x_0}{\mathrm{c}}.
\end{aligned}$$

Sentence of type: $\ll$Event $\mathcal{A}$ occurs in point $\underline{x} \gg$ will be written the followig way: $\ll \mathcal{A}(\underline{x}) \gg$".

Events of type $^\circ \ll \mathcal{A}(\underline{x}) \gg$ are called *dot events.* All dot events and all events received from dot events by operations of addition, multiplication and addition, are *physical events*.

$\mathcal{A}(D)$ means: $(\mathcal{A}(\underline{x})\, \&^\circ \ll (\underline{x}) \in D \gg)$.

Let P be the probability function.

A function $p_{\mathcal{A}}(\underline{x})$ is called *absolute probability density* of event $\mathcal{A}$ if for any domain D: if $D \subseteq R^{\mu+1}$ then

$$\int_D d^{\mu+1}\underline{x} \cdot p_{\mathcal{A}}(\underline{x}) = \mathrm{P}(\mathcal{A}(D)).$$

If J is Jackobian of transformation

$$\begin{aligned}
x_0 \to x'_0 &= \frac{x_0 - vx_k}{\sqrt{1-v^2}},\\
x_k \to x'_k &= \frac{x_k - vx_0}{\sqrt{1-v^2}},\\
x_j \to x'_j &= x_j \text{ for } j \neq k
\end{aligned} \tag{2.1}$$

then

$$J = \frac{\partial(x'_0, \mathbf{x}')}{\partial(x_0, \mathbf{x})} = 1.$$

Hence, absolute probability density is invariant under the Lorentz transformations.

If

$$\rho_{\mathcal{A}}(x_0, \mathbf{x}) := \frac{p_{\mathcal{A}}(x_0, \mathbf{x})}{\int d^\mu \mathbf{y} \cdot p_{\mathcal{A}}(x_0, \mathbf{y})},$$

then $\rho_{\mathcal{A}}(x_0, \mathbf{x})$ is *a probability density of event* $\mathcal{A}$ in an instant x_0.

In transformations (2.1):

$$\begin{aligned}
&\rho_{\mathcal{A}}(x_0, \mathbf{x}) \to \rho'_{\mathcal{A}}(x_0, \mathbf{x}) =\\
= \;& \frac{p_{\mathcal{A}}(x_0, \mathbf{x})}{\int d^\mu \mathbf{y} \cdot p_{\mathcal{A}}(x_0 + v(y_k - x_k), \mathbf{y})}.
\end{aligned}$$

Therefore, $\rho_{\mathcal{A}}$ is not invariant under these transformations.

Further there are considered events $\mathcal{A}(\underline{x})$ such that $\rho_{\mathcal{A}}$ is the zero component of some $3+1$-vector field $\underline{j}_{\mathcal{A}}$

$(\underline{j}_{\mathcal{A}} = (j_{\mathcal{A},0}, \mathbf{j}_{\mathcal{A}}) = (j_{\mathcal{A},0}, j_{\mathcal{A},1}, j_{\mathcal{A},2}, j_{\mathcal{A},3}))$.

Hence, there exist real functions $j_{\mathcal{A},k}(\underline{x})$ such that[15]

$$\rho_{\mathcal{A}} = \frac{j_{\mathcal{A},0}}{\mathrm{c}}$$

and under transformations (2.1):

$$\begin{aligned} j_{\mathcal{A},0} \to j'_{\mathcal{A},0} &= \frac{j_{\mathcal{A},0} - v j_{\mathcal{A},k}}{\sqrt{1-v^2}}, \\ j_{\mathcal{A},k} \to j'_{\mathcal{A},k} &= \frac{j_{\mathcal{A},k} - v j_{\mathcal{A},0}}{\sqrt{1-v^2}}, \\ j_{\mathcal{A},s} \to j'_{\mathcal{A},s} &= j_{\mathcal{A},s} \text{ for } s \neq k. \end{aligned}$$

Denote:

$$1_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 0_2 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \beta^{[0]} := -\begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 1_2 \end{bmatrix} = -1_4,$$

the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A set $\widetilde{C}$ of complex $n \times n$ matrices is called *a Clifford set* [16] *of rank n* [15] if the following conditions are fulfilled:

if $\alpha_k \in \widetilde{C}$ and $\alpha_r \in \widetilde{C}$ then $\alpha_k\alpha_r + \alpha_r\alpha_k = 2\delta_{k,r}$;

if $\alpha_k\alpha_r + \alpha_r\alpha_k = 2\delta_{k,r}$ for all elements α_r of set $\widetilde{C}$ then $\alpha_k \in \widetilde{C}$.

If $n = 4$ then a Clifford set either contains 3 matrices (*a Clifford triplet*) or contains 5 matrices (*a Clifford pentad*).

Here exist only six Clifford pentads [15]: one *light pentad* β:

$$\begin{gathered} \beta^{[1]} := \begin{bmatrix} \sigma_1 & 0_2 \\ 0_2 & -\sigma_1 \end{bmatrix}, \beta^{[2]} := \begin{bmatrix} \sigma_2 & 0_2 \\ 0_2 & -\sigma_2 \end{bmatrix}, \\ \beta^{[3]} := \begin{bmatrix} \sigma_3 & 0_2 \\ 0_2 & -\sigma_3 \end{bmatrix}, \end{gathered} \tag{2.2}$$

$$\gamma^{[0]} := \begin{bmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{bmatrix}, \tag{2.3}$$

$$\beta^{[4]} := \mathrm{i} \cdot \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix}; \tag{2.4}$$

three *chromatic* pentads:

---

[15] c = 299792458

[16] William Kingdon Clifford (4 May 1845 3 March 1879) was an English mathematician and philosopher.

the red pentad ζ:

$$\zeta^{[1]}=\begin{bmatrix}-\sigma_1 & 0_2\\ 0_2 & \sigma_1\end{bmatrix},\zeta^{[2]}=\begin{bmatrix}\sigma_2 & 0_2\\ 0_2 & \sigma_2\end{bmatrix},\zeta^{[3]}=\begin{bmatrix}-\sigma_3 & 0_2\\ 0_2 & -\sigma_3\end{bmatrix}, \tag{2.5}$$

$$\gamma_\zeta^{[0]}=\begin{bmatrix}0_2 & -\sigma_1\\ -\sigma_1 & 0_2\end{bmatrix}, \zeta^{[4]}=\mathrm{i}\begin{bmatrix}0_2 & \sigma_1\\ -\sigma_1 & 0_2\end{bmatrix}; \tag{2.6}$$

the green pentad η:

$$\eta^{[1]}=\begin{bmatrix}-\sigma_1 & 0_2\\ 0_2 & -\sigma_1\end{bmatrix},\eta^{[2]}=\begin{bmatrix}-\sigma_2 & 0_2\\ 0_2 & \sigma_2\end{bmatrix},\eta^{[3]}=\begin{bmatrix}\sigma_3 & 0_2\\ 0_2 & \sigma_3\end{bmatrix}, \tag{2.7}$$

$$\gamma_\eta^{[0]}=\begin{bmatrix}0_2 & -\sigma_2\\ -\sigma_2 & 0_2\end{bmatrix}, \eta^{[4]}=\mathrm{i}\begin{bmatrix}0_2 & \sigma_2\\ -\sigma_2 & 0_2\end{bmatrix}; \tag{2.8}$$

the blue pentad θ:

$$\theta^{[1]}=\begin{bmatrix}\sigma_1 & 0_2\\ 0_2 & \sigma_1\end{bmatrix},\theta^{[2]}=\begin{bmatrix}-\sigma_2 & 0_2\\ 0_2 & -\sigma_2\end{bmatrix},\theta^{[3]}=\begin{bmatrix}-\sigma_3 & 0_2\\ 0_2 & \sigma_3\end{bmatrix}, \tag{2.9}$$

$$\gamma_\theta^{[0]}=\begin{bmatrix}0_2 & -\sigma_3\\ -\sigma_3 & 0_2\end{bmatrix},\theta^{[4]}=\mathrm{i}\begin{bmatrix}0_2 & \sigma_3\\ -\sigma_3 & 0_2\end{bmatrix}; \tag{2.10}$$

two *gustatory* pentads:
the sweet pentad $\underline{\Delta}$:

$$\underline{\Delta}^{[1]}=\begin{bmatrix}0_2 & -\sigma_1\\ -\sigma_1 & 0_2\end{bmatrix},\underline{\Delta}^{[2]}=\begin{bmatrix}0_2 & -\sigma_2\\ -\sigma_2 & 0_2\end{bmatrix},\underline{\Delta}^{[3]}=\begin{bmatrix}0_2 & -\sigma_3\\ -\sigma_3 & 0_2\end{bmatrix},$$

$$\underline{\Delta}^{[0]}=\begin{bmatrix}-1_2 & 0_2\\ 0_2 & 1_2\end{bmatrix},\underline{\Delta}^{[4]}=\mathrm{i}\begin{bmatrix}0_2 & 1_2\\ -1_2 & 0_2\end{bmatrix};$$

the bitter pentad $\underline{\Gamma}$:

$$\underline{\Gamma}^{[1]}=\mathrm{i}\begin{bmatrix}0_2 & -\sigma_1\\ \sigma_1 & 0_2\end{bmatrix},\underline{\Gamma}^{[2]}=\mathrm{i}\begin{bmatrix}0_2 & -\sigma_2\\ \sigma_2 & 0_2\end{bmatrix},\underline{\Gamma}^{[3]}=\mathrm{i}\begin{bmatrix}0_2 & -\sigma_3\\ \sigma_3 & 0_2\end{bmatrix},$$

$$\underline{\Gamma}^{[0]}=\begin{bmatrix}-1_2 & 0_2\\ 0_2 & 1_2\end{bmatrix},\underline{\Gamma}^{[4]}=\begin{bmatrix}0_2 & 1_2\\ 1_2 & 0_2\end{bmatrix}.$$

Further we do not consider gustatory pentads since these pentads are not used yet in the contemporary physics.

Let us consider the following set of four real equations with eight real unknowns: b^2 with $b>0$, α, β, χ, θ, γ, υ, λ:

$$\left\{\begin{array}{c} b^2=\rho_{\mathcal{A}},\\ b^2\left(\cos^2(\alpha)\sin(2\beta)\cos(\theta-\gamma)-\sin^2(\alpha)\sin(2\chi)\cos(\upsilon-\lambda)\right)=-\frac{j_{\mathcal{A},1}}{\mathrm{c}},\\ b^2\left(\cos^2(\alpha)\sin(2\beta)\sin(\theta-\gamma)-\sin^2(\alpha)\sin(2\chi)\sin(\upsilon-\lambda)\right)=-\frac{j_{\mathcal{A},2}}{\mathrm{c}},\\ b^2\left(\cos^2(\alpha)\cos(2\beta)-\sin^2(\alpha)\cos(2\chi)\right)=-\frac{j_{\mathcal{A},3}}{\mathrm{c}}. \end{array}\right| \tag{2.11}$$

This set has solutions for any $\rho_{\mathcal{A}}$ and $j_{\mathcal{A},k}$. For example one of these solutions is the following:

1. A value of b^2 obtain from first equation.

2. Let

$$u_{\mathcal{A},k} := \frac{j_{\mathcal{A},k}}{\rho_{\mathcal{A}}}. \tag{2.12}$$

In this case:

$$\left\{\begin{array}{c} \cos^2(\alpha)\sin(2\beta)\cos(\theta-\gamma)-\sin^2(\alpha)\sin(2\chi)\cos(\upsilon-\lambda)=-\frac{u_{\mathcal{A},1}}{c}, \\ \cos^2(\alpha)\sin(2\beta)\sin(\theta-\gamma)-\sin^2(\alpha)\sin(2\chi)\sin(\upsilon-\lambda)=-\frac{u_{\mathcal{A},2}}{c}, \\ \cos^2(\alpha)\cos(2\beta)-\sin^2(\alpha)\cos(2\chi)=-\frac{u_{\mathcal{A},3}}{c}. \end{array}\right|$$

3. Let $\beta = \chi$.

In that case:

$$\left\{\begin{array}{c} \left(\cos^2(\alpha)\cos(\theta-\gamma)-\sin^2(\alpha)\cos(\upsilon-\lambda)\right)\sin(2\beta)=-\frac{u_{\mathcal{A},1}}{c}, \\ \left(\cos^2(\alpha)\sin(\theta-\gamma)-\sin^2(\alpha)\sin(\upsilon-\lambda)\right)\sin(2\beta)=-\frac{u_{\mathcal{A},2}}{c}, \\ \left(\cos^2(\alpha)-\sin^2(\alpha)\right)\cos(2\beta)=-\frac{u_{\mathcal{A},3}}{c}. \end{array}\right|$$

4. Let $(\theta-\gamma) = (\upsilon-\lambda)$.

In that case:

$$\left\{\begin{array}{c} \cos(2\alpha)\cos(\theta-\gamma)\sin(2\beta)=-\frac{u_{\mathcal{A},1}}{c}, \\ \cos(2\alpha)\sin(\theta-\gamma)\sin(2\beta)=-\frac{u_{\mathcal{A},2}}{c}, \\ \cos(2\alpha)\cos(2\beta)=-\frac{u_{\mathcal{A},3}}{c}. \end{array}\right|$$

5. Let us raise to the second power the first and the second equations:

$$\left\{\begin{array}{c} \cos^2(2\alpha)\cos^2(\theta-\gamma)\sin^2(2\beta)=\left(-\frac{u_{\mathcal{A},1}}{c}\right)^2, \\ \cos^2(2\alpha)\sin^2(\theta-\gamma)\sin^2(2\beta)=\left(-\frac{u_{\mathcal{A},2}}{c}\right)^2, \\ \cos(2\alpha)\cos(2\beta)=-\frac{u_{\mathcal{A},3}}{c}. \end{array}\right|$$

and let us summat these two equations:

$$\left\{\begin{array}{c} \sin^2(2\beta)\cos^2(2\alpha)\left(\cos^2(\theta-\gamma)+\sin^2(\theta-\gamma)\right) \\ =\left(-\frac{u_{\mathcal{A},1}}{c}\right)^2+\left(-\frac{u_{\mathcal{A},2}}{c}\right)^2, \\ \cos(2\alpha)\cos(2\beta)=-\frac{u_{\mathcal{A},3}}{c}. \end{array}\right|$$

Hence:

$$\left\{\begin{array}{c} \sin^2(2\beta)\cos^2(2\alpha)=\left(-\frac{u_{\mathcal{A},1}}{c}\right)^2+\left(-\frac{u_{\mathcal{A},2}}{c}\right)^2, \\ \cos(2\alpha)\cos(2\beta)=-\frac{u_{\mathcal{A},3}}{c}. \end{array}\right|$$

6. Let us raise to the second power the second equation and add this equation to the previous one:

$$\left\{\begin{array}{c} \sin^2(2\beta)\cos^2(2\alpha)=\left(-\frac{u_{\mathcal{A},1}}{c}\right)^2+\left(-\frac{u_{\mathcal{A},2}}{c}\right)^2, \\ \cos^2(2\alpha)\cos^2(2\beta)=\left(-\frac{u_{\mathcal{A},3}}{c}\right)^2 \end{array}\right|$$

$$\left(\sin^2(2\beta)+\cos^2(2\beta)\right)\cos^2(2\alpha)=\left(-\frac{u_{\mathcal{A},1}}{c}\right)^2+\left(-\frac{u_{\mathcal{A},2}}{c}\right)^2+\left(-\frac{u_{\mathcal{A},3}}{c}\right)^2,$$

$$\cos^2(2\alpha)=\left(-\frac{u_{\mathcal{A},1}}{c}\right)^2+\left(-\frac{u_{\mathcal{A},2}}{c}\right)^2+\left(-\frac{u_{\mathcal{A},3}}{c}\right)^2, \tag{2.13}$$

We receive $\cos^2(2\alpha)$.

7. From

$$\cos^2(2\alpha)\cos^2(2\beta)=\left(-\frac{u_{\mathcal{A},3}}{c}\right)^2$$

we receive $\cos^2(2\beta)$.

8. From

$$\cos^2(2\alpha)\cos^2(\theta-\gamma)\sin^2(2\beta)=\left(-\frac{u_{\mathcal{A},1}}{c}\right)^2$$

we receive $\cos^2(\theta-\gamma)$.

———————————————————-

If

$$\begin{aligned}
\varphi_1 &:= b\exp(\mathrm{i}\gamma)\cos(\beta)\cos(\alpha),\\
\varphi_2 &:= b\exp(\mathrm{i}\theta)\sin(\beta)\cos(\alpha),\\
\varphi_3 &:= b\exp(\mathrm{i}\lambda)\cos(\chi)\sin(\alpha),\\
\varphi_4 &:= b\exp(\mathrm{i}\upsilon)\sin(\chi)\sin(\alpha)
\end{aligned} \tag{2.14}$$

then you can calculate that

$$\begin{aligned}
\rho_{\mathcal{A}} &= \sum_{s=1}^{4}\varphi_s^*\varphi_s, \\
\frac{j_{\mathcal{A},\alpha}}{c} &= -\sum_{k=1}^{4}\sum_{s=1}^{4}\varphi_s^*\beta_{s,k}^{[\alpha]}\varphi_k
\end{aligned} \tag{2.15}$$

with $\alpha\in\{1,2,3\}$.

Let $h=6.6260755\cdot 10^{-34}$ and $\underline{\Omega}$ ($\underline{\Omega}\subset R^{1+3}$) be the domain such that: if $\underline{x}\in\underline{\Omega}$ then $|x_r|\leq\frac{c\pi}{h}$ for $r\in\{0,1,2,3\}$.

And let Ω ($\Omega\subset\mathbf{R}^3$) be the domain such that: if $\mathbf{x}\in\Omega$ then $|x_r|\leq\frac{c\pi}{h}$ for $r\in\{1,2,3\}$.

Let $\Re_{\underline{\Omega}}$ be the set of functions such that for each element $\varphi(\underline{x})$ of this set: if $\underline{x}\notin\underline{\Omega}$ then $\varphi(\underline{x})=0$.

Hence:

$$\int_{(\underline{D})}d\underline{x}\cdot\varphi(\underline{x})=\int_{-\frac{c\pi}{h}}^{\frac{c\pi}{h}}dx_0\int_{-\frac{c\pi}{h}}^{\frac{c\pi}{h}}dx_1\int_{-\frac{c\pi}{h}}^{\frac{c\pi}{h}}dx_2\int_{-\frac{c\pi}{h}}^{\frac{c\pi}{h}}dx_3\cdot\varphi(\underline{x})$$

for every domain $\underline{D}$ ($\underline{D}\subset R^{1+3}$).

Let for each element $\varphi(\underline{x})$ of $\Re_{\underline{\Omega}}$ here exists a number J_φ such that

$$J_\varphi = \int_{(\underline{\Omega})} d\underline{x} \cdot \varphi^*(\underline{x})\, \varphi(\underline{x}).$$

And let $\Re_\Omega$ be the set of functions such that for each element $\varphi(t,\mathbf{x})$ of this set: if $\mathbf{x} \notin \Omega$ then $\varphi(t,\mathbf{x}) = 0$.

Hence:

$$\int_{(\mathbf{D})} d\mathbf{x} \cdot \varphi(t,\mathbf{x}) = \int_{-\frac{c\pi}{h}}^{\frac{c\pi}{h}} dx_1 \int_{-\frac{c\pi}{h}}^{\frac{c\pi}{h}} dx_2 \int_{-\frac{c\pi}{h}}^{\frac{c\pi}{h}} dx_3 \cdot \varphi(t,\mathbf{x})$$

for every domain $\mathbf{D}$ ($\mathbf{D} \subset R^3$).

Let for each element $\varphi(t,\mathbf{x})$ of $\Re_\Omega$ here exists a number $\mathbf{J}_\varphi$ such that

$$\mathbf{J}_\varphi = \int_{(\Omega)} d\underline{x} \cdot \varphi^*(t,\mathbf{x})\, \varphi(t,\mathbf{x}).$$

Let $\widetilde{u} \in \Re_{\underline{\Omega}}$ and $\widetilde{v} \in \Re_{\underline{\Omega}}$ and denote:

$$\widetilde{u} * \widetilde{v} := \int_{(\underline{\Omega})} d\underline{x} \cdot \widetilde{u}^*(\underline{x})\, \widetilde{v}(\underline{x}). \tag{2.16}$$

And let $\widetilde{\mathbf{u}} \in \Re_\Omega$ and $\widetilde{\mathbf{v}} \in \Re_\Omega$ and denote:

$$\widetilde{\mathbf{u}} * \widetilde{\mathbf{v}} := \int_{(\Omega)} d\mathbf{x} \cdot \widetilde{\mathbf{u}}^*(t,\mathbf{x})\, \widetilde{\mathbf{v}}(t,\mathbf{x}).$$

In that case operations $\widetilde{u} * \widetilde{v}$ and $\widetilde{\mathbf{u}} * \widetilde{\mathbf{v}}$ fulfil all four properties of a scalar product:

1. $(\widetilde{u} * \widetilde{v})^* = \int_{(\underline{\Omega})} d\underline{x} \cdot (\widetilde{u}^*(\underline{x})\, \widetilde{v}(\underline{x}))^* = \int_{(\underline{\Omega})} d\underline{x} \cdot \widetilde{v}^*(\underline{x})\, \widetilde{u}(\underline{x}) = \widetilde{v} * \widetilde{u}$;

2. $\widetilde{u} * (\widetilde{v} + f) = \int_{(\underline{\Omega})} d\underline{x} \cdot \widetilde{u}(\underline{x}) * (\widetilde{v}(\underline{x}) + f(\underline{x})) =$
$= \int_{(\underline{\Omega})} d\underline{x} \cdot \widetilde{u}(\underline{x}) * \widetilde{v}(\underline{x}) + \int_{(\widetilde{\Omega})} d\underline{x} \cdot \widetilde{u}(\underline{x}) * f(\underline{x}) = \widetilde{u} * \widetilde{v} + \widetilde{u} * f$;

3. if z is a complex number then
$\widetilde{u} * (z\widetilde{v}) = \int_{(\underline{\Omega})} d\underline{x} \cdot \widetilde{u}^*(\underline{x})\, (z\widetilde{v}(\underline{x})) = z \int_{(\underline{\Omega})} d\underline{x} \cdot \widetilde{u}^*(\underline{x})\, \widetilde{v}(\underline{x}) = z(\widetilde{u} * \widetilde{v})$;

4. $\widetilde{u} * \widetilde{u} = \int_{(\underline{\Omega})} d\underline{x} \cdot \widetilde{u}^*(\underline{x})\, \widetilde{u}(\underline{x}) = \int_{(\underline{\Omega})} d\underline{x} \cdot |\widetilde{u}(\underline{x})|^2 \geq 0$;

Therefore, these operations are scalar products on $\Re_{\underline{\Omega}}$ and $\Re_\Omega$ and, therefore, these linear spaces are unitary spaces.

Denote:

$$\begin{aligned} \varsigma_{w,\mathbf{p}}(t,\mathbf{x}) &:= \left\{ \begin{array}{c} \frac{h}{2\pi} \exp(\mathrm{i}hwt) \left(\frac{h}{2\pi c}\right)^{\frac{3}{2}} \exp\left(-\mathrm{i}\frac{h}{c}\mathbf{px}\right) \text{ if } \underline{x} \in \underline{\Omega}; \\ 0, \text{ otherwise.} \end{array} \right| \text{ and} \\ \varsigma_{\mathbf{p}}(\mathbf{x}) &:= \left\{ \begin{array}{c} \left(\frac{h}{2\pi c}\right)^{\frac{3}{2}} \exp\left(-\mathrm{i}\frac{h}{c}\mathbf{px}\right) \text{ if } \mathbf{x} \in \Omega; \\ 0, \text{ otherwise.} \end{array} \right| \end{aligned} \tag{2.17}$$

with natural w, p_1, p_2, p_3 (here: $\mathbf{p}\langle p_1, p_2, p_3\rangle$ and $\mathbf{px} = p_1x_1 + p_2x_2 + p_3x_3$).

Since

$$\begin{aligned}\int_{-\frac{c\pi}{h}}^{\frac{c\pi}{h}} \exp\left(\mathrm{i}k\frac{h}{c}x\right)\exp\left(-\mathrm{i}n\frac{h}{c}x\right)dx &= \frac{2\pi c}{h}\frac{\sin\pi(n-k)}{\pi(n-k)} \text{ and} \\ \frac{\sin\pi(n-k)}{\pi(n-k)} := \delta_{n,k} &= \left\{\begin{array}{c} 1, \text{ if } n=k, \\ 0, \text{ otherwise.}\end{array}\right. \end{aligned}$$

then

$$\varsigma_{\underline{p}} * \varsigma_{\underline{k}} = \delta_{\underline{p}} := \delta_{p_0,k_0}\delta_{p_1,k_1}\delta_{p_2,k_2}\delta_{p_3,k_3}.$$

and

$$\varsigma_{\mathbf{p}} * \varsigma_{\mathbf{k}} = \delta_{\mathbf{p}} := \delta_{p_1,k_1}\delta_{p_2,k_2}\delta_{p_3,k_3}.$$

Hence, functions $\varsigma_{\underline{p}}$ and $\varsigma_{\underline{k}}$ are orthogonal and normalized and ς_p and ς_k, too. Moreover if $\widetilde{u} \in \Re_{\underline{\Omega}}$ then (*Fourier series* of function $\widetilde{u}$)[17]

$$\widetilde{u}(t,x_1,x_2,x_3) = \sum_{w=-\infty}^{\infty}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{k_3=-\infty}^{\infty} a_{w,k_1,k_2,k_3}\varsigma_{w,k_1,k_2,k_3}(t,x_1,x_2,x_3)$$

with

$$a_{w,k_1,k_2,k_3} := \varsigma_{w,k_1,k_2,k_3} * \widetilde{u},$$

and if $\widetilde{\mathbf{u}} \in \Re_{\Omega}$ then

$$\widetilde{\mathbf{u}}(t,x_1,x_2,x_3) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{k_3=-\infty}^{\infty} a_{k_1,k_2,k_3}(t)\,\varsigma_{k_1,k_2,k_3}(x_1,x_2,x_3)$$

with

$$a_{k_1,k_2,k_3}(t) := \varsigma_{k_1,k_2,k_3} * \widetilde{\mathbf{u}}(t),$$

Therefore, functions $\varsigma_{w,k_1,k_2,k_3}$ form an orthonormalized basis in space $\Re_{\underline{\Omega}}$ and functions ς_{k_1,k_2,k_3} form an orthonormalized basis in space $\Re_{\Omega}$.

Let $j \in \{1,2,3,4\}$, $k \in \{1,2,3,4\}$ and denote:

$$\sum_{\mathbf{k}} := \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{k_3=-\infty}^{\infty}.$$

Let a Fourier series for $\varphi_j(t,\mathbf{x})$ (2.14) has the following form:

$$\varphi_j(t,\mathbf{x}) = \sum_{w=-\infty}^{\infty}\sum_{\mathbf{p}} c_{j,w,\mathbf{p}}\varsigma_{w,\mathbf{p}}(t,\mathbf{x}). \tag{2.18}$$

If denote: $\varphi_{j,w,\mathbf{p}}(t,\mathbf{x}) := c_{j,w,\mathbf{p}}\varsigma_{w,\mathbf{p}}(t,\mathbf{x})$ then a Fourier series for $\varphi_j(t,\mathbf{x})$ has got the following form:

[17]Jean Baptiste Joseph Fourier (21 March 1768 16 May 1830) was a French mathematician and physicist

$$\varphi_j(t,\mathbf{x}) = \sum_{w=-\infty}^{\infty}\sum_{\mathbf{p}}\varphi_{j,w,\mathbf{p}}(t,\mathbf{x}). \tag{2.19}$$

Let $\langle t,\mathbf{x}\rangle$ be any space-time point.
Denote:

$$A_k := \varphi_{k,w,\mathbf{p}}|_{\langle t,\mathbf{x}\rangle} \tag{2.20}$$

the value of function $\varphi_{k,w,\mathbf{p}}$ in this point, and:

$$C_j := \left(\frac{1}{\mathrm{c}}\partial_t\varphi_{j,w,\mathbf{p}} - \sum_{s=1}^{4}\sum_{\alpha=1}^{3}\beta_{j,s}^{[\alpha]}\partial_\alpha\varphi_{s,w,\mathbf{p}}\right)|_{\langle t,\mathbf{x}\rangle} \tag{2.21}$$

the value of function $\left(\frac{1}{\mathrm{c}}\partial_t\varphi_{j,w,\mathbf{p}} - \sum_{s=1}^{4}\sum_{\alpha=1}^{3}\beta_{j,s}^{[\alpha]}\partial_\alpha\varphi_{s,w,\mathbf{p}}\right)$.
Here A_k and C_j are complex numbers. Hence, the following set of equations:

$$\left\{\begin{array}{l} \sum_{k=1}^{4} z_{j,k,w,\mathbf{p}}A_k = C_j, \\ z_{j,k,w,\mathbf{p}}^{*} = -z_{k,j,w,\mathbf{p}} \end{array}\right| \tag{2.22}$$

is a system of 14 algebraic equation with complex unknowns $z_{k,j,w,\mathbf{p}}$ [18]:

[18] A calculation of the $\varphi_{j,w,\mathbf{p}}$ partial derivative of t is the following:

$$\begin{gathered}
\partial_t\varphi_{j,w,\mathbf{p}} = \partial_t c_{j,w,\mathbf{p}}\varsigma_{w,\mathbf{p}}(t,\mathbf{x}) \\
= c_{j,w,\mathbf{p}}\partial_t\left(\frac{\mathrm{h}}{2\pi}\exp(\mathrm{ih}wt)\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{px}\right)\right) = \\
= c_{j,w,\mathbf{p}}\frac{\mathrm{h}}{2\pi}\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{px}\right)\partial_t\left(\exp(\mathrm{ih}wt)\right) \\
= c_{j,w,\mathbf{p}}\frac{\mathrm{h}}{2\pi}\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{px}\right)\mathrm{ih}w\exp(\mathrm{ih}wt) = \\
= \mathrm{ih}wc_{j,w,\mathbf{p}}\left(\frac{\mathrm{h}}{2\pi}\exp(\mathrm{ih}wt)\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{px}\right)\right) \\
= \mathrm{ih}wc_{j,w,\mathbf{p}}\varsigma_{w,\mathbf{p}}(t,\mathbf{x}).
\end{gathered}$$

Hence

$$\partial_t\varphi_{j,w,\mathbf{p}} = \mathrm{ih}w\varphi_{j,w,\mathbf{p}}.$$

Similarly for $k \neq 0$:

$$\begin{aligned}
\partial_k\varphi_{j,w,\mathbf{p}} &= \partial_k c_{j,w,\mathbf{p}}\varsigma_{w,\mathbf{p}}(t,\mathbf{x}) \\
&= c_{j,w,\mathbf{p}}\partial_k\left(\frac{\mathrm{h}}{2\pi}\exp(\mathrm{ih}wt)\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{px}\right)\right) = \\
&= c_{j,w,\mathbf{p}}\frac{\mathrm{h}}{2\pi}\exp(\mathrm{ih}wt)\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\partial_k\left(\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{px}\right)\right) = \\
&= -\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}p_k c_{j,w,\mathbf{p}}\frac{\mathrm{h}}{2\pi}\exp(\mathrm{ih}wt)\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{px}\right) = \\
&= -\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}p_k c_{j,w,\mathbf{p}}\varsigma_{w,\mathbf{p}}(t,\mathbf{x}).
\end{aligned}$$

Hence:

$$
\begin{aligned}
&z_{1,1,w,\mathbf{p}}A_1+z_{1,2,w,\mathbf{p}}A_2+z_{1,3,w,\mathbf{p}}A_3+z_{1,4,w,\mathbf{p}}A_4=\\
&=\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(w+p_3\right)A_1+\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(p_1-ip_2\right)A_2,\\
&z_{2,1,w,\mathbf{p}}A_1+z_{2,2,w,\mathbf{p}}A_2+z_{2,3,w,\mathbf{p}}A_3+z_{2,4,w,\mathbf{p}}A_4=\\
&=\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(w-p_3\right)A_2+\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(p_1+ip_2\right)A_1,\\
&z_{3,1,w,\mathbf{p}}A_1+z_{3,2,w,\mathbf{p}}A_2+z_{3,3,w,\mathbf{p}}A_3+z_{3,4,w,\mathbf{p}}A_4=\\
&=\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(w-p_3\right)A_3-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(p_1-ip_2\right)A_4,\\
&z_{4,1,w,\mathbf{p}}A_1+z_{4,2,w,\mathbf{p}}A_2+z_{4,3,w,\mathbf{p}}A_3+z_{4,4}A_{4,w,\mathbf{p}}=\\
&=\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(w+p_3\right)A_4-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(p_1+ip_2\right)A_3,\\
&z^*_{1,1,w,\mathbf{p}}=-z_{1,1,w,\mathbf{p}},\\
&z^*_{1,2,w,\mathbf{p}}=-z_{2,1,w,\mathbf{p}},\\
&z^*_{1,3,w,\mathbf{p}}=-z_{3,1,w,\mathbf{p}},\\
&z^*_{1,4,w,\mathbf{p}}=-z_{4,1,w,\mathbf{p}},\\
&z^*_{2,2,w,\mathbf{p}}=-z_{2,2,w,\mathbf{p}},\\
&z^*_{2,3,w,\mathbf{p}}=-z_{3,2,w,\mathbf{p}},\\
&z^*_{2,4,w,\mathbf{p}}=-z_{4,2,w,\mathbf{p}},\\
&z^*_{3,3,w,\mathbf{p}}=-z_{3,3,w,\mathbf{p}},\\
&z^*_{3,4,w,\mathbf{p}}=-z_{4,3,w,\mathbf{p}},\\
&z^*_{4,4,w,\mathbf{p}}=-z_{4,4,w,\mathbf{p}}.
\end{aligned}
$$

This system can be transformed as system of 8 linear real equations with 16 real un-

---

$$\partial_k\varphi_{j,w,\mathbf{p}}=-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}p_k\varphi_{j,w,\mathbf{p}}.$$

Therefore,

$$C_j=\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(w\varphi_{j,w,\mathbf{p}}+\sum_{s=1}^{4}\sum_{\alpha=1}^{3}\beta^{[\alpha]}_{j,s}p_\alpha\varphi_{s,w,\mathbf{p}}\right)|_{\langle t,\mathbf{x}\rangle}.$$

knowns $x_{s,k} := \mathrm{Re}\left(z_{s,k,w,\mathbf{p}}\right)$ for $s<k$ and $y_{s,k} := \mathrm{Im}\left(z_{s,k,w,\mathbf{p}}\right)$ for $s \leq k$:

$$\left\{\begin{array}{c}-y_{1,1}b_1+x_{1,2}a_2-y_{1,2}b_2+x_{1,3}a_3-y_{1,3}b_3+x_{1,4}a_4-y_{1,4}b_4\\ =-\frac{\mathrm{h}}{\mathrm{c}}wb_1-\frac{\mathrm{h}}{\mathrm{c}}p_3b_1-\frac{\mathrm{h}}{\mathrm{c}}p_1b_2+\frac{\mathrm{h}}{\mathrm{c}}p_2a_2,\\ y_{1,1}a_1+x_{1,2}b_2+y_{1,2}a_2+x_{1,3}b_3+y_{1,3}a_3+x_{1,4}b_4+y_{1,4}a_4\\ =\frac{\mathrm{h}}{\mathrm{c}}wa_1+hp_3a_1+\frac{\mathrm{h}}{\mathrm{c}}p_1a_2+hp_2b_2,\\ -x_{1,2}a_1-y_{1,2}b_1-y_{2,2}b_2+x_{2,3}a_3-y_{2,3}b_3+x_{2,4}a_4-y_{2,4}b_4\\ =-\frac{\mathrm{h}}{\mathrm{c}}wb_2-\frac{\mathrm{h}}{\mathrm{c}}p_1b_1-\frac{\mathrm{h}}{\mathrm{c}}p_2a_1+\frac{\mathrm{h}}{\mathrm{c}}p_3b_2,\\ -x_{1,2}b_1+y_{1,2}a_1+y_{2,2}a_2+x_{2,3}b_3+y_{2,3}a_3+x_{2,4}b_4+y_{2,4}a_4\\ =\frac{\mathrm{h}}{\mathrm{c}}wa_2+\frac{\mathrm{h}}{\mathrm{c}}p_1a_1-\frac{\mathrm{h}}{\mathrm{c}}p_2b_1-\frac{\mathrm{h}}{\mathrm{c}}p_3a_2,\\ -x_{1,3}a_1-y_{1,3}b_1-x_{2,3}a_2-y_{2,3}b_2-y_{3,3}b_3+x_{3,4}a_4-y_{3,4}b_4\\ =-\frac{\mathrm{h}}{\mathrm{c}}wb_3+\frac{\mathrm{h}}{\mathrm{c}}p_3b_3+\frac{\mathrm{h}}{\mathrm{c}}p_1b_4-\frac{\mathrm{h}}{\mathrm{c}}p_2a_4,\\ -x_{1,3}b_1+y_{1,3}a_1-x_{2,3}b_2+y_{2,3}a_2+y_{3,3}a_3+x_{3,4}b_4+y_{3,4}a_4\\ =\frac{\mathrm{h}}{\mathrm{c}}wa_3-\frac{\mathrm{h}}{\mathrm{c}}p_3a_3-\frac{\mathrm{h}}{\mathrm{c}}p_1a_4-\frac{\mathrm{h}}{\mathrm{c}}p_2b_4,\\ -x_{1,4}a_1-y_{1,4}b_1-x_{2,4}a_2-y_{2,4}b_2-x_{3,4}a_3-y_{3,4}b_3-y_{4,4}b_4\\ =-\frac{\mathrm{h}}{\mathrm{c}}wb_4+\frac{\mathrm{h}}{\mathrm{c}}p_1b_3+\frac{\mathrm{h}}{\mathrm{c}}p_2a_3-\frac{\mathrm{h}}{\mathrm{c}}p_3b_4,\\ -x_{1,4}b_1+y_{1,4}a_1-x_{2,4}b_2+y_{2,4}a_2-x_{3,4}b_3+y_{3,4}a_3+y_{4,4}a_4\\ =\frac{\mathrm{h}}{\mathrm{c}}wa_4-\frac{\mathrm{h}}{\mathrm{c}}p_1a_3+\frac{\mathrm{h}}{\mathrm{c}}p_2b_3+\frac{\mathrm{h}}{\mathrm{c}}p_3a_4;\end{array}\right|$$

(here $a_k = \mathrm{Re}A_k$ and $b_k = \mathrm{Im}A_k$.)

This system has solutions in accordance with the Kronecker-Capelli theorem. Hence, such complex numbers $z_{j,k,w,\mathbf{p}}|_{\langle t,\mathbf{x}\rangle}$ exist in all points $\langle t,\mathbf{x}\rangle$.

From (2.22), (2.20), (2.21):

$$\sum_{k=1}^{4} z_{j,k,w,\mathbf{p}}\varphi_{k,w,\mathbf{p}}|_{\langle t,\mathbf{x}\rangle} = \left(\frac{1}{\mathrm{c}}\partial_t\varphi_{j,w,\mathbf{p}} - \sum_{s=1}^{4}\sum_{\alpha=1}^{3}\beta_{j,s}^{[\alpha]}\partial_\alpha\varphi_{s,w,\mathbf{p}}\right)|_{\langle t,\mathbf{x}\rangle},$$

that is

$$\frac{1}{\mathrm{c}}\partial_t\varphi_{j,w,\mathbf{p}} = \sum_{k=1}^{4}\left(\sum_{\alpha=1}^{3}\beta_{j,k}^{[\alpha]}\partial_\alpha\varphi_{k,w,\mathbf{p}} + z_{j,k,w,\mathbf{p}}\varphi_{k,w,\mathbf{p}}\right) \tag{2.23}$$

in every point $\langle t,\mathbf{x}\rangle$.

Let $\kappa_{w,\mathbf{p}}$ be the linear operators on a linear space, spanned of basic functions $\varsigma_{w,\mathbf{p}}(t,\mathbf{x})$, such that

$$\kappa_{w,\mathbf{p}}\varsigma_{w',\mathbf{p}'} \stackrel{def}{=} \left\{\begin{array}{c}\varsigma_{w',\mathbf{p}'}, \text{ if } w=w', \mathbf{p}=\mathbf{p}';\\ 0, \text{ if } w\neq w' \text{ and/or } \mathbf{p}\neq\mathbf{p}'.\end{array}\right|$$

Let

$$Q_{j,k}|_{\langle t,\mathbf{x}\rangle} := \sum_{w,\mathbf{p}}\left(z_{j,k,w,\mathbf{p}}|_{\langle t,\mathbf{x}\rangle}\right)\kappa_{w,\mathbf{p}}$$

in every point $\langle t,\mathbf{x}\rangle$.

Therefore, from (2.19) and (2.23), for every function φ_j here exists an operator $Q_{j,k}$ such that a dependence of φ_j on t is described by the following differential equations [19]:

$$\partial_t \varphi_j = \mathrm{c} \sum_{k=1}^{4} \left(\beta_{j,k}^{[1]} \partial_1 + \beta_{j,k}^{[2]} \partial_2 + \beta_{j,k}^{[3]} \partial_3 + Q_{j,k} \right) \varphi_k. \tag{2.24}$$

and $Q_{j,k}^{*} = \sum_{w,\mathbf{p}} \left(z_{j,k,w,\mathbf{p}}^{*} |_{\langle t,\mathbf{x} \rangle} \right) \kappa_{w,\mathbf{p}} = -Q_{k,j}$.

In that case if

$$\widehat{H}_{j,k} := \mathrm{ic} \left(\beta_{j,k}^{[1]} \partial_1 + \beta_{j,k}^{[2]} \partial_2 + \beta_{j,k}^{[3]} \partial_3 + Q_{j,k} \right)$$

then $\widehat{H}$ is called *a Hamiltonian*[20] of a moving with equation (2.24).

A matrix form of formula (2.24) is the following:

$$\partial_t \varphi = \mathrm{c} \left(\beta^{[1]} \partial_1 + \beta^{[2]} \partial_2 + \beta^{[3]} \partial_3 + \widehat{Q} \right) \varphi \tag{2.25}$$

with

$$\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}$$

and

$$\widehat{Q} = \begin{bmatrix} \mathrm{i}\vartheta_{1,1} & \mathrm{i}\vartheta_{1,2} - \varpi_{1,2} & \mathrm{i}\vartheta_{1,3} - \varpi_{1,3} & \mathrm{i}\vartheta_{1,4} - \varpi_{1,4} \\ \mathrm{i}\vartheta_{1,2} + \varpi_{1,2} & \mathrm{i}\vartheta_{2,2} & \mathrm{i}\vartheta_{2,3} - \varpi_{2,3} & \mathrm{i}\vartheta_{2,4} - \varpi_{2,4} \\ \mathrm{i}\vartheta_{1,3} + \varpi_{1,3} & \mathrm{i}\vartheta_{2,3} + \varpi_{2,3} & \mathrm{i}\vartheta_{3,3} & \mathrm{i}\vartheta_{3,4} - \varpi_{3,4} \\ \mathrm{i}\vartheta_{1,4} + \varpi_{1,4} & \mathrm{i}\vartheta_{2,4} + \varpi_{2,4} & \mathrm{i}\vartheta_{3,4} + \varpi_{3,4} & \mathrm{i}\vartheta_{4,4} \end{bmatrix} \tag{2.26}$$

with $\varpi_{s,k} = \mathrm{Re}\left(Q_{s,k}\right)$ and $\vartheta_{s,k} = \mathrm{Im}\left(Q_{s,k}\right)$. Matrix φ is called *a state vector* of the event $\mathcal{A}$ probability.

An operator $\widehat{U}\left(t,t_0\right)$ with a domain and with a range of values on the set of state vectors is called *an evolution operator* if each state vector φ fulfils the following condition:

$$\varphi(t) = \widehat{U}\left(t,t_0\right) \varphi\left(t_0\right). \tag{2.27}$$

Let us denote:

$$\widehat{H}_d := \mathrm{c} \sum_{s=1}^{3} \mathrm{i}\beta^{[s]} \partial_s.$$

In that case

$$\widehat{H} = \widehat{H}_d + \mathrm{ic}\widehat{Q}$$

according the Hamiltonian definition:

[19]This set of equations is similar to the Dirac equation with the mass matrix [16], [17], [18]. I choose a form of this set of equations in order to describe the behavior of $\rho_{\wp}(t,\mathbf{x})$ by spinors and by Clifford's set elements.

[20]Sir William Rowan Hamilton (4 August 1805 2 September 1865) was an Irish physicist, astronomer, and mathematician, who made important contributions to classical mechanics, optics, and algebra.

$$\widehat{H} = \mathrm{ic}\left(\beta^{[1]}\partial_1 + \beta^{[2]}\partial_2 + \beta^{[3]}\partial_3 + Q\right).$$

From (2.25):

$$\mathrm{i}\partial_t \varphi = \widehat{H}\varphi.$$

.

Hence:

$$\mathrm{i}\partial_t \varphi = \left(\widehat{H}_d + \mathrm{ic}\widehat{Q}\right)\varphi.$$

This differential equation has the following solution:

$$\frac{\partial\varphi}{\varphi} = -\mathrm{i}\left(\widehat{H}_d + \mathrm{ic}\widehat{Q}\right)\partial t,$$

$$\int_{t=t_0}^{t} \frac{\partial\varphi}{\varphi} = -\mathrm{i}\int_{t=t_0}^{t}\left(\widehat{H}_d + \mathrm{ic}\widehat{Q}\right)\partial t,$$

$$\ln\varphi(t) - \ln\varphi(t_0) = \left(-\mathrm{i}\int_{t=t_0}^{t}\widehat{H}_d\partial t - \mathrm{iic}\int_{t=t_0}^{t}\widehat{Q}\partial t\right).$$

Since $\widehat{H}_d$ does not depend on time then

$$\int_{t=t_0}^{t}\widehat{H}_d\partial t = \widehat{H}_d(t - t_0).$$

Hence, according logarithm properties:

$$\ln\frac{\varphi(t)}{\varphi(t_0)} = \left(-\mathrm{i}\widehat{H}_d(t - t_0) + \mathrm{c}\int_{t=t_0}^{t}\widehat{Q}\partial t\right).$$

Therefore,[21]:

$$\varphi(t) = \varphi(t_0)\exp\left(-\mathrm{i}\widehat{H}_d(t - t_0) + \mathrm{c}\int_{t=t_0}^{t}\widehat{Q}\partial t\right).$$

Hence, from (2.27):

$$\widehat{U}(t, t_0) = \exp\left(-\mathrm{i}\widehat{H}_d(t - t_0) + \mathrm{c}\int_{t=t_0}^{t}\widehat{Q}\partial t\right)$$

---

[21]For an operator $\widehat{S}$:

$$\exp\left(\widehat{S}\right) := \widehat{1} + \widehat{S} + \frac{1}{2}\widehat{S}^2 + \frac{1}{3!}\widehat{S}^3 + \cdots + \frac{1}{n!}\widehat{S}^n + \cdots$$

with $\widehat{S}^2 := \widehat{S}\widehat{S}$ and $\widehat{S}^{r+1} := \widehat{S}^r\,\widehat{S}$.
Here $\widehat{1}$ is *the unit operator* such that for every $\widetilde{u}$: $\widehat{1}\widetilde{u} = \widetilde{u}$.

A Fourier series for $\varphi_j(t,\mathbf{x})$ in $\Re_\Omega$ has the following shape:

$$\varphi_j(t_0,\mathbf{x}) = \sum_{\mathbf{p}} c_{j,\mathbf{p}}(t_0)\,\varsigma_{\mathbf{p}}(t_0,\mathbf{x})$$

with (2.17)

$$\varsigma_{\mathbf{p}}(\mathbf{x}) := \left\{ \begin{array}{c} \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{p}\mathbf{x}\right) \text{ if } \mathbf{x}\in\Omega; \\ 0, \text{ otherwise} \end{array} \right|$$

and with (2.19)

$$c_{j,\mathbf{p}}(t_0) = \varsigma_{\mathbf{p}}(\mathbf{x}) * \varphi_j(t_0,\mathbf{x}).$$

That is in a matrix form:

$$c_{\mathbf{p}}(t_0) = \int_{(\Omega)} d\mathbf{x}_0 \cdot \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{p}\mathbf{x}_0\right)\varphi(t_0,\mathbf{x}_0)$$

Hence,

$$\varphi(t_0,\mathbf{x}) = \sum_{\mathbf{p}} \int_{(\blacksquare)} d\mathbf{x}_0 \cdot \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{p}\mathbf{x}_0\right)\varphi(t_0,\mathbf{x}_0)\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{p}\mathbf{x}\right).$$

That is:

$$\varphi(t_0,\mathbf{x}) = \int_{(\Omega)} d\mathbf{x}_0 \cdot \left(\sum_{\mathbf{p}} \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{p}(\mathbf{x}-\mathbf{x}_0)\right)\right)\varphi(t_0,\mathbf{x}_0).$$

Therefore,

$$\varphi(t,\mathbf{x}) = \int_{(\Omega)} d\mathbf{x}_0 \cdot \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \left(\begin{array}{c} \sum_{\mathbf{p}} \exp\left(-\mathrm{i}\widehat{H}_d(t-t_0) + \mathrm{c}\int_{t=t_0}^{t}\widehat{Q}\partial t\right)\cdot \\ \cdot\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{p}(\mathbf{x}-\mathbf{x}_0)\right) \end{array} \right)\varphi(t_0,\mathbf{x}_0).$$

An operator

$$K(t-t_0,\mathbf{x}-\mathbf{x}_0,t,t_0) := \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \left(\begin{array}{c} \sum_{\mathbf{p}} \exp\left(-\mathrm{i}\widehat{H}_d(t-t_0) + \mathrm{c}\int_{t=t_0}^{t}\widehat{Q}\partial t\right)\cdot \\ \cdot\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{p}(\mathbf{x}-\mathbf{x}_0)\right) \end{array} \right)$$

is called *propagator* of the event $\mathcal{A}$ probability.
Hence:

$$\varphi(t,\mathbf{x}) = \int_{(\Omega)} d\mathbf{x}_0 \cdot K(t-t_0,\mathbf{x}-\mathbf{x}_0,t,t_0)\,\varphi(t_0,\mathbf{x}_0). \tag{2.28}$$

A propagator has the following property:

$$K(t-t_0,\mathbf{x}-\mathbf{x}_0,t,t_0) = \int d\mathbf{x}_1 \cdot K(t-t_1,\mathbf{x}-\mathbf{x}_1,t,t_1)\,K(t_1-t_0,\mathbf{x}_1-\mathbf{x}_0,t_1,t_0).$$

2.2. Double-Slit Experiment

In a vacuum (Figure 1, Figure 2, fig 22): Here transmitter s of electrons, wall w, and the electrons detecting black screen d are placed[21].

Electrons are emitted one by one from the source s. When an electron hits against screen d then a bright spot arises in the hit place of d..

1. Let slit a be opened in wall w (Figure 1). An electron flies out from s, passes by a, and is detected by d.

If such operation will be reiterated N of times then N bright spots shall arise on d against slit a in the vicinity of point y_a.

2. Let slit b be opened in wall w (Figure 2). An electron flies out from s, passes by b, and is detected by d.

If such operation will be reiterated N of times then N bright spots shall arise on d against slit b in the vicinity of point y_b.

3. Let both slits be opened. In that case do you expect a result as on fig. 22? But no. We get result as on Figure 4[22][22].

For instance, such experiment was realized at Hitachi by A. Tonomura, J. Endo, T. Matsuda, T. Kawasaki and H. Ezawa in 1989. Here was presumed that interference fringes are produced only when two electrons pass through both slits simultaneously. If there were two electrons from the source s at the same time, such interference might happen. But this cannot occur, because here is no more than one electron from this source at one time. Please keep watching the experiment a little longer. When a large number of electrons is accumulated, something like regular fringes begin to appear in the perpendicular direction as Figure 5(c) shows. Clear interference fringes can be seen in the last scene of the experiment after 20 minutes (Figure 5(d)). It should also be noted that the fringes are made up of bright spots, each of which records the detection of an electron. We have reached a mysterious conclusion. Although electrons were sent one by one, interference fringes could be observed. These interference fringes are formed only when electron waves pass through on both slits at the same time but nothing other than this. Whenever electrons are observed, they are always detected as individual particles. When accumulated, however, interference fringes are formed. Please recall that at any one instant here was at most one electron from s. We have reached a conclusion which is far from what our common sense tells us.

4. But nevertheless, across which slit the electron had slipped?

Let (Figure 6) two detectors d_a and d_b and a photon source sf be added to devices of Figure 4.

An electron, slipped across slit a, is lighten by source sf, and detector d_a snaps into action. And an electron, slipped across slit b, is lighten by source sf, and detector d_b snaps into action.

If photon source sf lights all N electrons, slipped across slits, then we received the picture of Figure 3.

If source sf is faint then only a little part of N electrons, slipped across slits, are noticed

[22]Single-electron events build up over a 20 minute exposure to form an interference pattern in this double-slit experiment by Akira Tonomura and co-workers. Figure 5(a) 8 electrons; Figure 5(b) 270 electrons; Figure 5(c) 2000 electrons; Figure 5(d) 60,000. A video of this experiment will soon be available on the web (www.hqrd.hitachi.co.jp/em/doubleslit.html).

by detectors d_a and d_b. In that case electrons, noticed by detectors d_a and d_b, make picture of Figure 3, and all unnoticed electrons make picture of Figure 4. In result here the Figure 6 picture is received.

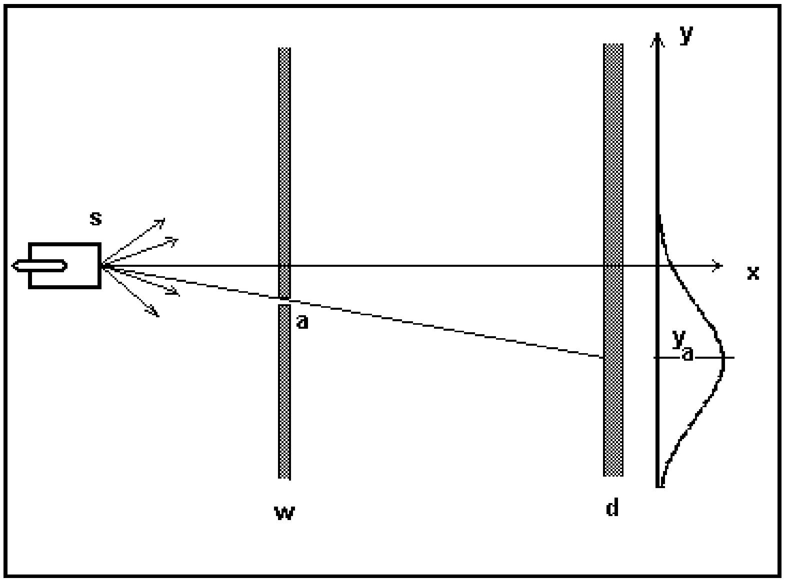

Figure 1:

Let us try to interpret these experiments by events and probabilities.

Denote the source s coordinates as $\langle x_0, y_0\rangle$, the slit a coordinates as $\langle x_a, y_a\rangle$, the slit b coordinates as $\langle x_b, y_b\rangle$. Here $x_a = x_b$, and the wall w equation is $x = x_a$. Denote the screen d equation as $x = x_d$.

Denote
an event, expressed by sentence: ≪electron is detected in point $\langle t, x, y\rangle$≫, as $\mathcal{C}(t,x,y)$,
an event, expressed by sentence ≪slit a is open≫, as $\mathcal{A}$,
and an event, expressed by sentence ≪slit b is open≫, as $\mathcal{B}$.

Let t_0 be an time instant of an electron emission from source s. Since s is a dotlike source then a state vector $\varphi_{\mathcal{C}}$ in instant t_0 has the following form:

$$\varphi_{\mathcal{C}}(t,x,y)\mid_{t=t_0} = \varphi_{\mathcal{C}}(t_0,x,y)\,\delta(x-x_0)\,\delta(y-y_0). \tag{2.29}$$

Let t_w be an time instant such that if event $\mathcal{C}(t,x,y)$ occurs in that instant then $\mathcal{C}(t,x,y)$ occurs on wall w.

Let t_d be an time instant of a electron detecting by screen d.

1. Let slit a be opened in wall w (Figure 1).

In that case the $\mathcal{C}(t,x,y)$ probabilities propagator $K_{\mathcal{C}\mathcal{A}}(t-t_0, x-x_s, y-y_s)$ in instant t_w should be of the following shape:

$$K_{\mathcal{C}\mathcal{A}}(t-t_0, x-x_s, y-y_s)\mid_{t=t_w}$$

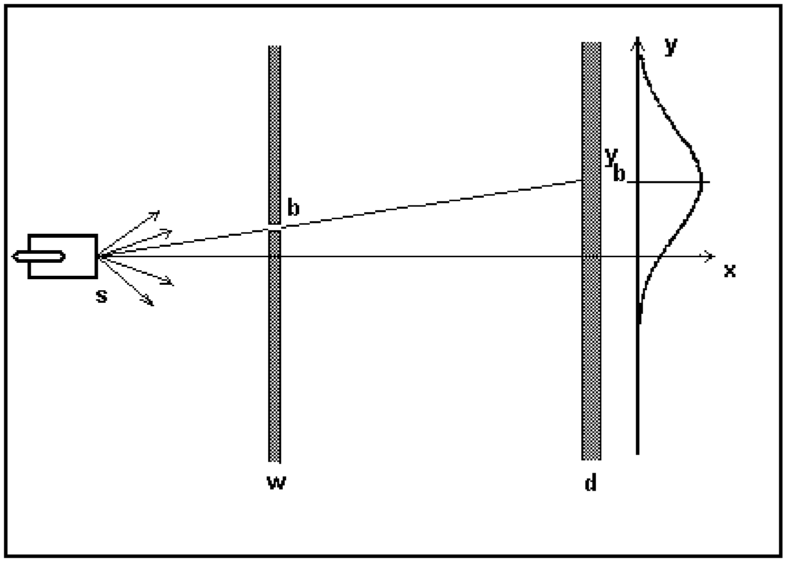

Figure 2:

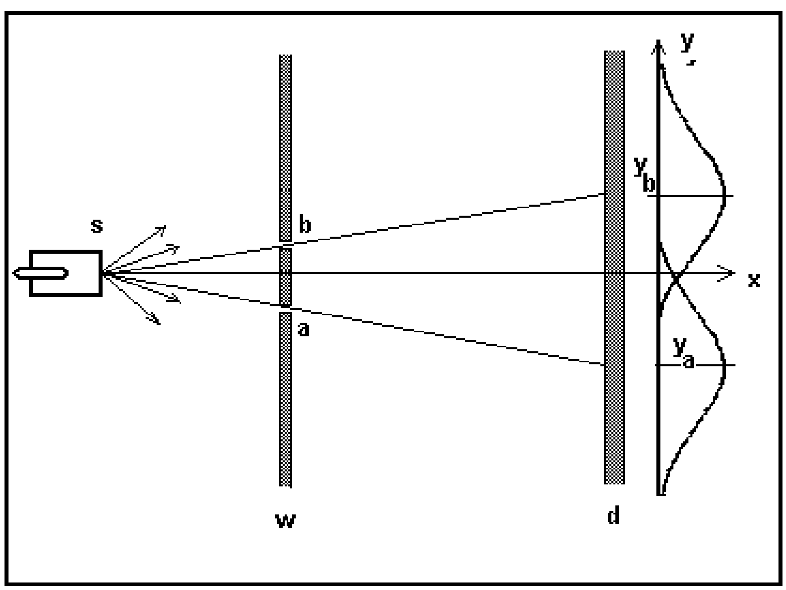

Figure 3:

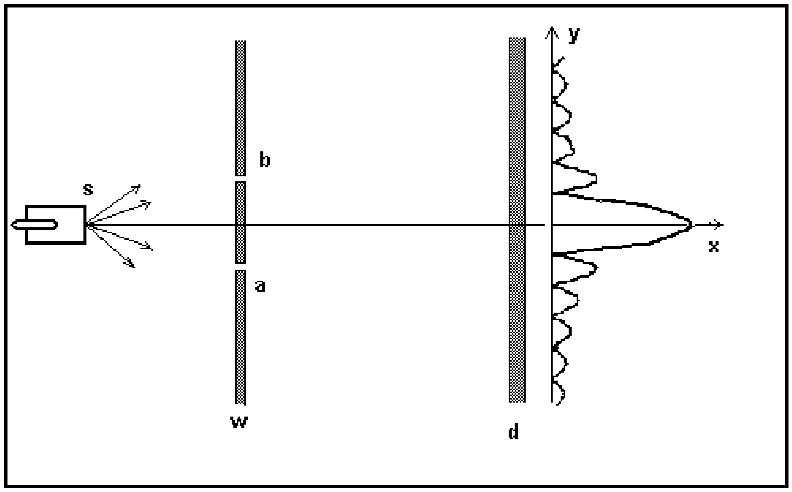

Figure 4:

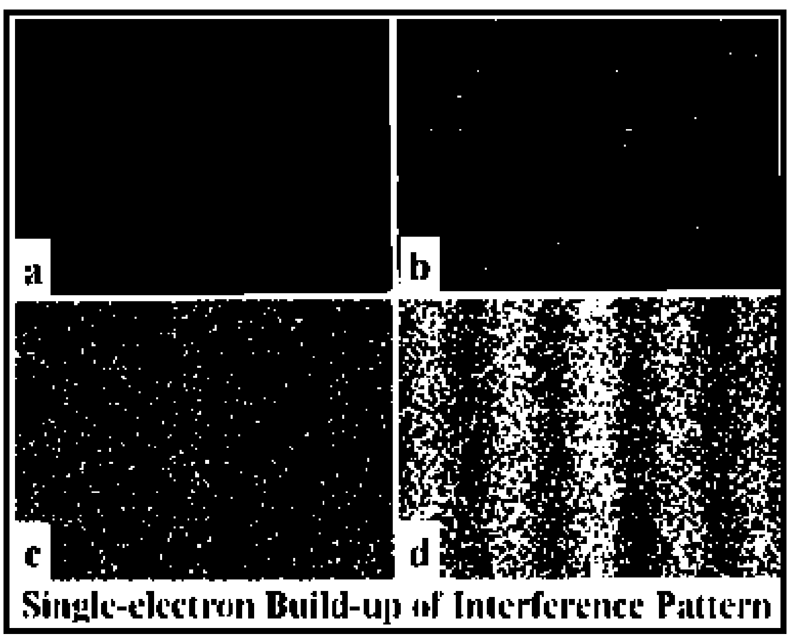

Figure 5:

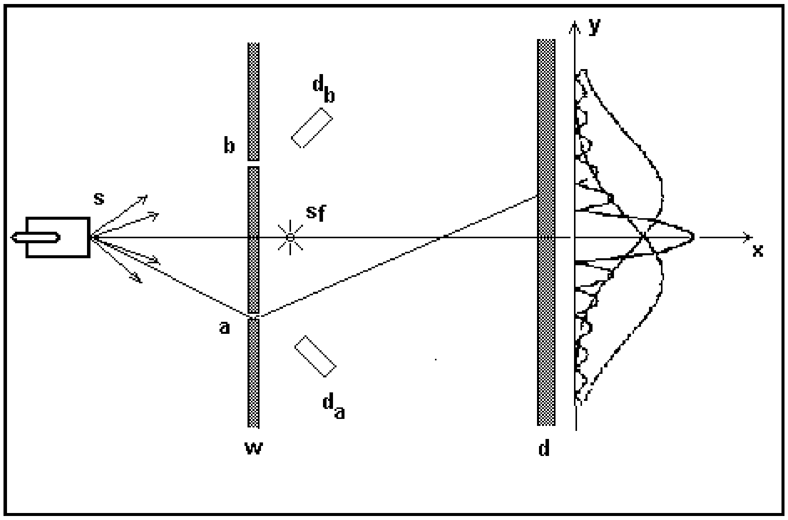

Figure 6:

$$= \quad K_{C\mathcal{A}}(t_w - t_0, x - x_s, y - y_s)\,\delta(x - x_a)\,\delta(y - y_a).$$

According the propagator property:

$$\begin{aligned} & K(t - t_0, x - x_s, y - y_s) = \\ = \quad & \int_R dx_1 \int_R dy_1 \cdot K(t - t_1, x - x_1, y - y_1)\,K(t_1 - t_0, x_1 - x_s, y_1 - y_s). \end{aligned}$$

Hence:

$$\begin{aligned} & K_{C\mathcal{A}}(t_d - t_0, x_d - x_s, y_d - y_s) = \\ = \quad & \int_R dx \int_R dy \cdot K_{C\mathcal{A}}(t_d - t_w, x_d - x, y_d - y) \\ & K_{C\mathcal{A}}(t_w - t_0, x - x_s, y - y_s)\,\delta(x - x_a)\,\delta(y - y_a). \end{aligned}$$

Therefore, according properties of δ-function:

$$\begin{aligned} & K_{C\mathcal{A}}(t_d - t_0, x_d - x_s, y_d - y_s) = \\ = \quad & K_{C\mathcal{A}}(t_d - t_w, x_d - x_a, y_d - y_a)\,K_{C\mathcal{A}}(t_w - t_0, x_a - x_s, y_a - y_s). \end{aligned}$$

The state vector for the event $C(t, x, y)$ in condition $\mathcal{A}$ probability has the following form (2.28):

$$\varphi_{C\mathcal{A}}(t_d, x_d, y_d) = \int dx_s \int dy_s \cdot K_{C\mathcal{A}}(t_d - t_0, x_d - x_s, y_d - y_s)\,\varphi_C(t_0, x_s, y_s).$$

Hence, from (2.29):

$$\begin{aligned}\varphi_{C\mathcal{A}}(t_d,x_d,y_d) &= \int dx_s \int dy_s \cdot K_{C\mathcal{A}}(t_d-t_0,x_d-x_s,y_d-y_s)\\ &\quad \varphi_C(t_0,x_s,y_s)\,\delta(x_s-x_0)\,\delta(y_s-y_0).\end{aligned}$$

That is:

$$\begin{aligned}&\varphi_{C\mathcal{A}}(t_d,x_d,y_d)\\ =\ &\int dx_s \int dy_s \cdot K_{C\mathcal{A}}(t_d-t_w,x_d-x_a,y_d-y_a)\,K_{C\mathcal{A}}(t_w-t_0,x_a-x_s,y_a-y_s)\\ &\varphi_C(t_0,x_s,y_s)\,\delta(x_s-x_0)\,\delta(y_s-y_0).\end{aligned}$$

Hence, according properties of δ-function:

$$\begin{aligned}&\varphi_{C\mathcal{A}}(t_d,x_d,y_d)\\ =\ &K_{C\mathcal{A}}(t_d-t_w,x_d-x_a,y_d-y_a)\,K_{C\mathcal{A}}(t_w-t_0,x_a-x_0,y_a-y_0)\,\varphi_C(t_0,x_0,y_0).\end{aligned}$$

In accordance with (2.15):

$$\rho_{C\mathcal{A}}(t_d,x_d,y_d) = \varphi^{\dagger}_{C\mathcal{A}}(t_d,x_d,y_d)\,\varphi_{C\mathcal{A}}(t_d,x_d,y_d).$$

Therefore, a probability to detect the electron in vicinity $\Delta x\Delta y$ of point $\langle x_d, y_d\rangle$ in instant t in condition $\mathcal{A}$ equals to the following:

$$P_a(t_d,x_d,y_d) := \mathrm{P}(C(t_d,\Delta x\Delta y)/\mathcal{A}) = \rho_{C\mathcal{A}}(t_d,x_d,y_d)\,\Delta x\Delta y.$$

2. Let slit b be opened in wall w (Figure 2).

In that case the $C(t,x,y)$ probabilities propagator $K_{C\mathcal{B}}(t-t_0,x-x_s,y-y_s)$ in instant t_w should be of the following shape:

$$\begin{aligned}&K_{C\mathcal{B}}(t-t_0,x-x_s,y-y_s)\,|_{t=t_w}\\ =\ &K_{C\mathcal{B}}(t_w-t_0,x-x_s,y-y_s)\,\delta(x-x_b)\,\delta(y-y_b).\end{aligned}$$

Hence, according the propagator property::

$$\begin{aligned}&K_{C\mathcal{B}}(t_d-t_0,x_d-x_s,y_d-y_s) =\\ =\ &\int_R dx \int_R dy \cdot K_{C\mathcal{B}}(t_d-t_w,x_d-x,y_d-y)\\ &K_{C\mathcal{B}}(t_w-t_0,x-x_s,y-y_s)\,\delta(x-x_b)\,\delta(y-y_b).\end{aligned}$$

Therefore, according properties of δ-function:

$$\begin{aligned}&K_{C\mathcal{B}}(t_d-t_0,x_d-x_s,y_d-y_s) =\\ =\ &K_{C\mathcal{B}}(t_d-t_w,x_d-x_b,y_d-y_b)\,K_{C\mathcal{B}}(t_w-t_0,x_b-x_s,y_b-y_s).\end{aligned}$$

The state vector for the event $\mathcal{C}(t,x,y)$ in condition $\mathcal{B}$ probability has the following form (2.28):

$$\varphi_{\mathcal{CB}}(t_d,x_d,y_d)=\int dx_s\int dy_s\cdot K_{\mathcal{CB}}(t_d-t_0,x_d-x_s,y_d-y_s)\,\varphi_{\mathcal{C}}(t_0,x_s,y_s).$$

Hence, from (2.29):

$$\begin{aligned}\varphi_{\mathcal{CB}}(t_d,x_d,y_d) &= \int dx_s\int dy_s\cdot K_{\mathcal{CB}}(t_d-t_0,x_d-x_s,y_d-y_s)\\ &\quad \varphi_{\mathcal{C}}(t_0,x_s,y_s)\,\delta(x_s-x_0)\,\delta(y_s-y_0).\end{aligned}$$

That is:

$$\begin{aligned}&\varphi_{\mathcal{CB}}(t_d,x_d,y_d)\\ =\ &\int dx_s\int dy_s\cdot K_{\mathcal{CB}}(t_d-t_w,x_d-x_b,y_d-y_b)\,K_{\mathcal{CB}}(t_w-t_0,x_b-x_s,y_b-y_s)\\ &\varphi_{\mathcal{C}}(t_0,x_s,y_s)\,\delta(x_s-x_0)\,\delta(y_s-y_0).\end{aligned}$$

Hence, according properties of δ-function:

$$\begin{aligned}&\varphi_{\mathcal{CB}}(t_d,x_d,y_d)\\ =\ &K_{\mathcal{CB}}(t_d-t_w,x_d-x_b,y_d-y_b)\,K_{\mathcal{CB}}(t_w-t_0,x_b-x_0,y_b-y_0)\,\varphi_{\mathcal{C}}(t_0,x_0,y_0).\end{aligned}$$

In accordance with (2.15):

$$\rho_{\mathcal{CB}}(t_d,x_d,y_d)=\varphi^{\dagger}_{\mathcal{CB}}(t_d,x_d,y_d)\,\varphi_{\mathcal{CB}}(t_d,x_d,y_d).$$

Therefore, a probability to detect the electron in vicinity $\Delta x\Delta y$ of point $\langle x_d,y_d\rangle$ in instant t in condition $\mathcal{B}$ equals to the following:

$$P_b(t_d,x_d,y_d):=\mathrm{P}(\mathcal{C}(t_d,\Delta x\Delta y)/\mathcal{B})=\rho_{\mathcal{CB}}(t_d,x_d,y_d)\,\Delta x\Delta y.$$

3. Let both slits and a and b are opened (Figure 4).

In that case the $\mathcal{C}(t,x,y)$ probabilities propagator $K_{\mathcal{CAB}}(t-t_0,x-x_s,y-y_s)$ in instant t_w should be of the following shape:

$$\begin{aligned}&K_{\mathcal{CAB}}(t-t_0,x-x_s,y-y_s)\,|_{t=t_w}=\\ &=K_{\mathcal{CAB}}(t_w-t_0,x-x_s,y-y_s)\,(\delta(x-x_a)\,\delta(y-y_a)+\delta(x-x_b)\,\delta(y-y_b)).\end{aligned}$$

Hence, according the propagator property::

$$\begin{aligned}&K_{\mathcal{CAB}}(t_d-t_0,x_d-x_s,y_d-y_s)=\\ &=\int_R dx\int_R dy\cdot K_{\mathcal{CAB}}(t_d-t_w,x_d-x,y_d-y)\\ &\quad K_{\mathcal{CAB}}(t_w-t_0,x-x_s,y-y_s)\cdot\\ &\quad\cdot(\delta(x-x_a)\,\delta(y-y_a)+\delta(x-x_b)\,\delta(y-y_b)).\end{aligned}$$

Hence,

$$\begin{gathered}
K_{\mathcal{CAB}}(t_d - t_0, x_d - x_s, y_d - y_s) = \\
\int_R dx \int_R dy \cdot K_{\mathcal{CAB}}(t_d - t_w, x_d - x, y_d - y)\ K_{\mathcal{CAB}}(t_w - t_0, x - x_s, y - y_s) \cdot \\
\cdot \delta(x - x_a)\, \delta(y - y_a) \\
+ \int_R dx \int_R dy \cdot K_{\mathcal{CAB}}(t_d - t_w, x_d - x, y_d - y)\ K_{\mathcal{CAB}}(t_w - t_0, x - x_s, y - y_s) \cdot \\
\cdot \delta(x - x_b)\, \delta(y - y_b).
\end{gathered}$$

Hence, according properties of δ-function:

$$\begin{gathered}
K_{\mathcal{CAB}}(t_d - t_0, x_d - x_s, y_d - y_s) = \\
K_{\mathcal{CAB}}(t_d - t_w, x_d - x_a, y_d - y_a)\ K_{\mathcal{CAB}}(t_w - t_0, x_a - x_s, y_a - y_s) \\
+ K_{\mathcal{CAB}}(t_d - t_w, x_d - x_b, y_d - y_b)\ K_{\mathcal{CAB}}(t_w - t_0, x_b - x_s, y_b - y_s)
\end{gathered}.$$

The state vector for the event $\mathcal{C}(t, x, y)$ in condition $\mathcal{A}$ and $\mathcal{B}$ probability has the following form (2.28):

$$\varphi_{\mathcal{CAB}}(t_d, x_d, y_d) = \int dx_s \int dy_s \cdot K_{\mathcal{CAB}}(t_d - t_0, x_d - x_s, y_d - y_s)\, \varphi_{\mathcal{C}}(t_0, x_s, y_s).$$

Hence, from (2.29):

$$\begin{aligned}
\varphi_{\mathcal{CAB}}(t_d, x_d, y_d) \quad = \quad & \int dx_s \int dy_s \cdot K_{\mathcal{CAB}}(t_d - t_0, x_d - x_s, y_d - y_s) \\
& \varphi_{\mathcal{C}}(t_0, x_s, y_s)\, \delta(x_s - x_0)\, \delta(y_s - y_0).
\end{aligned}$$

That is:

$$\begin{gathered}
\varphi_{\mathcal{CAB}}(t_d, x_d, y_d) = \int dx_s \int dy_s \cdot \\
\cdot \left(\begin{array}{c} K_{\mathcal{CAB}}(t_d - t_w, x_d - x_a, y_d - y_a)\ K_{\mathcal{CAB}}(t_w - t_0, x_a - x_s, y_a - y_s) \\ + K_{\mathcal{CAB}}(t_d - t_w, x_d - x_b, y_d - y_b)\ K_{\mathcal{CAB}}(t_w - t_0, x_b - x_s, y_b - y_s) \end{array} \right) \\
\varphi_{\mathcal{C}}(t_0, x_s, y_s)\, \delta(x_s - x_0)\, \delta(y_s - y_0).
\end{gathered}$$

Hence, according properties of δ-function:

$$\begin{gathered}
\varphi_{\mathcal{CAB}}(t_d, x_d, y_d) = \\
= \left(\begin{array}{c} K_{\mathcal{CAB}}(t_d - t_w, x_d - x_a, y_d - y_a)\ K_{\mathcal{CAB}}(t_w - t_0, x_a - x_0, y_a - y_0) \\ + K_{\mathcal{CAB}}(t_d - t_w, x_d - x_b, y_d - y_b)\ K_{\mathcal{CAB}}(t_w - t_0, x_b - x_0, y_b - y_0) \end{array} \right) \\
\varphi_{\mathcal{C}}(t_0, x_0, y_0).
\end{gathered}$$

That is:

$$\begin{gathered}
\varphi_{\mathcal{CAB}}(t_d, x_d, y_d) = \\
= K_{\mathcal{CAB}}(t_d - t_w, x_d - x_a, y_d - y_a)\ K_{\mathcal{CAB}}(t_w - t_0, x_a - x_0, y_a - y_0)\, \varphi_{\mathcal{C}}(t_0, x_0, y_0) \\
+ K_{\mathcal{CAB}}(t_d - t_w, x_d - x_b, y_d - y_b)\ K_{\mathcal{CAB}}(t_w - t_0, x_b - x_0, y_b - y_0)\, \varphi_{\mathcal{C}}(t_0, x_0, y_0).
\end{gathered}$$

Therefore,

$$\varphi_{\mathcal{CAB}}(t_d, x_d, y_d) = \varphi_{\mathcal{CA}}(t_d, x_d, y_d) + \varphi_{\mathcal{CB}}(t_d, x_d, y_d).$$

And in accordance with (2.15):

$$\rho_{\mathcal{CAB}}(t_d, x_d, y_d) = \varphi^{\dagger}_{\mathcal{CAB}}(t_d, x_d, y_d)\,\varphi_{\mathcal{CAB}}(t_d, x_d, y_d),$$

i.e.

$$\rho_{\mathcal{CAB}} = (\varphi_{\mathcal{CA}} + \varphi_{\mathcal{CB}})^{\dagger}(\varphi_{\mathcal{CA}} + \varphi_{\mathcal{CB}})$$

Since state vectors $\varphi_{\mathcal{CA}}$ and $\varphi_{\mathcal{CB}}$ are not numbers with the same number signs then in the general case:

$$(\varphi_{\mathcal{CA}} + \varphi_{\mathcal{CB}})^{\dagger}(\varphi_{\mathcal{CA}} + \varphi_{\mathcal{CB}}) \neq \varphi^{\dagger}_{\mathcal{CA}}\varphi_{\mathcal{CA}} + \varphi^{\dagger}_{\mathcal{CB}}\varphi_{\mathcal{CB}}.$$

Therefore, since a probability to detect the electron in vicinity $\Delta x \Delta y$ of point $\langle x_d, y_d \rangle$ in instant t in condition $\mathcal{AB}$ equals:

$$P_{ab}(t_d, x_d, y_d) := \mathrm{P}(\mathcal{C}(t_d, \Delta x \Delta y) / \mathcal{AB}) = \rho_{\mathcal{CAB}}(t_d, x_d, y_d)\,\Delta x \Delta y$$

then

$$P_{ab}(t_d, x_d, y_d) \neq P_a(t_d, x_d, y_d) + P_b(t_d, x_d, y_d).$$

Hence, we have the fig.23 picture instead of the Figure 3 picture.

4. Let us consider devices of Figure 6.

Denote event, expressed by sentence "detector d_a snaps into action", as $\mathcal{D}_a$ and event, expressed by sentence "detector d_b snaps into action", as $\mathcal{D}_b$. Since event $\mathcal{C}(t, x, y)$ is a dotlike event then events $\mathcal{D}_a$ and $\mathcal{D}_b$ are exclusive events.

According the property 10 of operations on events:

$$(\mathcal{D}_a + \mathcal{D}_b) + \overline{(\mathcal{D}_a + \mathcal{D}_b)} = \mathcal{T},$$

according the property 6 of operations on events:

$$\overline{(\mathcal{D}_a + \mathcal{D}_b)} = \overline{\mathcal{D}}_a \overline{\mathcal{D}}_b,$$

Hence:

$$\mathcal{D}_a + \mathcal{D}_b + \overline{\mathcal{D}}_a \overline{\mathcal{D}}_b = \mathcal{T}.$$

According the property 5 of operations on events:

$$\mathcal{C} = \mathcal{CT} = \mathcal{C}\left(\mathcal{D}_a + \mathcal{D}_b + \overline{\mathcal{D}}_a \overline{\mathcal{D}}_b\right).$$

According the property 3 of operations on events:

$$\mathcal{C} = \mathcal{C}\mathcal{D}_a + \mathcal{C}\mathcal{D}_b + \mathcal{C}\overline{\mathcal{D}}_a \overline{\mathcal{D}}_b.$$

Therefore, according the probabilities addition formula for exclusive events:

$$\mathrm{P}\left(\mathcal{C}\left(t_d\right)\right)=\mathrm{P}\left(\mathcal{C}\left(t_d\right)\mathcal{D}_a\right)+\mathrm{P}\left(\mathcal{C}\left(t_d\right)\mathcal{D}_b\right)+\mathrm{P}\left(\mathcal{C}\left(t_d\right)\overline{\mathcal{D}}_a\overline{\mathcal{D}}_b\right).$$

But

$$\begin{array}{rcl}\mathrm{P}\left(\mathcal{C}\left(t_d\right)\mathcal{D}_a\right) & = & P_a\left(t_d\right),\\ \mathrm{P}\left(\mathcal{C}\left(t_d\right)\mathcal{D}_b\right) & = & P_b\left(t_d\right),\\ \mathrm{P}\left(\mathcal{C}\left(t_d\right)\overline{\mathcal{D}}_a\overline{\mathcal{D}}_b\right) & = & P_{ab}\left(t_d\right),\end{array}$$

and we receive the Figure 6 picture.

Thus, here are no paradoxes for the event-probability interpretation of these experiments. We should depart from notion of a continuously existing electron and consider an elementary particle an ensemble of events connected by probability. Its like the fact that physical particle exists only at the instant when it is involved in some event. A particle doesnt exist in any other time, but theres a probability that something will happen to it. Thus, if nothing happens with the particle between the event of creating it and the event of detecting it the behavior of the particle is the behavior of probability between the point of creating and the point of detecting it with the presence of interference.

But what is with Wilson cloud chamber where the particle has a clear trajectory and no interference?

In that case these trajectories are not totally continuous lines. Every point of ionization has neighboring point of ionization, and there are no events between these points.

Consequently, physical particle is moving because corresponding probability propagates in the space between points of ionization. Consequently, particle is an ensemble of events, connected by probability. And charges, masses, moments, etc. represent statistical parameters of these probability waves, propagated in the space-time. It explains all paradoxes of quantum physics. Schrodingers cat lives easy without any superposition of states until the micro event awaited by all occures. And the wave function disappears without any collapse in the moment when an event probability disappears after the event occurs.

Hence, entanglement concerns not particles but probabilities. That is when event of the measuring of spin of Alices electron occurs then probability for these entangled electrons is changed instantly on whole space. Therefore, nonlocality acts for probabilities, not for particles. But probabilities can not transmit any information

2.3. Lepton Hamiltonian

Let $\vartheta_{s,k}$ and $\varpi_{s,k}$ be terms of $\widehat{Q}$ (2.26) and let Θ_0, Θ_3, Υ_0 and Υ_3 be a solution of the following equations set:

$$\left\{\begin{array}{c}-\Theta_0+\Theta_3-\Upsilon_0+\Upsilon_3=\vartheta_{1,1};\\ -\Theta_0-\Theta_3-\Upsilon_0-\Upsilon_3=\vartheta_{2,2};\\ -\Theta_0-\Theta_3+\Upsilon_0+\Upsilon_3=\vartheta_{3,3};\\ -\Theta_0+\Theta_3+\Upsilon_0-\Upsilon_3=\vartheta_{4,4}\end{array}\right|,$$

and Θ_1, Υ_1, Θ_2, Υ_2, M_0, M_4, $M_{\zeta,0}$, $M_{\zeta,4}$, $M_{\eta,0}$, $M_{\eta,4}$, $M_{\theta,0}$, $M_{\theta,4}$ be solutions of the following sets of equations:

$$\left\{\begin{array}{c} \Theta_1+\Upsilon_1=\vartheta_{1,2}; \\ -\Theta_1+\Upsilon_1=\vartheta_{3,4}; \end{array}\right|$$

$$\left\{\begin{array}{c} -\Theta_2-\Upsilon_2=\varpi_{1,2}; \\ \Theta_2-\Upsilon_2=\varpi_{3,4}; \end{array}\right|$$

$$\left\{\begin{array}{c} M_0+M_{\theta,0}=\vartheta_{1,3}; \\ M_0-M_{\theta,0}=\vartheta_{2,4}; \end{array}\right|$$

$$\left\{\begin{array}{c} M_4+M_{\theta,4}=\varpi_{1,3}; \\ M_4-M_{\theta,4}=\varpi_{2,4}; \end{array}\right|$$

$$\left\{\begin{array}{c} M_{\zeta,0}-M_{\eta,4}=\vartheta_{1,4}; \\ M_{\zeta,0}+M_{\eta,4}=\vartheta_{2,3}; \end{array}\right|$$

$$\left\{\begin{array}{c} M_{\zeta,4}-M_{\eta,0}=\varpi_{1,4}; \\ M_{\zeta,4}+M_{\eta,0}=\varpi_{2,3} \end{array}\right|.$$

Thus the columns of $\widehat{Q}$ are the following:
the first and the second columns:

$$\begin{array}{cc} -\mathrm{i}\Theta_0+\mathrm{i}\Theta_3-\mathrm{i}\Upsilon_0+\mathrm{i}\Upsilon_3 & \mathrm{i}\Theta_1+\mathrm{i}\Upsilon_1+\Theta_2+\Upsilon_2 \\ \mathrm{i}\Theta_1+\mathrm{i}\Upsilon_1-\Theta_2-\Upsilon_2 & -\mathrm{i}\Theta_0-\mathrm{i}\Theta_3-\mathrm{i}\Upsilon_0-\mathrm{i}\Upsilon_3 \\ \mathrm{i}M_0+\mathrm{i}M_{\theta,0}+M_4+M_{\theta,4} & \mathrm{i}M_{\zeta,0}+\mathrm{i}M_{\eta,4}+M_{\zeta,4}+M_{\eta,0} \\ \mathrm{i}M_{\zeta,0}-\mathrm{i}M_{\eta,4}+M_{\zeta,4}-M_{\eta,0} & \mathrm{i}M_0-\mathrm{i}M_{\theta,0}+M_4-M_{\theta,4} \end{array},$$

the third and the fourth columns:

$$\begin{array}{cc} \mathrm{i}M_0+\mathrm{i}M_{\theta,0}-M_4-M_{\theta,4} & \mathrm{i}M_{\zeta,0}-\mathrm{i}M_{\eta,4}-M_{\zeta,4}+M_{\eta,0} \\ \mathrm{i}M_{\zeta,0}+\mathrm{i}M_{\eta,4}-M_{\zeta,4}-M_{\eta,0} & \mathrm{i}M_0-\mathrm{i}M_{\theta,0}-M_4+M_{\theta,4} \\ -\mathrm{i}\Theta_0-\mathrm{i}\Theta_3+\mathrm{i}\Upsilon_0+\mathrm{i}\Upsilon_3 & -\mathrm{i}\Theta_1+\mathrm{i}\Upsilon_1-\Theta_2+\Upsilon_2 \\ -\mathrm{i}\Theta_1+\mathrm{i}\Upsilon_1+\Theta_2-\Upsilon_2 & -\mathrm{i}\Theta_0+\mathrm{i}\Theta_3+\mathrm{i}\Upsilon_0-\mathrm{i}\Upsilon_3 \end{array}.$$

Hence,

$$\begin{gathered} \widehat{Q}= \\ =\mathrm{i}\Theta_0\beta^{[0]}+\mathrm{i}\Upsilon_0\beta^{[0]}\gamma^{[5]}+ \\ +\mathrm{i}\Theta_1\beta^{[1]}+\mathrm{i}\Upsilon_1\beta^{[1]}\gamma^{[5]}+ \\ +\mathrm{i}\Theta_2\beta^{[2]}+\mathrm{i}\Upsilon_2\beta^{[2]}\gamma^{[5]}+ \\ +\mathrm{i}\Theta_3\beta^{[3]}+\mathrm{i}\Upsilon_3\beta^{[3]}\gamma^{[5]}+ \\ +\mathrm{i}M_0\gamma^{[0]}+\mathrm{i}M_4\beta^{[4]}- \\ -\mathrm{i}M_{\zeta,0}\gamma_\zeta^{[0]}+\mathrm{i}M_{\zeta,4}\zeta^{[4]}- \\ -\mathrm{i}M_{\eta,0}\gamma_\eta^{[0]}-\mathrm{i}M_{\eta,4}\eta^{[4]}+ \\ +\mathrm{i}M_{\theta,0}\gamma_\theta^{[0]}+\mathrm{i}M_{\theta,4}\theta^{[4]}. \end{gathered}$$

Therefore, from (2.25):

$$\frac{1}{c}\partial_t\varphi - \left(i\Theta_0\beta^{[0]} + i\Upsilon_0\beta^{[0]}\gamma^{[5]}\right)\varphi = \left(\begin{array}{c} \sum_{\nu=1}^{3}\beta^{[\nu]}\left(\partial_\nu + i\Theta_\nu + i\Upsilon_\nu\gamma^{[5]}\right) + \\ +iM_0\gamma^{[0]} + iM_4\beta^{[4]} - \\ -iM_{\zeta,0}\gamma_\zeta^{[0]} + iM_{\zeta,4}\zeta^{[4]} - \\ -iM_{\eta,0}\gamma_\eta^{[0]} - iM_{\eta,4}\eta^{[4]} + \\ +iM_{\theta,0}\gamma_\theta^{[0]} + iM_{\theta,4}\theta^{[4]} \end{array}\right)\varphi. \tag{2.30}$$

with

$$\gamma^{[5]} := \left[\begin{array}{cc} 1_2 & 0_2 \\ 0_2 & -1_2 \end{array}\right]. \tag{2.31}$$

Because

$$\zeta^{[k]} + \eta^{[k]} + \theta^{[k]} = -\beta^{[k]}$$

with $k \in \{1,2,3\}$ then from (2.30):

$$\begin{aligned} &\left(\begin{array}{c} -\left(\partial_0 + i\Theta_0 + i\Upsilon_0\gamma^{[5]}\right) + \sum_{k=1}^{3}\beta^{[k]}\left(\partial_k + i\Theta_k + i\Upsilon_k\gamma^{[5]}\right) \\ +2\left(iM_0\gamma^{[0]} + iM_4\beta^{[4]}\right) \end{array}\right)\varphi + \\ &+\left(\begin{array}{c} -\left(\partial_0 + i\Theta_0 + i\Upsilon_0\gamma^{[5]}\right) - \sum_{k=1}^{3}\zeta^{[k]}\left(\partial_k + i\Theta_k + i\Upsilon_k\gamma^{[5]}\right) \\ +2\left(-iM_{\zeta,0}\gamma_\zeta^{[0]} + iM_{\zeta,4}\zeta^{[4]}\right) \end{array}\right)\varphi + \\ &+\left(\begin{array}{c} \left(\partial_0 + i\Theta_0 + i\Upsilon_0\gamma^{[5]}\right) - \sum_{k=1}^{3}\eta^{[k]}\left(\partial_k + i\Theta_k + i\Upsilon_k\gamma^{[5]}\right) \\ +2\left(-iM_{\eta,0}\gamma_\eta^{[0]} - iM_{\eta,4}\eta^{[4]}\right) \end{array}\right)\varphi + \\ &+\left(\begin{array}{c} -\left(\partial_0 + i\Theta_0 + i\Upsilon_0\gamma^{[5]}\right) - \sum_{k=1}^{3}\theta^{[k]}\left(\partial_k + i\Theta_k + i\Upsilon_k\gamma^{[5]}\right) \\ +2\left(iM_{\theta,0}\gamma_\theta^{[0]} + iM_{\theta,4}\theta^{[4]}\right) \end{array}\right)\varphi = 0. \end{aligned}$$

In (2.30) summands

$$\begin{array}{c} -iM_{\zeta,0}\gamma_\zeta^{[0]} + iM_{\zeta,4}\zeta^{[4]} - \\ -iM_{\eta,0}\gamma_\eta^{[0]} - iM_{\eta,4}\eta^{[4]} + \\ +iM_{\theta,0}\gamma_\theta^{[0]} + iM_{\theta,4}\theta^{[4]} \end{array}$$

contain elements of chromatic pentads and

$$\sum_{k=1}^{3}\beta^{[k]}\left(\partial_k + i\Theta_k + i\Upsilon_k\gamma^{[5]}\right) + iM_0\gamma^{[0]} + iM_4\beta^{[4]}$$

contains only elements of the light pentads. The following sum

$$\widehat{H}_l := c\sum_{k=1}^{3}\beta^{[k]}\left(i\partial_k - \Theta_k - \Upsilon_k\gamma^{[5]}\right) - cM_0\gamma^{[0]} - cM_4\beta^{[4]} \tag{2.32}$$

is called *lepton Hamiltonian*.
And the following equation:

$$\left(\sum_{k=0}^{3}\beta^{[k]}\left(\mathrm{i}\partial_k-\Theta_k-\Upsilon_k\gamma^{[5]}\right)-M_0\gamma^{[0]}-M_4\beta^{[4]}\right)\widetilde{\varphi}=0 \tag{2.33}$$

is called *lepton moving equation*
If like to (2.15):

$$\varphi^{\dagger}\gamma^{[0]}\varphi:=-\frac{j_{\mathcal{A},0}}{\mathrm{c}}\text{ and }\varphi^{\dagger}\beta^{[4]}\varphi:=-\frac{j_{\mathcal{A},4}}{\mathrm{c}}$$

and:

$$\rho_{\mathcal{A}}u_{\mathcal{A},4}:=j_{\mathcal{A},4}\text{ and }\rho_{\mathcal{A}}u_{\mathcal{A},5}:=j_{\mathcal{A},5} \tag{2.34}$$

then from (2.14):

$$-\frac{u_{\mathcal{A},5}}{\mathrm{c}}=\sin 2\alpha\left(\begin{array}{c}\sin\beta\sin\chi\cos(-\theta+\upsilon)\\+\cos\beta\cos\chi\cos(\gamma-\lambda)\end{array}\right),$$
$$-\frac{u_{\mathcal{A},4}}{\mathrm{c}}=\sin 2\alpha\left(\begin{array}{c}-\sin\beta\sin\chi\sin(-\theta+\upsilon)\\+\cos\beta\cos\chi\sin(\gamma-\lambda)\end{array}\right).$$

Hence, from (2.11):

$$u_{\mathcal{A},1}^2+u_{\mathcal{A},2}^2+u_{\mathcal{A},3}^2+u_{\mathcal{A},4}^2+u_{\mathcal{A},5}^2=\mathrm{c}^2.$$

Thus, of only all five elements of a Clifford pentad lends an entire kit of velocity components and, for completeness, yet two ”space” coordinates x_5 and x_4 should be added to our three x_1,x_2,x_3. These additional coordinates can be selected such that

$$-\frac{\pi\mathrm{c}}{\mathrm{h}}\leq x_5\leq\frac{\pi\mathrm{c}}{\mathrm{h}},-\frac{\pi\mathrm{c}}{\mathrm{h}}\leq x_4\leq\frac{\pi\mathrm{c}}{\mathrm{h}}.$$

Coordinates x_4 and x_5 are not of any events coordinates. Hence, our devices do not detect of its as space coordinates.

Let us denote:

$$\widetilde{\varphi}(t,x_1,x_2,x_3,x_5,x_4):=\varphi(t,x_1,x_2,x_3)\cdot$$
$$\cdot(\exp(\mathrm{i}(x_5M_0(t,x_1,x_2,x_3)+x_4M_4(t,x_1,x_2,x_3)))).$$

In this case equation of moving with lepton Hamiltonian (2.32) shape is the following:

$$\left(\sum_{k=0}^{3}\beta^{[0]}\left(\mathrm{i}\partial_k-\Theta_k-\Upsilon_k\gamma^{[5]}\right)-\gamma^{[0]}\mathrm{i}\partial_5-\beta^{[4]}\mathrm{i}\partial_4\right)\widetilde{\varphi}=0 \tag{2.35}$$

Let g_1 be the positive real number and for $\mu\in\{0,1,2,3\}$: F_μ and B_μ be the solutions of the following system of the equations:

$$\left\{\begin{array}{c}-0.5g_1B_\mu+F_\mu=-\Theta_\mu-\Upsilon_\mu;\\-g_1B_\mu+F_\mu=-\Theta_\mu+\Upsilon_\mu.\end{array}\right|$$

Let *charge matrix* be denoted as the following:

$$Y := -\begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 2\cdot 1_2 \end{bmatrix}. \tag{2.36}$$

Thus[23]:

$$\begin{aligned}
& -\Theta_\mu - \Upsilon_\mu \gamma^{[5]} = \\
= & -\Theta_\mu 1_4 - \Upsilon_\mu \gamma^{[5]} = \\
= & -\Theta_\mu \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 1_2 \end{bmatrix} - \Upsilon_\mu \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{bmatrix} = \\
= & -\left(\begin{bmatrix} \Theta_\mu 1_2 & 0_2 \\ 0_2 & \Theta_\mu 1_2 \end{bmatrix} + \begin{bmatrix} \Upsilon_\mu 1_2 & 0_2 \\ 0_2 & -\Upsilon_\mu 1_2 \end{bmatrix} \right) = \\
= & \begin{bmatrix} (-\Theta_\mu - \Upsilon_\mu) 1_2 & 0_2 \\ 0_2 & (-\Theta_\mu + \Upsilon_\mu) 1_2 \end{bmatrix} =
\end{aligned}$$

[23] If products $AB_{j,s}$ exist for all j, s then

$$A \begin{bmatrix} B_{0,0} & B_{0,1} & \cdots & B_{0,n} \\ B_{1,0} & B_{1,1} & \cdots & B_{1,n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m,0} & B_{m,1} & \cdots & B_{m,n} \end{bmatrix} := \begin{bmatrix} AB_{0,0} & AB_{0,1} & \cdots & AB_{0,n} \\ AB_{1,0} & AB_{1,1} & \cdots & AB_{1,n} \\ \cdots & \cdots & \cdots & \cdots \\ AB_{m,0} & AB_{m,1} & \cdots & AB_{m,n} \end{bmatrix}$$

and if products $B_{j,s}A$ exist for all j, s then

$$\begin{bmatrix} B_{0,0} & B_{0,1} & \cdots & B_{0,n} \\ B_{1,0} & B_{1,1} & \cdots & B_{1,n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m,0} & B_{m,1} & \cdots & B_{m,n} \end{bmatrix} A := \begin{bmatrix} B_{0,0}A & B_{0,1}A & \cdots & B_{0,n}A \\ B_{1,0}A & B_{1,1}A & \cdots & B_{1,n}A \\ \cdots & \cdots & \cdots & \cdots \\ B_{m,0}A & B_{m,1}A & \cdots & B_{m,n}A \end{bmatrix}. \tag{2.37}$$

If A and all $B_{j,s}$ are $k \times k$ matrices then

$$A + \begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} & \cdots & B_{0,n} \\ B_{1,0} & B_{1,1} & B_{1,2} & \cdots & B_{1,n} \\ B_{2,0} & B_{2,1} & B_{2,2} & \cdots & B_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{n,0} & B_{n,1} & B_{n,2} & \cdots & B_{n,n} \end{bmatrix} :=$$

$$:= A1_{nk} + \begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} & \cdots & B_{0,n} \\ B_{1,0} & B_{1,1} & B_{1,2} & \cdots & B_{1,n} \\ B_{2,0} & B_{2,1} & B_{2,2} & \cdots & B_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{n,0} & B_{n,1} & B_{n,2} & \cdots & B_{n,n} \end{bmatrix} =$$

$$= \begin{bmatrix} B_{0,0}+A & B_{0,1} & B_{0,2} & \cdots & B_{0,n} \\ B_{1,0} & B_{1,1}+A & B_{1,2} & \cdots & B_{1,n} \\ B_{2,0} & B_{2,1} & B_{2,2}+A & \cdots & B_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{n,0} & B_{n,1} & B_{n,2} & \cdots & B_{n,n}+A \end{bmatrix}. \tag{2.38}$$

$$= \left[\begin{array}{cc} (-0.5g_1B_\mu + F_\mu)\,1_2 & 0_2 \\ 0_2 & (-g_1B_\mu + F_\mu)\,1_2 \end{array}\right].$$

And

$$\begin{aligned}
&F_\mu + 0.5g_1YB_\mu = \\
&= F_\mu 1_4 + 0.5g_1YB_\mu \\
&= F_\mu \left[\begin{array}{cc} 1_2 & 0_2 \\ 0_2 & 1_2 \end{array}\right] + 0.5g_1\left(-\left[\begin{array}{cc} 1_2 & 0_2 \\ 0_2 & 2\cdot 1_2 \end{array}\right]\right)B_\mu = \\
&= \left[\begin{array}{cc} F_\mu 1_2 & 0_2 \\ 0_2 & F_\mu 1_2 \end{array}\right] - \left[\begin{array}{cc} 0.5g_1B_\mu 1_2 & 0_2 \\ 0_2 & 0.5g_1B_\mu 2\cdot 1_2 \end{array}\right] = \\
&= \left[\begin{array}{cc} F_\mu 1_2 - 0.5g_1B_\mu 1_2 & 0_2 \\ 0_2 & F_\mu 1_2 - g_1B_\mu\cdot 1_2 \end{array}\right].
\end{aligned}$$

Hence,

$$-\Theta_\mu - \Upsilon_\mu\gamma^{[5]} = F_\mu + 0.5g_1YB_\mu$$

and from (2.35):

$$\left(\sum_{k=0}^{3}\beta^{[k]}\left(\mathrm{i}\partial_k + F_k + 0.5g_1YB_k\right) - \gamma^{[0]}\mathrm{i}\partial_5 - \beta^{[4]}\mathrm{i}\partial_4\right)\widetilde{\varphi} = 0 \qquad (2.39)$$

Let $\chi(t, x_1, x_2, x_3)$ be the real function and:

$$\widetilde{U}(\chi) := \left[\begin{array}{cc} \exp\left(\mathrm{i}\frac{\chi}{2}\right)1_2 & 0_2 \\ 0_2 & \exp(\mathrm{i}\chi)\,1_2 \end{array}\right]. \qquad (2.40)$$

In that case for $\mu \in \{0,1,2,3\}$:

$$\begin{aligned}
\partial_\mu\widetilde{U} &= \partial_\mu\left[\begin{array}{cc} \exp\left(\mathrm{i}\frac{\chi}{2}\right)1_2 & 0_2 \\ 0_2 & \exp(\mathrm{i}\chi)\,1_2 \end{array}\right] \\
&= \left[\begin{array}{cc} \partial_\mu\exp\left(\mathrm{i}\frac{\chi}{2}\right)1_2 & \partial_\mu 0_2 \\ \partial_\mu 0_2 & \partial_\mu\exp(\mathrm{i}\chi)\,1_2 \end{array}\right] \\
&= \left[\begin{array}{cc} \mathrm{i}\frac{\partial_\mu\chi}{2}\exp\left(\mathrm{i}\frac{\chi}{2}\right)1_2 & 0_2 \\ 0_2 & \mathrm{i}\partial_\mu\chi\exp(\mathrm{i}\chi)\,1_2 \end{array}\right] \\
&= \mathrm{i}\frac{\partial_\mu\chi}{2}\left[\begin{array}{cc} \exp\left(\mathrm{i}\frac{\chi}{2}\right)1_2 & 0_2 \\ 0_2 & 2\exp(\mathrm{i}\chi)\,1_2 \end{array}\right],
\end{aligned}$$

and

$$\begin{aligned}
Y\widetilde{U} &= -\left[\begin{array}{cc} 1_2 & 0_2 \\ 0_2 & 2\cdot 1_2 \end{array}\right]\left[\begin{array}{cc} \exp\left(\mathrm{i}\frac{\chi}{2}\right)1_2 & 0_2 \\ 0_2 & \exp(\mathrm{i}\chi)\,1_2 \end{array}\right] \\
&= -\left[\begin{array}{cc} \exp\left(\mathrm{i}\frac{\chi}{2}\right)1_2 & 0_2 \\ 0_2 & 2\exp(\mathrm{i}\chi)\,1_2 \end{array}\right].
\end{aligned}$$

Hence:

$$\partial_\mu \widetilde{U} = -\mathrm{i}\frac{\partial_\mu \chi}{2} Y \widetilde{U}. \tag{2.41}$$

Moreover you can calculate that:

$$\begin{array}{rcl}
\widetilde{U}^\dagger \gamma^{[0]} \widetilde{U} & = & \gamma^{[0]} \cos\frac{\chi}{2} + \beta^{[4]} \sin\frac{\chi}{2}, \\
\widetilde{U}^\dagger \beta^{[4]} \widetilde{U} & = & \beta^{[4]} \cos\frac{\chi}{2} - \gamma^{[0]} \sin\frac{\chi}{2}, \\
\widetilde{U}^\dagger \widetilde{U} & = & 1_4, \\
\widetilde{U}^\dagger Y \widetilde{U} & = & Y, \\
\beta^{[k]} \widetilde{U} & = & \widetilde{U} \beta^{[k]}
\end{array}$$

for $k \in \{0, 1, 2, 3\}$
Let

$$\begin{array}{rcl}
x'_4 & = & x_4 \cos\frac{\chi}{2} - x_5 \sin\frac{\chi}{2}, \\
x'_5 & = & x_5 \cos\frac{\chi}{2} + x_4 \sin\frac{\chi}{2}.
\end{array}$$

In that case by the partial derivate definition for any function u:

$$\begin{array}{rcl}
\partial_4 u & = & \partial'_4 u \cdot \partial_4 x'_4 + \partial'_5 u \cdot \partial_4 x'_5 = \partial'_4 u \cdot \cos\frac{\chi}{2} + \partial'_5 u \cdot \sin\frac{\chi}{2}, \\
\partial_5 u & = & \partial'_4 u \cdot \partial_5 x'_4 + \partial'_5 u \cdot \partial_5 x'_5 = \partial'_4 u \cdot \left(-\sin\frac{\chi}{2}\right) + \partial'_5 u \cdot \cos\frac{\chi}{2}.
\end{array} \tag{2.42}$$

Let $\partial_4 \chi = 0$ and $\partial_5 \chi = 0$; hence, $\partial_4 U = U \partial_4$ and $\partial_5 U = U \partial_5$.
From (2.39):

$$\left(\sum_{s=0}^{3} \beta^{[s]} \left(\mathrm{i}\partial_s + F_s + 0.5 g_1 Y B_s\right) - \gamma^{[0]} \mathrm{i}\partial_5 - \beta^{[4]} \mathrm{i}\partial_4\right) \widetilde{\varphi} = 0. \tag{2.43}$$

Let

$$B'_\mu = B_\mu - \frac{1}{g_1} \partial_\mu \chi.$$

According to (2.42) and since $\widetilde{U}^\dagger \widetilde{U} = 1_4$ and $\widetilde{U}^\dagger Y \widetilde{U} = Y$ then

$$\left(\begin{array}{c}
\sum_{s=0}^{3} \beta^{[s]} \left(\mathrm{i}\partial_s + F_s + 0.5 g_1 \widetilde{U}^\dagger Y \widetilde{U} \left(B'_s + \frac{1}{g_1} \partial_s \chi\right)\right) - \\
-\gamma^{[0]} \mathrm{i} \left(-\sin\frac{\chi}{2} \partial'_4 + \cos\frac{\chi}{2} \partial'_5\right) - \beta^{[4]} \mathrm{i} \left(\cos\frac{\chi}{2} \partial'_4 + \sin\frac{\chi}{2} \partial'_5\right)
\end{array}\right) \widetilde{U}^\dagger \widetilde{U} \widetilde{\varphi} = 0.$$

Hence:

$$\left(\begin{array}{c}\sum_{s=0}^{3}\beta^{[s]}\left(\mathrm{i}\partial_s+F_s+0.5g_1\widetilde{U}^{\dagger}Y\widetilde{U}\left(B'_s+\frac{1}{g_1}\partial_s\chi\right)\right)-\\-\left(-\gamma^{[0]}\sin\frac{\chi}{2}+\beta^{[4]}\cos\frac{\chi}{2}\right)\mathrm{i}\partial'_4-\left(\gamma^{[0]}\cos\frac{\chi}{2}+\beta^{[4]}\sin\frac{\chi}{2}\right)\mathrm{i}\partial'_5\end{array}\right)\widetilde{U}^{\dagger}\widetilde{U}\widetilde{\varphi}=0.$$

Since $\widetilde{U}$ is a linear operator then

$$\left(\begin{array}{c}\sum_{s=0}^{3}\beta^{[s]}\left(\mathrm{i}\partial_s+F_s+0.5g_1\widetilde{U}^{\dagger}Y\widetilde{U}\left(B'_s+\frac{1}{g_1}\partial_s\chi\right)\right)\widetilde{U}^{\dagger}-\\-\left(-\gamma^{[0]}\sin\frac{\chi}{2}+\beta^{[4]}\cos\frac{\chi}{2}\right)\mathrm{i}\partial'_4\widetilde{U}^{\dagger}-\left(\gamma^{[0]}\cos\frac{\chi}{2}+\beta^{[4]}\sin\frac{\chi}{2}\right)\mathrm{i}\partial'_5\widetilde{U}^{\dagger}\end{array}\right)\widetilde{U}\widetilde{\varphi}=0$$

and since $\partial_4 U=U\partial_4$ and $\partial_5 U=U\partial_5$ then

$$\left(\begin{array}{c}\sum_{s=0}^{3}\beta^{[s]}\left(\mathrm{i}\partial_s\widetilde{U}^{\dagger}+F_s\widetilde{U}^{\dagger}+0.5g_1\widetilde{U}^{\dagger}Y\widetilde{U}\widetilde{U}^{\dagger}\left(B'_s+\frac{1}{g_1}\partial_s\chi\right)\right)-\\-\left(-\gamma^{[0]}\widetilde{U}^{\dagger}\sin\frac{\chi}{2}+\beta^{[4]}\widetilde{U}^{\dagger}\cos\frac{\chi}{2}\right)\mathrm{i}\partial'_4\\-\left(\gamma^{[0]}\widetilde{U}^{\dagger}\cos\frac{\chi}{2}+\beta^{[4]}\widetilde{U}^{\dagger}\sin\frac{\chi}{2}\right)\mathrm{i}\partial'_5\end{array}\right)\widetilde{U}\widetilde{\varphi}=0. \tag{2.44}$$

Since

$$\begin{aligned}\widetilde{U}^{\dagger}\gamma^{[0]}\widetilde{U}&=\gamma^{[0]}\cos\frac{\chi}{2}+\beta^{[4]}\sin\frac{\chi}{2},\\\widetilde{U}^{\dagger}\beta^{[4]}\widetilde{U}&=\beta^{[4]}\cos\frac{\chi}{2}-\gamma^{[0]}\sin\frac{\chi}{2}\end{aligned}$$

then

$$\begin{aligned}\widetilde{U}^{\dagger}\gamma^{[0]}\widetilde{U}\widetilde{U}^{\dagger}&=\gamma^{[0]}\widetilde{U}^{\dagger}\cos\frac{\chi}{2}+\beta^{[4]}\widetilde{U}^{\dagger}\sin\frac{\chi}{2},\\\widetilde{U}^{\dagger}\beta^{[4]}\widetilde{U}\widetilde{U}^{\dagger}&=\beta^{[4]}\widetilde{U}^{\dagger}\cos\frac{\chi}{2}-\gamma^{[0]}\widetilde{U}^{\dagger}\sin\frac{\chi}{2},\end{aligned}$$

Hence,

$$\begin{aligned}\widetilde{U}^{\dagger}\gamma^{[0]}&=\gamma^{[0]}\widetilde{U}^{\dagger}\cos\frac{\chi}{2}+\beta^{[4]}\widetilde{U}^{\dagger}\sin\frac{\chi}{2},\\\widetilde{U}^{\dagger}\beta^{[4]}&=\beta^{[4]}\widetilde{U}^{\dagger}\cos\frac{\chi}{2}-\gamma^{[0]}\widetilde{U}^{\dagger}\sin\frac{\chi}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma^{[0]}\widetilde{U}^{\dagger}&=\widetilde{U}^{\dagger}\gamma^{[0]}\cos\tfrac{\chi}{2}-\widetilde{U}^{\dagger}\beta^{[4]}\sin\tfrac{\chi}{2},\\\beta^{[4]}\widetilde{U}^{\dagger}&=\widetilde{U}^{\dagger}\gamma^{[0]}\sin\tfrac{\chi}{2}+\widetilde{U}^{\dagger}\beta^{[4]}\cos\tfrac{\chi}{2}.\end{aligned}$$

Thus, from (2.44):

$$\left(\begin{array}{c} \sum_{s=0}^{3}\beta^{[s]}\left(\mathrm{i}\partial_s\widetilde{U}^{\dagger}+F_s\widetilde{U}^{\dagger}+0.5g_1\widetilde{U}^{\dagger}Y\widetilde{U}\widetilde{U}^{\dagger}\left(B'_s+\frac{1}{g_1}\partial_s\chi\right)\right)- \\ -\left(\begin{array}{c} -\left(\widetilde{U}^{\dagger}\gamma^{[0]}\cos\frac{\chi}{2}-\widetilde{U}^{\dagger}\beta^{[4]}\sin\frac{\chi}{2}\right)\sin\frac{\chi}{2} \\ +\left(\widetilde{U}^{\dagger}\gamma^{[0]}\sin\frac{\chi}{2}+\widetilde{U}^{\dagger}\beta^{[4]}\cos\frac{\chi}{2}\right)\cos\frac{\chi}{2} \end{array}\right)\mathrm{i}\partial'_4 \\ -\left(\begin{array}{c} \left(\widetilde{U}^{\dagger}\gamma^{[0]}\cos\frac{\chi}{2}-\widetilde{U}^{\dagger}\beta^{[4]}\sin\frac{\chi}{2}\right)\cos\frac{\chi}{2} \\ +\left(\widetilde{U}^{\dagger}\gamma^{[0]}\sin\frac{\chi}{2}+\widetilde{U}^{\dagger}\beta^{[4]}\cos\frac{\chi}{2}\right)\sin\frac{\chi}{2} \end{array}\right)\mathrm{i}\partial'_5 \end{array}\right)\widetilde{U}\widetilde{\varphi}=0.$$

Hence:

$$\left(\begin{array}{c} \sum_{s=0}^{3}\beta^{[s]}\left(\mathrm{i}\partial_s\widetilde{U}^{\dagger}+F_s\widetilde{U}^{\dagger}+0.5g_1\widetilde{U}^{\dagger}Y\left(B'_s+\frac{1}{g_1}\partial_s\chi\right)\right)- \\ -\widetilde{U}^{\dagger}\beta^{[4]}\mathrm{i}\partial'_4-\ \widetilde{U}^{\dagger}\gamma^{[0]}\mathrm{i}\partial'_5 \end{array}\right)\widetilde{U}\widetilde{\varphi}=0. \tag{2.45}$$

Since (2.41):

$$\partial_\mu\widetilde{U}=-\mathrm{i}\frac{\partial_\mu\chi}{2}Y\widetilde{U}$$

then for $s\in\{0,1,2,3\}$:

$$\partial_s\widetilde{U}^{\dagger}=\mathrm{i}\frac{\partial_s\chi}{2}\widetilde{U}^{\dagger}Y^{\dagger}=\mathrm{i}\frac{\partial_s\chi}{2}Y\widetilde{U}^{\dagger}.$$

Therefore,

$$\begin{gathered}\partial_s\left(\widetilde{U}^{\dagger}\widetilde{U}\widetilde{\varphi}\right)=\partial_s\left(\widetilde{U}^{\dagger}\left(\widetilde{U}\widetilde{\varphi}\right)\right)= \\ =\left(\partial_s\widetilde{U}^{\dagger}\right)\left(\widetilde{U}\widetilde{\varphi}\right)+\widetilde{U}^{\dagger}\partial_s\left(\widetilde{U}\widetilde{\varphi}\right)=\mathrm{i}\frac{\partial_s\chi}{2}Y\widetilde{U}^{\dagger}\left(\widetilde{U}\widetilde{\varphi}\right)+\widetilde{U}^{\dagger}\partial_s\left(\widetilde{U}\widetilde{\varphi}\right)= \\ =\left(\mathrm{i}\frac{\partial_s\chi}{2}Y\widetilde{U}^{\dagger}+\widetilde{U}^{\dagger}\partial_s\right)\left(\widetilde{U}\widetilde{\varphi}\right).\end{gathered}$$

Since $Y\widetilde{U}^{\dagger}=\widetilde{U}^{\dagger}Y$ then

$$\mathrm{i}\frac{\partial_s\chi}{2}Y\widetilde{U}^{\dagger}+\widetilde{U}^{\dagger}\partial_s=\widetilde{U}^{\dagger}\mathrm{i}\frac{\partial_s\chi}{2}Y+\widetilde{U}^{\dagger}\partial_s.$$

Hence,

$$\mathrm{i}\partial_s\widetilde{U}^{\dagger}=-\widetilde{U}^{\dagger}\frac{\partial_s\chi}{2}Y+\widetilde{U}^{\dagger}\mathrm{i}\partial_s.$$

Therefore, from (2.45):

$$\left(\begin{array}{c} \sum_{s=0}^{3}\beta^{[s]}\left(-\widetilde{U}^{\dagger}\frac{\partial_s\chi}{2}Y+\widetilde{U}^{\dagger}\mathrm{i}\partial_s+F_s\widetilde{U}^{\dagger}+0.5g_1\widetilde{U}^{\dagger}Y\left(B'_s+\frac{1}{g_1}\partial_s\chi\right)\right)- \\ -\widetilde{U}^{\dagger}\beta^{[4]}\mathrm{i}\partial'_4-\ \widetilde{U}^{\dagger}\gamma^{[0]}\mathrm{i}\partial'_5 \end{array}\right)\widetilde{U}\widetilde{\varphi}=0.$$

Hence:

$$\left(\begin{array}{c}\sum_{s=0}^{3}\beta^{[s]}\left(\widetilde{U}^{\dagger}\mathrm{i}\partial_s+\widetilde{U}^{\dagger}F_s'+0.5g_1\widetilde{U}^{\dagger}YB_s'\right)- \\ -\widetilde{U}^{\dagger}\beta^{[4]}\mathrm{i}\partial_4'-\ \widetilde{U}^{\dagger}\gamma^{[0]}\mathrm{i}\partial_5'\end{array}\right)\widetilde{U}\widetilde{\varphi}=0$$

with $F_s' := \widetilde{U}F_s\widetilde{U}^{\dagger}$.
Since $\beta^{[s]}\widetilde{U}=\widetilde{U}\beta^{[s]}$ for $s\in\{0,1,2,3\}$ then

$$\left(\begin{array}{c}\sum_{s=0}^{3}\widetilde{U}^{\dagger}\beta^{[s]}\left(\mathrm{i}\partial_s+\widetilde{U}^{\dagger}F_s'+0.5g_1\widetilde{U}^{\dagger}YB_s'\right)- \\ -\widetilde{U}^{\dagger}\beta^{[4]}\mathrm{i}\partial_4'-\ \widetilde{U}^{\dagger}\gamma^{[0]}\mathrm{i}\partial_5'\end{array}\right)\widetilde{U}\widetilde{\varphi}=0.$$

Hence, if denote: $\widetilde{\varphi}' := \widetilde{U}\widetilde{\varphi}$ then since $\widetilde{U}$ is a linear operator then:

$$\widetilde{U}^{\dagger}\left(\sum_{s=0}^{3}\beta^{[s]}\left(\mathrm{i}\partial_s+F_s'+0.5g_1YB_s'\right)-\beta^{[4]}\mathrm{i}\partial_4'-\ \gamma^{[0]}\mathrm{i}\partial_5'\right)\widetilde{\varphi}'=0.$$

That is

$$\left(\sum_{s=0}^{3}\beta^{[s]}\left(\mathrm{i}\partial_s+F_s'+0.5g_1YB_s'\right)-\beta^{[4]}\mathrm{i}\partial_4'-\ \gamma^{[0]}\mathrm{i}\partial_5'\right)\widetilde{\varphi}'=0.$$

Compare with (2.43).
Thus, this Equation of moving is invariant under the following transformations:

$$\begin{aligned}
&x_4\to x_4'=x_4\cos\frac{\chi}{2}-x_5\sin\frac{\chi}{2};\\
&x_5\to x_5'=x_5\cos\frac{\chi}{2}+x_4\sin\frac{\chi}{2};\\
&x_\mu\to x_\mu'=x_\mu \text{ for } \mu\in\{0,1,2,3\};\\
&\widetilde{\varphi}\to\widetilde{\varphi}'=\widetilde{U}\widetilde{\varphi},\\
&B_\mu\to B_\mu'=B_\mu-\frac{1}{g_1}\partial_\mu\chi,\\
&F_\mu\to F_\mu'=\widetilde{U}F_s\widetilde{U}^{\dagger}.
\end{aligned}\tag{2.46}$$

Therefore, B_μ is like to the B-boson field of Standard Model[24] [20]. field.

2.4. Masses

Let

$$\varepsilon_1=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\varepsilon_2=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\varepsilon_3=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\varepsilon_4=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.\tag{2.47}$$

[24]Sheldon Lee Glashow (born December 5, 1932) is a American theoretical physicist.

Functions of type :

$$\frac{\mathrm{h}}{2\pi \mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(sx_4+nx_5\right)\right)\varepsilon_k \tag{2.48}$$

with an integer n and s form orthonormal basis of some unitary space $\Im$ with scalar product of the following shape:

$$(\widetilde{\varphi},\widetilde{\chi}) := \int_{-\frac{\pi c}{h}}^{\frac{\pi c}{h}} dx_5 \int_{-\frac{\pi c}{h}}^{\frac{\pi c}{h}} dx_4 \cdot \widetilde{\varphi}^{\dagger}\widetilde{\chi} \tag{2.49}$$

(compare with (2.16)).
In that case from (2.15):

$$\begin{aligned} (\widetilde{\varphi},\widetilde{\varphi}) &= \rho_{\mathcal{A}}, \\ \left(\widetilde{\varphi},\beta^{[s]}\widetilde{\varphi}\right) &= -\frac{j_{\mathcal{A},k}}{\mathrm{c}}. \end{aligned} \tag{2.50}$$

for $s \in \{1,2,3\}$
Let[25]

$$N_{\vartheta}\left(t,x_1,x_2,x_3\right) := \mathrm{trunc}\left(\frac{\mathrm{c}M_0}{\mathrm{h}}\right), N_{\varpi}\left(t,x_1,x_2,x_3\right) := \mathrm{trunc}\left(\frac{\mathrm{c}M_4}{\mathrm{h}}\right).$$

Hence, functions $N_{\vartheta}\left(t,x_1,x_2,x_3\right)$ and $N_{\varpi}\left(t,x_1,x_2,x_3\right)$ have got integer values.
In that case to high precision:

$$\widetilde{\varphi} = \varphi\left(t,x_1,x_2,x_3\right)\cdot\exp\left(-\mathrm{i}\left(x_5\frac{\mathrm{h}}{\mathrm{c}}N_{\vartheta}\left(t,x_1,x_2,x_3\right)+x_4\frac{\mathrm{h}}{\mathrm{c}}N_{\varpi}\left(t,x_1,x_2,x_3\right)\right)\right)$$

and Fourier series for $\widetilde{\varphi}$ is of the following form:

$$\widetilde{\varphi}\left(t,x_1,x_2,x_3,x_5,x_4\right) = \varphi\left(t,x_1,x_2,x_3\right)\cdot\sum_{n,s}\delta_{-n,N_{\vartheta}(t,\mathbf{x})}\delta_{-s,N_{\varpi}(t,\mathbf{x})}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(nx_5+sx_4\right)\right)$$

with

$$\begin{aligned} \delta_{-n,N_{\vartheta}} &= \frac{\mathrm{h}}{2\pi\mathrm{c}}\int_{-\frac{\pi c}{h}}^{\frac{\pi c}{h}}\exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(nx_5\right)\right)\exp\left(\mathrm{i}N_{\vartheta}\frac{\mathrm{h}}{\mathrm{c}}x_5\right)dx_5 = \frac{\sin\pi\left(n+N_{\vartheta}\right)}{\pi\left(n+N_{\vartheta}\right)}, \\ \delta_{-s,N_{\varpi}} &= \frac{\mathrm{h}}{2\pi\mathrm{c}}\int_{-\frac{\pi c}{h}}^{\frac{\pi c}{h}}\exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(sx_4\right)\right)\exp\left(\mathrm{i}N_{\varpi}\frac{\mathrm{h}}{\mathrm{c}}x_4\right)dx_4 = \frac{\sin\pi\left(s+N_{\varpi}\right)}{\pi\left(s+N_{\varpi}\right)} \end{aligned}$$

with integer n and s.
If denote:

[25] Function $\mathrm{trunc}(x)$ returns the integer part of a real number x by removing the fractional part. For example:

$$\mathrm{trunc}(-2.0857) = -2.$$

$$f(t,\mathbf{x},-n,-s) := \varphi(t,\mathbf{x})\,\delta_{n,N_\vartheta(t,\mathbf{x})}\delta_{s,N_\varpi(t,\mathbf{x})}$$

then

$$\begin{array}{c}\widetilde{\varphi}(t,\mathbf{x},x_5,x_4) = \\ = \sum_{n,s} f(t,\mathbf{x},n,s)\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(nx_5+sx_4)\right).\end{array} \tag{2.51}$$

The integer numbers n and s are denoted *mass numbers*.

From properties of δ: in every point $\langle t,\mathbf{x}\rangle$: either

$$\widetilde{\varphi}(t,\mathbf{x},x_5,x_4) = 0$$

or integer numbers n_0 and s_0 exist for which:

$$\begin{array}{c}\widetilde{\varphi}(t,\mathbf{x},x_5,x_4) = \\ = f(t,\mathbf{x},n_0,s_0)\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(n_0x_5+s_0x_4)\right).\end{array} \tag{2.52}$$

Here if

$$m_0 := \sqrt{n_0^2+s_0^2}$$

and

$$m := \frac{\mathrm{h}^2}{\mathrm{c}^2}m_0 \tag{2.53}$$

then m is denoted *mass* of $\widetilde{\varphi}$.

That is for every space-time point: either this point is empty or single mass is placed in this point.

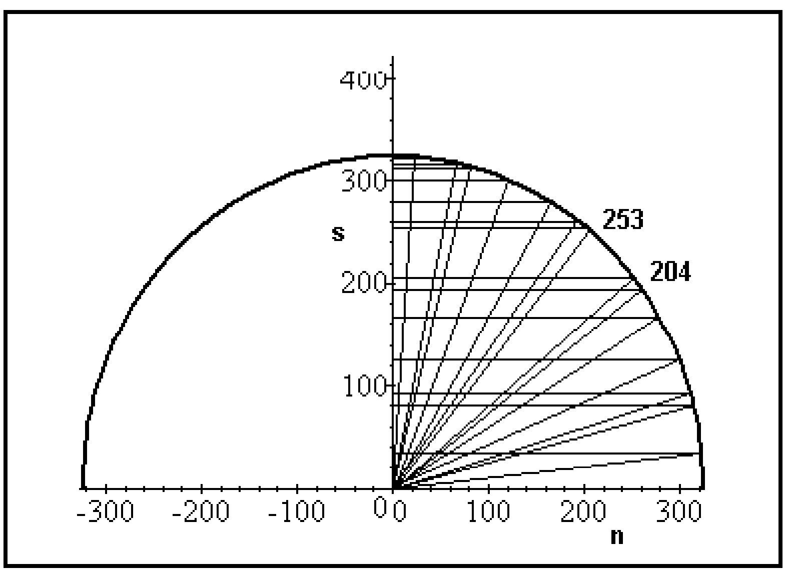

Figure 7:

Equation of moving (2.39) under the transformation (2.46) has the following form:

$$\sum_{n',s'}\left(\begin{array}{c}\beta^{[0]}\left(\mathrm{i}\frac{1}{\mathrm{c}}\partial_t+F_\mu+0.5g_1YB_\mu\gamma^{[5]}\right)\\+\sum_{\mu=1}^{3}\beta^{[\mu]}\left(\mathrm{i}\partial_\mu+F_\mu+0.5g_1YB_\mu\gamma^{[5]}\right)+\gamma^{[0]}\mathrm{i}\partial'_5+\beta^{[4]}\mathrm{i}\partial'_4\end{array}\right)\cdot\\ \cdot\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(n'x_5+s'x_4\right)\right)\widetilde{U}f=0$$

with:

$$n'=n\cos\frac{\chi}{2}-s\sin\frac{\chi}{2},$$
$$s'=n\sin\frac{\chi}{2}+s\cos\frac{\chi}{2}.$$

But s and n are integer numbers and s' and n' must be integer numbers, too.

A triplet $\langle\lambda;n,s\rangle$ of integer numbers is called *a Pythagorean triple* [19] if

$$\lambda^2=n^2+s^2.$$

Let ε be the tiny positive real number. I call an integer number λ as *a father number with precise* ε if for each real number χ and for every Pythagorean triple $\langle\lambda;n,s\rangle$ here exists a Pythagorean triple $\langle\lambda;n',s'\rangle$ such that:

$$\begin{array}{rcl}\left|-s\sin\frac{\chi}{2}+n\cos\frac{\chi}{2}-n'\right| & < & \lambda\varepsilon,\\ \left|s\cos\frac{\chi}{2}+n\sin\frac{\chi}{2}-s'\right| & < & \lambda\varepsilon.\end{array}$$

For example: number 325 is a father number for the following Pythagorean triples (Figure 7):

$$\begin{array}{l}\langle325;323,36\rangle,\\ \langle325;315,80\rangle,\\ \langle325;312,91\rangle,\\ \langle325;300,125\rangle,\\ \langle325;280,165\rangle,\\ \langle325;260,195\rangle,\\ \langle325;253,204\rangle,\end{array}$$

Here ε is maximal ratio value of difference between adjacent s values to father number. That is here $\varepsilon=\frac{253-204}{325}=0.15$. But for any value of precise ε here exists a fitting father number in long distant domain of the natural numerical line. But I can not calculate it since more high-end machine than my computer is needed for such calculation.

The nearest-neighbors to 325 father numbers are numbers 333 and 337. But these father numbers have got one at a time triple. Hence, fathers, having many "children", are isolated numbers on the natural numerical line. I suspect that these numbers are fathers of particles families.

Here are three families (generations) according to the Standard Model of particle physics [20]:

$$\left[\begin{array}{ccc}\begin{pmatrix}\nu_e\\ e^-\end{pmatrix} & \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix} & \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}\\ \begin{pmatrix}u\\ d\end{pmatrix} & \begin{pmatrix}c\\ s\end{pmatrix} & \begin{pmatrix}t\\ b\end{pmatrix}\end{array}\right].$$

Each generation is divided into two leptons:

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}, \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}, \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix},$$

and two quarks:

$$\begin{pmatrix}u\\ d\end{pmatrix}, \begin{pmatrix}c\\ s\end{pmatrix}, \begin{pmatrix}t\\ b\end{pmatrix}.$$

The two leptons may be divided into one electron-like (e^- - electron, μ^- - μ–lepton, τ^- - τ-lepton) and neutrino (ν_e, ν_μ, ν_τ); the two quarks may be divided into one down-type (d, s, b) and one up-type (u, c, t). The first generation consists of the electron, electron neutrino and the down and up quarks. The second generation consists of the muon, muon neutrino and the strange and charm quarks. The third generation consists of the tau lepton, tau neutrino and the bottom and top quarks. Each member of a higher generation has greater mass than the corresponding particle of the previous generation. For example: the first-generation electron has a mass of only 0.511 MeV, the second-generation muon has a mass of 106 MeV, and the third-generation tau lepton has a mass of 1777 MeV (almost twice as heavy as a proton). All ordinary atoms are made of particles from the first generation. Electrons surround a nucleus made of protons and neutrons, which contain up and down quarks. The second and third generations of charged particles do not occur in normal matter and are only seen in extremely high-energy environments. Neutrinos of all generations stream throughout the universe but rarely interact with normal matter.

2.5. One-Mass State

Let form of (2.51) be the following:

$$\widetilde{\varphi}(t,\mathbf{x},x_5,x_4) = \exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}nx_5\right)\sum_{k=1}^{4} f_k(t,\mathbf{x},n,0)\,\varepsilon_k.$$

In that case the Hamiltonian has the following form (from (2.39)):

$$\widehat{H} = \mathrm{c}\left(\sum_{k=1}^{3}\beta^{[k]}\mathrm{i}\partial_k + \frac{\mathrm{h}}{\mathrm{c}}n\gamma^{[0]} + \widehat{G}\right)$$

with

$$\widehat{G} := \sum_{\mu=0}^{3}\beta^{[\mu]}\left(F_\mu + 0.5g_1YB_\mu\right).$$

Let

$$\omega(\mathbf{k}) := \sqrt{\mathbf{k}^2 + n^2} = \sqrt{k_1^2 + k_2^2 + k_3^2 + n^2}$$

and

$$e_1(\mathbf{k}) := \frac{1}{2\sqrt{\omega(\mathbf{k})(\omega(\mathbf{k}) + n)}} \begin{bmatrix} \omega(\mathbf{k}) + n + k_3 \\ k_1 + \mathrm{i}k_2 \\ \omega(\mathbf{k}) + n - k_3 \\ -k_1 - \mathrm{i}k_2 \end{bmatrix}. \tag{2.54}$$

Let

$$\widehat{H}_0 := \mathrm{c}\sum_{s=1}^{3} \beta^{[s]}\mathrm{i}\partial_s + \mathrm{h}n\gamma^{[0]}. \tag{2.55}$$

Since (2.53):

$$\mathrm{h}n = m\frac{\mathrm{c}^2}{\mathrm{h}}$$

then equation of moving with Hamiltonian $\widehat{H}_0$ has the following form:

$$\boxed{\frac{1}{\mathrm{c}}\mathrm{i}\partial_t\varphi = \left(\sum_{s=1}^{3} \beta^{[s]}\mathrm{i}\partial_s + m\frac{\mathrm{c}}{\mathrm{h}}\gamma^{[0]}\right)\varphi.} \tag{2.56}$$

This is the Dirac equation (Paul Dirac[26] formulated it in 1928).
Let us denote

$$\gamma^{[s]} := \gamma^{[0]}\beta^{[s]}$$

for $s \neq 0$.
Let us calculate:

$$\begin{aligned} & \gamma^{[s]}\gamma^{[j]} + \gamma^{[j]}\gamma^{[s]} \\ = \; & \gamma^{[0]}\beta^{[s]}\gamma^{[0]}\beta^{[j]} + \gamma^{[0]}\beta^{[j]}\gamma^{[0]}\beta^{[s]} = \\ = \; & -\gamma^{[0]}\gamma^{[0]}\beta^{[s]}\beta^{[j]} - \gamma^{[0]}\gamma^{[0]}\beta^{[j]}\beta^{[s]} = \\ = \; & -\left(\beta^{[s]}\beta^{[j]} + \beta^{[j]}\beta^{[s]}\right) = -2\delta_{j,s} \end{aligned}$$

for $s \neq 0$ and $j \neq 0$.
and

$$\gamma^{[s]}\gamma^{[0]} + \gamma^{[0]}\gamma^{[s]} = \gamma^{[0]}\beta^{[s]}\gamma^{[0]} + \gamma^{[0]}\gamma^{[0]}\beta^{[s]} = -\beta^{[s]} + \beta^{[s]} = 0$$

for $s \neq 0$.
From (2.56):

[26]Paul Adrien Maurice Dirac (1902 – 1984) was an English theoretical physicist who made fundamental contributions to the early development of both quantum mechanics and quantum electrodynamics.

$$\left(\frac{1}{\mathrm{c}}\mathrm{i}\gamma^{[0]}\partial_t - \sum_{s=1}^{3}\gamma^{[s]}\mathrm{i}\partial_s - m\frac{\mathrm{c}}{\mathrm{h}}\right)\varphi = 0.$$

Let us multiply both parts of this equation on

$$\left(\frac{1}{\mathrm{c}}\mathrm{i}\gamma^{[0]}\partial_t - \sum_{s'=1}^{3}\gamma^{[s']}\mathrm{i}\partial_{s'} + m\frac{\mathrm{c}}{\mathrm{h}}\right):$$

$$\left(\frac{1}{\mathrm{c}}\mathrm{i}\gamma^{[0]}\partial_t - \sum_{s'=1}^{3}\gamma^{[s']}\mathrm{i}\partial_{s'} + m\frac{\mathrm{c}}{\mathrm{h}}\right)\left(\frac{1}{\mathrm{c}}\mathrm{i}\gamma^{[0]}\partial_t - \sum_{s=1}^{3}\gamma^{[s]}\mathrm{i}\partial_s - m\frac{\mathrm{c}}{\mathrm{h}}\right)\varphi = 0.$$

Hence,

$$\left(\begin{array}{c} -\frac{1}{\mathrm{c}^2}\partial_t^2 \\ -\sum_{s=1}^{3}\frac{1}{\mathrm{c}}\mathrm{i}\gamma^{[0]}\partial_t\gamma^{[s]}\mathrm{i}\partial_s - \sum_{s'=1}^{3}\gamma^{[s']}\mathrm{i}\partial_{s'}\frac{1}{\mathrm{c}}\mathrm{i}\gamma^{[0]}\partial_t \\ -\frac{1}{\mathrm{c}}\mathrm{i}\gamma^{[0]}\partial_t m\frac{\mathrm{c}}{\mathrm{h}} + m\frac{\mathrm{c}}{\mathrm{h}}\frac{1}{\mathrm{c}}\mathrm{i}\gamma^{[0]}\partial_t \\ +\sum_{s'=1}^{3}\gamma^{[s']}\mathrm{i}\partial_{s'}\sum_{s=1}^{3}\gamma^{[s]}\mathrm{i}\partial_s \\ +\sum_{s'=1}^{3}\gamma^{[s']}\mathrm{i}\partial_{s'}m\frac{\mathrm{c}}{\mathrm{h}} - \sum_{s=1}^{3}m\frac{\mathrm{c}}{\mathrm{h}}\gamma^{[s]}\mathrm{i}\partial_s \\ -\frac{m^2\mathrm{c}^2}{\mathrm{h}^2} \end{array}\right)\varphi = 0.$$

Hence,

$$\left(\begin{array}{c} -\frac{1}{\mathrm{c}^2}\partial_t^2 \\ +\sum_{s'=1}^{3}\gamma^{[s']}\mathrm{i}\partial_{s'}\sum_{s=1}^{3}\gamma^{[s]}\mathrm{i}\partial_s \\ -\frac{m^2\mathrm{c}^2}{\mathrm{h}^2} \end{array}\right)\varphi = 0$$

Since

$$\begin{aligned}
& \sum_{s'=1}^{3}\gamma^{[s']}\mathrm{i}\partial_{s'}\sum_{s=1}^{3}\gamma^{[s]}\mathrm{i}\partial_s \\
= \quad & -\sum_{s=1}^{3}\sum_{s'=1}^{3}\gamma^{[s']}\gamma^{[s]}\partial_{s'}\partial_s = \\
= \quad & -\left(\begin{array}{c} \gamma^{[1]}\gamma^{[1]}\partial_1\partial_1 + \gamma^{[2]}\gamma^{[1]}\partial_2\partial_1 + \gamma^{[3]}\gamma^{[1]}\partial_3\partial_1 \\ +\gamma^{[1]}\gamma^{[2]}\partial_1\partial_2 + \gamma^{[2]}\gamma^{[2]}\partial_2\partial_2 + \gamma^{[3]}\gamma^{[2]}\partial_3\partial_2 \\ +\gamma^{[1]}\gamma^{[3]}\partial_1\partial_3 + \gamma^{[2]}\gamma^{[3]}\partial_2\partial_3 + \gamma^{[3]}\gamma^{[3]}\partial_3\partial_3 \end{array}\right) = \\
= \quad & -\left(\begin{array}{c} -\partial_1\partial_1 \\ +\gamma^{[2]}\gamma^{[1]}\partial_2\partial_1 + \gamma^{[1]}\gamma^{[2]}\partial_1\partial_2 \\ +\gamma^{[3]}\gamma^{[1]}\partial_3\partial_1 + \gamma^{[1]}\gamma^{[3]}\partial_1\partial_3 \\ -\partial_2\partial_2 \\ +\gamma^{[3]}\gamma^{[2]}\partial_3\partial_2 + \gamma^{[2]}\gamma^{[3]}\partial_2\partial_3 \\ -\partial_3\partial_3 \end{array}\right).
\end{aligned}$$

Hence,

$$\sum_{s'=1}^{3}\gamma^{[s']}\mathrm{i}\partial_{s'}\sum_{s=1}^{3}\gamma^{[s]}\mathrm{i}\partial_{s}=\partial_1\partial_1+\partial_2\partial_2+\partial_3\partial_3=\sum_{s=1}^{3}\partial_s^2.$$

Thus,

$$\boxed{\left(-\frac{1}{\mathrm{c}^2}\partial_t^2+\sum_{s=1}^{3}\partial_s^2-\frac{m^2\mathrm{c}^2}{\mathrm{h}^2}\right)\varphi=0.} \tag{2.57}$$

This is the Klein-Gordon[27][28] equation for a free particle with mass m.
Let us calculate:

$$\begin{aligned}
&\widehat{H}_0 e_1(\mathbf{k})\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)=\\
&=\left(\mathrm{c}\textstyle\sum_{s=1}^{3}\beta^{[s]}\mathrm{i}\partial_s+\mathrm{h}n\gamma^{[0]}\right)\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}e_1(\mathbf{k})\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)=\\
&==\mathrm{c}\textstyle\sum_{s=1}^{3}\beta^{[s]}\mathrm{i}\partial_s e_1(\mathbf{k})\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)+\\
&+\mathrm{h}n\gamma^{[0]}e_1(\mathbf{k})\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)=\\
&=\mathrm{c}\textstyle\sum_{s=1}^{3}\beta^{[s]}\mathrm{i}e_1(\mathbf{k})\partial_s\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)+\\
&+\mathrm{h}n\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)\gamma^{[0]}e_1(\mathbf{k})=\\
&=\mathrm{c}\textstyle\sum_{s=1}^{3}\beta^{[s]}\mathrm{i}e_1(\mathbf{k})\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}k_s\right)\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}k_s\right)+\\
&+\mathrm{h}n\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)\gamma^{[0]}e_1(\mathbf{k})=\\
&=\textstyle\sum_{s=1}^{3}(-\mathrm{i}\mathrm{h}k_s)\beta^{[s]}\mathrm{i}e_1(\mathbf{k})\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)+\\
&+\mathrm{h}n\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)\gamma^{[0]}e_1(\mathbf{k})=\\
&=\mathrm{h}\left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\tfrac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)\left(\textstyle\sum_{s=1}^{3}k_s\beta^{[s]}+n\gamma^{[0]}\right)e_1(\mathbf{k})=
\end{aligned}$$

$$\begin{aligned}
= {} & \mathrm{h}\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)\begin{bmatrix} k_3 & k_1-\mathrm{i}k_2 & n & 0\\ k_1+\mathrm{i}k_2 & -k_3 & 0 & n\\ n & 0 & -k_3 & -k_1+\mathrm{i}k_2\\ 0 & n & -k_1-\mathrm{i}k_2 & k_3\end{bmatrix}\cdot\\
& \cdot\frac{1}{2\sqrt{\omega(\mathbf{k})(\omega(\mathbf{k})+n)}}\begin{bmatrix}\omega(\mathbf{k})+n+k_3\\ k_1+\mathrm{i}k_2\\ \omega(\mathbf{k})+n-k_3\\ -k_1-\mathrm{i}k_2\end{bmatrix}\\
= {} & \mathrm{h}\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)\begin{bmatrix}k_3\omega(\mathbf{k})+k_3^2+k_1^2+k_2^2+n\omega(\mathbf{k})+n^2\\ k_1\omega(\mathbf{k})+ik_2\omega(\mathbf{k})\\ n\omega(\mathbf{k})+n^2-k_3\omega(\mathbf{k})+k_3^2+k_1^2+k_2^2\\ -k_1\omega(\mathbf{k})-ik_2\omega(\mathbf{k})\end{bmatrix}\\
= {} & \omega(\mathbf{k})\frac{1}{2\sqrt{\omega(\mathbf{k})(\omega(\mathbf{k})+n)}}\begin{bmatrix}k_3+n+\omega(\mathbf{k})\\ k_1+ik_2\\ n+\omega(\mathbf{k})-k_3\\ -k_1-ik_2\end{bmatrix}\mathrm{h}\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right).
\end{aligned}$$

[27] Oskar Klein, 1894-1977
[28] Walter Gordon, 1893-1939

Therefore,

$$\widehat{H}_0 e_1(\mathbf{k})\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)=\mathrm{h}\omega(\mathbf{k})\,e_1(\mathbf{k})\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right). \tag{2.58}$$

Hence, function $e_1(\mathbf{k})\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)$ is an eigenvector of $\widehat{H}_0$ with eigenvalue

$$\mathrm{h}\omega(\mathbf{k})=\mathrm{h}\sqrt{\mathbf{k}^2+n^2}.$$

Similarly, function $e_2(\mathbf{k})\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)$ with

$$e_2(\mathbf{k}):=\frac{1}{2\sqrt{\omega(\mathbf{k})(\omega(\mathbf{k})+n)}}\begin{bmatrix} k_1-\mathrm{i}k_2 \\ \omega(\mathbf{k})+n-k_3 \\ -k_1+\mathrm{i}k_2 \\ \omega(\mathbf{k})+n+k_3 \end{bmatrix} \tag{2.59}$$

is eigenvector of $\widehat{H}_0$ with eigenvalue $\mathrm{h}\omega(\mathbf{k})=\mathrm{h}\sqrt{\mathbf{k}^2+n^2}$, too, and functions

$$e_3(\mathbf{k})\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)\text{ and }e_4(\mathbf{k})\left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)$$

with

$$e_3(\mathbf{k}):=\frac{1}{2\sqrt{\omega(\mathbf{k})(\omega(\mathbf{k})+n)}}\begin{bmatrix} -\omega(\mathbf{k})-n+k_3 \\ k_1+\mathrm{i}k_2 \\ \omega(\mathbf{k})+n+k_3 \\ k_1+\mathrm{i}k_2 \end{bmatrix} \tag{2.60}$$

and

$$e_4(\mathbf{k}):=\frac{1}{2\sqrt{\omega(\mathbf{k})(\omega(\mathbf{k})+n)}}\begin{bmatrix} k_1-\mathrm{i}k_2 \\ -\omega(\mathbf{k})-n-k_3 \\ k_1-\mathrm{i}k_2 \\ \omega(\mathbf{k})+n-k_3 \end{bmatrix} \tag{2.61}$$

are eigenvectors of $\widehat{H}_0$ with eigenvalue $-\mathrm{h}\omega(\mathbf{k})$.

Here $e_\mu(\mathbf{k})$ with $\mu\in\{1,2,3,4\}$ form an orthonormal basis in the space spanned on vectors ε_μ (2.47).

2.6. Creating and Annihilation Operators

Let $\mathfrak{H}$ be some unitary space. Let $\widetilde{0}$ be the zero element of $\mathfrak{H}$. That is any element $\widetilde{F}$ of $\mathfrak{H}$ obeys to the following conditions:

$0\widetilde{F}=\widetilde{0}$, $\widetilde{0}+\widetilde{F}=\widetilde{F}$, $\widetilde{0}^\dagger\widetilde{F}=\widetilde{F}$, $\widetilde{0}^\dagger=\widetilde{0}$.

Let $\widehat{0}$ be the zero operator on $\mathfrak{H}$. That is any element $\widetilde{F}$ of $\mathfrak{H}$ obeys to the following condition:

$\widehat{0}\widetilde{F} = 0\widetilde{F}$, and if $\widehat{b}$ is any operator on $\mathfrak{H}$ then

$\widehat{0}+\widehat{b} = \widehat{b}+\widehat{0} = \widehat{b}$, $\widehat{0}\widehat{b} = \widehat{b}\widehat{0} = \widehat{0}$.

Let $\widehat{1}$ be the identy operator on $\mathfrak{H}$. That is any element $\widetilde{F}$ of $\mathfrak{H}$ obeys to the following condition:
$\widehat{1}\widetilde{F} = 1\widetilde{F} = \widetilde{F}$, and if $\widehat{b}$ is any operator on $\mathfrak{H}$ then
$\widehat{1}\widehat{b} = \widehat{b}\widehat{1} = \widehat{b}$.

Let linear operators $b_{s,\mathbf{k}}$ ($s \in \{1,2,3,4\}$) act on all elements of this space. And let these operators fulfill the following conditions:

$$\left\{b_{s,\mathbf{k}}^{\dagger}, b_{s',\mathbf{k}'}\right\} := b_{s,\mathbf{k}}^{\dagger} b_{s',\mathbf{k}'} + b_{s',\mathbf{k}'} b_{s,\mathbf{k}}^{\dagger} = \left(\frac{}{2\pi}\right)^3 \delta_{\mathbf{k},\mathbf{k}'} \delta_{s,s'} \widehat{1},$$

$$\left\{b_{s,\mathbf{k}}, b_{s',\mathbf{k}'}\right\} = b_{s,\mathbf{k}} b_{s',\mathbf{k}'} + b_{s',\mathbf{k}'} b_{s,\mathbf{k}} = \left\{b_{s,\mathbf{k}}^{\dagger}, b_{s',\mathbf{k}'}^{\dagger}\right\} = \widehat{0}.$$

Hence,

$$b_{s,\mathbf{k}} b_{s,\mathbf{k}} = b_{s,\mathbf{k}}^{\dagger} b_{s,\mathbf{k}}^{\dagger} = \widehat{0}.$$

There exists element $\widetilde{F}_0$ of $\mathfrak{H}$ such that $\widetilde{F}_0^{\dagger}\widetilde{F}_0 = 1$ and for any $b_{s,\mathbf{k}}$: $b_{s,\mathbf{k}}\widetilde{F}_0 = \widetilde{0}$. Hence, $\widetilde{F}_0^{\dagger} b_{s,\mathbf{k}}^{\dagger} = \widetilde{0}$.

Let

$$\psi_s(\mathbf{x}) := \sum_{\mathbf{k}} \sum_{r=1}^{4} b_{r,\mathbf{k}} e_{r,s}(\mathbf{k}) \exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}\mathbf{x}\right).$$

Because

$$\sum_{r=1}^{4} e_{r,s}(\mathbf{k})\, e_{r,s'}(\mathbf{k}) = \delta_{s,s'}$$

and

$$\sum_{\mathbf{k}} \exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}\left(\mathbf{x}-\mathbf{x}'\right)\right) = \left(\frac{2\pi\mathrm{c}}{\mathrm{h}}\right)^3 \delta\left(\mathbf{x}-\mathbf{x}'\right)$$

then

$$\begin{aligned} \left\{\psi_s^{\dagger}(\mathbf{x}), \psi_{s'}\left(\mathbf{x}'\right)\right\} &:= \psi_s^{\dagger}(\mathbf{x})\,\psi_{s'}\left(\mathbf{x}'\right) + \psi_{s'}\left(\mathbf{x}'\right)\psi_s^{\dagger}(\mathbf{x}) \\ &= \delta\left(\mathbf{x}-\mathbf{x}'\right)\delta_{s,s'}\widehat{1}. \end{aligned}$$

And these operators obey the following conditions:

$\psi_s(\mathbf{x})\,\widetilde{F}_0 = \widetilde{0}$, $\{\psi_s(\mathbf{x}), \psi_{s'}(\mathbf{x}')\} = \left\{\psi_s^{\dagger}(\mathbf{x}), \psi_{s'}^{\dagger}(\mathbf{x}')\right\} = \widehat{0}$.

Hence,

$\psi_s(\mathbf{x})\,\psi_{s'}(\mathbf{x}') = \psi_s^{\dagger}(\mathbf{x})\,\psi_{s'}^{\dagger}(\mathbf{x}') = \widehat{0}$.

Let

$$\Psi(t,\mathbf{x}) := \sum_{s=1}^{4} \varphi_s(t,\mathbf{x})\,\psi_s^{\dagger}(\mathbf{x})\,\widetilde{F}_0. \tag{2.62}$$

These function obey the following condition:

$$\Psi^{\dagger}\left(t,\mathbf{x}'\right)\Psi\left(t,\mathbf{x}\right)=\varphi^{\dagger}\left(t,\mathbf{x}'\right)\varphi\left(t,\mathbf{x}\right)\delta\left(\mathbf{x}-\mathbf{x}'\right).$$

Hence,

$$\int d\mathbf{x}'\cdot\Psi^{\dagger}\left(t,\mathbf{x}'\right)\Psi\left(t,\mathbf{x}\right)=\rho\left(t,\mathbf{x}\right). \tag{2.63}$$

Let a Fourier series of $\varphi_{s}\left(t,\mathbf{x}\right)$ has the following form:

$$\varphi_{s}\left(t,\mathbf{x}\right)=\sum_{\mathbf{p}}\sum_{r=1}^{4}c_{r}\left(t,\mathbf{p}\right)e_{r,s}\left(\mathbf{p}\right)\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{p}\mathbf{x}\right).$$

In that case:

$$\underline{\Psi}\left(t,\mathbf{p}\right):=\left(\frac{2\pi\mathrm{c}}{\mathrm{h}}\right)^{3}\sum_{r=1}^{4}c_{r}\left(t,\mathbf{p}\right)b_{r,\mathbf{p}}^{\dagger}\widetilde{F}_{0}.$$

If

$$\mathcal{H}_{0}\left(\mathbf{x}\right):=\psi^{\dagger}\left(\mathbf{x}\right)\widehat{H}_{0}\psi\left(\mathbf{x}\right) \tag{2.64}$$

then $\mathcal{H}_{0}\left(\mathbf{x}\right)$ is called a Hamiltonian $\widehat{H}_{0}$ density.

Because

$$\widehat{H}_{0}\varphi\left(t,\mathbf{x}\right)=\mathrm{i}\frac{\partial}{\partial t}\varphi\left(t,\mathbf{x}\right)$$

then

$$\int d\mathbf{x}'\cdot\mathcal{H}_{0}\left(\mathbf{x}'\right)\Psi\left(t,\mathbf{x}\right)=\mathrm{i}\frac{\partial}{\partial t}\Psi\left(t,\mathbf{x}\right). \tag{2.65}$$

Therefore, if

$$\widehat{\mathbb{H}}:=\int d\mathbf{x}'\cdot\mathcal{H}_{0}\left(\mathbf{x}'\right)$$

then $\widehat{\mathbb{H}}$ acts similar to the Hamiltonian on space $\mathfrak{H}$.

And if

$$E_{\Psi}\left(\widetilde{F}_{0}\right):=\sum_{\mathbf{p}}\underline{\Psi}^{\dagger}\left(t,\mathbf{p}\right)\widehat{\mathbb{H}}\underline{\Psi}\left(t,\mathbf{p}\right)$$

then $E_{\Psi}\left(\widetilde{F}_{0}\right)$ is an energy of Ψ on vacuum $\widetilde{F}_{0}$.

Let us consider operator $\widehat{N}_{a}\left(\mathbf{x}_{0}\right):=\psi_{a}^{\dagger}\left(\mathbf{x}_{0}\right)\psi_{a}\left(\mathbf{x}_{0}\right)$.

Let us calculate an average value of this operator:

$$\left\langle\widehat{N}_{a}\left(\mathbf{x}_{0}\right)\right\rangle_{\Psi}:=\int_{\Omega}d\mathbf{x}\cdot\widehat{N}_{a}\left(\mathbf{x}_{0}\right)\rho\left(t,\mathbf{x}\right).$$

In accordance with (2.63):

$$\left\langle\widehat{N}_{a}\left(\mathbf{x}_{0}\right)\right\rangle_{\Psi}=\int_{\Omega}d\mathbf{x}\int_{\Omega}d\mathbf{x}'\cdot\Psi^{\dagger}\left(t,\mathbf{x}'\right)\psi_{a}^{\dagger}\left(\mathbf{x}_{0}\right)\psi_{a}\left(\mathbf{x}_{0}\right)\Psi\left(t,\mathbf{x}\right).$$

Since in accordance with (2.62):

$$\Psi(t,\mathbf{x}) = \sum_{j=1}^{4} \varphi_j(t,\mathbf{x})\,\psi_j^{\dagger}(\mathbf{x})\,\widetilde{F}_0.$$

then

$$\begin{aligned}
&\left\langle \widehat{N}_a(\mathbf{x}_0)\right\rangle_{\Psi} = \\
= &\int_{\Omega} d\mathbf{x} \int_{\Omega} d\mathbf{x}' \cdot \sum_{s=1}^{4} \varphi_s^{*}\left(t,\mathbf{x}'\right) \widetilde{F}_0^{\dagger} \psi_s\left(\mathbf{x}'\right) \psi_a^{\dagger}(\mathbf{x}_0)\,\psi_a(\mathbf{x}_0) \sum_{j=1}^{4} \varphi_j(t,\mathbf{x})\,\psi_j^{\dagger}(\mathbf{x})\,\widetilde{F}_0 = \\
= &\int_{\Omega} d\mathbf{x} \int_{\Omega} d\mathbf{x}' \cdot \sum_{s=1}^{4} \sum_{j=1}^{4} \varphi_s^{*}\left(t,\mathbf{x}'\right) \varphi_j(t,\mathbf{x})\,\widetilde{F}_0^{\dagger} \psi_s\left(\mathbf{x}'\right) \psi_a^{\dagger}(\mathbf{x}_0)\,\psi_a(\mathbf{x}_0)\,\psi_j^{\dagger}(\mathbf{x})\,\widetilde{F}_0.
\end{aligned}$$

Since

$$\psi_a^{\dagger}(\mathbf{x}_0)\,\psi_s\left(\mathbf{x}'\right) + \psi_s\left(\mathbf{x}'\right) \psi_a^{\dagger}(\mathbf{x}_0) = \delta\left(\mathbf{x}_0 - \mathbf{x}'\right) \delta_{s,a}\widehat{1}$$

then

$$\begin{aligned}
\left\langle \widehat{N}_a(\mathbf{x}_0)\right\rangle_{\Psi} = &\int_{\Omega} d\mathbf{x} \int_{\Omega} d\mathbf{x}' \cdot \sum_{s=1}^{4} \sum_{j=1}^{4} \varphi_s^{*}\left(t,\mathbf{x}'\right) \varphi_j(t,\mathbf{x}) \cdot \\
&\cdot \widetilde{F}_0^{\dagger}\left(\delta\left(\mathbf{x}_0 - \mathbf{x}'\right) \delta_{s,a}\widehat{1} - \psi_a^{\dagger}(\mathbf{x}_0)\,\psi_s\left(\mathbf{x}'\right)\right) \psi_a(\mathbf{x}_0)\,\psi_j^{\dagger}(\mathbf{x})\,\widetilde{F}_0 \\
= &\int_{\Omega} d\mathbf{x} \int_{\Omega} d\mathbf{x}' \cdot \sum_{s=1}^{4} \sum_{j=1}^{4} \varphi_s^{*}\left(t,\mathbf{x}'\right) \varphi_j(t,\mathbf{x}) \cdot \\
&\cdot \left(\delta\left(\mathbf{x}_0 - \mathbf{x}'\right) \delta_{s,a}\widetilde{F}_0^{\dagger}\widehat{1} - \widetilde{F}_0^{\dagger}\psi_a^{\dagger}(\mathbf{x}_0)\,\psi_s\left(\mathbf{x}'\right)\right) \psi_a(\mathbf{x}_0)\,\psi_j^{\dagger}(\mathbf{x})\,\widetilde{F}_0.
\end{aligned}$$

Since $\widetilde{F}_0^{\dagger}\widehat{1} = \widetilde{F}_0^{\dagger}$ and $\widetilde{F}_0^{\dagger}\psi_a^{\dagger}(\mathbf{x}_0) = \widetilde{0}$ then

$$\begin{aligned}
&\left\langle \widehat{N}_a(\mathbf{x}_0)\right\rangle_{\Psi} = \\
= &\int_{\Omega} d\mathbf{x} \int_{\Omega} d\mathbf{x}' \cdot \sum_{s=1}^{4} \sum_{j=1}^{4} \varphi_s^{*}\left(t,\mathbf{x}'\right) \varphi_j(t,\mathbf{x})\,\delta\left(\mathbf{x}_0 - \mathbf{x}'\right) \delta_{s,a}\widetilde{F}_0^{\dagger}\psi_a(\mathbf{x}_0)\,\psi_j^{\dagger}(\mathbf{x})\,\widetilde{F}_0.
\end{aligned}$$

According with properties of δ-function and δ:

$$\left\langle \widehat{N}_a(\mathbf{x}_0)\right\rangle_{\Psi} = \int_{\Omega} d\mathbf{x} \cdot \sum_{j=1}^{4} \varphi_a^{*}(t,\mathbf{x}_0)\,\varphi_j(t,\mathbf{x})\,\widetilde{F}_0^{\dagger}\psi_a(\mathbf{x}_0)\,\psi_j^{\dagger}(\mathbf{x})\,\widetilde{F}_0.$$

Since

$$\psi_j^{\dagger}(\mathbf{x})\,\psi_a(\mathbf{x}_0) + \psi_a(\mathbf{x}_0)\,\psi_j^{\dagger}(\mathbf{x}) = \delta(\mathbf{x}_0 - \mathbf{x})\,\delta_{j,a}\widehat{1}$$

then

$$
\begin{aligned}
&\left\langle \widehat{N}_a(\mathbf{x}_0)\right\rangle_{\Psi} = \\
&= \int_{\Omega} d\mathbf{x}\cdot\sum_{j=1}^{4}\varphi_a^*(t,\mathbf{x}_0)\varphi_j(t,\mathbf{x})\widetilde{F}_0^{\dagger}\left(\delta(\mathbf{x}_0-\mathbf{x})\delta_{j,a}\widehat{1}-\psi_j^{\dagger}(\mathbf{x})\psi_a(\mathbf{x}_0)\right)\widetilde{F}_0 \\
&= \int_{\Omega} d\mathbf{x}\cdot\sum_{j=1}^{4}\varphi_a^*(t,\mathbf{x}_0)\varphi_j(t,\mathbf{x})\left(\delta(\mathbf{x}_0-\mathbf{x})\delta_{j,a}\widetilde{F}_0^{\dagger}\widehat{1}\widetilde{F}_0-\widetilde{F}_0^{\dagger}\psi_j^{\dagger}(\mathbf{x})\psi_a(\mathbf{x}_0)\widetilde{F}_0\right) \\
&= \int_{\Omega} d\mathbf{x}\cdot\sum_{j=1}^{4}\varphi_a^*(t,\mathbf{x}_0)\varphi_j(t,\mathbf{x})\left(\delta(\mathbf{x}_0-\mathbf{x})\delta_{j,a}\widetilde{F}_0^{\dagger}\widetilde{F}_0-\widetilde{0}^{\dagger}\widetilde{0}\right). \\
&= \int_{\Omega} d\mathbf{x}\cdot\sum_{j=1}^{4}\varphi_a^*(t,\mathbf{x}_0)\varphi_j(t,\mathbf{x})\left(\delta(\mathbf{x}_0-\mathbf{x})\delta_{j,a}1-0\right) \\
&= \int_{\Omega} d\mathbf{x}\cdot\sum_{j=1}^{4}\varphi_a^*(t,\mathbf{x}_0)\varphi_j(t,\mathbf{x})\delta(\mathbf{x}_0-\mathbf{x})\delta_{j,a}.
\end{aligned}
$$

Thus:

$$
\boxed{\left\langle \widehat{N}_a(\mathbf{x}_0)\right\rangle_{\Psi} = \varphi_a^*(t,\mathbf{x}_0)\varphi_a(t,\mathbf{x}_0).} \tag{2.66}
$$

That is operator $\widehat{N}_a(\mathbf{x}_0)$ brings the a-component of the event probability density.
Let $\Psi_a(t,\mathbf{x}) := \psi_a(\mathbf{x}_0)\Psi(t,\mathbf{x})$.
In that case

$$
\begin{aligned}
\left\langle \widehat{N}_a(\mathbf{x}_0)\right\rangle_{\Psi_a} &= \int_{\Omega} d\mathbf{x}\int_{\Omega} d\mathbf{x}'\cdot\Psi^{\dagger}(t,\mathbf{x})\psi_a^{\dagger}(\mathbf{x}_0)\psi_a^{\dagger}(\mathbf{x}_0) \\
&\quad \psi_a(\mathbf{x}_0)\psi_a(\mathbf{x}_0)\psi_a(\mathbf{x}_0)\Psi(t,\mathbf{x}).
\end{aligned}
$$

Since

$$
\psi_a(\mathbf{x}_0)\psi_a(\mathbf{x}_0) = \widehat{0}
$$

then

$$
\left\langle \widehat{N}_a(\mathbf{x}_0)\right\rangle_{\Psi_a} = 0.
$$

Therefore $\psi_a(\mathbf{x}_0)$ ”annihilates” the a of the event-probability density.

2.7. Particles and Antiparticles

Operator $\widehat{\mathbb{H}}$ obeys the following condition:

$$
\widehat{\mathbb{H}} = \left(\frac{2\pi c}{h}\right)^3\sum_{\mathbf{k}} h\omega(\mathbf{k})\left(\sum_{r=1}^{2} b_{r,\mathbf{k}}^{\dagger}b_{r,\mathbf{k}}-\sum_{r=3}^{4} b_{r,\mathbf{k}}^{\dagger}b_{r,\mathbf{k}}\right).
$$

This operator is not positive defined and in this case

$$E_{\Psi}\left(\widetilde{F}_0\right)=\left(\frac{2\pi c}{h}\right)^3\sum_{\mathbf{p}}h\omega\left(\mathbf{p}\right)\left(\sum_{r=1}^{2}\left|c_r\left(t,\mathbf{p}\right)\right|^2-\sum_{r=3}^{4}\left|c_r\left(t,\mathbf{p}\right)\right|^2\right).$$

This problem is usually solved in the following way [25, p.54]:
Let:

$$\begin{aligned}
v_1\left(\mathbf{k}\right) &:= \gamma^{[0]}e_3\left(\mathbf{k}\right),\\
v_2\left(\mathbf{k}\right) &:= \gamma^{[0]}e_4\left(\mathbf{k}\right),\\
d_{1,\mathbf{k}} &:= -b^{\dagger}_{3,-\mathbf{k}},\\
d_{2,\mathbf{k}} &:= -b^{\dagger}_{4,-\mathbf{k}}.
\end{aligned}$$

In that case:

$$\begin{aligned}
e_3\left(\mathbf{k}\right) &= -v_1\left(-\mathbf{k}\right),\\
e_4\left(\mathbf{k}\right) &= -v_2\left(-\mathbf{k}\right),\\
b_{3,\mathbf{k}} &= -d^{\dagger}_{1,-\mathbf{k}},\\
b_{4,\mathbf{k}} &= -d^{\dagger}_{2,-\mathbf{k}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi_s\left(\mathbf{x}\right) &:= \sum_{\mathbf{k}}\sum_{r=1}^{2}\left(b_{r,\mathbf{k}}e_{r,s}\left(\mathbf{k}\right)\exp\left(-i\frac{h}{c}\mathbf{kx}\right)+\right.\\
&\left.+d^{\dagger}_{r,\mathbf{k}}v_{r,s}\left(\mathbf{k}\right)\exp\left(i\frac{h}{c}\mathbf{kx}\right)\right)
\end{aligned}$$

$$\begin{aligned}
\widehat{\mathbb{H}} &= \left(\frac{2\pi c}{h}\right)^3\sum_{\mathbf{k}}h\omega\left(\mathbf{k}\right)\sum_{r=1}^{2}\left(b^{\dagger}_{r,\mathbf{k}}b_{r,\mathbf{k}}+d^{\dagger}_{r,\mathbf{k}}d_{r,\mathbf{k}}\right)\\
&-2\sum_{\mathbf{k}}h\omega\left(\mathbf{k}\right)\widehat{1}.
\end{aligned}$$

The first term on the right side of this equality is positive defined. This term is taken as the desired Hamiltonian. The second term of this equality is infinity constant. And this infinity is deleted (?!) [25, p.58]

But in this case $d_{r,\mathbf{k}}\widetilde{F}_0\neq\widetilde{0}$. In order to satisfy such condition, the vacuum element $\widetilde{F}_0$ must be replaced by the following:

$$\widetilde{F}_0\rightarrow\widetilde{\Phi}_0:=\prod_{\mathbf{k}}\prod_{r=3}^{4}\left(\frac{2\pi c}{h}\right)^3b^{\dagger}_{r,\mathbf{k}}\widetilde{F}_0.$$

But in this case:

$$\psi_s\left(\mathbf{x}\right)\widetilde{\Phi}_0\neq\widetilde{0}.$$

And condition (2.65) isn't carried out.

In order to satisfy such condition, operators $\psi_s(\mathbf{x})$ must be replaced by the following:

$$\psi_s(\mathbf{x}) \rightarrow \phi_s(\mathbf{x}) :=$$
$$:= \sum_{\mathbf{k}} \sum_{r=1}^{2} \left(b_{r,\mathbf{k}} e_{r,s}(\mathbf{k}) \exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right) + d_{r,\mathbf{k}} v_r(\mathbf{k}) \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}\right)\right).$$

Hence,

$$\widehat{\mathbb{H}} = \int d\mathbf{x} \cdot \mathcal{H}(\mathbf{x}) = \int d\mathbf{x} \cdot \phi^{\dagger}(\mathbf{x}) \widehat{H}_0 \phi(\mathbf{x}) =$$
$$= \left(\frac{2\pi\mathrm{c}}{\mathrm{h}}\right)^3 \sum_{\mathbf{k}} \mathrm{h}\omega(\mathbf{k}) \sum_{r=1}^{2} \left(b^{\dagger}_{r,\mathbf{k}} b_{r,\mathbf{k}} - d^{\dagger}_{r,\mathbf{k}} d_{r,\mathbf{k}} \right).$$

And again we get negative energy.

Let's consider the meaning of such energy: An event with positive energy transfers this energy photons which carries it on recorders observers. Observers know that this event occurs, not before it happens. But event with negative energy should absorb this energy from observers. Consequently, observers know that this event happens before it happens. This contradicts Theorem 1.5.2. Therefore, events with negative energy do not occur.

Hence, over vacuum $\widetilde{\Phi}_0$ single fermions can exist, but there is no single antifermions.

A two-particle state is defined the following field operator [28]:

$$\psi_{s_1,s_2}(\mathbf{x},\mathbf{y}) := \begin{vmatrix} \phi_{s_1}(\mathbf{x}) & \phi_{s_2}(\mathbf{x}) \\ \phi_{s_1}(\mathbf{y}) & \phi_{s_2}(\mathbf{y}) \end{vmatrix}.$$

In that case:

$$\widehat{\mathbb{H}} = 2\mathrm{h}\left(\frac{2\pi\mathrm{c}}{\mathrm{h}}\right)^6 \left(\widehat{\mathbb{H}}_a + \widehat{\mathbb{H}}_b\right)$$

where

$$\begin{aligned}
\widehat{\mathbb{H}}_a :&= \sum_{\mathbf{k}} \sum_{\mathbf{p}} \left(\omega(\mathbf{k}) - \omega(\mathbf{p})\right) \sum_{r=1}^{2} \sum_{j=1}^{2} \times \\
&\times \Big\{ v^{\dagger}_j(-\mathbf{k}) v_j(-\mathbf{p}) e^{\dagger}_r(\mathbf{p}) e_r(\mathbf{k}) \times \\
&\times \left(+b^{\dagger}_{r,\mathbf{p}} d^{\dagger}_{j,-\mathbf{k}} d_{j,-\mathbf{p}} b_{r,\mathbf{k}} \right) + \\
&+ \left(+d^{\dagger}_{r,-\mathbf{p}} b^{\dagger}_{j,\mathbf{k}} b_{j,\mathbf{k}} d_{r,-\mathbf{p}} \right) + \\
&+ v^{\dagger}_j(-\mathbf{p}) v_j(-\mathbf{k}) e^{\dagger}_r(\mathbf{k}) e_r(\mathbf{p}) \times \\
&\times \left(-b^{\dagger}_{r,\mathbf{k}} d^{\dagger}_{j,-\mathbf{p}} d_{j,-\mathbf{k}} b_{r,\mathbf{p}} \right) + \\
&+ \left(-b^{\dagger}_{r,\mathbf{p}} d^{\dagger}_{j,-\mathbf{k}} d_{j,-\mathbf{k}} b_{r,\mathbf{p}} \right) \Big\}
\end{aligned}$$

and

$$\widehat{\mathbb{H}}_b := \sum_{\mathbf{k}} \sum_{\mathbf{p}} \left(\omega(\mathbf{k}) + \omega(\mathbf{p})\right) \sum_{r=1}^{2} \sum_{j=1}^{2} \times$$

$$
\begin{aligned}
&\times\left\{v_j^\dagger(-\mathbf{p})\,v_j(-\mathbf{k})\,v_r^\dagger(-\mathbf{k})\,v_r(-\mathbf{p})\times\right.\\
&\times\left(-d_{r,-\mathbf{k}}^\dagger d_{j,-\mathbf{p}}^\dagger d_{j,-\mathbf{k}} d_{r,-\mathbf{p}}\right)+\\
&+\left(-d_{r,-\mathbf{p}}^\dagger d_{j,-\mathbf{k}}^\dagger d_{j,-\mathbf{k}} d_{r,-\mathbf{p}}\right)\\
&+e_r^\dagger(\mathbf{k})\,e_r(\mathbf{p})\,e_j^\dagger(\mathbf{p})\,e_j(\mathbf{k})\times\\
&\times\left(+b_{r,\mathbf{k}}^\dagger b_{j,\mathbf{p}}^\dagger b_{j,\mathbf{k}} b_{r,\mathbf{p}}\right)+\\
&\left.+\left(+b_{r,\mathbf{p}}^\dagger b_{j,\mathbf{k}}^\dagger b_{j,\mathbf{k}} b_{r,\mathbf{p}}\right)\right\}.
\end{aligned}
$$

If velocities are small then the following formula is fair.

$$
\widehat{\mathbb{H}} = 4\mathrm{h}\left(\frac{2\pi\mathrm{c}}{\mathrm{h}}\right)^6\left(\widehat{\mathbb{H}}_a+\widehat{\mathbb{H}}_b\right)
$$

where

$$
\begin{aligned}
\widehat{\mathbb{H}}_a \quad : \quad &= \sum_{\mathbf{k}}\sum_{\mathbf{p}}(\omega(\mathbf{k})-\omega(\mathbf{p}))\times\\
&\times\sum_{r=1}^{2}\sum_{j=1}^{2}\left(d_{j,-\mathbf{p}}^\dagger b_{r,\mathbf{k}}^\dagger b_{r,\mathbf{k}} d_{j,-\mathbf{p}} - b_{j,\mathbf{p}}^\dagger d_{r,-\mathbf{k}}^\dagger d_{r,-\mathbf{k}} b_{j,\mathbf{p}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{\mathbb{H}}_b \quad : \quad &= \sum_{\mathbf{k}}\sum_{\mathbf{p}}(\omega(\mathbf{k})+\omega(\mathbf{p}))\times\\
&\times\sum_{j=1}^{2}\sum_{r=1}^{2}\left(b_{j,\mathbf{p}}^\dagger b_{r,\mathbf{k}}^\dagger b_{r,\mathbf{k}} b_{j,\mathbf{p}} - d_{j,-\mathbf{p}}^\dagger d_{r,-\mathbf{k}}^\dagger d_{r,-\mathbf{k}} d_{j,-\mathbf{p}}\right).
\end{aligned}
$$

Therefore, in any case events with pairs of fermions and events with fermion-antifermion pairs can occur, but events with pairs of antiftrmions can not happen.

Therefore, an antifermion can exists only with a fermion.

Chapter 3

Fields

3.1. Electroweak Fields

In 1963 American physicist Sheldon Glashow[1] [44] proposed that the weak nuclear force and electricity and magnetism could arise from a partially unified electroweak theory. But "... there is major problem: all the fermions and gauge bosons are massless, while experiment shows otherwise. Why not just add in mass terms explicitly? That will not work, since the associated terms break SU(2) or gauge invariances. For fermions, the mass term should be $m\overline{\psi}\psi$?

$$\begin{aligned} m\overline{\psi}\psi &= m\overline{\psi}\left(P_L + P_R\right)\psi = \\ &= m\left(\overline{\psi}\left(P_L P_L\right)\psi + \overline{\psi}\left(P_R P_R\right)\psi\right) \\ &= m\left(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R\right). \end{aligned}$$

However, the left-handed fermion are put into SU(2) doublets and the right-handed ones into SU(2) singlets, so $\overline{\psi}_R\psi_L$ and $\overline{\psi}_L\psi_R$ are not SU(2) singlets and would not give an SU(2) invariant Lagrangian. Similarly, the expected mass terms for the gauge bosons,

$$\frac{1}{2}m_B^2 B^\mu B_\mu$$

plus similar terms for other, are clearly not invariant under gauge transformations $B_\mu \rightarrow B'_\mu = B_\mu - \partial_\mu\chi/g$, The only direct way to preserve the gauge invariance and SU(2) invariance of Lagrangian is to set $m = 0$ for all quarks, leptons and gauge bosons:. There is a way to solve this problem, called the Higgs mechanism" [35].

No. The Dirac Lagrangian for a free fermion can have of the following form:

$$\mathcal{L}_f := \psi^\dagger\left(\beta^{[0]}\partial_0 + \beta^{[1]}\partial_1 + \beta^{[2]}\partial_2 + \beta^{[3]}\partial_3 + \mathrm{i}m\gamma^{[0]}\right)\psi.$$

Indeed, this Lagrangian is not invariant under the SU(2) transformation. But it is beautiful and truncating its mass term is not good idea.

Further you will see, how it is possible to keep this beauty.

[1] Sheldon Lee Glashow (born December 5, 1932) is an American theoretical physicist.

3.1.1. The Bi-mass State

Let us consider [23], [24] the subspace $\Im_{\mathbf{J}}$ of the space $\Im$ spanned of the following subbasis (2.48):

$$\mathbf{J} := \left\langle \begin{array}{c} \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(s_0x_4\right)\right)\varepsilon_1, \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(s_0x_4\right)\right)\varepsilon_2, \\ \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(s_0x_4\right)\right)\varepsilon_3, \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(s_0x_4\right)\right)\varepsilon_4, \\ \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(n_0x_5\right)\right)\varepsilon_1, \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(n_0x_5\right)\right)\varepsilon_2, \\ \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(n_0x_5\right)\right)\varepsilon_3, \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(n_0x_5\right)\right)\varepsilon_4 \end{array} \right\rangle \tag{3.1}$$

with some integer numbers s_0 and n_0.

Let U be any linear transformation of space $\Im_{\mathbf{J}}$ such that for every $\widetilde{\varphi}$: if $\widetilde{\varphi} \in \Im_{\mathbf{J}}$ then (2.49, 2.50, 2.16):

$$\begin{array}{rcl} \left(U\widetilde{\varphi}, U\widetilde{\varphi}\right) & = & \rho_{\mathcal{A}}, \\ \left(U\widetilde{\varphi}, \beta^{[s]}U\widetilde{\varphi}\right) & = & -\frac{j_{\mathcal{A},s}}{\mathrm{c}} \end{array} \tag{3.2}$$

for $s \in \{1,2,3\}$.

In that case:

$$U^{\dagger}\beta^{[\mu]}U = \beta^{[\mu]}$$

for $\mu \in \{0,1,2,3\}$.

Such transformation has a matrix of the following shape:

$$U := \left[\begin{array}{cccc} \left(a" + b"\mathrm{i}\right)1_2 & 0_2 & \left(c" + \mathrm{i}g"\right)1_2 & 0_2 \\ 0_2 & \left(a` + b`\mathrm{i}\right)1_2 & 0_2 & \left(c` + \mathrm{i}g`\right)1_2 \\ \left(u" + \mathrm{i}v"\right)1_2 & 0_2 & \left(k" + \mathrm{i}s"\right)1_2 & 0_2 \\ 0_2 & \left(u` + \mathrm{i}v`\right)1_2 & 0_2 & \left(k` + \mathrm{i}s`\right)1_2 \end{array}\right].$$

with real functions

$a"(t,\mathbf{x})$, $b"(t,\mathbf{x})$, $c"(t,\mathbf{x})$, $g"(t,\mathbf{x})$, $u"(t,\mathbf{x})$, $v"(t,\mathbf{x})$, $k"(t,\mathbf{x})$, $s"(t,\mathbf{x})$,
$a`(t,\mathbf{x})$, $b`(t,\mathbf{x})$, $c`(t,\mathbf{x})$, $g`(t,\mathbf{x})$, $u`(t,\mathbf{x})$, $v`(t,\mathbf{x})$, $k`(t,\mathbf{x})$, $s`(t,\mathbf{x})$.

These functions fulfil the following conditions:

$$\begin{array}{c} v"^2 + b"^2 + u"^2 + a"^2 = 1, \\ c"^2 + g"^2 + k"^2 + s"^2 = 1, \end{array}$$

$$s" = -\frac{a"g"u" - u"b"c" + a"c"v" + b"g"v"}{u"^2 + v"^2},$$

$$k" = \frac{-u"a"c" - u"b"g" + v"a"g" - b"c"v"}{u"^2 + v"^2}.$$

$$\begin{array}{c} v`^2 + b`^2 + u`^2 + a`^2 = 1, \\ c`^2 + g`^2 + k`^2 + s`^2 = 1, \end{array}$$

$$s` = -\frac{a`g`u` - u`b`c` + a`c`v` + b`g`v`}{u`^2 + v`^2},$$

$$k` = \frac{-u`a`c` - u`b`g` + v`a`g` - b`c`v`}{u`^2 + v`^2}.$$

U has 4 eigenvalues: $\exp(i\alpha_1)$, $\exp(i\alpha_2)$, $\exp(i\alpha_3)$, $\exp(i\alpha_4)$ for 8 orthonormalized eigenvectors:

$$\varepsilon_{1,1}, \varepsilon_{1,2}, \varepsilon_{2,1}, \varepsilon_{2,2}, \varepsilon_{3,1}, \varepsilon_{3,2}, \varepsilon_{4,1}, \varepsilon_{4,2}.$$

Let

$$K := \left[\begin{array}{cccccccc} \varepsilon_{1,1} & \varepsilon_{1,2} & \varepsilon_{2,1} & \varepsilon_{2,2} & \varepsilon_{3,1} & \varepsilon_{3,2} & \varepsilon_{4,1} & \varepsilon_{4,2} \end{array}\right].$$

Let θ_1, θ_2, θ_3, θ_4 be solution of the following system of equations:

$$\left\{\begin{array}{l} \theta_1 + \theta_2 + \theta_3 + \theta_4 = \alpha_1, \\ \theta_1 + \theta_2 - \theta_3 - \theta_4 = \alpha_1, \\ \theta_1 - \theta_2 + \theta_3 - \theta_4 = \alpha_1, \\ \theta_1 - \theta_2 - \theta_3 + \theta_4 = \alpha_1. \end{array}\right|$$

and

$$U_1 := \exp(\mathrm{i}\theta_1),$$

$$U_2 := K \left[\begin{array}{cc} \exp(\mathrm{i}\theta_2) 1_4 & 0_4 \\ 0_4 & \exp(-\mathrm{i}\theta_2) 1_4 \end{array}\right] K^\dagger,$$

$$U_3 := K \left[\begin{array}{cccc} \exp(\mathrm{i}\theta_3) 1_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & \exp(-\mathrm{i}\theta_3) 1_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & \exp(\mathrm{i}\theta_3) 1_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & \exp(-\mathrm{i}\theta_3) 1_2 \end{array}\right] K^\dagger,$$

$$U_4 := K \left[\begin{array}{cccc} \exp(\mathrm{i}\theta_4) 1_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & \exp(-\mathrm{i}\theta_4) 1_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & \exp(-\mathrm{i}\theta_4) 1_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & \exp(\mathrm{i}\theta_4) 1_2 \end{array}\right] K^\dagger.$$

In this case:

$$U_1 U_2 U_3 U_4 = U$$

and

$$U_2 = \begin{bmatrix} \exp(\mathrm{i}\theta_2)\,1_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & \exp(-\mathrm{i}\theta_2)\,1_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & \exp(\mathrm{i}\theta_2)\,1_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & \exp(-\mathrm{i}\theta_2)\,1_2 \end{bmatrix}.$$

Besides

$$U_1U_2 = \begin{bmatrix} e^{\mathrm{i}(\theta_1+\theta_2)} & 0 & 0 & 0 \\ 0 & e^{\mathrm{i}(\theta_1-\theta_2)} & 0 & 0 \\ 0 & 0 & e^{\mathrm{i}(\theta_1+\theta_2)} & 0 \\ 0 & 0 & 0 & e^{\mathrm{i}(\theta_1-\theta_2)} \end{bmatrix}.$$

Let χ and ς be thesolution of the following set of equations:

$$\left\{ \begin{array}{c} 0.5\chi+\varsigma=\theta_1+\theta_2, \\ \chi+\varsigma=\theta_1-\theta_2, \end{array} \right|$$

i.e.:

$$\begin{gathered} \chi=-4\theta_2, \\ \varsigma=\theta_1+3\theta_2. \end{gathered}$$

Let

$$U^{[e]} := \exp(\mathrm{i}\varsigma)$$

and (2.40)

$$\widetilde{U} = \begin{bmatrix} \exp\left(\mathrm{i}\frac{\chi}{2}\right)1_2 & 0_2 \\ 0_2 & \exp(\mathrm{i}\chi)\,1_2 \end{bmatrix}.$$

In that case:

$$\widetilde{U}U^{[e]}1_8 = U_1U_2.$$

Here real functions
$a(t,\mathbf{x})$, $b(t,\mathbf{x})$, $c(t,\mathbf{x})$, $g(t,\mathbf{x})$, $u(t,\mathbf{x})$, $v(t,\mathbf{x})$, $k(t,\mathbf{x})$,$s(t,\mathbf{x})$
exist such that:

$$U_3U_4 = \begin{bmatrix} (a+\mathrm{i}b)\,1_2 & 0_2 & (c+\mathrm{i}g)\,1_2 & 0_2 \\ 0_2 & (u+\mathrm{i}v)\,1_2 & 0_2 & (k+\mathrm{i}s)\,1_2 \\ (-c+\mathrm{i}g)\,1_2 & 0_2 & (a-\mathrm{i}b)\,1_2 & 0_2 \\ 0_2 & (-k+\mathrm{i}s)\,1_2 & 0_2 & (u-\mathrm{i}v)\,1_2 \end{bmatrix}$$

and

$$\begin{gathered} a^2+b^2+c^2+g^2=1, \\ u^2+v^2+r^2+s^2=1. \end{gathered}$$

If

$$U^{(+)} := \begin{bmatrix} 1_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & (u+\mathrm{i}v)\,1_2 & 0_2 & (k+\mathrm{i}s)\,1_2 \\ 0_2 & 0_2 & 1_2 & 0_2 \\ 0_2 & (-k+\mathrm{i}s)\,1_2 & 0_2 & (u-\mathrm{i}v)\,1_2 \end{bmatrix} \tag{3.3}$$

and

$$U^{(-)} := \begin{bmatrix} (a+\mathrm{i}b)\,1_2 & 0_2 & (c+\mathrm{i}g)\,1_2 & 0_2 \\ 0_2 & 1_2 & 0_2 & 0_2 \\ (-c+\mathrm{i}g)\,1_2 & 0_2 & (a-\mathrm{i}b)\,1_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 1_2 \end{bmatrix} \tag{3.4}$$

then

$$U_3U_4 = U^{(-)}U^{(+)} = U^{(+)}U^{(-)}.$$

Let us consider $U^{(-)}$.
Let:

$$\ell_\circ := \frac{1}{2\sqrt{(1-a^2)}} \begin{bmatrix} \left(b+\sqrt{(1-a^2)}\right)1_4 & (q-\mathrm{i}c)\,1_4 \\ (q+\mathrm{i}c)\,1_4 & \left(\sqrt{(1-a^2)}-b\right)1_4 \end{bmatrix} \tag{3.5}$$

and

$$\ell_* := \frac{1}{2\sqrt{(1-a^2)}} \begin{bmatrix} \left(\sqrt{(1-a^2)}-b\right)1_4 & (-q+\mathrm{i}c)\,1_4 \\ (-q-\mathrm{i}c)\,1_4 & \left(b+\sqrt{(1-a^2)}\right)1_4 \end{bmatrix}. \tag{3.6}$$

These operators are fulfilled to the following conditions:

$$\begin{aligned} \ell_\circ\ell_\circ = \ell_\circ,\ \ell_*\ell_* &= \ell_*; \\ \ell_\circ\ell_* = 0 &= \ell_*\ell_\circ, \\ (\ell_\circ-\ell_*)\,(\ell_\circ-\ell_*) &= 1_8, \\ \ell_\circ+\ell_* &= 1_8, \end{aligned}$$

$$\begin{gathered} \ell_\circ\gamma^{[0]} = \gamma^{[0]}\ell_\circ,\ \ell_*\gamma^{[0]} = \gamma^{[0]}\ell_*, \\ \ell_\circ\beta^{[4]} = \beta^{[4]}\ell_\circ,\ \ell_*\beta^{[4]} = \beta^{[4]}\ell_* \end{gathered}$$

and

$$\begin{aligned} U^{(-)\dagger}\gamma^{[0]}U^{(-)} &= a\gamma^{[0]} - (\ell_\circ-\ell_*)\sqrt{1-a^2}\beta^{[4]}, \\ U^{(-)\dagger}\beta^{[4]}U^{(-)} &= a\beta^{[4]} + (\ell_\circ-\ell_*)\sqrt{1-a^2}\gamma^{[0]}. \end{aligned} \tag{3.7}$$

From (2.39) the lepton equation of motion is the following:

$$\left(\sum_{\mu=0}^{3}\beta^{[\mu]}\left(\mathrm{i}\partial_\mu+F_\mu+0.5g_1YB_\mu\right)+\gamma^{[0]}\mathrm{i}\partial_5+\beta^{[4]}\mathrm{i}\partial_4\right)U^{(-)\dagger}U^{(-)}\widetilde{\varphi}=0.$$

If

$$\partial_k U^{(-)\dagger} = U^{(-)\dagger}\partial_k \tag{3.8}$$

for $k \in \{0,1,2,3,4,5\}$ then

$$\left(\begin{array}{c} U^{(-)\dagger}\mathrm{i}\sum_{\mu=0}^{3}\beta^{[\mu]}\left(\mathrm{i}\partial_\mu + F_\mu + 0.5g_1YB_\mu\right) \\ +\gamma^{[0]}U^{(-)\dagger}\mathrm{i}\partial_5 + \beta^{[4]}U^{(-)\dagger}\mathrm{i}\partial_4 \end{array} \right) U^{(-)}\widetilde{\varphi} = 0.$$

Hence, from (3.7):

$$U^{(-)\dagger}\left(\begin{array}{c} \sum_{\mu=0}^{3}\beta^{[\mu]}\left(\mathrm{i}\partial_\mu + F_\mu + 0.5g_1YB_\mu\right) \\ +\gamma^{[0]}\mathrm{i}\left(a\partial_5 - (\ell_\circ - \ell_*)\sqrt{1-a^2}\partial_4\right) \\ +\beta^{[4]}\mathrm{i}\left(\sqrt{1-a^2}\,(\ell_\circ - \ell_*)\,\partial_5 + a\partial_4\right) \end{array} \right) U^{(-)}\widetilde{\varphi} = 0.$$

Thus, if denote:

$$x_4' = (\ell_\circ + \ell_*)\,ax_4 + (\ell_\circ - \ell_*)\sqrt{1-a^2}x_5,$$
$$x_5' = (\ell_\circ + \ell_*)\,ax_5 - (\ell_\circ - \ell_*)\sqrt{1-a^2}x_4$$

then

$$\left(\sum_{\mu=0}^{3}\beta^{[\mu]}\left(\mathrm{i}\partial_\mu + F_\mu + 0.5g_1YB_\mu\right) + \left(\gamma^{[0]}\mathrm{i}\partial_5' + \beta^{[4]}\mathrm{i}\partial_4'\right)\right)\widetilde{\varphi}' = 0 \tag{3.9}$$

with

$$\widetilde{\varphi}' = U^{(-)}\widetilde{\varphi}.$$

That is the lepton Hamiltonian is invariant for the following global transformation:

$$\begin{aligned} &\widetilde{\varphi} \to \widetilde{\varphi}' = U^{(-)}\widetilde{\varphi}, \\ &x_4 \to x_4' = (\ell_\circ + \ell_*)\,ax_4 + (\ell_\circ - \ell_*)\sqrt{1-a^2}x_5, \\ &x_5 \to x_5' = (\ell_\circ + \ell_*)\,ax_5 - (\ell_\circ - \ell_*)\sqrt{1-a^2}x_4, \\ &x_\mu \to x_\mu' = x_\mu. \end{aligned} \tag{3.10}$$

3.1.2. Neutrino

Wolfgang Pauli postulated the neutrino in 1930 to explain the energy spectrum of beta decays, the decay of a neutron into a proton and an electron. Clyde Cowan, Frederick Reines found the neutrino experimentally in 1955. Enrico Fermi[2] developed the first theory describing neutrino interactions and denoted this particles as *neutrino* in 1933. In 1962 Leon M. Lederman, Melvin Schwartz and Jack Steinberger showed that more than one type of neutrino exists. Bruno Pontecorvo[3] suggested a practical method for investigating neutrino

[2]Enrico Fermi (29 September 1901 28 November 1954) was an Italian-born, naturalized American physicist particularly known for his work on the development of the first nuclear reactor, Chicago Pile-1, and for his contributions to the development of quantum theory, nuclear and particle physics, and statistical mechanics.

[3]Bruno Pontecorvo (Marina di Pisa, Italy, August 22, 1913 – Dubna, Russia, September 24, 1993) was an Italian-born atomic physicist, an early assistant of Enrico Fermi and then the author of numerous studies in high energy physics, especially on neutrinos.

masses in 1957, over the subsequent 10 years he developed the mathematical formalism and the modern formulation of vacuum oscillations...

Let $\Im_{e\nu}$ be the unitary space, spanned by the following basis:

$$\mathbf{J}_{e\nu} := \left\langle \begin{array}{c} \frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_1, \\ \frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_2, \\ \frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_3, \\ \frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_4, \\ \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(n_0x_5)\right)\varepsilon_1, \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(n_0x_5)\right)\varepsilon_2, \\ \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(n_0x_5)\right)\varepsilon_3, \frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(n_0x_5)\right)\varepsilon_4 \end{array} \right\rangle. \tag{3.11}$$

Let $\Im_e$ be the subspace of the space $\Im_{e\nu}$ such that if $\widetilde{\varphi} \in \Im_e$ then $\widetilde{\varphi}$ has the following shape:

$$\widetilde{\varphi}(t,\mathbf{x},x_5,x_4) = \exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}n_0x_5\right)\sum_{k=1}^{4} f_k(t,\mathbf{x},n_0,0)\,\varepsilon_k$$

That is $\widetilde{\varphi}$ has the following matrix in the basis $\mathbf{J}_{e\nu}$:

$$\widetilde{\varphi} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}. \tag{3.12}$$

Let us consider the following Hamiltonian on $\Im_e$:

$$\widehat{H}_{0,4} := \mathrm{c}\left(\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r + \gamma^{[0]}\mathrm{i}\partial_5 + \beta^{[4]}\mathrm{i}\partial_4\right): \tag{3.13}$$

$$\begin{gathered} \widehat{H}_{0,4}\widetilde{\varphi} = \mathrm{c}\left(\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r + \gamma^{[0]}\mathrm{i}\partial_5 + \beta^{[4]}\mathrm{i}\partial_4\right)\widetilde{\varphi} = \\ = \sum_{r=1}^{3}\beta^{[r]}\mathrm{ci}\partial_r\widetilde{\varphi} + \\ +\gamma^{[0]}\mathrm{ci}\partial_5\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}n_0x_5\right)\sum_{k=1}^{4} f_k(t,\mathbf{x},n_0,0)\,\varepsilon_k + \\ +\beta^{[4]}\mathrm{ci}\partial_4\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}n_0x_5\right)\sum_{k=1}^{4} f_k(t,\mathbf{x},n_0,0)\,\varepsilon_k = \\ = \sum_{r=1}^{3}\beta^{[r]}\mathrm{ci}\partial_r\widetilde{\varphi} + \\ +\gamma^{[0]}\mathrm{ci}\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}n_0\right)\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}n_0x_5\right)\sum_{k=1}^{4} f_k(t,\mathbf{x},n_0,0)\,\varepsilon_k + \\ +0 = \\ = \sum_{r=1}^{3}\beta^{[r]}\mathrm{ci}\partial_r\widetilde{\varphi} + \mathrm{h}n_0\gamma^{[0]}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}n_0x_5\right)\sum_{k=1}^{4} f_k(t,\mathbf{x},n_0,0)\,\varepsilon. = \\ = \sum_{r=1}^{3}\beta^{[r]}\mathrm{ci}\partial_r\widetilde{\varphi} + \mathrm{h}n_0\gamma^{[0]}\widetilde{\varphi}. \end{gathered}$$

Hence, on this space:

$$\widehat{H}_{0,4}=\widehat{H}_0:=\mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r+\mathrm{h}n_0\gamma^{[0]}. \tag{3.14}$$

Let $\Im_\circ$ be the subspace of the space $\Im_{e\nu}$ such that if $\widetilde{\varphi}_\circ\in\Im_\circ$ then $\widetilde{\varphi}_\circ=\ell_\circ\,\widetilde{\varphi}$ and $\widetilde{\varphi}\in\Im_e$, and if $\widetilde{\varphi}\in\Im_e$ then $(\ell_\circ\widetilde{\varphi})\in\Im_\circ$. If $\widetilde{\varphi}_\circ=\ell_\circ\widetilde{\varphi}$ then in the basis $\mathbf{J}_{e\nu}$:

$$\widetilde{\varphi}_\circ=\frac{1}{2\sqrt{(1-a^2)}}\left[\begin{array}{c}-(-q+ic)\,f_1\\-(-q+ic)\,f_2\\-(-q+ic)\,f_3\\-(-q+ic)\,f_4\\-\left(-\sqrt{(1-a^2)}+b\right)f_1\\-\left(-\sqrt{(1-a^2)}+b\right)f_2\\-\left(-\sqrt{(1-a^2)}+b\right)f_3\\-\left(-\sqrt{(1-a^2)}+b\right)f_4\end{array}\right].$$

Let us consider the Hamiltonian $\widehat{H}_{0,4}$ mode of behavior on the space $\Im_\circ$:
Hence,

$$\begin{array}{c}\widehat{H}_{0,4}\widetilde{\varphi}_\circ=\mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r\widetilde{\varphi}_\circ+\\+\gamma^{[0]}\mathrm{i}\mathrm{c}\frac{(q-ic)}{2\sqrt{(1-a^2)}}\frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\times\\\times\left(\begin{array}{c}\partial_5\left(\begin{array}{c}f_1\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_1+\\+f_2\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_2+\\+f_3\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_3+\\+f_4\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_4\end{array}\right)+\\+\left(\sqrt{(1-a^2)}-b\right)\frac{\mathrm{h}}{2\pi\mathrm{c}}\partial_5\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(n_0x_5)\right)\cdot\\\cdot(f_1\varepsilon_1+f_2\varepsilon_2+f_3\varepsilon_3+f_4\varepsilon_4)\end{array}\right)+\end{array}$$

$$\begin{array}{c}+\beta^{[4]}\mathrm{i}\mathrm{c}\frac{(q-ic)}{2\sqrt{(1-a^2)}}\frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\times\\\times\left(\begin{array}{c}\partial_4\left(\begin{array}{c}f_1\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_1+\\+f_2\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_2+\\+f_3\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_3+\\+f_4\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_4\end{array}\right)+\\+\left(\sqrt{(1-a^2)}-b\right)\partial_4\frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(n_0x_5)\right)\cdot\\\cdot(f_1\varepsilon_1+f_2\varepsilon_2+f_3\varepsilon_3+f_4\varepsilon_4)\end{array}\right).\end{array}$$

$$\widehat{H}_{0,4}\widetilde{\varphi}_\circ = \mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r\widetilde{\varphi}_\circ +$$
$$+\gamma^{[0]}\mathrm{ic}\frac{(q-ic)}{2\sqrt{(1-a^2)}}\frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\times$$
$$\times\left(\begin{array}{c}\partial_5\left(\begin{array}{c} f_1\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_1+\\ +f_2\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_2+\\ +f_3\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_3+\\ +f_4\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_4\end{array}\right)+\\ +\left(\sqrt{(1-a^2)}-b\right)\frac{\mathrm{h}}{2\pi\mathrm{c}}\partial_5\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(n_0x_5\right)\right)\cdot\\ \cdot\left(f_1\varepsilon_1+f_2\varepsilon_2+f_3\varepsilon_3+f_4\varepsilon_4\right)\end{array}\right)+$$

$$+\beta^{[4]}\mathrm{ic}\frac{(q-ic)}{2\sqrt{(1-a^2)}}\frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\times$$
$$\times\left(\begin{array}{c}\partial_4\left(\begin{array}{c} f_1\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_1+\\ +f_2\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_2+\\ +f_3\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_3+\\ +f_4\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_4\end{array}\right)+\\ +\left(\sqrt{(1-a^2)}-b\right)\partial_4\frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(n_0x_5\right)\right)\cdot\\ \cdot\left(f_1\varepsilon_1+f_2\varepsilon_2+f_3\varepsilon_3+f_4\varepsilon_4\right)\end{array}\right).$$

Therefore,

$$\widehat{H}_{0,4}\widetilde{\varphi}_\circ = \mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r\widetilde{\varphi}_\circ +$$
$$+\gamma^{[0]}\mathrm{ic}\frac{\sqrt{1-a^2}-b}{2\sqrt{1-a^2}}\times$$
$$\times\left(0+\frac{\mathrm{h}}{2\pi\mathrm{c}}\partial_5\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\left(n_0x_5\right)\right)\left(f_1\varepsilon_1+f_2\varepsilon_2+f_3\varepsilon_3+f_4\varepsilon_4\right)\right)+$$
$$+\beta^{[4]}\mathrm{ic}\frac{q-ic}{2\sqrt{1-a^2}}\frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\times$$
$$\times\left(\begin{array}{c}\left(\begin{array}{c} f_1\left(\partial_4\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\partial_4\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_1+\\ +f_2\left(\partial_4\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\partial_4\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_2+\\ +f_3\left(\partial_4\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\partial_4\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_3+\\ +f_4\left(\partial_4\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\partial_4\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_4\end{array}\right)+\\ +0\end{array}\right).$$

Hence,

$$\widehat{H}_{0,4}\widetilde{\varphi}_\circ = \mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r\widetilde{\varphi}_\circ +$$
$$+\gamma^{[0]}\mathrm{ic}\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}n_0\right)\frac{\sqrt{1-a^2}-b}{2\sqrt{1-a^2}}\frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}n_0x_5\right)\times$$
$$\times\left(f_1\varepsilon_1+f_2\varepsilon_2+f_3\varepsilon_3+f_4\varepsilon_4\right)+$$
$$+\beta^{[4]}\mathrm{ic}\frac{\mathrm{h}}{\mathrm{c}}n_0\frac{q-ic}{2\sqrt{1-a^2}}\frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\times$$
$$\times\left(\begin{array}{c} f_1\left(\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_1+\\ +f_2\left(\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_2+\\ +f_3\left(\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_3+\\ +f_4\left(\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_4\end{array}\right).$$

Therefore,

$$\begin{gathered}
\widehat{H}_{0,4}\widetilde{\varphi}_{\circ} = \mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r\widetilde{\varphi}_{\circ}+ \\
+\mathrm{h}n_0\gamma^{[0]}\frac{\sqrt{1-a^2}-b}{2\sqrt{1-a^2}}\frac{\mathrm{h}}{2\pi\mathrm{c}}\exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}n_0x_5\right)\times \\
\times\left(f_1\varepsilon_1+f_2\varepsilon_2+f_3\varepsilon_3+f_4\varepsilon_4\right)+ \\
+\mathrm{h}n_0\beta^{[4]}\mathrm{i}\frac{q-ic}{2\sqrt{1-a^2}}\frac{\mathrm{h}}{2\pi\mathrm{c}}\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}}\times \\
\times\left(\begin{array}{c}
f_1\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_1+ \\
+f_2\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)+\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_2- \\
-f_3\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_3- \\
-f_4\left(\cosh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)-\sinh\left(\frac{\mathrm{h}}{\mathrm{c}}n_0x_4\right)\right)\varepsilon_4
\end{array}\right).
\end{gathered}$$

Hence, in basis $\mathbf{J}_{e\nu}$:

$$\begin{gathered}
\widehat{H}_{0,4}\widetilde{\varphi}_{\circ} = \mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r\widetilde{\varphi}_{\circ}+\mathrm{h}n_0\times \\
\times\left(\gamma^{[0]}\frac{\sqrt{1-a^2}-b}{2\sqrt{1-a^2}}\begin{bmatrix}0\\0\\0\\0\\f_1\\f_2\\f_3\\f_4\end{bmatrix}+\beta^{[4]}\mathrm{i}\frac{q-ic}{2\sqrt{1-a^2}}\begin{bmatrix}f_1\\f_2\\-f_3\\-f_4\\0\\0\\0\\0\end{bmatrix}\right)= \\
= \mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r\widetilde{\varphi}_{\circ}+\mathrm{h}n_0\times \\
\times\left(\gamma^{[0]}\frac{\sqrt{1-a^2}-b}{2\sqrt{1-a^2}}\begin{bmatrix}0\\0\\0\\0\\f_1\\f_2\\f_3\\f_4\end{bmatrix}+\beta^{[4]}\mathrm{i}\frac{q-ic}{2\sqrt{1-a^2}}\gamma^{[5]}\begin{bmatrix}f_1\\f_2\\f_3\\f_4\\0\\0\\0\\0\end{bmatrix}\right).
\end{gathered}$$

with

$$\gamma^{[5]} := \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}.$$

Since

$$\beta^{[4]}\mathrm{i}\gamma^{[5]} = \gamma^{[0]}$$

then

$$\widehat{H}_{0,4}\widetilde{\varphi}_{\circ} = \mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r\widetilde{\varphi}_{\circ} + \mathrm{h}n_0 \times$$

$$\times\left(\gamma^{[0]}\frac{1}{2\sqrt{1-a^2}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \left(\sqrt{1-a^2}-b\right)f_1 \\ \left(\sqrt{1-a^2}-b\right)f_2 \\ \left(\sqrt{1-a^2}-b\right)f_3 \\ \left(\sqrt{1-a^2}-b\right)f_4 \end{bmatrix} + \gamma^{[0]}\frac{1}{2\sqrt{1-a^2}}\begin{bmatrix} (q-ic)f_1 \\ (q-ic)f_2 \\ (q-ic)f_3 \\ (q-ic)f_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right).$$

Therefore,

$$\widehat{H}_{0,4}\widetilde{\varphi}_{\circ} = \mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r\widetilde{\varphi}_{\circ} + \mathrm{h}n_0\gamma^{[0]}\frac{1}{2\sqrt{1-a^2}}\begin{pmatrix} -(-q+ic)f_1, \\ -(-q+ic)f_2, \\ -(-q+ic)f_3, \\ -(-q+ic)f_4, \\ -\left(-\sqrt{1-a^2}+b\right)f_1, \\ -\left(-\sqrt{1-a^2}+b\right)f_2, \\ -\left(-\sqrt{1-a^2}+b\right)f_3, \\ -\left(-\sqrt{1-a^2}+b\right)f_4 \end{pmatrix}.$$

Hence,

$$\widehat{H}_{0,4}\widetilde{\varphi}_{\circ} = \mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r\widetilde{\varphi}_{\circ} + \mathrm{h}n_0\gamma^{[0]}\widetilde{\varphi}_{\circ}.$$

Thus, in space $\Im_e$:

$$\widehat{H}_{0,4} = \widehat{H}_0 = \mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_r + \mathrm{h}n_0\gamma^{[0]},$$

too.

Let $\Im_*$ be the subspace of the space $\Im_{ev}$ such that if $\widetilde{\varphi}_* \in \Im_*$ then $\widetilde{\varphi}_* = \ell_*\,\widetilde{\varphi}$ and $\widetilde{\varphi} \in \Im_e$, and if $\widetilde{\varphi} \in \Im_e$ then $(\ell_*\widetilde{\varphi}) \in \Im_*$. If $\widetilde{\varphi}_* = \ell_*\widetilde{\varphi}$ (3.12) then in the basis $\mathbf{J}_{ev}$:

$$\widetilde{\varphi}_* = \frac{1}{2\sqrt{(1-a^2)}}\begin{bmatrix} (-q+ic)f_1 \\ (-q+ic)f_2 \\ (-q+ic)f_3 \\ (-q+ic)f_4 \\ \left(b+\sqrt{1-a^2}\right)f_1 \\ \left(b+\sqrt{1-a^2}\right)f_2 \\ \left(b+\sqrt{1-a^2}\right)f_3 \\ \left(b+\sqrt{1-a^2}\right)f_4 \end{bmatrix}.$$

Similarly to $\widetilde{\varphi}_{\circ}$ you can calculate that

$$\widehat{H}_{0,4}\widetilde{\varphi}_{*}=\widehat{H}_{0}\widetilde{\varphi}_{*}=\mathrm{c}\sum_{r=1}^{3}\beta^{[r]}\mathrm{i}\partial_{r}\widetilde{\varphi}_{*}+\mathrm{h}n_{0}\gamma^{[0]}\widetilde{\varphi}_{*}.,$$

too.
Let

$$e_{1L}(\mathbf{k}):=\left[\begin{array}{c}\omega(\mathbf{k})+n_0+k_3\\ k_1+\mathrm{i}k_2\end{array}\right], e_{1R}(\mathbf{k}):=\left[\begin{array}{c}\omega(\mathbf{k})+n_0-k_3\\ -k_1-\mathrm{i}k_2\end{array}\right],$$

$$e_{2L}(\mathbf{k}):=\left[\begin{array}{c}k_1-\mathrm{i}k_2\\ \omega(\mathbf{k})+n_0-k_3\end{array}\right], e_{2R}(\mathbf{k}):=\left[\begin{array}{c}-k_1+\mathrm{i}k_2\\ \omega(\mathbf{k})+n_0+k_3\end{array}\right],$$

$$\begin{array}{c}e_{3L}(\mathbf{k}):=-e_{1R}(\mathbf{k}), e_{3R}(\mathbf{k}):=e_{1L}(\mathbf{k}),\\ e_{4L}(\mathbf{k}):=-e_{2R}(\mathbf{k}), e_{4R}(\mathbf{k}):=e_{2L}(\mathbf{k}).\end{array}$$

with

$$\omega(\mathbf{k}):=\sqrt{n_0^2+k_1^2+k_2^2+k_3^2}$$

(n_0, k_1, k_2,k_3 are real numbers).
In this case:

$$e_s(\mathbf{k})=\frac{1}{2\sqrt{\omega(\mathbf{k})(\omega(\mathbf{k})+n_0)}}\left[\begin{array}{c}e_{sL}(\mathbf{k})\\ e_{sR}(\mathbf{k})\end{array}\right].$$

Let

$$\underline{e}_s(\mathbf{k}):=\left[\begin{array}{c}\overrightarrow{0}_4\\ e_s(\mathbf{k})\end{array}\right]$$

here $s\in\{1,2,3,4\}$.
And let:

$$\begin{array}{rl} & \underline{e}_{\circ s}(\mathbf{k}):=\ell_{\circ}\underline{e}_s(\mathbf{k})\\ = & \dfrac{1}{\sqrt{2\left(\sqrt{1-a^2}-b\right)\sqrt{1-a^2}}}\left[\begin{array}{c}(q-\mathrm{i}c)\,e_s(\mathbf{k})\\ \left(\sqrt{1-a^2}-b\right)e_s(\mathbf{k})\end{array}\right],\end{array}$$

$$\begin{array}{rl} & \underline{e}_{*s}(\mathbf{k}):=\ell_{*}\underline{e}_s(\mathbf{k})\\ = & \dfrac{1}{\sqrt{2\left(\sqrt{1-a^2}+b\right)\sqrt{1-a^2}}}\left[\begin{array}{c}(-q+\mathrm{i}c)\,e_s(\mathbf{k})\\ \left(b+\sqrt{1-a^2}\right)e_s(\mathbf{k})\end{array}\right].\end{array}$$

Denote

$$\widehat{H}_0(\mathbf{k}) := \sum_{r=1}^{3} \beta^{[r]} k_r = \begin{bmatrix} k_3 & k_1 - \mathrm{i}k_2 & n_0 & 0 \\ k_1 + \mathrm{i}k_2 & -k_3 & 0 & n_0 \\ n_0 & 0 & -k_3 & -k_1 + \mathrm{i}k_2 \\ 0 & n_0 & -k_1 - \mathrm{i}k_2 & k_3 \end{bmatrix}.$$

In that case

$$\begin{aligned} &\widehat{H}_0 \underline{e}_{\circ 1}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right) = \\ = \;& \mathrm{h}\widehat{H}_0(\mathbf{k}) \underline{e}_{\circ 1}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right) \\ = \;& \mathrm{h}\omega(\mathbf{k}) \underline{e}_{\circ 1}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right). \end{aligned}$$

Therefore, $\underline{e}_{\circ 1}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right)$ is an eigenvector of $\widehat{H}_0$ with the eigenvalue $\mathrm{h}\omega(\mathbf{k})$. Similarly you can calculate that

$\underline{e}_{\circ 2}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right)$, $\underline{e}_{*1}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right)$, $\underline{e}_{*2}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right)$,
are eigenvectors of $\widehat{H}_0$ with the same eigenvalue, and
$\underline{e}_{\circ 3}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right)$, $\underline{e}_{\circ 4}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right)$,
$\underline{e}_{*3}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right)$, $\underline{e}_{*4}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right)$
are an eigenvectors of $\widehat{H}_0$ with the eigenvalue $(-\mathrm{h}\omega(\mathbf{k}))$.

Vectors $\underline{e}_{\circ s}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right)$, $\underline{e}_{*s}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{\frac{3}{2}} \exp\left(\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\right)$ with $s \in \{1,2,3,4\}$ form an orthonormalized basis in the space $\Im_{ev}$ (3.11) and

$$\sum_{s=1}^{4} \left(\underline{e}^{*}_{*s,r}(\mathbf{k}) \underline{e}_{*s,r'}(\mathbf{k}) + \underline{e}^{*}_{\circ s,r}(\mathbf{k}) \underline{e}_{\circ s,r'}(\mathbf{k})\right) = \delta_{r,r'} \tag{3.15}$$

for $r, r' \in \{1,2,3,4,5,6,7,8\}$.

Let

$$\begin{aligned} &\underline{e}'_{*s}(\mathbf{k}) := U^{(-)} \underline{e}_{*s}(\mathbf{k}) \\ = \;& \frac{1}{\sqrt{2\left(\sqrt{1-a^2}+b\right)\sqrt{1-a^2}}} \frac{1}{2\sqrt{\omega(\mathbf{k})(\omega(\mathbf{k})+n_0)}} \\ & \cdot \begin{bmatrix} \left(a - \mathrm{i}\sqrt{1-a^2}\right)(-q+\mathrm{i}c)\, e_{sL}(\mathbf{k}) \\ (-q+\mathrm{i}c)\, e_{sR}(\mathbf{k}) \\ \left(a - \mathrm{i}\sqrt{1-a^2}\right)\left(\sqrt{1-a^2}+b\right) e_{sL}(\mathbf{k}) \\ \left(\sqrt{(1-a^2)}+b\right) e_{sR}(\mathbf{k}) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned}&\underline{e}'_{\circ s}(\mathbf{k}) := U^{(-)}\underline{e}_{\circ s}(\mathbf{k})\\ = \quad &\frac{1}{\sqrt{2\left(\sqrt{1-a^2}-b\right)\sqrt{1-a^2}}}\frac{1}{2\sqrt{\omega(\mathbf{k})(\omega(\mathbf{k})+n_0)}}\\ &\cdot\begin{bmatrix}\left(a+\mathrm{i}\sqrt{1-a^2}\right)(q-\mathrm{i}c)\,e_{sL}(\mathbf{k})\\ (q-\mathrm{i}c)\,e_{sR}(\mathbf{k})\\ \left(a+\mathrm{i}\sqrt{1-a^2}\right)\left(\sqrt{1-a^2}-b\right)e_{sL}(\mathbf{k})\\ \left(\sqrt{(1-a^2)}-b\right)e_{sR}(\mathbf{k})\end{bmatrix}.\end{aligned}$$

For these vectors:

$$\sum_{r=1}^{4}\left(\underline{e}'^{*}_{*r,j}(\mathbf{k})\,\underline{e}'_{*r,j'}(\mathbf{k})+\underline{e}'^{*}_{\circ r,j}(\mathbf{k})\,\underline{e}'_{\circ r,j'}(\mathbf{k})\right)=\delta_{j,j'}$$

and since $U^{(-)\dagger}U^{(-)}=1_8$ then $\underline{e}'_{\circ s}(\mathbf{k})$ and $\underline{e}'_{*s}(\mathbf{k})$ form an orthonormalized basis in the space $\mathfrak{I}_{ev}$, too.

Let

$$\underline{e}'_r(\mathbf{k}) := U^{(-)}\underline{e}_r(\mathbf{k}) = \frac{1}{2\sqrt{\omega(\mathbf{k})(\omega(\mathbf{k})+n_0)}}\begin{bmatrix}(c+\mathrm{i}q)\,e_{rL}(\mathbf{k})\\ \overrightarrow{0}_2\\ (a-\mathrm{i}b)\,e_{rL}(\mathbf{k})\\ e_{rR}(\mathbf{k})\end{bmatrix}. \tag{3.16}$$

In that case:

$$\underline{e}'_r(\mathbf{k}) = \frac{1}{\sqrt{2}}\left(\sqrt{1-\frac{b}{\sqrt{1-a^2}}}\,\underline{e}'_{\circ r}(\mathbf{k})+\sqrt{1+\frac{b}{\sqrt{1-a^2}}}\,\underline{e}'_{*r}(\mathbf{k})\right).$$

Let for $j,j'\in\{1,2,3,4,5,6,7,8\}$:

$$\begin{gathered}\left\{\psi^{\dagger}_{j'}(\mathbf{y}),\psi_j(\mathbf{x})\right\}=\delta(\mathbf{y}-\mathbf{x})\,\delta_{j',j}\widehat{1},\\ \left\{\psi^{\dagger}_{j'}(\mathbf{y}),\psi^{\dagger}_j(\mathbf{x})\right\}=\widehat{0}=\left\{\psi_{j'}(\mathbf{y}),\psi_j(\mathbf{x})\right\}\end{gathered}$$

and let

$$\begin{aligned}b_{\circ r,\mathbf{k}} &:= \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3\int_{(\Omega)}d\mathbf{x}\cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}\mathbf{x}}\sum_{j'=1}^{8}\underline{e}^{*}_{\circ r,j'}(\mathbf{k})\,\psi_{j'}(\mathbf{x}),\\ b_{*r,\mathbf{k}} &:= \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3\int_{(\Omega)}d\mathbf{x}\cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}\mathbf{x}}\sum_{j'=1}^{8}\underline{e}^{*}_{*r,j'}(\mathbf{k})\,\psi_{j'}(\mathbf{x}).\end{aligned}$$

In that case:

$$\begin{aligned}
&\sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \left(\sum_{r=1}^{4} \underline{e}_{\circ r,j}(\mathbf{k})\, b_{\circ r,\mathbf{k}} + \sum_{r=1}^{4} \underline{e}_{*r,j}(\mathbf{k})\, b_{*r,\mathbf{k}} \right) \\
= \;& \sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \left(\begin{array}{c} \sum_{r=1}^{4} \underline{e}_{\circ r,j}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{3} \cdot \\ \cdot \int_{(\Omega)} d\mathbf{x}' \cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}'} \sum_{j'=1}^{8} \underline{e}^{*}_{\circ r,j'}(\mathbf{k})\, \psi_{j'}(\mathbf{x}') \\ + \sum_{r=1}^{4} \underline{e}_{*r,j}(\mathbf{k}) \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{3} \cdot \\ \cdot \int_{(\Omega)} d\mathbf{x}' \cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}'} \sum_{j'=1}^{8} \underline{e}^{*}_{*r,j'}(\mathbf{k})\, \psi_{j'}(\mathbf{x}') \end{array} \right) \\
= \;& \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{3} \sum_{\mathbf{k}} \int_{(\Omega)} d\mathbf{x}' \cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}'} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \cdot \\
& \cdot \sum_{r=1}^{4} \sum_{j'=1}^{8} \left(\underline{e}_{\circ r,j}(\mathbf{k})\, \underline{e}^{*}_{\circ r,j'}(\mathbf{k}) + \underline{e}_{*r,j}(\mathbf{k})\, \underline{e}^{*}_{*r,j'}(\mathbf{k}) \right) \psi_{j'}(\mathbf{x}').
\end{aligned}$$

In accordance with (3.15):

$$\begin{aligned}
&\sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \left(\sum_{r=1}^{4} \underline{e}_{\circ r,j}(\mathbf{k})\, b_{\circ r,\mathbf{k}} + \sum_{r=1}^{4} \underline{e}_{*r,j}(\mathbf{k})\, b_{*r,\mathbf{k}} \right) \\
= \;& \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{3} \int_{(\Omega)} d\mathbf{x}' \cdot \sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}(\mathbf{x}-\mathbf{x}')} \sum_{j'=1}^{8} \delta_{j,j'} \psi_{j'}(\mathbf{x}').
\end{aligned}$$

Hence, since

$$\sum_{\mathbf{k}} e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}(\mathbf{x}'-\mathbf{x})} = \left(\frac{h}{2\pi\mathrm{c}}\right)^{3} \delta(\mathbf{x}'-\mathbf{x})$$

and according properties of δ:

$$\begin{aligned}
&\sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \left(\sum_{r=1}^{4} \underline{e}_{\circ r,j}(\mathbf{k})\, b_{\circ r,\mathbf{k}} + \sum_{r=1}^{4} \underline{e}_{*r,j}(\mathbf{k})\, b_{*r,\mathbf{k}} \right) \\
= \;& \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^{3} \int_{(\Omega)} d\mathbf{x}' \cdot \left(\frac{h}{2\pi\mathrm{c}}\right)^{3} \delta(\mathbf{x}'-\mathbf{x})\, \psi_{j}(\mathbf{x}') \\
= \;& \int_{(\Omega)} d\mathbf{x}' \cdot \delta(\mathbf{x}'-\mathbf{x})\, \psi_{j}(\mathbf{x}') = \psi_{j}(\mathbf{x}).
\end{aligned}$$

Thus:

$$\boxed{\sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \left(\sum_{r=1}^{4} \underline{e}_{\circ r,j}(\mathbf{k})\, b_{\circ r,\mathbf{k}} + \sum_{r=1}^{4} \underline{e}_{*r,j}(\mathbf{k})\, b_{*r,\mathbf{k}} \right) = \psi_{j}(\mathbf{x}).} \tag{3.17}$$

Let

$$\psi(\mathbf{x}) := \frac{\mathrm{h}}{2\pi\mathrm{c}} \times$$
$$\times \left(\begin{array}{c} \sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}} \left(\begin{array}{c} \left(\begin{array}{c} \cosh\left(\frac{\mathrm{h}}{\mathrm{c}} n_0 x_4\right) + \\ + \sinh\left(\frac{\mathrm{h}}{\mathrm{c}} n_0 x_4\right) \end{array} \right) \sum_{r=1}^{2} \psi_r(\mathbf{x})\,\varepsilon_r + \\ + \left(\begin{array}{c} \cosh\left(\frac{\mathrm{h}}{\mathrm{c}} n_0 x_4\right) - \\ - \sinh\left(\frac{\mathrm{h}}{\mathrm{c}} n_0 x_4\right) \end{array} \right) \sum_{r=3}^{4} \psi_r(\mathbf{x})\,\varepsilon_r \end{array} \right) + \\ + \exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(n_0 x_4)\right) \sum_{r=1}^{4} \psi_{r+4}(\mathbf{x})\,\varepsilon_r \end{array} \right). \tag{3.18}$$

That is in basis $\mathbf{J}_{e\nu}$ (3.11):

$$\psi(\mathbf{x}) = \left[\begin{array}{c} \psi_1(\mathbf{x}) \\ \psi_2(\mathbf{x}) \\ \psi_3(\mathbf{x}) \\ \psi_4(\mathbf{x}) \\ \psi_5(\mathbf{x}) \\ \psi_6(\mathbf{x}) \\ \psi_7(\mathbf{x}) \\ \psi_8(\mathbf{x}) \end{array} \right].$$

That is in this basis:

$$b_{\circ r,\mathbf{k}} := \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \int_{(\Omega)} d\mathbf{x} \cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \underline{e}^{\dagger}_{\circ r,j}(\mathbf{k})\,\psi(\mathbf{x}),$$
$$b_{*r,\mathbf{k}} := \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \int_{(\Omega)} d\mathbf{x} \cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \underline{e}^{\dagger}_{*r,j}(\mathbf{k})\,\psi(\mathbf{x}).$$

Let

$$\psi'(\mathbf{x}) := U^{(-)}\psi(\mathbf{x}).$$

In that case:

$$b'_{\circ r,\mathbf{k}} := \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \int_{(\Omega)} d\mathbf{x} \cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \underline{e}'^{\dagger}_{\circ r,j}(\mathbf{k})\,\psi'(\mathbf{x}),$$
$$b'_{*r,\mathbf{k}} := \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \int_{(\Omega)} d\mathbf{x} \cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \underline{e}'^{\dagger}_{*r,j}(\mathbf{k})\,\psi'(\mathbf{x}).$$

Hence:

$$b'_{\circ r,\mathbf{k}} = \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \int_{(\Omega)} d\mathbf{x} \cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \left(U^{(-)}\underline{e}_{\circ r,j}(\mathbf{k})\right)^{\dagger} \left(U^{(-)}\psi(\mathbf{x})\right),$$
$$b'_{*r,\mathbf{k}} = \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \int_{(\Omega)} d\mathbf{x} \cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \left(U^{(-)}\underline{e}_{*r,j}(\mathbf{k})\right)^{\dagger} \left(U^{(-)}\psi(\mathbf{x})\right).$$

Since $U^{(-)\dagger}U^{(-)} = 1_8$ then

$$\begin{aligned} b'_{\circ r,\mathbf{k}} &= \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \int_{(\Omega)} d\mathbf{x}\cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \underline{e}^{\dagger}_{\circ r,j}(\mathbf{k})\,\psi(\mathbf{x}), \\ b'_{*r,\mathbf{k}} &= \left(\frac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \int_{(\Omega)} d\mathbf{x}\cdot e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \underline{e}^{\dagger}_{*r,j}(\mathbf{k})\,\psi(\mathbf{x}). \end{aligned}$$

That is:

$$b'_{\circ r,\mathbf{k}} = b_{\circ r,\mathbf{k}} \text{ and } b'_{*r,\mathbf{k}} = b_{*r,\mathbf{k}}.$$

And from (3.17):

$$\psi'_j(\mathbf{x}) = \sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \sum_{r=1}^{4} \left(\underline{e}'_{\circ r,j}(\mathbf{k})\,b_{\circ r,\mathbf{k}} + \underline{e}'_{*r,j}(\mathbf{k})\,b_{*r,\mathbf{k}}\right). \tag{3.19}$$

For operators $b_{\circ r,\mathbf{k}}$ and $b_{*r,\mathbf{k}}$:

$$\boxed{\begin{aligned} \left\{b^{\dagger}_{\circ r',\mathbf{k}'}, b_{\circ r,\mathbf{k}}\right\} &= \left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \delta_{r,r'}\delta_{\mathbf{k},\mathbf{k}'}\widehat{1}, \\ \left\{b^{\dagger}_{*r',\mathbf{k}'}, b_{*r,\mathbf{k}}\right\} &= \left(\tfrac{\mathrm{h}}{2\pi\mathrm{c}}\right)^3 \delta_{r,r'}\delta_{\mathbf{k},\mathbf{k}'}\widehat{1}, \\ \left\{b^{\dagger}_{\circ r',\mathbf{k}'}, b_{*r,\mathbf{k}}\right\} &= \widehat{0}, \\ \left\{b^{\dagger}_{*r',\mathbf{k}'}, b_{\circ r,\mathbf{k}}\right\} &= \widehat{0}, \\ \left\{b^{\dagger}_{\circ r',\mathbf{k}'}, b^{\dagger}_{\circ r,\mathbf{k}}\right\} &= \widehat{0}, \\ \left\{b^{\dagger}_{*r',\mathbf{k}'}, b^{\dagger}_{*r,\mathbf{k}}\right\} &= \widehat{0}, \\ \left\{b_{\circ r',\mathbf{k}'}, b_{\circ r,\mathbf{k}}\right\} &= \widehat{0}, \\ \left\{b_{*r',\mathbf{k}'}, b_{*r,\mathbf{k}}\right\} &= \widehat{0}, \\ \left\{b^{\dagger}_{*r',\mathbf{k}'}, b^{\dagger}_{\circ r,\mathbf{k}}\right\} &= \widehat{0}. \end{aligned}} \tag{3.20}$$

Let

$$b_{r,\mathbf{k}} := \sqrt{2}\left(1-a^2\right)^{\frac{1}{4}} \left(\frac{1}{\sqrt{\sqrt{1-a^2}-b}}b_{\circ r,\mathbf{k}} + \frac{1}{\sqrt{\sqrt{1-a^2}+b}}b_{*r,\mathbf{k}}\right).$$

In that case:

$$\begin{aligned} &\underline{e}'_{\circ r}(\mathbf{k})\,b_{\circ r,\mathbf{k}} + \underline{e}'_{*r}(\mathbf{k})\,b_{*r,\mathbf{k}} = \\ = \;& \frac{1}{\sqrt{2}}\left(\sqrt{1-\frac{b}{\sqrt{1-a^2}}}\underline{e}'_{\circ r}(\mathbf{k}) + \sqrt{1+\frac{b}{\sqrt{1-a^2}}}\underline{e}'_{*r}(\mathbf{k})\right) b_{r,\mathbf{k}} \\ & -\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\underline{e}'_{\circ r}(\mathbf{k})\,b_{*r,\mathbf{k}} - \sqrt{\frac{b+\sqrt{1-a^2}}{b-\sqrt{1-a^2}}}\underline{e}'_{*r}(\mathbf{k})\,b_{\circ r,\mathbf{k}} \end{aligned}$$

And from (3.16):

$$\begin{aligned}
& \underline{e}'_{\circ r}\left(\mathbf{k}\right) b_{\circ r,\mathbf{k}} + \underline{e}'_{*r}\left(\mathbf{k}\right) b_{*r,\mathbf{k}} \\
= & \underline{e}'_{r}\left(\mathbf{k}\right) b_{r,\mathbf{k}} \\
& -\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\underline{e}'_{\circ r}\left(\mathbf{k}\right) b_{*r,\mathbf{k}} \\
& -\sqrt{\frac{b+\sqrt{1-a^2}}{b-\sqrt{1-a^2}}}\underline{e}'_{*r}\left(\mathbf{k}\right) b_{\circ r,\mathbf{k}}.
\end{aligned}$$

For $b_{r,\mathbf{k}}$:

$$\boxed{\begin{array}{c}
\left\{b^{\dagger}_{r',\mathbf{k}'}, b_{r,\mathbf{k}}\right\} = 4\frac{b^2+c^2+q^2}{c^2+q^2}\left(\frac{h}{2\pi c}\right)^3 \delta_{r,r'}\delta_{\mathbf{k},\mathbf{k}'}\widehat{1}, \\
\left\{b^{\dagger}_{r',\mathbf{k}'}, b^{\dagger}_{r,\mathbf{k}}\right\} = \widehat{0}, \\
\left\{b_{r',\mathbf{k}'}, b_{r,\mathbf{k}}\right\} = \widehat{0}.
\end{array}} \tag{3.21}$$

From (3.19):

$$\begin{aligned}
\psi'_j\left(\mathbf{x}\right) = & \sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \sum_{r=1}^{4} \underline{e}'_{r,j}\left(\mathbf{k}\right) b_{r,\mathbf{k}} \\
& -\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \sum_{r=1}^{4} \underline{e}'_{\circ r,j}\left(\mathbf{k}\right) b_{*r,\mathbf{k}} \\
& -\sqrt{\frac{b+\sqrt{1-a^2}}{b-\sqrt{1-a^2}}}\sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \sum_{r=1}^{4} \underline{e}'_{*r,j}\left(\mathbf{k}\right) b_{\circ r,\mathbf{k}}.
\end{aligned}$$

Let:

$$\chi\left(\mathbf{x}\right) := \sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \sum_{r=1}^{4} \underline{e}'_{r}\left(\mathbf{k}\right) b_{r,\mathbf{k}}, \tag{3.22}$$

$$\chi_{*j}\left(\mathbf{x}\right) := \sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \sum_{r=1}^{4} \underline{e}'_{\circ r,j}\left(\mathbf{k}\right) b_{*r,\mathbf{k}}$$

$$\chi_{\circ j}\left(\mathbf{x}\right) := \sqrt{\frac{b+\sqrt{1-a^2}}{b-\sqrt{1-a^2}}}\sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \sum_{r=1}^{4} \underline{e}'_{*r,j}\left(\mathbf{k}\right) b_{\circ r,\mathbf{k}}.$$

In that case:

$$\psi'_j\left(\mathbf{x}\right) = \chi_j\left(\mathbf{x}\right) - \chi_{*j}\left(\mathbf{x}\right) - \chi_{\circ j}\left(\mathbf{x}\right).$$

Let

$$\widehat{H}_0' := U^{(-)}\widehat{H}_0 U^{(-)\dagger}.$$

For this Hamiltonian:

$$\begin{aligned}
&\int_{(\Omega)} d\mathbf{x}\cdot\chi_*^\dagger(\mathbf{x})\,\widehat{H}_0'\psi'(\mathbf{x})\\
=\;&\int_{(\Omega)} d\mathbf{x}\cdot\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}'} e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}'\mathbf{x}}\sum_{r'=1}^{4} b_{*r',\mathbf{k}'}^{\dagger}\underline{e}_{\circ r'}^{\prime\dagger}(\mathbf{k}')\cdot\\
&\cdot\widehat{H}_0'\sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}\mathbf{x}}\sum_{r=1}^{4}\left(\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)\\
=\;&\int_{(\Omega)} d\mathbf{x}\cdot\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}}\sum_{\mathbf{k}'} e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}'\mathbf{x}}\sum_{r'=1}^{4} b_{*r',\mathbf{k}'}^{\dagger}\underline{e}_{\circ r'}^{\prime\dagger}(\mathbf{k}')\cdot\\
&\cdot\widehat{H}_0' e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}\mathbf{x}}\begin{pmatrix}\sum_{r=1}^{2}\left(\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)+\\ +\sum_{r=3}^{4}\left(\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)\end{pmatrix}\\
=\;&\int_{(\Omega)} d\mathbf{x}\cdot\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}}\sum_{\mathbf{k}'} e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}'\mathbf{x}}e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}\mathbf{x}}\sum_{r'=1}^{4} b_{*r',\mathbf{k}'}^{\dagger}\underline{e}_{\circ r'}^{\prime\dagger}(\mathbf{k}')\cdot\\
&\cdot\mathrm{h}\widehat{H}_0'(\mathbf{k})\begin{pmatrix}\sum_{r=1}^{2}\left(\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)+\\ +\sum_{r=3}^{4}\left(\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)\end{pmatrix}.
\end{aligned}$$

Hence:

$$\begin{aligned}
&\int_{(\Omega)} d\mathbf{x}\cdot\chi_*^\dagger(\mathbf{x})\,\widehat{H}_0'\psi'(\mathbf{x})\\
=\;&\int_{(\Omega)} d\mathbf{x}\cdot\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}}\sum_{\mathbf{k}'} e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}'\mathbf{x}}e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}\mathbf{x}}\sum_{r'=1}^{4} b_{*r',\mathbf{k}'}^{\dagger}\underline{e}_{\circ r'}^{\prime\dagger}(\mathbf{k}')\cdot\\
&\cdot\mathrm{h}\begin{pmatrix}\omega(\mathbf{k})\sum_{r=1}^{2}\left(\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)-\\ -\omega(\mathbf{k})\sum_{r=3}^{4}\left(\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)\end{pmatrix}\\
=\;&\int_{(\Omega)} d\mathbf{x}\cdot\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}}\mathrm{h}\omega(\mathbf{k})\sum_{\mathbf{k}'} e^{\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}'\mathbf{x}}e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{k}\mathbf{x}}\cdot\\
&\cdot\sum_{r'=1}^{4} b_{*r',\mathbf{k}'}^{\dagger}\underline{e}_{\circ r'}^{\prime\dagger}(\mathbf{k}')\begin{pmatrix}\sum_{r=1}^{2}\left(\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)-\\ -\sum_{r=3}^{4}\left(\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)\end{pmatrix}\\
=\;&\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}}\mathrm{h}\omega(\mathbf{k})\sum_{\mathbf{k}'}\left(\int d\mathbf{x}\cdot e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(\mathbf{k}-\mathbf{k}')\mathbf{x}}\right)\cdot\\
&\cdot\sum_{r'=1}^{4} b_{*r',\mathbf{k}'}^{\dagger}\begin{pmatrix}\sum_{r=1}^{2}\left(\underline{e}_{\circ r'}^{\prime\dagger}(\mathbf{k}')\,\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{\circ r'}^{\prime\dagger}(\mathbf{k}')\,\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)-\\ -\sum_{r=3}^{4}\left(\underline{e}_{\circ r'}^{\prime\dagger}(\mathbf{k}')\,\underline{e}_{\circ r}'(\mathbf{k})\,b_{\circ r,\mathbf{k}}+\underline{e}_{\circ r'}^{\prime\dagger}(\mathbf{k}')\,\underline{e}_{*r}'(\mathbf{k})\,b_{*r,\mathbf{k}}\right)\end{pmatrix}.
\end{aligned}$$

Since

$$\int_{(\Omega)} d\mathbf{x}\cdot e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(\mathbf{k}-\mathbf{k}')\mathbf{x}} = \left(\frac{2\pi\mathrm{c}}{\mathrm{h}}\right)^3 \delta_{\mathbf{k},\mathbf{k}'}$$

then

$$\begin{aligned}
&\int_{(\Omega)} d\mathbf{x}\cdot\chi_*^{\dagger}(\mathbf{x})\widehat{H}_0'\psi'(\mathbf{x})\\
=\ &\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}}\mathrm{h}\omega(\mathbf{k})\sum_{\mathbf{k}'}\left(\frac{2\pi\mathrm{c}}{\mathrm{h}}\right)^3\delta_{\mathbf{k},\mathbf{k}'}\cdot\\
&\cdot\sum_{r'=1}^{4} b^{\dagger}_{*r',\mathbf{k}'}\left(\begin{array}{c}\sum_{r=1}^{2}\left(\underline{e}'^{\dagger}_{\circ r'}(\mathbf{k}')\underline{e}'_{\circ r}(\mathbf{k})b_{\circ r,\mathbf{k}}+\underline{e}'^{\dagger}_{\circ r'}(\mathbf{k}')\underline{e}'_{*r}(\mathbf{k})b_{*r,\mathbf{k}}\right)-\\ -\sum_{r=3}^{4}\left(\underline{e}'^{\dagger}_{\circ r'}(\mathbf{k}')\underline{e}'_{\circ r}(\mathbf{k})b_{\circ r,\mathbf{k}}+\underline{e}'^{\dagger}_{\circ r'}(\mathbf{k}')\underline{e}'_{*r}(\mathbf{k})b_{*r,\mathbf{k}}\right)\end{array}\right).
\end{aligned}$$

In accordance with properties of δ:

$$\begin{aligned}
&\int_{(\Omega)} d\mathbf{x}\cdot\chi_*^{\dagger}(\mathbf{x})\widehat{H}_0'\psi'(\mathbf{x})\\
=\ &\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}}\mathrm{h}\omega(\mathbf{k})\left(\frac{2\pi\mathrm{c}}{\mathrm{h}}\right)^3\cdot\\
&\cdot\sum_{r'=1}^{4} b^{\dagger}_{*r',\mathbf{k}}\left(\begin{array}{c}\sum_{r=1}^{2}\left(\underline{e}'^{\dagger}_{\circ r'}(\mathbf{k})\underline{e}'_{\circ r}(\mathbf{k})b_{\circ r,\mathbf{k}}+\underline{e}'^{\dagger}_{\circ r'}(\mathbf{k})\underline{e}'_{*r}(\mathbf{k})b_{*r,\mathbf{k}}\right)-\\ -\sum_{r=3}^{4}\left(\underline{e}'^{\dagger}_{\circ r'}(\mathbf{k})\underline{e}'_{\circ r}(\mathbf{k})b_{\circ r,\mathbf{k}}+\underline{e}'^{\dagger}_{\circ r'}(\mathbf{k})\underline{e}'_{*r}(\mathbf{k})b_{*r,\mathbf{k}}\right)\end{array}\right).
\end{aligned}$$

Since

$$\begin{aligned}
\underline{e}'^{\dagger}_{\circ r'}(\mathbf{k}')\underline{e}'_{\circ r}(\mathbf{k}) &= \delta_{r,r'},\\
\underline{e}'^{\dagger}_{\circ r'}(\mathbf{k}')\underline{e}'_{*r}(\mathbf{k}) &= 0
\end{aligned}$$

then

$$\begin{aligned}
&\int_{(\Omega)} d\mathbf{x}\cdot\chi_*^{\dagger}(\mathbf{x})\widehat{H}_0'\psi'(\mathbf{x})\\
=\ &\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}}\mathrm{h}\omega(\mathbf{k})\left(\frac{2\pi\mathrm{c}}{\mathrm{h}}\right)^3\cdot\\
&\cdot\sum_{r'=1}^{4} b^{\dagger}_{*r',\mathbf{k}}\left(\sum_{r=1}^{2}\left(\delta_{r,r'}b_{\circ r,\mathbf{k}}+0b_{*r,\mathbf{k}}\right)-\sum_{r=3}^{4}\left(\delta_{r,r'}b_{\circ r,\mathbf{k}}+0b_{*r,\mathbf{k}}\right)\right)\\
=\ &\left(\frac{2\pi\mathrm{c}}{\mathrm{h}}\right)^3\sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}}\sum_{\mathbf{k}}\mathrm{h}\omega(\mathbf{k})\cdot
\end{aligned}$$

$$\cdot \sum_{r'=1}^{4} b^{\dagger}_{*r',\mathbf{k}} \left(\sum_{r=1}^{2} \delta_{r,r'} b_{\circ r,\mathbf{k}} - \sum_{r=3}^{4} \delta_{r,r'} b_{\circ r,\mathbf{k}} \right)$$

$$= \left(\frac{2\pi c}{h}\right)^3 \sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}} \cdot$$

$$\cdot \sum_{\mathbf{k}} h\omega(\mathbf{k}) \left(\sum_{r=1}^{2} b^{\dagger}_{*r,\mathbf{k}} b_{\circ r,\mathbf{k}} - \sum_{r=3}^{4} b^{\dagger}_{*r,\mathbf{k}} b_{\circ r,\mathbf{k}} \right).$$

Therefore,

$$\int_{(\Omega)} d\mathbf{x} \cdot \chi^{\dagger}_{*}(\mathbf{x}) \widehat{H}'_0 \psi'(\mathbf{x})$$

$$= \left(\frac{2\pi c}{h}\right)^3 \sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}} \sum_{\mathbf{k}} h\omega(\mathbf{k}) \left(\sum_{r=1}^{2} b^{\dagger}_{*r,\mathbf{k}} b_{\circ r,\mathbf{k}} - \sum_{r=3}^{4} b^{\dagger}_{*r,\mathbf{k}} b_{\circ r,\mathbf{k}} \right).$$

Similarly you can calculate that

$$\int d\mathbf{x} \cdot \chi^{\dagger}_{\circ}(\mathbf{x}) \widehat{H}'_0 \psi'(\mathbf{x})$$

$$= \left(\frac{2\pi c}{h}\right)^3 \sqrt{\frac{b-\sqrt{1-a^2}}{b+\sqrt{1-a^2}}} \sum_{\mathbf{k}} h\omega(\mathbf{k}) \left(\sum_{r=1}^{2} b^{\dagger}_{\circ r,\mathbf{k}} b_{*r,\mathbf{k}} - \sum_{r=3}^{4} b^{\dagger}_{\circ r,\mathbf{k}} b_{*r,\mathbf{k}} \right).$$

Since

$$\widetilde{\Psi}(t,\mathbf{p}) = \left(\frac{2\pi c}{h}\right)^3 \sum_{r=1}^{4} c_r(t,\mathbf{p}) b^{\dagger}_{r,\mathbf{p}} \widetilde{F}_0$$

and (3.20)

$$\left\{ b^{\dagger}_{*r',\mathbf{k}'}, b_{\circ r,\mathbf{k}} \right\} = \widehat{0},\ \left\{ b^{\dagger}_{*r',\mathbf{k}'}, b^{\dagger}_{*r,\mathbf{k}} \right\} = \widehat{0},\ \left\{ b^{\dagger}_{*r',\mathbf{k}'}, b^{\dagger}_{\circ r,\mathbf{k}} \right\} = \widehat{0}.$$

then

$$b^{\dagger}_{*r,\mathbf{k}} b_{\circ r,\mathbf{k}} \widetilde{\Psi} = -b_{\circ r,\mathbf{k}} \left(\frac{2\pi c}{h}\right)^3 \sum_{r=1}^{4} c_r(t,\mathbf{p}) b^{\dagger}_{*r,\mathbf{k}} b^{\dagger}_{r,\mathbf{p}} \widetilde{F}_0 = \widetilde{0}.$$

Similarly

$$b^{\dagger}_{\circ r,\mathbf{k}} b_{*r,\mathbf{k}} \widetilde{\Psi} = \widetilde{0}.$$

Hence,

$$\boxed{\int_{(\Omega)} d\mathbf{x} \cdot \psi'^{\dagger}(\mathbf{x}) \widehat{H}'_0 \psi'(\mathbf{x}) \Psi(t,\mathbf{x}_0) = \int_{(\Omega)} d\mathbf{x} \cdot \chi^{\dagger}(\mathbf{x}) \widehat{H}'_0 \chi(\mathbf{x}) \Psi(t,\mathbf{x}_0).}$$

Thus, the function $\psi'(\mathbf{x})$ can be substituted for the function $\chi(\mathbf{x})$ in calculations of a probabilities evolution.

Let

$$\nu_{n_0,(s)}(\mathbf{k}) := \begin{bmatrix} (c+iq)\,e_{sL}(\mathbf{k}) \\ \overrightarrow{0}_2 \end{bmatrix}, \; l_{n_0,(s)}(\mathbf{k}) := \begin{bmatrix} (a-ib)\,e_{sL}(\mathbf{k}) \\ e_{sR}(\mathbf{k}) \end{bmatrix}.$$

Hence, from (3.16):

$$\underline{e}'_s(\mathbf{k}) = \frac{\mathrm{h}}{2\pi\mathrm{c}} \times$$
$$\times \left(\sqrt{\frac{2\pi n_0}{\sinh(2n_0\pi)}} \left(\begin{array}{c} \left(\begin{array}{c} \cosh\left(\frac{\mathrm{h}}{\mathrm{c}} n_0 x_4\right) + \\ +\sinh\left(\frac{\mathrm{h}}{\mathrm{c}} n_0 x_4\right) \end{array} \right) (c+iq)\,e_{sL}(\mathbf{k}) + \\ + \left(\begin{array}{c} \cosh\left(\frac{\mathrm{h}}{\mathrm{c}} n_0 x_4\right) - \\ -\sinh\left(\frac{\mathrm{h}}{\mathrm{c}} n_0 x_4\right) \end{array} \right) \overrightarrow{0}_2 \end{array} \right) + \atop + \exp\left(-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}(n_0 x_4)\right) l_{n_0,(s)}(\mathbf{k}) \right).$$

Therefore, in basis $\mathbf{J}_{e\nu}$:

$$\underline{e}'_s(\mathbf{k}) = \begin{bmatrix} \nu_{n_0,(s)}(\mathbf{k}) \\ l_{n_0,(s)}(\mathbf{k}) \end{bmatrix}.$$

Therefore, from (3.22):

$$\chi(\mathbf{x}) = \sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \sum_{s=1}^{4} \begin{bmatrix} \nu_{n_0,(s)}(\mathbf{k}) \\ l_{n_0,(s)}(\mathbf{k}) \end{bmatrix} b_{s,\mathbf{k}},$$

Let

$$\widetilde{\nu}_{n_0}(\mathbf{x}) := \sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \sum_{s=1}^{2} \nu_{n_0,(s)}(\mathbf{k})\, b_s(\mathbf{k}),$$

$$\widetilde{l}_{n_0}(\mathbf{x}) := \sum_{\mathbf{k}} e^{-\mathrm{i}\frac{\mathrm{h}}{\mathrm{c}}\mathbf{kx}} \sum_{s=1}^{4} l_{n_0,(s)}(\mathbf{k})\, b_s(\mathbf{k}).$$

Hence, in basis $\mathbf{J}_{e\nu}$:

$$\chi(\mathbf{x}) = \begin{bmatrix} \widetilde{\nu}_{n_0}(\mathbf{x}) \\ \widetilde{l}_{n_0}(\mathbf{x}) \end{bmatrix}. \tag{3.23}$$

Let:

$$\widehat{H}_{l,0} := \mathrm{c}\sum_{r=1}^{3} \beta^{[r]}\mathrm{i}\partial_r + \mathrm{h}n_0\left(a\gamma^{[0]} - b\beta^{[4]}\right),$$

$$\widehat{H}_{\nu,0} := \mathrm{c}\sum_{r=1}^{3} \beta^{[r]}\mathrm{i}\partial_r + \mathrm{h}n_0\left(a\gamma^{[0]} + b\beta^{[4]}\right),$$

$$\widehat{H}_{\nu,l} := (c+\mathrm{i}q)\begin{bmatrix} 0 & 0 & n_0 & 0 \\ 0 & 0 & 0 & n_0 \\ -n_0 & 0 & 0 & 0 \\ 0 & -n_0 & 0 & 0 \end{bmatrix},$$

$$\widehat{H}_{l,\nu} := (c - \mathrm{i}q) \begin{bmatrix} 0 & 0 & -n_0 & 0 \\ 0 & 0 & 0 & -n_0 \\ n_0 & 0 & 0 & 0 \\ 0 & n_0 & 0 & 0 \end{bmatrix}.$$

In that case in basis $\mathbf{J}_{e\nu}$:

$$\widehat{H}'_0 = \begin{bmatrix} \widehat{H}_{\nu,0} & \widehat{H}_{\nu,l} \\ \widehat{H}_{l,\nu} & \widehat{H}_{l,0} \end{bmatrix}.$$

Let

$$\widehat{H}_{l,0}(\mathbf{k}) := \sum_{r=1}^{3} \beta^{[r]} k_r + n_0 \left(a\gamma^{[0]} - b\beta^{[4]} \right),$$

$$\widehat{H}_{\nu,0}(\mathbf{k}) := \sum_{r=1}^{3} \beta^{[r]} k_r + n_0 \left(a\gamma^{[0]} + b\beta^{[4]} \right).$$

In that case

$$\widehat{H}'_0(\mathbf{k}) = \begin{bmatrix} \widehat{H}_{\nu,0}(\mathbf{k}) & \widehat{H}_{\nu,l} \\ \widehat{H}_{l,\nu} & \widehat{H}_{l,0}(\mathbf{k}) \end{bmatrix}$$

An neutrino and it's lepton are tied by the follows equations:

$$\widehat{H}_{\nu,0}(\mathbf{k}) \nu_{n_0,(s)}(\mathbf{k}) + \widehat{H}_{\nu,l} l_{n_0,(s)}(\mathbf{k}) = \omega(\mathbf{k}) \nu_{n_0,(s)}(\mathbf{k})$$

for $s \in \{1,2\}$ and

$$\widehat{H}_{\nu,0}(\mathbf{k}) \nu_{n_0,(s)}(\mathbf{k}) + \widehat{H}_{\nu,l} l_{n_0,(s)}(\mathbf{k}) = -\omega(\mathbf{k}) \nu_{n_0,(s)}(\mathbf{k})$$

for $s \in \{3,4\}$.

I suppose that such neutrino can fly 1.5 cm. [36] and give birth to it's leptons.

3.1.3. Electroweak Transformations

During the 1960s Sheldon Lee Glashow, Abdus Salam, and Steven Weinberg independently discovered that they could construct a gauge-invariant theory of the weak force, provided that they also included the electromagnetic force.

The existence of the force carriers, the neutral Z particles and the charged W particles, was verified experimentally in 1983 in high-energy proton-antiproton collisions at the European Organization for Nuclear Research (CERN).

Let (3.8) does not hold true, that is $U^{(-)}$ depends on $\underline{x}$. And let denote:

$$K := \sum_{\mu=0}^{3} \beta^{[\mu]} (F_\mu + 0.5 g_1 Y B_\mu). \tag{3.24}$$

In that case from (2.39) the equation of moving is of following form:

$$\left(K+\sum_{\mu=0}^{3}\beta^{[\mu]}\mathrm{i}\partial_{\mu}+\gamma^{[0]}\mathrm{i}\partial_{5}+\beta^{[4]}\mathrm{i}\partial_{4}\right)\widetilde{\varphi}=0. \tag{3.25}$$

Let us consider for this Hamiltonian the following transformations:

$$\begin{aligned}
&\widetilde{\varphi}\to\widetilde{\varphi}':=U^{(-)}\widetilde{\varphi},\\
&x_4\to x_4':=(\ell_\circ+\ell_*)\,ax_4+(\ell_\circ-\ell_*)\sqrt{1-a^2}x_5,\\
&x_5\to x_5':=(\ell_\circ+\ell_*)\,ax_5-(\ell_\circ-\ell_*)\sqrt{1-a^2}x_4,\\
&x_\mu\to x_\mu':=x_\mu,\text{ for }\mu\in\{0,1,2,3\},\\
&K\to K'=U^{(-)}KU^{(-)\dagger}-\mathrm{i}\sum_{\mu=0}^{3}\beta^{[\mu]}\left(\partial_\mu U^{(-)}\right)U^{(-)\dagger}
\end{aligned} \tag{3.26}$$

with

$$\partial_4 U^{(-)}=U^{(-)}\partial_4 \text{ and } \partial_5 U^{(-)}=U^{(-)}\partial_5:$$

Since

$$(\ell_\circ-\ell_*)(\ell_\circ-\ell_*)=1_8$$

then

$$\begin{aligned}
x_4 &= ax_4'-(\ell_\circ-\ell_*)\sqrt{1-a^2}x_5' \text{ and}\\
x_5 &= (\ell_\circ-\ell_*)\sqrt{1-a^2}x_4'+ax_5'.
\end{aligned}$$

Since for any f:

$$\begin{aligned}
\partial_4' f &= \partial_4 f\cdot\partial_4' x_4+\partial_5 f\cdot\partial_4' x_5,\\
\partial_5' f &= \partial_4 f\cdot\partial_5' x_4+\partial_5 f\cdot\partial_5' x_5
\end{aligned}$$

then

$$\begin{aligned}
\partial_4' f &= \partial_4 f\cdot a+\partial_5 f\cdot(\ell_\circ-\ell_*)\sqrt{1-a^2},\\
\partial_5' f &= \partial_4 f\cdot\left(-(\ell_\circ-\ell_*)\sqrt{1-a^2}\right)+\partial_5 f\cdot a.
\end{aligned}$$

Therefore, if

$$\left(K'+\sum_{\mu=0}^{3}\beta^{[\mu]}\mathrm{i}\partial_{\mu}+\gamma^{[0]}\mathrm{i}\partial_5'+\beta^{[4]}\mathrm{i}\partial_4'\right)U^{(-)}\widetilde{\varphi}=0$$

then

$$\left(\begin{array}{c} U^{(-)}KU^{(-)\dagger}-\mathrm{i}\sum_{\mu=0}^{3}\beta^{[\mu]}\left(\partial_{\mu}U^{(-)}\right)U^{(-)\dagger} \\ +\sum_{\mu=0}^{3}\beta^{[\mu]}\mathrm{i}\partial_{\mu}+\gamma^{[0]}\mathrm{i}\left(\left(-\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\right)\partial_{4}+a\partial_{5}\right) \\ +\beta^{[4]}\mathrm{i}\left(a\partial_{4}+\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\partial_{5}\right) \end{array}\right)U^{(-)}\widetilde{\varphi}=0.$$

Hence,

$$\left(\begin{array}{c} U^{(-)}KU^{(-)\dagger}U^{(-)}-\mathrm{i}\sum_{\mu=0}^{3}\beta^{[\mu]}\left(\partial_{\mu}U^{(-)}\right)U^{(-)\dagger}U^{(-)} \\ +\sum_{\mu=0}^{3}\beta^{[\mu]}\mathrm{i}\partial_{\mu}U^{(-)}+\gamma^{[0]}U^{(-)}\mathrm{i}\left(\left(-\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\right)\partial_{4}+a\partial_{5}\right) \\ +\beta^{[4]}U^{(-)}\mathrm{i}\left(a\partial_{4}+\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\partial_{5}\right) \end{array}\right)\widetilde{\varphi}=0$$

since $U^{(-)}$ is a linear operator such that $\partial_4 U^{(-)} = U^{(-)}\partial_4$ and $\partial_5 U^{(-)} = U^{(-)}\partial_5$.

Since

$$U^{(-)\dagger}U^{(-)} = 1_8,$$

for any f:

$$\partial_{\mu}\left(U^{(-)}f\right)=\left(\partial_{\mu}U^{(-)}\right)f+\left(U^{(-)}\partial_{\mu}f\right)=\left(\left(\partial_{\mu}U^{(-)}\right)+U^{(-)}\partial_{\mu}\right)f,$$

and

$$\begin{aligned} \gamma^{[0]}U^{(-)} &= U^{(-)}\left(a\gamma^{[0]}-\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\beta^{[4]}\right), \\ \beta^{[4]}U^{(-)} &= U^{(-)}\left(a\beta^{[4]}+\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\gamma^{[0]}\right) \end{aligned}$$

then

$$\left(\begin{array}{c} U^{(-)}K-\mathrm{i}\sum_{\mu=0}^{3}\beta^{[\mu]}\left(\partial_{\mu}U^{(-)}\right) \\ +\sum_{\mu=0}^{3}\beta^{[\mu]}\mathrm{i}\left(\left(\partial_{\mu}U^{(-)}\right)+U^{(-)}\partial_{\mu}\right) \\ +U^{(-)}\left(a\gamma^{[0]}-\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\beta^{[4]}\right)\times \\ \times\mathrm{i}\left(\left(-\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\right)\partial_{4}+a\partial_{5}\right) \\ +U^{(-)}\left(a\beta^{[4]}+\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\gamma^{[0]}\right)\times \\ \times\mathrm{i}\left(a\partial_{4}+\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\partial_{5}\right) \end{array}\right)\widetilde{\varphi}=0.$$

Therefore,

$$\left(\begin{array}{c} U^{(-)}K-\mathrm{i}\sum_{\mu=0}^{3}\beta^{[\mu]}\left(\partial_{\mu}U^{(-)}\right) \\ +\sum_{\mu=0}^{3}\beta^{[\mu]}\mathrm{i}\left(\left(\partial_{\mu}U^{(-)}\right)+U^{(-)}\partial_{\mu}\right) \\ +\mathrm{i}U^{(-)}\left(\begin{array}{c} \left(a\gamma^{[0]}-\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\beta^{[4]}\right)\times \\ \times\left(-\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\partial_{4}+a\partial_{5}\right) \\ +\left(a\beta^{[4]}+\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\gamma^{[0]}\right)\times \\ \times\left(a\partial_{4}+\left(\ell_{\circ}-\ell_{*}\right)\sqrt{1-a^{2}}\partial_{5}\right) \end{array}\right) \end{array}\right)\widetilde{\varphi}=0,$$

$$\left(U^{(-)}K+\sum_{\mu=0}^{3}\beta^{[\mu]}\mathrm{i}U^{(-)}\partial_\mu+\mathrm{i}U^{(-)}\left(+\gamma^{[0]}\partial_5+\beta^{[4]}\partial_4\right)\right)\widetilde{\varphi}=0,$$

Hence,

$$U^{(-)}\left(K+\sum_{\mu=0}^{3}\beta^{[\mu]}\mathrm{i}\partial_\mu+\mathrm{i}\left(+\gamma^{[0]}\partial_5+\beta^{[4]}\partial_4\right)\right)\widetilde{\varphi}=0$$

since $\beta^{[\mu]}U^{(-)}=U^{(-)}\beta^{[\mu]}$ for $\mu\in\{0,1,2,3\}$. Compare with (3.25).

Therefore, this equation of moving is invariant under the transformation (3.26).

Let g_2 be some positive real number.

If design (here: a,b,c,q form $U^{(-)}$ in (3.4)):

$$\begin{array}{l}
W_{0,\mu}:=-2\frac{1}{g_2q}\left(\begin{array}{c}q\left(\partial_\mu a\right)b-q\left(\partial_\mu b\right)a+\left(\partial_\mu c\right)q^2+\\+a\left(\partial_\mu a\right)c+b\left(\partial_\mu b\right)c+c^2\left(\partial_\mu c\right)\end{array}\right)\\
W_{1,\mu}:=-2\frac{1}{g_2q}\left(\begin{array}{c}\left(\partial_\mu a\right)a^2-bq\left(\partial_\mu c\right)+a\left(\partial_\mu b\right)b+\\+a\left(\partial_\mu c\right)c+q^2\left(\partial_\mu a\right)+c\left(\partial_\mu b\right)q\end{array}\right)\\
W_{2,\mu}:=-2\frac{1}{g_2q}\left(\begin{array}{c}q\left(\partial_\mu a\right)c-a\left(\partial_\mu a\right)b-b^2\left(\partial_\mu b\right)-\\-c\left(\partial_\mu c\right)b-\left(\partial_\mu b\right)q^2-\left(\partial_\mu c\right)qa\end{array}\right)
\end{array}$$

and

$$W_\mu:=\left[\begin{array}{cccc}W_{0,\mu}1_2 & 0_2 & \left(W_{1,\mu}-\mathrm{i}W_{2,\mu}\right)1_2 & 0_2\\ 0_2 & 0_2 & 0_2 & 0_2\\ \left(W_{1,\mu}+\mathrm{i}W_{2,\mu}\right)1_2 & 0_2 & -W_{0,\mu}1_2 & 0_2\\ 0_2 & 0_2 & 0_2 & 0_2\end{array}\right] \tag{3.27}$$

then

$$-\mathrm{i}\left(\partial_\mu U^{(-)}\right)U^{(-)\dagger}=\frac{1}{2}g_2W_\mu, \tag{3.28}$$

and from (3.24), (3.25):

$$\left(\begin{array}{c}\sum_{\mu=0}^{3}\beta^{[\mu]}\mathrm{i}\left(\partial_\mu-\mathrm{i}0.5g_1B_\mu Y-\mathrm{i}\frac{1}{2}g_2W_\mu-\mathrm{i}F_\mu\right)\\+\gamma^{[0]}\mathrm{i}\partial'_5+\beta^{[4]}\mathrm{i}\partial'_4\end{array}\right)\widetilde{\varphi}'=0. \tag{3.29}$$

Let (3.4) $a'(t,\mathbf{x})$, $b'(t,\mathbf{x})$, $c'(t,\mathbf{x})$, $q'(t,\mathbf{x})$ are real functions and:

$$U':=\left[\begin{array}{cccc}\left(a'+\mathrm{i}b'\right)1_2 & 0_2 & \left(c'+\mathrm{i}g'\right)1_2 & 0_2\\ 0_2 & 1_2 & 0_2 & 0_2\\ \left(-c'+\mathrm{i}g'\right)1_2 & 0_2 & \left(a'-\mathrm{i}b'\right)1_2 & 0_2\\ 0_2 & 0_2 & 0_2 & 1_2\end{array}\right].$$

In this case if

$$U'':=U'U^{(-)}$$

then there exist real functions $a''(t,\mathbf{x})$, $b''(t,\mathbf{x})$, $c''(t,\mathbf{x})$, $q''(t,\mathbf{x})$ such that U'' has the similar shape:

$$U'' = \begin{bmatrix} (a''+\mathrm{i}b'')\,1_2 & 0_2 & (c''+\mathrm{i}g'')\,1_2 & 0_2 \\ 0_2 & 1_2 & 0_2 & 0_2 \\ (-c''+\mathrm{i}g'')\,1_2 & 0_2 & (a''-\mathrm{i}b'')\,1_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 1_2 \end{bmatrix}.$$

Let:

$$W''_\mu := -\frac{2\mathrm{i}}{g_2}\left(\partial_\mu\left(U'U^{(-)}\right)\right)\left(U'U^{(-)}\right)^\dagger,$$

Hence,

$$\begin{aligned} W''_\mu &= -\frac{2\mathrm{i}}{g_2}\left(\partial_\mu U'\right)U^{(-)}\left(U'U^{(-)}\right)^\dagger - \frac{2\mathrm{i}}{g_2}U'\partial_\mu U^{(-)}\left(U'U^{(-)}\right)^\dagger \\ &= -\frac{2\mathrm{i}}{g_2}\left(\partial_\mu U'\right)U^{(-)}U^{(-)\dagger}U'^\dagger - \frac{2\mathrm{i}}{g_2}U'\left(\partial_\mu U^{(-)}\right)U^{(-)\dagger}U'^\dagger \\ &= -\frac{2\mathrm{i}}{g_2}\left(\partial_\mu U'\right)U'^\dagger - \frac{2\mathrm{i}}{g_2}U'\left(\partial_\mu U^{(-)}\right)U^{(-)\dagger}U'^\dagger. \end{aligned}$$

Since from (3.28):

$$W_\mu = -\mathrm{i}\frac{2}{g_2}\left(\partial_\mu U^{(-)}\right)U^{(-)\dagger}$$

then

$$\begin{aligned} W''_\mu &= -\frac{2\mathrm{i}}{g_2}\left(\partial_\mu U'\right)U'^\dagger - \frac{2\mathrm{i}}{g_2}U'\left(\left(\partial_\mu U^{(-)}\right)U^{(-)\dagger}\right)U'^\dagger \\ &= -\frac{2\mathrm{i}}{g_2}\left(\partial_\mu U'\right)U'^\dagger + U'W_\mu U'^\dagger. \end{aligned}$$

Therefore, if

$$\ell''_\circ := \frac{1}{2\sqrt{(1-a''^2)}}\begin{bmatrix} \left(b''+\sqrt{(1-a''^2)}\right)1_4 & (q''-\mathrm{i}c'')\,1_4 \\ (q''+\mathrm{i}c'')\,1_4 & \left(\sqrt{(1-a''^2)}-b''\right)1_4 \end{bmatrix},$$

$$\ell''_* := \frac{1}{2\sqrt{(1-a''^2)}}\begin{bmatrix} \left(\sqrt{(1-a''^2)}-b''\right)1_4 & (-q''+\mathrm{i}c'')\,1_4 \\ (-q''-\mathrm{i}c'')\,1_4 & \left(b''+\sqrt{(1-a''^2)}\right)1_4 \end{bmatrix}.$$

then under the following transformation

$$\begin{aligned} &\widetilde{\varphi} \to \widetilde{\varphi}'' := U''\widetilde{\varphi}, \\ &x_4 \to x''_4 := \left(\ell''_\circ + \ell''_*\right)a''x_4 + \left(\ell''_\circ - \ell''_*\right)\sqrt{1-a''^2}x_5, \\ &x_5 \to x''_5 := \left(\ell''_\circ + \ell''_*\right)a''x_5 - \left(\ell''_\circ - \ell''_*\right)\sqrt{1-a''^2}x_4, \\ &x_\mu \to x''_\mu := x_\mu, \text{ for } \mu \in \{0,1,2,3\}, \\ &K \to K'' := \sum_{\mu=0}^{3}\beta^{[\mu]}\left(F_\mu + 0.5g_1YB_\mu + \frac{1}{2}g_2W''_\mu\right) \end{aligned} \tag{3.30}$$

fields W''_μ and W_μ are tied by the following equation

$$\boxed{W''_\mu = U' W_\mu U'^\dagger - \frac{2\mathrm{i}}{g_2}\left(\partial_\mu U'\right) U'^\dagger} \tag{3.31}$$

like in Standard Model.

From (3.28):

$$W_\mu = -\mathrm{i}\frac{2}{g_2}\left(\partial_\mu U^{(-)}\right) U^{(-)\dagger}.$$

Let us calculate:

$$\begin{aligned}
&\partial_\mu W_\nu - \partial_\nu W_\mu = \\
= \ &\partial_\mu \left(-\mathrm{i}\frac{2}{g_2}\left(\partial_\nu U^{(-)}\right) U^{(-)\dagger}\right) - \partial_\nu \left(-\mathrm{i}\frac{2}{g_2}\left(\partial_\mu U^{(-)}\right) U^{(-)\dagger}\right) = \\
= \ &-\mathrm{i}\frac{2}{g_2}\left(\begin{array}{c} \left(\partial_\mu \partial_\nu U^{(-)}\right) U^{(-)\dagger} + \left(\partial_\nu U^{(-)}\right)\left(\partial_\mu U^{(-)\dagger}\right) \\ -\left(\partial_\nu \partial_\mu U^{(-)}\right) U^{(-)\dagger} - \left(\partial_\mu U^{(-)}\right)\left(\partial_\nu U^{(-)\dagger}\right) \end{array}\right).
\end{aligned}$$

Since

$$\partial_\mu \partial_\nu U^{(-)} = \partial_\nu \partial_\mu U^{(-)}$$

then

$$\begin{aligned}
&\partial_\mu W_\nu - \partial_\nu W_\mu = \\
= \ &-\mathrm{i}\frac{2}{g_2}\left(\left(\partial_\nu U^{(-)}\right)\left(\partial_\mu U^{(-)\dagger}\right) - \left(\partial_\mu U^{(-)}\right)\left(\partial_\nu U^{(-)\dagger}\right)\right).
\end{aligned} \tag{3.32}$$

And let us calculate:

$$\begin{aligned}
&W_\mu W_\nu - W_\nu W_\mu = \\
= \ &\left(-\mathrm{i}\frac{2}{g_2}\left(\partial_\mu U^{(-)}\right) U^{(-)\dagger}\right)\left(-\mathrm{i}\frac{2}{g_2}\left(\partial_\nu U^{(-)}\right) U^{(-)\dagger}\right) - \\
&-\left(-\mathrm{i}\frac{2}{g_2}\left(\partial_\nu U^{(-)}\right) U^{(-)\dagger}\right)\left(-\mathrm{i}\frac{2}{g_2}\left(\partial_\mu U^{(-)}\right) U^{(-)\dagger}\right) \\
= \ &-\frac{4}{g_2^2}\left(\begin{array}{c} \left(\partial_\mu U^{(-)}\right) U^{(-)\dagger}\left(\partial_\nu U^{(-)}\right) U^{(-)\dagger} \\ -\left(\partial_\nu U^{(-)}\right) U^{(-)\dagger}\left(\partial_\mu U^{(-)}\right) U^{(-)\dagger} \end{array}\right).
\end{aligned}$$

Since

$$U^{(-)} U^{(-)\dagger} = 1_8$$

then

$$\partial_\mu \left(U^{(-)} U^{(-)\dagger}\right) = 0, \text{ and } \partial_\nu \left(U^{(-)} U^{(-)\dagger}\right) = 0,$$

Hence,

$$\left(\partial_\mu U^{(-)}\right)U^{(-)\dagger}+U^{(-)}\partial_\mu U^{(-)\dagger}=0,\text{ and }\left(\partial_\nu U^{(-)}\right)U^{(-)\dagger}+U^{(-)}\partial_\nu U^{(-)\dagger}=0$$

Hence,

$$\left(\partial_\mu U^{(-)}\right)U^{(-)\dagger}=-U^{(-)}\partial_\mu U^{(-)\dagger}\text{ and }\left(\partial_\nu U^{(-)}\right)U^{(-)\dagger}=-U^{(-)}\partial_\nu U^{(-)\dagger}.$$

Therefore,

$$\begin{aligned} & W_\mu W_\nu - W_\nu W_\mu = \\ = & -\frac{4}{g_2^2}\begin{pmatrix} -\left(\partial_\mu U^{(-)}\right)U^{(-)\dagger}U^{(-)}\partial_\nu U^{(-)\dagger}+ \\ +\left(\partial_\nu U^{(-)}\right)U^{(-)\dagger}U^{(-)}\partial_\mu U^{(-)\dagger} \end{pmatrix} \\ = & -\frac{4}{g_2^2}\left(-\left(\partial_\mu U^{(-)}\right)\left(\partial_\nu U^{(-)\dagger}\right)+\left(\partial_\nu U^{(-)}\right)\left(\partial_\mu U^{(-)\dagger}\right)\right) \end{aligned}$$

since

$$U^{(-)\dagger}U^{(-)}=1_8.$$

Therefore, in accordance with (3.32):

$$\boxed{\partial_\mu W_\nu-\partial_\nu W_\mu=\mathrm{i}\tfrac{g_2}{2}\left(W_\mu W_\nu-W_\nu W_\mu\right).} \tag{3.33}$$

In accordance with (3.27) matrix $W_\mu W_\nu - W_\nu W_\mu$ has the following columns:
the first and the second columns are the following:

$$\begin{array}{cc} 2\mathrm{i}W_{1,\mu}W_{2,\nu}-2\mathrm{i}W_{2,\mu}W_{1,\nu} & 0_2 \\ 0_2 & 0_2 \\ 2W_{0,\nu}W_{1,\mu}+2\mathrm{i}W_{0,\nu}W_{2,\mu}-2W_{0,\mu}W_{1,\nu}-2\mathrm{i}W_{0,\mu}W_{2,\nu} & 0_2 \\ 0_2 & 0_2 \end{array},$$

the third and the fourth columns are the following:

$$\begin{array}{cc} 2W_{0,\mu}W_{1,\nu}-2\mathrm{i}W_{0,\mu}W_{2,\nu}-2W_{0,\nu}W_{1,\mu}+2\mathrm{i}W_{0,\nu}W_{2,\mu} & 0_2 \\ 0_2 & 0_2 \\ -2\mathrm{i}W_{1,\mu}W_{2,\nu}+2\mathrm{i}W_{2,\mu}W_{1,\nu} & 0_2 \\ 0_2 & 0_2 \end{array}.$$

And matrix $\partial_\mu W_\nu - \partial_\nu W_\mu$ has the following columns:
the first and the second ones are the following:

$$\begin{array}{cc} \partial_\mu W_{0,\nu}-\partial_\nu W_{0,\mu} & 0_2 \\ 0_2 & 0_2 \\ \partial_\mu W_{1,\nu}+\mathrm{i}\partial_\mu W_{2,\nu}-\partial_\nu W_{1,\mu}-\mathrm{i}\partial_\nu W_{2,\mu} & 0_2 \\ 0_2 & 0_2 \end{array},$$

the third and the fourth columns are the following:

$$\begin{matrix} \partial_\mu W_{1,\nu} - \mathrm{i}\partial_\mu W_{2,\nu} - \partial_\nu W_{1,\mu} + \mathrm{i}\partial_\nu W_{2,\mu} & 0_2 \\ 0_2 & 0_2 \\ -\partial_\mu W_{0,\nu} + \partial_\nu W_{0,\mu} & 0_2 \\ 0_2 & 0_2 \end{matrix}.$$

Therefore, in accordance with (3.33):

$$\begin{aligned} & \mathrm{i}\frac{g_2}{2}\left(2\mathrm{i}W_{1,\mu}W_{2,\nu} - 2\mathrm{i}W_{2,\mu}W_{1,\nu}\right) \\ = \; & \partial_\mu W_{0,\nu} - \partial_\nu W_{0,\mu}, \\ & \mathrm{i}\frac{g_2}{2}\left(2W_{0,\nu}W_{1,\mu} + 2\mathrm{i}W_{0,\nu}W_{2,\mu} - 2W_{0,\mu}W_{1,\nu} - 2\mathrm{i}W_{0,\mu}W_{2,\nu}\right) \\ = \; & \partial_\mu W_{1,\nu} + \mathrm{i}\partial_\mu W_{2,\nu} - \partial_\nu W_{1,\mu} - \mathrm{i}\partial_\nu W_{2,\mu}, \\ & \mathrm{i}\frac{g_2}{2}\left(2W_{0,\mu}W_{1,\nu} - 2\mathrm{i}W_{0,\mu}W_{2,\nu} - 2W_{0,\nu}W_{1,\mu} + 2\mathrm{i}W_{0,\nu}W_{2,\mu}\right) \\ = \; & \partial_\mu W_{1,\nu} - \mathrm{i}\partial_\mu W_{2,\nu} - \partial_\nu W_{1,\mu} + \mathrm{i}\partial_\nu W_{2,\mu}, \\ & \mathrm{i}\frac{g_2}{2}\left(-2\mathrm{i}W_{1,\mu}W_{2,\nu} + 2\mathrm{i}W_{2,\mu}W_{1,\nu}\right) \\ = \; & -\partial_\mu W_{0,\nu} + \partial_\nu W_{0,\mu}. \end{aligned}$$

Hence,

$$\partial_\nu W_{0,\mu} = \partial_\mu W_{0,\nu} - g_2\left(W_{1,\mu}W_{2,\nu} - W_{1,\nu}W_{2,\mu}\right), \tag{3.34}$$

$$\partial_\nu W_{1,\mu} = \partial_\mu W_{1,\nu} - g_2\left(W_{2,\mu}W_{0,\nu} - W_{2,\nu}W_{0,\mu}\right), \tag{3.35}$$

$$\partial_\nu W_{2,\mu} = \partial_\mu W_{2,\nu} - g_2\left(W_{0,\mu}W_{1,\nu} - W_{0,\nu}W_{1,\mu}\right). \tag{3.36}$$

The derivative of (3.34) with respect to x_ν is of the following form:

$$\begin{gathered} \partial_\nu\partial_\nu W_{0,\mu} = \partial_\mu\partial_\nu W_{0,\nu} - \\ -g_2\begin{pmatrix} \left(\partial_\nu W_{1,\mu}\right)W_{2,\nu} + W_{1,\mu}\left(\partial_\nu W_{2,\nu}\right) \\ -\left(\partial_\nu W_{1,\nu}\right)W_{2,\mu} - W_{1,\nu}\left(\partial_\nu W_{2,\mu}\right) \end{pmatrix}. \end{gathered}$$

Let us substitute $\partial_\nu W_{1,\mu}$ and $\partial_\nu W_{2,\mu}$ for its expressions from (3.35) and (3.36):

$$\begin{gathered} \partial_\nu\partial_\nu W_{0,\mu} = \partial_\mu\partial_\nu W_{0,\nu} - \\ -g_2\begin{pmatrix} \left(\partial_\mu W_{1,\nu} - g_2\left(W_{2,\mu}W_{0,\nu} - W_{2,\nu}W_{0,\mu}\right)\right)W_{2,\nu} \\ +W_{1,\mu}\left(\partial_\nu W_{2,\nu}\right) - \left(\partial_\nu W_{1,\nu}\right)W_{2,\mu} \\ -W_{1,\nu}\left(\partial_\mu W_{2,\nu} - g_2\left(W_{0,\mu}W_{1,\nu} - W_{0,\nu}W_{1,\mu}\right)\right) \end{pmatrix} = \end{gathered}$$

$$\begin{gathered} = \partial_\mu\partial_\nu W_{0,\nu} \\ -g_2\begin{pmatrix} \left(\partial_\mu W_{1,\nu}\right)W_{2,\nu} - g_2\left(W_{2,\mu}W_{0,\nu}W_{2,\nu} - W_{2,\nu}W_{0,\mu}W_{2,\nu}\right) \\ +W_{1,\mu}\left(\partial_\nu W_{2,\nu}\right) - \left(\partial_\nu W_{1,\nu}\right)W_{2,\mu} \\ -W_{1,\nu}\partial_\mu W_{2,\nu} + g_2\left(W_{1,\nu}W_{0,\mu}W_{1,\nu} - W_{1,\nu}W_{0,\nu}W_{1,\mu}\right) \end{pmatrix} = \end{gathered}$$

$$\begin{gathered}= -g_2^2\left(W_{1,\nu}W_{1,\nu}+W_{2,\nu}W_{2,\nu}\right)W_{0,\mu}+ \\ +g_2^2\left(W_{1,\nu}W_{1,\mu}+W_{2,\mu}W_{2,\nu}\right)W_{0,\nu} \\ -g_2\left(\begin{array}{c}\left(\partial_\mu W_{1,\nu}\right)W_{2,\nu}-W_{1,\nu}\partial_\mu W_{2,\nu} \\ +W_{1,\mu}\left(\partial_\nu W_{2,\nu}\right)-\left(\partial_\nu W_{1,\nu}\right)W_{2,\mu}\end{array}\right)+ \\ +\partial_\mu\partial_\nu W_{0,\nu}\end{gathered}$$

Hence,

$$\begin{gathered}\partial_\nu\partial_\nu W_{0,\mu}= \\ = -g_2^2\left(W_{1,\nu}W_{1,\nu}+W_{2,\nu}W_{2,\nu}\right)W_{0,\mu}+ \\ +g_2^2\left(W_{1,\nu}W_{1,\mu}+W_{2,\mu}W_{2,\nu}\right)W_{0,\nu} \\ -g_2\left(\begin{array}{c}\left(\partial_\mu W_{1,\nu}\right)W_{2,\nu}-W_{1,\nu}\partial_\mu W_{2,\nu} \\ +W_{1,\mu}\left(\partial_\nu W_{2,\nu}\right)-\left(\partial_\nu W_{1,\nu}\right)W_{2,\mu}\end{array}\right)+ \\ +\partial_\mu\partial_\nu W_{0,\nu}.\end{gathered}$$

Therefore:

$$\begin{gathered}\partial_\nu\partial_\nu W_{0,\mu}= \\ = -g_2^2\left(W_{0,\nu}W_{0,\nu}+W_{1,\nu}W_{1,\nu}+W_{2,\nu}W_{2,\nu}\right)W_{0,\mu}+ \\ +g_2^2W_{0,\nu}W_{0,\nu}W_{0,\mu} \\ +g_2^2\left(W_{1,\nu}W_{1,\mu}+W_{2,\mu}W_{2,\nu}\right)W_{0,\nu} \\ -g_2\left(\begin{array}{c}\left(\partial_\mu W_{1,\nu}\right)W_{2,\nu}-W_{1,\nu}\partial_\mu W_{2,\nu} \\ +W_{1,\mu}\left(\partial_\nu W_{2,\nu}\right)-\left(\partial_\nu W_{1,\nu}\right)W_{2,\mu}\end{array}\right)+ \\ +\partial_\mu\partial_\nu W_{0,\nu}.\end{gathered}$$

Thus,

$$\begin{gathered}\partial_\nu\partial_\nu W_{0,\mu}= \\ = -g_2^2\left(W_{0,\nu}W_{0,\nu}+W_{1,\nu}W_{1,\nu}+W_{2,\nu}W_{2,\nu}\right)W_{0,\mu}+ \\ +g_2^2\left(W_{0,\nu}W_{0,\mu}+W_{1,\nu}W_{1,\mu}+W_{2,\mu}W_{2,\nu}\right)W_{0,\nu} \\ -g_2\left(\begin{array}{c}\left(\partial_\mu W_{1,\nu}\right)W_{2,\nu}-W_{1,\nu}\partial_\mu W_{2,\nu} \\ +W_{1,\mu}\left(\partial_\nu W_{2,\nu}\right)-\left(\partial_\nu W_{1,\nu}\right)W_{2,\mu}\end{array}\right)+ \\ +\partial_\mu\partial_\nu W_{0,\nu}.\end{gathered} \tag{3.37}$$

Since

$$\widetilde{W}_\nu^2 := W_{0,\nu}W_{0,\nu}+W_{1,\nu}W_{1,\nu}+W_{2,\nu}W_{2,\nu}$$

and

$$\left\langle \widetilde{W}_\nu|\widetilde{W}_\mu\right\rangle := W_{0,\nu}W_{0,\mu}+W_{1,\nu}W_{1,\mu}+W_{2,\mu}W_{2,\nu} = \left\langle \widetilde{W}_\nu|\widetilde{W}_\mu\right\rangle$$

for

$$\widetilde{W}_\mu = \begin{bmatrix} W_{0,\mu} \\ W_{1,\mu} \\ W_{2,\mu}\end{bmatrix} \text{ and } \widetilde{W}_\nu = \begin{bmatrix} W_{0,\nu} \\ W_{1,\nu} \\ W_{2,\nu}\end{bmatrix}$$

then

$$\begin{array}{c}\partial_\nu\partial_\nu W_{0,\mu}=-\left(g_2\widetilde{W}_\nu\right)^2W_{0,\mu}+\\ +g_2^2\left\langle\widetilde{W}_\nu|\widetilde{W}_\mu\right\rangle W_{0,\nu}\\ -g_2\left(\begin{array}{c}\left(\partial_\mu W_{1,\nu}\right)W_{2,\nu}-W_{1,\nu}\partial_\mu W_{2,\nu}\\ +W_{1,\mu}\left(\partial_\nu W_{2,\nu}\right)-\left(\partial_\nu W_{1,\nu}\right)W_{2,\mu}\end{array}\right)+\\ +\partial_\mu\partial_\nu W_{0,\nu}.\end{array}$$

Hence,

$$\begin{array}{c}\partial_0\partial_0 W_{0,\mu}=-\left(g_2\widetilde{W}_0\right)^2W_{0,\mu}+\\ +g_2^2\left\langle W_0|W_\mu\right\rangle W_{0,0}\\ -g_2\left(\begin{array}{c}\left(\partial_\mu W_{1,0}\right)W_{2,0}-W_{1,0}\partial_\mu W_{2,0}\\ +W_{1,\mu}\left(\partial_0 W_{2,0}\right)-\left(\partial_0 W_{1,0}\right)W_{2,\mu}\end{array}\right)+\\ +\partial_\mu\partial_0 W_{0,0}.\end{array}$$

Since $\partial_0=\frac{1}{\mathrm{c}}\partial_t$ then

$$\begin{array}{c}\frac{1}{\mathrm{c}^2}\partial_t^2 W_{0,\mu}=-\left(g_2\widetilde{W}_0\right)^2W_{0,\mu}+\\ +g_2^2\left\langle\widetilde{W}_0|\widetilde{W}_\mu\right\rangle W_{0,0}\\ -g_2\left(\begin{array}{c}\left(\partial_\mu W_{1,0}\right)W_{2,0}-W_{1,0}\partial_\mu W_{2,0}\\ +W_{1,\mu}\left(\partial_0 W_{2,0}\right)-\left(\partial_0 W_{1,0}\right)W_{2,\mu}\end{array}\right)+\\ +\partial_\mu\partial_0 W_{0,0}.\end{array}$$

And for $s\in\{1,2,3\}$:

$$\begin{array}{rcl}\partial_s\partial_s W_{0,\mu} & = & -\left(g_2\widetilde{W}_s\right)^2W_{0,\mu}\\ & & +g_2^2\left\langle\widetilde{W}_s|\widetilde{W}_\mu\right\rangle W_{0,s}\\ & & -g_2\left(\begin{array}{c}\left(\partial_\mu W_{1,s}\right)W_{2,s}-W_{1,s}\partial_\mu W_{2,s}\\ +W_{1,\mu}\left(\partial_s W_{2,s}\right)-\left(\partial_s W_{1,s}\right)W_{2,\mu}\end{array}\right)\\ & & +\partial_\mu\partial_s W_{0,s}.\end{array}$$

Therefore,

$$\begin{array}{c}-\frac{1}{\mathrm{c}^2}\partial_t^2 W_{0,\mu}+\sum_{s=1}^3\partial_s^2 W_{0,\mu}=\\ -\left(\begin{array}{c}-\left(g_2\widetilde{W}_0\right)^2W_{0,\mu}+g_2^2\left\langle\widetilde{W}_0|\widetilde{W}_\mu\right\rangle W_{0,0}\\ -g_2\left(\begin{array}{c}\left(\partial_\mu W_{1,0}\right)W_{2,0}-W_{1,0}\partial_\mu W_{2,0}\\ +W_{1,\mu}\left(\partial_0 W_{2,0}\right)-\left(\partial_0 W_{1,0}\right)W_{2,\mu}\end{array}\right)\\ +\partial_\mu\partial_0 W_{0,0}\end{array}\right)+\\ +\left(\begin{array}{c}\sum_{s=1}^3-\left(g_2\widetilde{W}_s\right)^2W_{0,\mu}\\ +g_2^2\left\langle\widetilde{W}_s|\widetilde{W}_\mu\right\rangle W_{0,s}\\ -g_2\left(\begin{array}{c}\left(\partial_\mu W_{1,s}\right)W_{2,s}-W_{1,s}\partial_\mu W_{2,s}\\ +W_{1,\mu}\left(\partial_s W_{2,s}\right)-\left(\partial_s W_{1,s}\right)W_{2,\mu}\end{array}\right)\\ +\partial_\mu\partial_s W_{0,s}\end{array}\right)..\end{array}$$

Hence,

$$-\frac{1}{c^2}\partial_t^2 W_{0,\mu} + \sum_{s=1}^3 \partial_s^2 W_{0,\mu} =$$
$$\left(g_2\widetilde{W}_0\right)^2 W_{0,\mu} - \sum_{s=1}^3 \left(g_2\widetilde{W}_s\right)^2 W_{0,\mu}$$
$$+g_2^2 \sum_{s=1}^3 \left\langle \widetilde{W}_s|\widetilde{W}_\mu \right\rangle W_{0,s} - g_2^2 \left\langle \widetilde{W}_0|\widetilde{W}_\mu \right\rangle W_{0,0}$$
$$+g_2 \left(\begin{array}{c} \left(\begin{array}{c} (\partial_\mu W_{1,0}) W_{2,0} - W_{1,0}\partial_\mu W_{2,0} \\ +W_{1,\mu}(\partial_0 W_{2,0}) - (\partial_0 W_{1,0}) W_{2,\mu} \end{array} \right) \\ -\sum_{s=1}^3 \left(\begin{array}{c} (\partial_\mu W_{1,s}) W_{2,s} - W_{1,s}\partial_\mu W_{2,s} \\ +W_{1,\mu}(\partial_s W_{2,s}) - (\partial_s W_{1,s}) W_{2,\mu} \end{array} \right) \end{array} \right)$$
$$+\partial_\mu \sum_{s=1}^3 \partial_s W_{0,s} - \partial_\mu \partial_0 W_{0,0} - .$$

Hence,

$$\boxed{\begin{array}{c} \left(-\frac{1}{c^2}\partial_t^2 + \sum_{s=1}^3 \partial_s^2\right) W_{0,\mu} = g_2^2 \left(\widetilde{W}_0^2 - \sum_{s=1}^3 \widetilde{W}_s^2\right) W_{0,\mu} + \\ +g_2^2 \left(\sum_{s=1}^3 \left\langle \widetilde{W}_s|\widetilde{W}_\mu \right\rangle W_{0,s} - \left\langle \widetilde{W}_0|\widetilde{W}_\mu \right\rangle W_{0,0}\right) \\ +g_2 \left(\begin{array}{c} \left(\begin{array}{c} (\partial_\mu W_{1,0}) W_{2,0} - W_{1,0}\partial_\mu W_{2,0} \\ +W_{1,\mu}(\partial_0 W_{2,0}) - (\partial_0 W_{1,0}) W_{2,\mu} \end{array} \right) \\ -\sum_{s=1}^3 \left(\begin{array}{c} (\partial_\mu W_{1,s}) W_{2,s} - W_{1,s}\partial_\mu W_{2,s} \\ +W_{1,\mu}(\partial_s W_{2,s}) - (\partial_s W_{1,s}) W_{2,\mu} \end{array} \right) \end{array} \right) \\ +\partial_\mu \sum_{s=1}^3 \partial_s W_{0,s} - \partial_\mu \partial_0 W_{0,0}. \end{array}} \tag{3.38}$$

This equation looks like to the Klein-Gordon equation[4] of field $W_{0,\mu}$ with mass

$$m = \frac{\mathrm{h}}{\mathrm{c}} g_2 \sqrt{\widetilde{W}_0^2 - \sum_{s=1}^3 \widetilde{W}_s^2} \tag{3.39}$$

and with additional terms of the $W_{0,\mu}$ interactions with others components of $\widetilde{W}$. You can receive similar equations for $W_{1,\mu}$ and for $W_{2,\mu}$.

If

$$\widetilde{W}_0' := \frac{\widetilde{W}_0 - \frac{v}{\mathrm{c}}\widetilde{W}_k}{\sqrt{1-\left(\frac{v}{\mathrm{c}}\right)^2}}, \widetilde{W}_k' := \frac{\widetilde{W}_k - \frac{v}{\mathrm{c}}\widetilde{W}_0}{\sqrt{1-\left(\frac{v}{\mathrm{c}}\right)^2}}, \widetilde{W}_k' := \widetilde{W}_k, \text{ if } s \neq k$$

then

$$\widetilde{W}_0'^2 - \sum_{s=1}^3 \widetilde{W}_s'^2$$
$$= \frac{\left(\widetilde{W}_0 - \frac{v}{\mathrm{c}}\widetilde{W}_k\right)^2}{1-\left(\frac{v}{\mathrm{c}}\right)^2} - \frac{\left(\widetilde{W}_k - \frac{v}{\mathrm{c}}\widetilde{W}_0\right)^2}{1-\left(\frac{v}{\mathrm{c}}\right)^2} - \sum_{s\neq k} \widetilde{W}_s'^2$$

[4] (2.57)

$$\left(-\frac{1}{\mathrm{c}^2}\partial_t^2 + \sum_{s=1}^3 \partial_s^2\right)\varphi = \frac{m^2\mathrm{c}^2}{\mathrm{h}^2}\varphi$$

$$= \frac{\widetilde{W}_0^2 + \left(\frac{v}{c}\right)^2 \widetilde{W}_k^2 - \widetilde{W}_k^2 - \left(\frac{v}{c}\right)^2 \widetilde{W}_0^2}{1 - \left(\frac{v}{c}\right)^2} - \sum_{s \neq k} \widetilde{W}_s'^2$$
$$= \frac{\left(1 - \left(\frac{v}{c}\right)^2\right) \widetilde{W}_0^2 - \left(1 - \left(\frac{v}{c}\right)^2\right) \widetilde{W}_k^2}{1 - \left(\frac{v}{c}\right)^2} - \sum_{s \neq k} \widetilde{W}_s'^2.$$

Hence,

$$\widetilde{W}_0'^2 - \sum_{s=1}^{3} \widetilde{W}_s'^2 = \widetilde{W}_0^2 - \sum_{s=1}^{3} \widetilde{W}_s^2.$$

Therefore, such ”mass” (3.39) is invariant for the Lorentz transformations:

You can calculate that it is invariant for the transformations of turns, too:

$$\left\{ \begin{array}{l} \widetilde{W}_r' = \widetilde{W}_r \cos\lambda - \widetilde{W}_s \sin\lambda. \\ \widetilde{W}_s' = \widetilde{W}_r \sin\lambda + \widetilde{W}_s \cos\lambda; \end{array} \right|$$

with a real number λ, and $r \in \{1,2,3\}$, $s \in \{1,2,3\}$; and it is invariant for a global weak isospin transformation U':

$$W_\nu' \to W_\nu'' = U' W_\nu U'^\dagger$$

but is not invariant for a local transformation (3.31). But local transformations for $W_{0,\mu}$, $W_{1,\mu}$ and $W_{2,\mu}$ is insignificant since all three particles are very short-lived with a mean life of about 3×10^{-25} seconds.

That is in (3.38) the form

$$m = \frac{\mathrm{h}}{\mathrm{c}} g_2 \sqrt{\widetilde{W}_0^2 - \sum_{s=1}^{3} \widetilde{W}_s^2}$$

varies in space, but locally acts like a mass - i.e. it does not allow to particles of this field to behave as a massless ones.

A mass of the W-boson was measured, for instant (Figure 29), between 1996 and 2000 at LEP[5] [26].

Let[6]

$$\begin{array}{c} \alpha := \arctan \frac{g_1}{g_2}, \\ Z_\mu := (W_{0,\mu} \cos\alpha - B_\mu \sin\alpha), \\ A_\mu := (B_\mu \cos\alpha + W_{0,\mu} \sin\alpha). \end{array} \tag{3.40}$$

In that case:

$$\sum_\nu g_{\nu,\nu} \partial_\nu \partial_\nu W_{0,\mu} = \cos\alpha \cdot \sum_\nu g_{\nu,\nu} \partial_\nu \partial_\nu Z_\mu + \sin\alpha \cdot \sum_\nu g_{\nu,\nu} \partial_\nu \partial_\nu A_\mu.$$

If

[5]The Large Electron-Positron Collider (LEP) (Figure 28) is largest particles accelerator (ring with a circumference of 27 kilometers built in a tunnel under the border of Switzerland and France.)

[6]here α is the Weinberg Angle. The experimental value of $\sin^2\alpha = 0.23124 \pm 0.00024$ [27].

$$\sum_{\nu} g_{\nu,\nu}\partial_{\nu}\partial_{\nu}A_{\mu} = 0$$

then

$$m_Z = \frac{m_W}{\cos\alpha}$$

with m_W from (3.39). It is like Standard Model.

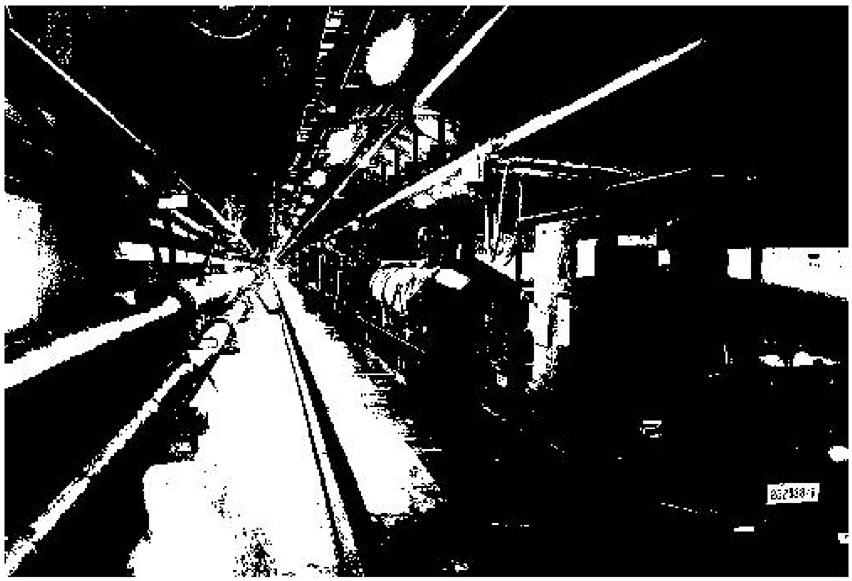

Figure 28:

The equation of moving (3.29) under $F_{\mu} = 0$ has the following form:

$$\left(\begin{array}{c} \sum_{\mu=0}^{3} \beta^{[\mu]}\mathrm{i}\left(\partial_{\mu} - \mathrm{i}0.5g_1B_{\mu}Y - \mathrm{i}\frac{1}{2}g_2W_{\mu}\right) \\ +\gamma^{[0]}\mathrm{i}\partial_5 + \beta^{[4]}\mathrm{i}\partial_4 \end{array} \right)\widetilde{\varphi} = 0. \quad (3.41)$$

Hence, in accordance with (3.27) and (2.36):

$$\begin{array}{c} \sum_{\mu=0}^{3} \beta^{[\mu]}\mathrm{i}\times \\ \times \left(\left(\begin{array}{c} \partial_{\mu} - \mathrm{i}0.5g_1B_{\mu}\left(-\left[\begin{array}{cc} 1_2 & 0_2 \\ 0_2 & 2\cdot 1_2 \end{array}\right]\right) - \\ -\mathrm{i}\frac{1}{2}g_2 \left[\begin{array}{cccc} W_{0,\mu}1_2 & 0_2 & (W_{1,\mu} - \mathrm{i}W_{2,\mu})\,1_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \\ (W_{1,\mu} + \mathrm{i}W_{2,\mu})\,1_2 & 0_2 & -W_{0,\mu}1_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \end{array}\right] \end{array} \right) \\ +\gamma^{[0]}\mathrm{i}\partial_5 + \beta^{[4]}\mathrm{i}\partial_4 \right) \\ \cdot\widetilde{\varphi} = 0. \end{array}$$

In accordance with (3.40) [29]:

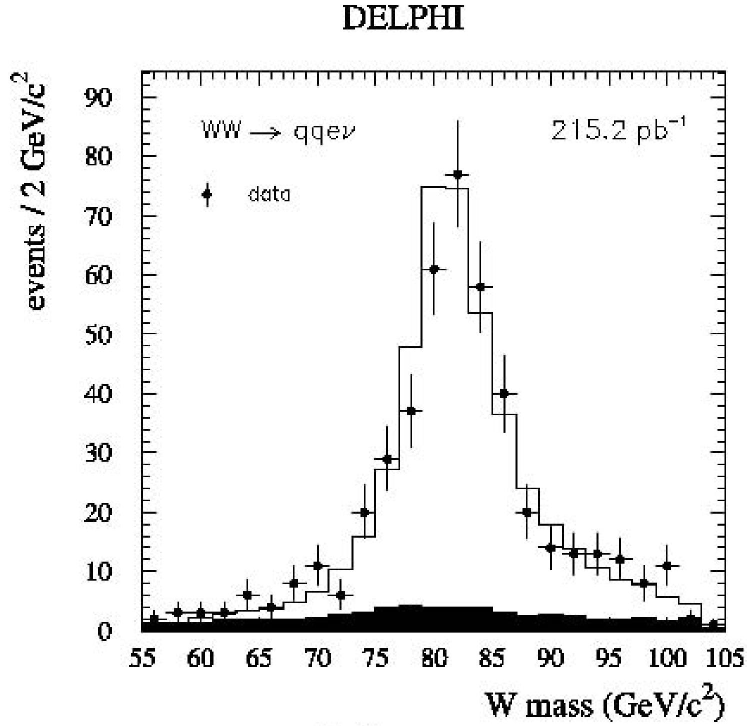

Figure 29:

$$\begin{aligned} B_\mu &= \left(-Z_\mu \frac{g_1}{\sqrt{g_1^2+g_2^2}} + A_\mu \frac{g_2}{\sqrt{g_1^2+g_2^2}} \right), \\ W_{0,\mu} &= \left(Z_\mu \frac{g_2}{\sqrt{g_1^2+g_2^2}} + A_\mu \frac{g_1}{\sqrt{g_1^2+g_2^2}} \right). \end{aligned}$$

Let (e is the *elementary charge*[7]: $\mathrm{e} = 1.60217733 \times 10^{-19}$ C).

$$e := \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}},$$

and let

$$\widehat{Z}_\mu := Z_\mu \frac{1}{\sqrt{g_2^2+g_1^2}} \begin{bmatrix} \left(g_2^2+g_1^2\right) 1_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 2g_1^2 1_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & \left(g_2^2-g_1^2\right) 1_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 2g_1^2 1_2 \end{bmatrix},$$

[7]Sir Joseph John "J. J." Thomson, (18 December 1856 - 30 August 1940) was a British physicist. He is credited for the discovery of the electron and of isotopes, and the invention of the mass spectrometer.

$$\widehat{W}_\mu := g_2 \begin{bmatrix} 0_2 & 0_2 & (W_{1,\mu} - \mathrm{i}W_{2,\mu})\,1_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \\ (W_{1,\mu} + \mathrm{i}W_{2,\mu})\,1_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \cdot 1_2 \end{bmatrix},$$

$$\widehat{A}_\mu := A_\mu \begin{bmatrix} 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 1_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 1_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 1_2 \end{bmatrix}.$$

In that case from (3.41):

$$\boxed{\left(\sum_{\mu=0}^{3} \beta^{[\mu]}\mathrm{i}\left(\partial_\mu + \mathrm{i}e\widehat{A}_\mu - \mathrm{i}0.5\left(\widehat{Z}_\mu + \widehat{W}_\mu\right)\right) + \gamma^{[0]}\mathrm{i}\partial_5 + \beta^{[4]}\mathrm{i}\partial_4\right)\widetilde{\varphi} = 0.} \tag{3.42}$$

Let in basis (3.11) (3.23) :

$$\widetilde{\varphi} = \begin{bmatrix} \varphi_\nu \\ \overrightarrow{0}_2 \\ \varphi_{e,L} \\ \varphi_{e,R} \end{bmatrix}.$$

In that case

$$\boxed{\left(\begin{array}{c} \sum_{\mu=0}^{3} \beta^{[\mu]}\mathrm{i}\left(\partial_\mu \widetilde{\varphi} + \mathrm{i}A_\mu e \begin{bmatrix} \varphi_{e,L} \\ \varphi_{e,R} \end{bmatrix} - \mathrm{i}0.5\left(\widehat{Z}_\mu + \widehat{W}_\mu\right)\widetilde{\varphi}\right) \\ + \left(\gamma^{[0]}\mathrm{i}\partial_5 + \beta^{[4]}\mathrm{i}\partial_4\right)\widetilde{\varphi} \end{array} \right) = 0.} \tag{3.43}$$

Here the vector field A_μ is *the electromagnetic potential* [8]. And $\left(\widehat{Z}_\mu + \widehat{W}_\mu\right)$ is *the weak interaction potential* Evidently neutrinos do not involve in the electromagnetic interactions.

3.2. Quarks and Gluons

The quark model was independently proposed by physicists Murray Gell-Mann[9] and George Zweig[10] in 1964.

[8]James Clerk Maxwell of Glenlair (13 June 1831 - 5 November 1879) was a Scottish physicist and mathematician. His most prominent achievement was formulating classical electromagnetic theory.

[9]Murray Gell-Mann (born September 15, 1929) is an American physicist and linguist

[10]George Zweig (born on May 30, 1937 in Moscow, Russia into a Jewish family) was originally trained as a particle physicist under Richard Feynman and later turned his attention to neurobiology. He spent a number of years as a Research Scientist at Los Alamos National Laboratory and MIT, but as of 2004, has gone on to work in the financial services industry.

The first direct experimental evidence of gluons was found in 1979 when three-jet events were observed at t he electron-positron collider PETRA. However, just before PETRA[11] appeared on the scene, the PLUTO experiment at DORIS[12] showed event topologies suggestive of a three-gluon decay.

The following part of (2.30):

$$\left(\begin{array}{c}\sum\limits_{k=0}^{3}\beta^{[k]}\left(-\mathrm{i}\partial_k+\Theta_k+\Upsilon_k\gamma^{[5]}\right)-\\ -M_{\zeta,0}\gamma_{\zeta}^{[0]}+M_{\zeta,4}\zeta^{[4]}+\\ -M_{\eta,0}\gamma_{\eta}^{[0]}-M_{\eta,4}\eta^{[4]}+\\ +M_{\theta,0}\gamma_{\theta}^{[0]}+M_{\theta,4}\theta^{[4]}\end{array}\right)\varphi=0. \tag{3.44}$$

is called *the chromatic equation of moving*.

Here (2.6), (2.8), (2.10):

$$\gamma_{\zeta}^{[0]}=-\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},\ \zeta^{[4]}=\begin{bmatrix}0&0&0&\mathrm{i}\\0&0&\mathrm{i}&0\\0&-\mathrm{i}&0&0\\-\mathrm{i}&0&0&0\end{bmatrix}$$

are mass elements of red pentad;

$$\gamma_{\eta}^{[0]}=\begin{bmatrix}0&0&0&\mathrm{i}\\0&0&-\mathrm{i}&0\\0&\mathrm{i}&0&0\\-\mathrm{i}&0&0&0\end{bmatrix},\ \eta^{[4]}=\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}$$

are mass elements of green pentad;

$$\gamma_{\theta}^{[0]}=\begin{bmatrix}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{bmatrix},\ \theta^{[4]}=\begin{bmatrix}0&0&-\mathrm{i}&0\\0&0&0&\mathrm{i}\\-\mathrm{i}&0&0&0\\0&\mathrm{i}&0&0\end{bmatrix}$$

are mass elements of blue pentad.

I call:

- $M_{\zeta,0}$, $M_{\zeta,4}$ *red lower and upper mass members*;
- $M_{\eta,0}$, $M_{\eta,4}$ *green lower and upper mass members*;
- $M_{\theta,0}$, $M_{\theta,4}$ *blue lower and upper mass members*.

[11]PETRA (or the Positron-Electron Tandem Ring Accelerator) is one of the particle accelerators at DESY in Hamburg, Germany.

[12]DORIS (Doppel-Ring-Speicher, "double-ring storage"), built between 1969 and 1974, was DESY's second circular accelerator and its first storage ring with a circumference of nearly 300 m.

The mass members of this equation form the following matrix sum:

$$\widehat{M} := \begin{pmatrix} -M_{\zeta,0}\gamma_{\zeta}^{[0]} + M_{\zeta,4}\zeta^{[4]} - \\ -M_{\eta,0}\gamma_{\eta}^{[0]} - M_{\eta,4}\eta^{[4]} + \\ +M_{\theta,0}\gamma_{\theta}^{[0]} + M_{\theta,4}\theta^{[4]} \end{pmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & -M_{\theta,0} & M_{\zeta,\eta,0} \\ 0 & 0 & M^{*}_{\zeta,\eta,0} & M_{\theta,0} \\ -M_{\theta,0} & M_{\zeta,\eta,0} & 0 & 0 \\ M^{*}_{\zeta,\eta,0} & M_{\theta,0} & 0 & 0 \end{bmatrix} + \mathrm{i} \begin{bmatrix} 0 & 0 & M_{\theta,4} & M^{*}_{\zeta,\eta,4} \\ 0 & 0 & M_{\zeta,\eta,4} & -M_{\theta,4} \\ -M_{\theta,4} & -M^{*}_{\zeta,\eta,4} & 0 & 0 \\ -M_{\zeta,\eta,4} & M_{\theta,4} & 0 & 0 \end{bmatrix}$$

with $M_{\zeta,\eta,0} := M_{\zeta,0} - \mathrm{i}M_{\eta,0}$ and $M_{\zeta,\eta,4} := M_{\zeta,4} - \mathrm{i}M_{\eta,4}$.
Elements of these matrices can be turned by formula of shape [28]:

$$\begin{bmatrix} \cos\frac{\theta}{2} & \mathrm{i}\sin\frac{\theta}{2} \\ \mathrm{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} Z & X - \mathrm{i}Y \\ X + \mathrm{i}Y & -Z \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & -\mathrm{i}\sin\frac{\theta}{2} \\ -\mathrm{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} =$$
$$= \begin{bmatrix} Z\cos\theta - Y\sin\theta & X - \mathrm{i}\begin{pmatrix} Y\cos\theta \\ +Z\sin\theta \end{pmatrix} \\ X + \mathrm{i}\begin{pmatrix} Y\cos\theta \\ +Z\sin\theta \end{pmatrix} & -Z\cos\theta + Y\sin\theta \end{bmatrix}.$$

Hence, if:

$$U_{2,3}(\alpha) := \begin{bmatrix} \cos\alpha & \mathrm{i}\sin\alpha & 0 & 0 \\ \mathrm{i}\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \mathrm{i}\sin\alpha \\ 0 & 0 & \mathrm{i}\sin\alpha & \cos\alpha \end{bmatrix}$$

and

$$\widehat{M}' := \begin{pmatrix} -M'_{\zeta,0}\gamma_{\zeta}^{[0]} + M'_{\zeta,4}\zeta^{[4]} - \\ -M'_{\eta,0}\gamma_{\eta}^{[0]} - M'_{\eta,4}\eta^{[4]} + \\ +M'_{\theta,0}\gamma_{\theta}^{[0]} + M'_{\theta,4}\theta^{[4]} \end{pmatrix} := U^{\dagger}_{2,3}(\alpha)\widehat{M}U_{2,3}(\alpha)$$

then

$$\begin{aligned} M'_{\zeta,0} &= M_{\zeta,0}\,, \\ M'_{\eta,0} &= M_{\eta,0}\cos 2\alpha + M_{\theta,0}\sin 2\alpha\,, \\ M'_{\theta,0} &= M_{\theta,0}\cos 2\alpha - M_{\eta,0}\sin 2\alpha\,, \\ M'_{\zeta,4} &= M_{\zeta,4}\,, \\ M'_{\eta,4} &= M_{\eta,4}\cos 2\alpha + M_{\theta,4}\sin 2\alpha\,, \\ M'_{\theta,4} &= M_{\theta,4}\cos 2\alpha - M_{\eta,4}\sin 2\alpha\,. \end{aligned}$$

Therefore, matrix $U_{2,3}(\alpha)$ makes an oscillation between green and blue chromatics.

Let us consider equation (2.30) under transformation $U_{2,3}(\alpha)$ where α is an arbitrary real function of time-space variables ($\alpha = \alpha(t, x_1, x_2, x_3)$):

$$U_{2,3}^{\dagger}(\alpha)\left(\frac{1}{c}\partial_t + i\Theta_0 + i\Upsilon_0\gamma^{[5]}\right)U_{2,3}(\alpha)\varphi =$$
$$= U_{2,3}^{\dagger}(\alpha)\left(\begin{array}{c}\sum_{\nu=1}^{3}\beta^{[\nu]}\left(\partial_\nu + i\Theta_\nu + i\Upsilon_\nu\gamma^{[5]}\right) + \\ + iM_0\gamma^{[0]} + iM_4\beta^{[4]} + \widehat{M}\end{array}\right)U_{2,3}(\alpha)\varphi.$$

Because

$$U_{2,3}^{\dagger}(\alpha)U_{2,3}(\alpha) = 1_4,$$
$$U_{2,3}^{\dagger}(\alpha)\gamma^{[5]}U_{2,3}(\alpha) = \gamma^{[5]},$$
$$U_{2,3}^{\dagger}(\alpha)\gamma^{[0]}U_{2,3}(\alpha) = \gamma^{[0]},$$
$$U_{2,3}^{\dagger}(\alpha)\beta^{[4]}U_{2,3}(\alpha) = \beta^{[4]},$$
$$U_{2,3}^{\dagger}(\alpha)\beta^{[1]} = \beta^{[1]}U_{2,3}^{\dagger}(\alpha),$$
$$U_{2,3}^{\dagger}(\alpha)\beta^{[2]} = \left(\beta^{[2]}\cos 2\alpha + \beta^{[3]}\sin 2\alpha\right)U_{2,3}^{\dagger}(\alpha),$$
$$U_{2,3}^{\dagger}(\alpha)\beta^{[3]} = \left(\beta^{[3]}\cos 2\alpha - \beta^{[2]}\sin 2\alpha\right)U_{2,3}^{\dagger}(\alpha),$$

then

$$\left(\frac{1}{c}\partial_t + U_{2,3}^{\dagger}(\alpha)\frac{1}{c}\partial_t U_{2,3}(\alpha) + i\Theta_0 + i\Upsilon_0\gamma^{[5]}\right)\varphi =$$
$$= \left(\begin{array}{c}\beta^{[1]}\left(\partial_1 + U_{2,3}^{\dagger}(\alpha)\partial_1 U_{2,3}(\alpha) + i\Theta_1 + i\Upsilon_1\gamma^{[5]}\right) + \\ +\beta^{[2]}\left(\begin{array}{c}(\cos 2\alpha\cdot\partial_2 - \sin 2\alpha\cdot\partial_3) \\ +U_{2,3}^{\dagger}(\alpha)(\cos 2\alpha\cdot\partial_2 - \sin 2\alpha\cdot\partial_3)U_{2,3}(\alpha) \\ +i(\Theta_2\cos 2\alpha - \Theta_3\sin 2\alpha) \\ +i\left(\Upsilon_2\gamma^{[5]}\cos 2\alpha - \Upsilon_3\gamma^{[5]}\sin 2\alpha\right)\end{array}\right) \\ +\beta^{[3]}\left(\begin{array}{c}(\cos 2\alpha\cdot\partial_3 + \sin 2\alpha\cdot\partial_2) \\ +U_{2,3}^{\dagger}(\alpha)(\cos 2\alpha\cdot\partial_3 + \sin 2\alpha\cdot\partial_2)U_{2,3}(\alpha) \\ +i(\Theta_2\sin 2\alpha + \Theta_3\cos 2\alpha) \\ +i\left(\Upsilon_3\gamma^{[5]}\cos 2\alpha + \Upsilon_2\gamma^{[5]}\sin 2\alpha\right)\end{array}\right) \\ +iM_0\gamma^{[0]} + iM_4\beta^{[4]} + \widehat{M}'\end{array}\right)\varphi. \tag{3.45}$$

Let x_2' and x_3' be elements of other coordinate system such that:

$$\begin{aligned}\partial_2' &:= (\cos 2\alpha\cdot\partial_2 - \sin 2\alpha\cdot\partial_3), \\ \partial_3' &:= (\cos 2\alpha\cdot\partial_3 + \sin 2\alpha\cdot\partial_2).\end{aligned} \tag{3.46}$$

Therefore, from (3.45):

$$\left(\frac{1}{c}\partial_t + U_{2,3}^{\dagger}(\alpha)\frac{1}{c}\partial_t U_{2,3}(\alpha) + i\Theta_0 + i\Upsilon_0\gamma^{[5]}\right)\varphi =$$

$$= \begin{pmatrix} \beta^{[1]}\left(\partial_1 + U_{2,3}^{\dagger}(\alpha)\partial_1 U_{2,3}(\alpha) + \mathrm{i}\Theta_1 + \mathrm{i}\Upsilon_1\gamma^{[5]}\right) \\ +\beta^{[2]}\left(\partial_2' + U_{2,3}^{\dagger}(\alpha)\partial_2' U_{2,3}(\alpha) + \mathrm{i}\Theta_2' + \mathrm{i}\Upsilon_2'\gamma^{[5]}\right) \\ +\beta^{[3]}\left(\partial_3' + U_{2,3}^{\dagger}(\alpha)\partial_3' U_{2,3}(\alpha) + \mathrm{i}\Theta_3' + \mathrm{i}\Upsilon_3'\gamma^{[5]}\right) \\ +\mathrm{i}M_0\gamma^{[0]} + \mathrm{i}M_4\beta^{[4]} + \widehat{M}' \end{pmatrix}\varphi.$$

with

$$\Theta_2' := \Theta_2\cos 2\alpha - \Theta_3\sin 2\alpha,$$
$$\Theta_3' := \Theta_2\sin 2\alpha + \Theta_3\cos 2\alpha,$$
$$\Upsilon_2' := \Upsilon_2\cos 2\alpha - \Upsilon_3\sin 2\alpha,$$
$$\Upsilon_3' := \Upsilon_3\cos 2\alpha + \Upsilon_2\sin 2\alpha.$$

Therefore, the oscillation between blue and green chromatics curves the space in the x_2, x_3 directions.

Similarly, matrix

$$U_{1,3}(\vartheta) := \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 & 0 \\ -\sin\vartheta & \cos\vartheta & 0 & 0 \\ 0 & 0 & \cos\vartheta & \sin\vartheta \\ 0 & 0 & -\sin\vartheta & \cos\vartheta \end{bmatrix}$$

with an arbitrary real function $\vartheta(t,x_1,x_2,x_3)$ describes the oscillation between blue and red chromatics which curves the space in the x_1, x_3 directions. And matrix

$$U_{1,2}(\varsigma) := \begin{bmatrix} e^{-\mathrm{i}\varsigma} & 0 & 0 & 0 \\ 0 & e^{\mathrm{i}\varsigma} & 0 & 0 \\ 0 & 0 & e^{-\mathrm{i}\varsigma} & 0 \\ 0 & 0 & 0 & e^{\mathrm{i}\varsigma} \end{bmatrix}$$

with an arbitrary real function $\varsigma(t,x_1,x_2,x_3)$ describes the oscillation between green and red chromatics which curves the space in the x_1, x_2 directions.

Now, let

$$U_{0,1}(\sigma) := \begin{bmatrix} \cosh\sigma & -\sinh\sigma & 0 & 0 \\ -\sinh\sigma & \cosh\sigma & 0 & 0 \\ 0 & 0 & \cosh\sigma & \sinh\sigma \\ 0 & 0 & \sinh\sigma & \cosh\sigma \end{bmatrix}.$$

and

$$\widehat{M}'' := \begin{pmatrix} -M_{\zeta,0}''\gamma_{\zeta}^{[0]} + M_{\zeta,4}''\zeta^{[4]} - \\ -M_{\eta,0}''\gamma_{\eta}^{[0]} - M_{\eta,4}''\eta^{[4]} + \\ +M_{\theta,0}''\gamma_{\theta}^{[0]} + M_{\theta,4}''\theta^{[4]} \end{pmatrix} := U_{0,1}^{\dagger}(\sigma)\widehat{M}U_{0,1}(\sigma)$$

then:

$$M_{\zeta,0}'' = M_{\zeta,0},$$
$$M_{\eta,0}'' = \left(M_{\eta,0}\cosh 2\sigma - M_{\theta,4}\sinh 2\sigma\right),$$

$$
\begin{aligned}
M''_{\theta,0} &= M_{\theta,0}\cosh 2\sigma + M_{\eta,4}\sinh 2\sigma\,,\\
M''_{\zeta,4} &= M_{\zeta,4}\,,\\
M''_{\eta,4} &= M_{\eta,4}\cosh 2\sigma + M_{\theta,0}\sinh 2\sigma\,,\\
M''_{\theta,4} &= M_{\theta,4}\cosh 2\sigma - M_{\eta,0}\sinh 2\sigma\,.
\end{aligned}
$$

Therefore, matrix $U_{0,1}(\sigma)$ makes an oscillation between green and blue chromatics with an oscillation between upper and lower mass members.

Let us consider equation (2.30) under transformation $U_{0,1}(\sigma)$ where σ is an arbitrary real function of time-space variables ($\sigma = \sigma(t, x_1, x_2, x_3)$):

$$
\begin{aligned}
&U^{\dagger}_{0,1}(\sigma)\left(\frac{1}{\mathrm{c}}\partial_t + \mathrm{i}\Theta_0 + \mathrm{i}\Upsilon_0\gamma^{[5]}\right)U_{0,1}(\sigma)\varphi =\\
&= U^{\dagger}_{0,1}(\sigma)\left(\begin{array}{c}\sum\limits_{\nu=1}^{3}\beta^{[\nu]}\left(\partial_\nu + \mathrm{i}\Theta_\nu + \mathrm{i}\Upsilon_\nu\gamma^{[5]}\right) + \\ + \mathrm{i}M_0\gamma^{[0]} + \mathrm{i}M_4\beta^{[4]} + \widehat{M}\end{array}\right)U_{0,1}(\sigma)\varphi\,.
\end{aligned}
$$

Since:

$$
\begin{aligned}
&U^{\dagger}_{0,1}(\sigma)U_{0,1}(\sigma) = \left(\cosh 2\sigma - \beta^{[1]}\sinh 2\sigma\right),\\
&U^{\dagger}_{0,1}(\sigma) = \left(\cosh 2\sigma + \beta^{[1]}\sinh 2\sigma\right)U^{-1}_{0,1}(\sigma),\\
&U^{\dagger}_{0,1}(\sigma)\beta^{[1]} = \left(\beta^{[1]}\cosh 2\sigma - \sinh 2\sigma\right)U^{-1}_{0,1}(\sigma),\\
&U^{\dagger}_{0,1}(\sigma)\beta^{[2]} = \beta^{[2]}U^{-1}_{0,1}(\sigma),\\
&U^{\dagger}_{0,1}(\sigma)\beta^{[3]} = \beta^{[3]}U^{-1}_{0,1}(\sigma),\\
&U^{\dagger}_{0,1}(\sigma)\gamma^{[0]}U_{0,1}(\sigma) = \gamma^{[0]},\\
&U^{\dagger}_{0,1}(\sigma)\beta^{[4]}U_{0,1}(\sigma) = \beta^{[4]},\\
&U^{-1}_{0,1}(\sigma)U_{0,1}(\sigma) = 1_4\,,\\
&U^{-1}_{0,1}(\sigma)\gamma^{[5]}U_{0,1}(\sigma) = \gamma^{[5]}\,,\\
&U^{\dagger}_{0,1}(\sigma)\gamma^{[5]}U_{0,1}(\sigma) = \gamma^{[5]}\left(\cosh 2\sigma - \beta^{[1]}\sinh 2\sigma\right),
\end{aligned}
$$

then

$$\left(\begin{array}{c} U_{0,1}^{-1}(\sigma)\left(\cosh 2\sigma\cdot\frac{1}{c}\partial_t+\sinh 2\sigma\cdot\partial_1\right)U_{0,1}(\sigma) \\ +\left(\cosh 2\sigma\cdot\frac{1}{c}\partial_t+\sinh 2\sigma\cdot\partial_1\right) \\ +\mathrm{i}\left(\Theta_0\cosh 2\sigma+\Theta_1\sinh 2\sigma\right) \\ +\mathrm{i}\left(\Upsilon_0\cosh 2\sigma+\sinh 2\sigma\cdot\Upsilon_1\right)\gamma^{[5]}- \\ -\beta^{[1]}\left(\begin{array}{c} U_{0,1}^{-1}(\sigma)\left(\cosh 2\sigma\cdot\partial_1+\sinh 2\sigma\cdot\frac{1}{c}\partial_t\right)U_{0,1}(\sigma) \\ +\left(\cosh 2\sigma\cdot\partial_1+\sinh 2\sigma\cdot\frac{1}{c}\partial_t\right) \\ +\mathrm{i}\left(\Theta_1\cosh 2\sigma+\Theta_0\sinh 2\sigma\right) \\ +\mathrm{i}\left(\Upsilon_1\cosh 2\sigma+\Upsilon_0\sinh 2\sigma\right)\gamma^{[5]} \end{array}\right) \\ -\beta^{[2]}\left(\partial_2+U_{0,1}^{-1}(\sigma)\left(\partial_2 U_{0,1}(\sigma)\right)+\mathrm{i}\Theta_2+\mathrm{i}\Upsilon_2\gamma^{[5]}\right) \\ -\beta^{[3]}\left(\partial_3+U_{0,1}^{-1}(\sigma)\left(\partial_3 U_{0,1}(\sigma)\right)+\mathrm{i}\Theta_3+\mathrm{i}\Upsilon_3\gamma^{[5]}\right) \\ -\mathrm{i}M_0\gamma^{[0]}-\mathrm{i}M_4\beta^{[4]}-\widehat{M}'' \end{array}\right)\varphi=0. \tag{3.47}$$

Let t' and x_1' be elements of other coordinate system such that:

$$\left.\begin{array}{l} \frac{\partial x_1}{\partial x_1'}=\cosh 2\sigma \\ \frac{\partial t}{\partial x_1'}=\frac{1}{c}\sinh 2\sigma \\ \frac{\partial x_1}{\partial t'}=c\sinh 2\sigma \\ \frac{\partial t}{\partial t'}=\cosh 2\sigma \\ \frac{\partial x_2}{\partial t'}=\frac{\partial x_3}{\partial t'}=\frac{\partial x_2}{\partial x_1'}=\frac{\partial x_3}{\partial x_1'}=0 \end{array}\right\}. \tag{3.48}$$

Hence:

$$\begin{aligned} \partial_t' &:= \frac{\partial}{\partial t'}=\frac{\partial}{\partial t}\frac{\partial t}{\partial t'}+\frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial t'}+\frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial t'}+\frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial t'}= \\ &=\cosh 2\sigma\cdot\frac{\partial}{\partial t}+c\sinh 2\sigma\cdot\frac{\partial}{\partial x_1}= \\ &=\cosh 2\sigma\cdot\partial_t+c\sinh 2\sigma\cdot\partial_1, \end{aligned}$$

that is

$$\frac{1}{c}\partial_t'=\frac{1}{c}\cosh 2\sigma\cdot\partial_t+\sinh 2\sigma\cdot\partial_1$$

and

$$\partial_1' := \frac{\partial}{\partial x_1'}=$$

$$= \frac{\partial}{\partial t}\frac{\partial t}{\partial x_1'} + \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x_1'} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x_1'} + \frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial x_1'} =$$
$$= \cosh 2\sigma \cdot \frac{\partial}{\partial x_1} + \sinh 2\sigma \cdot \frac{1}{\mathrm{c}}\frac{\partial}{\partial t} =$$
$$= \cosh 2\sigma \cdot \partial_1 + \sinh 2\sigma \cdot \frac{1}{\mathrm{c}}\partial_t \,.$$

Therefore, from (3.47):

$$\left(\begin{array}{c} \beta^{[0]}\left(\frac{1}{\mathrm{c}}\partial_t' + U_{0,1}^{-1}(\sigma)\frac{1}{\mathrm{c}}\partial_t' U_{0,1}(\sigma) + \mathrm{i}\Theta_0'' + \mathrm{i}\Upsilon_0''\gamma^{[5]}\right) \\ +\beta^{[1]}\left(\partial_1' + U_{0,1}^{-1}(\sigma)\partial_1' U_{0,1}(\sigma) + \mathrm{i}\Theta_1'' + \mathrm{i}\Upsilon_1''\gamma^{[5]}\right) \\ +\beta^{[2]}\left(\partial_2 + U_{0,1}^{-1}(\sigma)\partial_2 U_{0,1}(\sigma) + \mathrm{i}\Theta_2 + \mathrm{i}\Upsilon_2\gamma^{[5]}\right) \\ +\beta^{[3]}\left(\partial_3 + U_{0,1}^{-1}(\sigma)\partial_3 U_{0,1}(\sigma) + \mathrm{i}\Theta_3 + \mathrm{i}\Upsilon_3\gamma^{[5]}\right) \\ +\mathrm{i}M_0\gamma^{[0]} + \mathrm{i}M_4\beta^{[4]} + \widehat{M}'' \end{array} \right)\varphi = 0$$

with

$$\Theta_0'' := \Theta_0 \cosh 2\sigma + \Theta_1 \sinh 2\sigma \,,$$
$$\Theta_1'' := \Theta_1 \cosh 2\sigma + \Theta_0 \sinh 2\sigma \,,$$
$$\Upsilon_0'' := \Upsilon_0 \cosh 2\sigma + \sinh 2\sigma \cdot \Upsilon_1 \,,$$
$$\Upsilon_1'' := \Upsilon_1 \cosh 2\sigma + \Upsilon_0 \sinh 2\sigma \,.$$

Therefore, the oscillation between blue and green chromatics with the oscillation between upper and lower mass members curves the space in the t, x_1 directions.

Similarly, matrix

$$U_{0,2}(\phi) := \left[\begin{array}{cccc} \cosh\phi & \mathrm{i}\sinh\phi & 0 & 0 \\ -\mathrm{i}\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & \cosh\phi & -\mathrm{i}\sinh\phi \\ 0 & 0 & \mathrm{i}\sinh\phi & \cosh\phi \end{array} \right]$$

with an arbitrary real function $\phi(t, x_1, x_2, x_3)$ describes the oscillation between blue and red chromatics with the oscillation between upper and lower mass members curves the space in the t, x_2 directions. And matrix

$$U_{0,3}(\iota) := \left[\begin{array}{cccc} e^{\iota} & 0 & 0 & 0 \\ 0 & e^{-\iota} & 0 & 0 \\ 0 & 0 & e^{-\iota} & 0 \\ 0 & 0 & 0 & e^{\iota} \end{array} \right]$$

with an arbitrary real function $\iota(t, x_1, x_2, x_3)$ describes the oscillation between green and red chromatics with the oscillation between upper and lower mass members curves the space in the t, x_3 directions.

Now let

$$\widetilde{U}(\chi) := \begin{bmatrix} e^{i\chi} & 0 & 0 & 0 \\ 0 & e^{i\chi} & 0 & 0 \\ 0 & 0 & e^{2i\chi} & 0 \\ 0 & 0 & 0 & e^{2i\chi} \end{bmatrix}$$

and

$$\widehat{M}' := \begin{pmatrix} -M'_{\zeta,0}\gamma^{[0]}_{\zeta} + M'_{\zeta,4}\zeta^{[4]} - \\ -M'_{\eta,0}\gamma^{[0]}_{\eta} - M'_{\eta,4}\eta^{[4]} + \\ +M'_{\theta,0}\gamma^{[0]}_{\theta} + M'_{\theta,4}\theta^{[4]} \end{pmatrix} := \widetilde{U}^{\dagger}(\chi)\widehat{M}\widetilde{U}(\chi)$$

then:

$$\begin{aligned} M'_{\zeta,0} &= \left(M_{\zeta,0}\cos\chi - M_{\zeta,4}\sin\chi\right), \\ M'_{\zeta,4} &= \left(M_{\zeta,4}\cos\chi + M_{\zeta,0}\sin\chi\right), \\ M'_{\eta,4} &= \left(M_{\eta,4}\cos\chi - M_{\eta,0}\sin\chi\right), \\ M'_{\eta,0} &= \left(M_{\eta,0}\cos\chi + M_{\eta,4}\sin\chi\right), \\ M'_{\theta,0} &= \left(M_{\theta,0}\cos\chi + M_{\theta,4}\sin\chi\right), \\ M'_{\theta,4} &= \left(M_{\theta,4}\cos\chi - M_{\theta,0}\sin\chi\right). \end{aligned}$$

Therefore, matrix $\widetilde{U}(\chi)$ makes an oscillation between upper and lower mass members.

Let us consider equation (3.44) under transformation $\widetilde{U}(\chi)$ where χ is an arbitrary real function of time-space variables ($\chi = \chi(t, x_1, x_2, x_3)$):

$$\begin{aligned} &\widetilde{U}^{\dagger}(\chi)\left(\frac{1}{c}\partial_t + i\Theta_0 + i\Upsilon_0\gamma^{[5]}\right)\widetilde{U}(\chi)\varphi = \\ &= \widetilde{U}^{\dagger}(\chi)\left(\sum_{\nu=1}^{3}\beta^{[\nu]}\left(\partial_\nu + i\Theta_\nu + i\Upsilon_\nu\gamma^{[5]}\right) + \widehat{M}\right)\widetilde{U}(\chi)\varphi. \end{aligned}$$

Because

$$\gamma^{[5]}\widetilde{U}(\chi) = \widetilde{U}(\chi)\gamma^{[5]},$$

$$\begin{aligned} \beta^{[1]}\widetilde{U}(\chi) &= \widetilde{U}(\chi)\beta^{[1]}, \\ \beta^{[2]}\widetilde{U}(\chi) &= \widetilde{U}(\chi)\beta^{[2]}, \\ \beta^{[3]}\widetilde{U}(\chi) &= \widetilde{U}(\chi)\beta^{[3]}, \end{aligned}$$

$$\widetilde{U}^{\dagger}(\chi)\widetilde{U}(\chi) = 1_4,$$

then

$$\begin{aligned} &\left(\frac{1}{c}\partial_t + \frac{1}{c}\widetilde{U}^{\dagger}(\chi)\left(\partial_t\widetilde{U}(\chi)\right) + i\Theta_0 + i\Upsilon_0\gamma^{[5]}\right)\varphi = \\ &= \begin{pmatrix} \sum_{\nu=1}^{3}\beta^{[\nu]}\left(\partial_\nu + \widetilde{U}^{\dagger}(\chi)\left(\partial_\nu\widetilde{U}(\chi)\right) + i\Theta_\nu + i\Upsilon_\nu\gamma^{[5]}\right) \\ +\widetilde{U}^{\dagger}(\chi)\widehat{M}\widetilde{U}(\chi) \end{pmatrix}\varphi. \end{aligned}$$

Now let:

$$\widehat{U}(\kappa) := \begin{bmatrix} e^{\kappa} & 0 & 0 & 0 \\ 0 & e^{\kappa} & 0 & 0 \\ 0 & 0 & e^{2\kappa} & 0 \\ 0 & 0 & 0 & e^{2\kappa} \end{bmatrix}$$

and

$$\widehat{M}' := \begin{pmatrix} -M'_{\zeta,0}\gamma_{\zeta}^{[0]} + M'_{\zeta,4}\zeta^{[4]} - \\ -M'_{\eta,0}\gamma_{\eta}^{[0]} - M'_{\eta,4}\eta^{[4]} + \\ +M'_{\theta,0}\gamma_{\theta}^{[0]} + M'_{\theta,4}\theta^{[4]} \end{pmatrix} := \widehat{U}^{-1}(\kappa)\widehat{M}\widehat{U}(\kappa)$$

then:

$$\begin{aligned}
M'_{\theta,0} &= (M_{\theta,0}\cosh\kappa - \mathrm{i}M_{\theta,4}\sinh\kappa), \\
M'_{\theta,4} &= (M_{\theta,4}\cosh\kappa + \mathrm{i}M_{\theta,0}\sinh\kappa), \\
M'_{\eta,0} &= (M_{\eta,0}\cosh\kappa - \mathrm{i}M_{\eta,4}\sinh\kappa), \\
M'_{\eta,4} &= (M_{\eta,4}\cosh\kappa + \mathrm{i}M_{\eta,0}\sinh\kappa), \\
M'_{\zeta,0} &= \left(M_{\zeta,0}\cosh\kappa + \mathrm{i}M_{\zeta,4}\sinh\kappa\right), \\
M'_{\zeta,4} &= \left(M_{\zeta,4}\cosh\kappa - \mathrm{i}M_{\zeta,0}\sinh\kappa\right).
\end{aligned}$$

Therefore, matrix $\widehat{U}(\kappa)$ makes an oscillation between upper and lower mass members, too.

Let us consider equation (3.44) under transformation $\widehat{U}(\kappa)$ where κ is an arbitrary real function of time-space variables ($\kappa = \kappa(t, x_1, x_2, x_3)$):

$$\begin{aligned}
&\widehat{U}^{-1}(\kappa)\left(\frac{1}{\mathrm{c}}\partial_t + \mathrm{i}\Theta_0 + \mathrm{i}\Upsilon_0\gamma^{[5]}\right)\widehat{U}(\kappa)\varphi = \\
&= \widehat{U}^{-1}(\kappa)\left(\sum_{\nu=1}^{3}\beta^{[\nu]}\left(\partial_\nu + \mathrm{i}\Theta_\nu + \mathrm{i}\Upsilon_\nu\gamma^{[5]}\right) + \widehat{M}\right)\widehat{U}(\kappa)\varphi.
\end{aligned}$$

Because

$$\gamma^{[5]}\widehat{U}(\kappa) = \widehat{U}(\kappa)\gamma^{[5]},$$

$$\begin{aligned}
\widehat{U}^{-1}(\kappa)\beta^{[1]} &= \beta^{[1]}\widehat{U}^{-1}(\kappa), \\
\widehat{U}^{-1}(\kappa)\beta^{[2]} &= \beta^{[2]}\widehat{U}^{-1}(\kappa), \\
\widehat{U}^{-1}(\kappa)\beta^{[3]} &= \beta^{[3]}\widehat{U}^{-1}(\kappa),
\end{aligned}$$

$$\widehat{U}^{-1}(\kappa)\widehat{U}(\kappa) = 1_4,$$

then

$$\left(\frac{1}{\mathrm{c}}\partial_t + \widehat{U}^{-1}(\kappa)\left(\frac{1}{\mathrm{c}}\partial_t\widehat{U}(\kappa)\right) + \mathrm{i}\Theta_0 + \mathrm{i}\Upsilon_0\gamma^{[5]}\right)\varphi =$$

$$= \left(\begin{array}{c} \sum_{\nu=1}^{3} \beta^{[\nu]} \left(\partial_\nu + \widehat{U}^{-1}(\kappa) \left(\partial_\nu \widehat{U}(\kappa) \right) + \mathrm{i}\Theta_\nu + \mathrm{i}\Upsilon_\nu \gamma^{[5]} \right) + \\ + \widehat{U}^{-1}(\kappa) \widehat{M} \widehat{U}(\kappa) \end{array} \right) \varphi.$$

If denote:

$$\Lambda_1 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\Lambda_2 := \begin{bmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix},$$

$$\Lambda_3 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\Lambda_4 := \begin{bmatrix} 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix},$$

$$\Lambda_5 := \begin{bmatrix} -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{bmatrix},$$

$$\Lambda_6 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda_7 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$\Lambda_8 := \begin{bmatrix} \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 2\mathrm{i} & 0 \\ 0 & 0 & 0 & 2\mathrm{i} \end{bmatrix},$$

then

$$U_{0,1}^{-1}(\sigma)\left(\partial_s U_{0,1}(\sigma)\right) = \Lambda_1 \partial_s \sigma,$$

$$U_{2,3}^{-1}(\alpha)\left(\partial_s U_{2,3}(\alpha)\right) = \Lambda_2 \partial_s \alpha,$$

$$U_{1,3}^{-1}(\vartheta)(\partial_s U_{1,3}(\vartheta)) = \Lambda_3 \partial_s \vartheta,$$

$$U_{0,2}^{-1}(\phi)(\partial_s U_{0,2}(\phi)) = \Lambda_4 \partial_s \phi,$$

$$U_{1,2}^{-1}(\varsigma)(\partial_s U_{1,2}(\varsigma)) = \Lambda_5 \partial_s \varsigma,$$

$$U_{0,3}^{-1}(\iota)(\partial_s U_{0,3}(\iota)) = \Lambda_6 \partial_s \iota,$$

$$\widehat{U}^{-1}(\kappa)(\partial_s \widehat{U}(\kappa)) = \Lambda_7 \partial_s \kappa,$$

$$\widetilde{U}^{-1}(\chi)(\partial_s \widetilde{U}(\chi)) = \Lambda_8 \partial_s \chi.$$

Let $\grave{U}$ be the following set:

$$\grave{U} := \left\{ U_{0,1}, U_{2,3}, U_{1,3}, U_{0,2}, U_{1,2}, U_{0,3}, \widehat{U}, \widetilde{U} \right\}.$$

Because

$U_{2,3}^{-1}(\alpha)\Lambda_1 U_{2,3}(\alpha) = \Lambda_1$
$U_{1,3}^{-1}(\vartheta)\Lambda_1 U_{1,3}(\vartheta) = (\Lambda_1 \cos 2\vartheta + \Lambda_6 \sin 2\vartheta)$
$U_{0,2}^{-1}(\phi)\Lambda_1 U_{0,2}(\phi) = (\Lambda_1 \cosh 2\phi - \Lambda_5 \sinh 2\phi)$
$U_{1,2}^{-1}(\varsigma)\Lambda_1 U_{1,2}(\varsigma) = \Lambda_1 \cos 2\varsigma - \Lambda_4 \sin 2\varsigma$
$U_{0,3}^{-1}(\iota)\Lambda_1 U_{0,3}(\iota) = \Lambda_1 \cosh 2\iota + \Lambda_3 \sinh 2\iota$
$\widehat{U}^{-1}(\kappa)\Lambda_1 \widehat{U}(\kappa) = \Lambda_1$
$\widetilde{U}^{-1}(\chi)\Lambda_1 \widetilde{U}(\chi) = \Lambda_1$
========
$\widetilde{U}^{-1}(\chi)\Lambda_2 \widetilde{U}(\chi) = \Lambda_2$
$\widehat{U}^{-1}(\kappa)\Lambda_2 \widehat{U}(\kappa) = \Lambda_2$
$U_{0,3}^{-1}(\iota)\Lambda_2 U_{0,3}(\iota) = \Lambda_2 \cosh 2\iota - \Lambda_4 \sinh 2\iota$
$U_{1,2}^{-1}(\varsigma)\Lambda_2 U_{1,2}(\varsigma) = \Lambda_2 \cos 2\varsigma - \Lambda_3 \sin 2\varsigma$
$U_{0,2}^{-1}(\phi)\Lambda_2 U_{0,2}(\phi) = \Lambda_2 \cosh 2\phi + \Lambda_6 \sinh 2\phi$
$U_{1,3}^{-1}(\vartheta)\Lambda_2 U_{1,3}(\vartheta) = \Lambda_2 \cos 2\vartheta + \Lambda_5 \sin 2\vartheta$
$U_{0,1}^{-1}(\sigma)\Lambda_2 U_{0,1}(\sigma) = \Lambda_2$
=========
$U_{0,1}^{-1}(\sigma)\ \Lambda_3 U_{0,1}(\sigma) = \Lambda_3 \cosh 2\sigma - \Lambda_6 \sinh 2\sigma$
$U_{2,3}^{-1}(\alpha)\ \Lambda_3 U_{2,3}(\alpha) = \Lambda_3 \cos 2\alpha - \Lambda_5 \sin 2\alpha$
$U_{0,2}^{-1}(\phi)\ \Lambda_3 U_{0,2}(\phi) = \Lambda_3$
$U_{1,2}^{-1}(\varsigma)\ \Lambda_3 U_{1,2}(\varsigma) = \Lambda_3 \cos 2\varsigma + \Lambda_2 \sin 2\varsigma$
$U_{0,3}^{-1}(\iota)\ \Lambda_3 U_{0,3}(\iota) = \Lambda_3 \cosh 2\iota + \Lambda_1 \sinh 2\iota$
$\widehat{U}^{-1}(\kappa)\ \Lambda_3 \widehat{U}(\kappa) = \Lambda_3$
$\widetilde{U}^{-1}(\chi)\ \Lambda_3 \widetilde{U}(\chi) = \Lambda_3$

==========

$\widetilde{U}^{-1}(\chi)\ \Lambda_4\widetilde{U}(\chi)=\Lambda_4$

$\widehat{U}^{-1}(\kappa)\ \Lambda_4\widehat{U}(\kappa)=\Lambda_4$

$U_{0,3}^{-1}(\iota)\ \Lambda_4 U_{0,3}(\iota)=\Lambda_4\cosh 2\iota-\Lambda_2\sinh 2\iota$

$U_{1,2}^{-1}(\varsigma)\ \Lambda_4 U_{1,2}(\varsigma)=\Lambda_4\cos 2\varsigma+\Lambda_1\sin 2\varsigma$

$U_{1,3}^{-1}(\vartheta)\ \Lambda_4 U_{1,3}(\vartheta)=\Lambda_4$

$U_{2,3}^{-1}(\alpha)\ \Lambda_4 U_{2,3}(\alpha)=\Lambda_4\cos 2\alpha-\Lambda_6\sin 2\alpha$

$U_{0,1}^{-1}(\sigma)\ \Lambda_4 U_{0,1}(\sigma)=\Lambda_4\cosh 2\sigma+\Lambda_5\sinh 2\sigma$

==========

$U_{0,1}^{-1}(\sigma)\ \Lambda_5 U_{0,1}(\sigma)\ =\Lambda_5\cosh 2\sigma+\Lambda_4\sinh 2\sigma$

$U_{2,3}^{-1}(\alpha)\ \Lambda_5 U_{2,3}(\alpha)\ =\Lambda_5\cos 2\alpha+\Lambda_3\sin 2\alpha$

$U_{1,3}^{-1}(\vartheta)\ \Lambda_5 U_{1,3}(\vartheta)\ =(\Lambda_5\cos 2\vartheta-\Lambda_2\sin 2\vartheta)$

$U_{0,2}^{-1}(\phi)\ \Lambda_5 U_{0,2}(\phi)\ =\Lambda_5\cosh 2\phi-\Lambda_1\sinh 2\phi$

$U_{0,3}^{-1}(\iota)\ \Lambda_5 U_{0,3}(\iota)\ =\Lambda_5$

$\widehat{U}^{-1}(\kappa)\ \Lambda_5\widehat{U}(\kappa)\ =\Lambda_5$

$\widetilde{U}^{-1}(\chi)\Lambda_5\widetilde{U}(\chi)\ =\Lambda_5$

===========

$\widetilde{U}^{-1}(\chi)\Lambda_6\widetilde{U}(\chi)\ =\Lambda_6$

$\widehat{U}^{-1}(\kappa)\Lambda_6\widehat{U}(\kappa)\ =\Lambda_6$

$U_{1,2}^{-1}(\varsigma)\Lambda_6 U_{1,2}(\varsigma)\ =\Lambda_6$

$U_{0,2}^{-1}(\phi)\Lambda_6 U_{0,2}(\phi)\ =\Lambda_6\cosh 2\phi+\Lambda_2\sinh 2\phi$

$U_{1,3}^{-1}(\vartheta)\Lambda_6 U_{1,3}(\vartheta)\ =\Lambda_6\cos 2\vartheta-\Lambda_1\sin 2\vartheta$

$U_{2,3}^{-1}(\alpha)\Lambda_6 U_{2,3}(\alpha)\ =\Lambda_6\cos 2\alpha+\Lambda_4\sin 2\alpha$

$U_{0,1}^{-1}(\sigma)\Lambda_6 U_{0,1}(\sigma)=\Lambda_6\cosh 2\sigma-\Lambda_3\sinh 2\sigma$

========

$\widetilde{U}^{-1}(\chi)\ \Lambda_7\widetilde{U}(\chi)=\Lambda_7$

$U_{0,3}^{-1}(\iota)\ \Lambda_7 U_{0,3}(\iota)=\Lambda_7$

$U_{1,2}^{-1}(\varsigma)\ \Lambda_7 U_{1,2}(\varsigma)=\Lambda_7$

$U_{0,2}^{-1}(\phi)\ \Lambda_7 U_{0,2}(\phi)=\Lambda_7$

$U_{1,3}^{-1}(\vartheta)\ \Lambda_7 U_{1,3}(\vartheta)=\Lambda_7$

$U_{2,3}^{-1}(\alpha)\ \Lambda_7 U_{2,3}(\sigma)=\Lambda_7$

$U_{0,1}^{-1}(\sigma)\ \Lambda_7 U_{0,1}(\sigma)=\Lambda_7$

=========

$U_{0,1}^{-1}(\sigma)\ \Lambda_8 U_{0,1}(\sigma)=\Lambda_8$

$U_{2,3}^{-1}(\alpha)\ \Lambda_8 U_{2,3}(\alpha)=\Lambda_8$

$U_{1,3}^{-1}(\vartheta)\ \Lambda_8 U_{1,3}(\vartheta) = \Lambda_8$
$U_{0,2}^{-1}(\phi)\ \Lambda_8 U_{0,2}(\phi) = \Lambda_8$
$U_{1,2}^{-1}(\varsigma)\ \Lambda_8 U_{1,2}(\varsigma) = \Lambda_8$
$U_{0,3}^{-1}(\iota)\ \Lambda_8 U_{0,3}(\iota) = \Lambda_8$
$\widehat{U}^{-1}(\kappa)\ \Lambda_8 \widehat{U}(\kappa) = \Lambda_8$

then for every product U of $\grave{U}$'s elements real functions $G_s^r(t,x_1,x_2,x_3)$ exist such that

$$U^{-1}(\partial_s U) = \frac{g_3}{2}\sum_{r=1}^{8}\Lambda_r G_s^r$$

with some real constant g_3 (similar to 8 gluons).

3.3. Asymptoiic Freedom, Confinement, Cravitation

The Quarks Asymptotic Freedomin phenomenon and the Quarks Confinement phenomenon has been was discovered by J. Friedman[13], H. Kendall[14], R. Taylor[15] at SLAC in the late 1960s and early 1970s.

Researches of the phenomenon of gravitation were spent by Galileo Galilei[16] in the late 16th and early 17th centuries, by Isaac Newton[17] in 17th centuries, by A. Einstein[18] in 20th centuries.

From (3.48):

$$\begin{aligned}\frac{\partial t}{\partial t'} &= \cosh 2\sigma, \qquad (3.49)\\ \frac{\partial x}{\partial t'} &= c\sinh 2\sigma.\end{aligned}$$

Hence, if v is the velocity of a coordinate system $\{t',x'\}$ in the coordinate system $\{t,x\}$ then

$$\sinh 2\sigma = \frac{\left(\frac{v}{c}\right)}{\sqrt{1-\left(\frac{v}{c}\right)^2}},\ \cosh 2\sigma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}.$$

Therefore,

$$v = c\tanh 2\sigma. \qquad (3.50)$$

[13]Jerome Isaac Friedman (born March 28, 1930) is an American physicist.

[14]Henry Way Kendall (December 9, 1926 – February 15, 1999) was an American particle physicist

[15]Richard Edward Taylor (born November 2, 1929 in Medicine Hat, Alberta) is a Canadian-American professor (Emeritus) at Stanford University.

[16]Galileo Galilei (15 February 1564[4] – 8 January 1642), was an Italian physicist, mathematician, astronomer, and philosopher

[17]Sir Isaac Newton PRS (25 December 1642 – 20 March 1727 was an English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian

[18]Albert Einstein (14 March 1879 – 18 April 1955) was a German-born theoretical physicist

Let

$$2\sigma := \omega(x)\,\frac{t}{x}$$

with

$$\omega(x) = \frac{\lambda}{|x|},$$

where λ is a real constant with positive numerical value.

In that case

$$v(t,x) = \mathrm{c}\tanh\left(\frac{\lambda}{|x|}\frac{t}{x}\right). \tag{3.51}$$

and if g is an acceleration of system $\{t',x_1'\}$ as respects to system $\{t,x_1\}$ then

$$\mathrm{g}(t,x_1) = \frac{\partial v}{\partial t} = \frac{\mathrm{c}\omega(x_1)}{\left(\cosh^2\omega(x_1)\,\frac{t}{x_1}\right)x_1}.$$

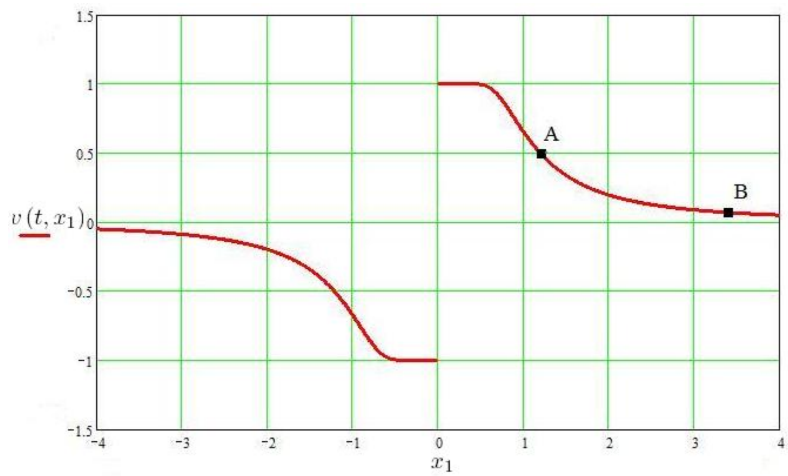

Figure 30:

Figure 30 shows the dependency of a system $\{t',x_1'\}$ velocity $v(t,x_1)$ on x_1 in system $\{t,x_1\}$.

This velocity in point A is not equal to one in point B. Hence, an oscillator, placed in B, has a nonzero velocity in respect to an observer, placed in point A. Therefore, from the Lorentz transformations, this oscillator frequency for observer, placed in point A, is less than own frequency of this oscillator (*red shift*).

Figure 31 shows a dependency of a system $\{t',x_1'\}$ acceleration $g(t,x_1)$ on x_1 in system $\{t,x_1\}$.

If an object immovable in system $\{t,x_1\}$ is placed in point K then in system $\{t',x_1'\}$ this object must move to the left with acceleration g and $\mathrm{g} \simeq \frac{\lambda}{x_1^2}$.

I call:

- interval from S to ∞ the *Newton Gravity Zone*,

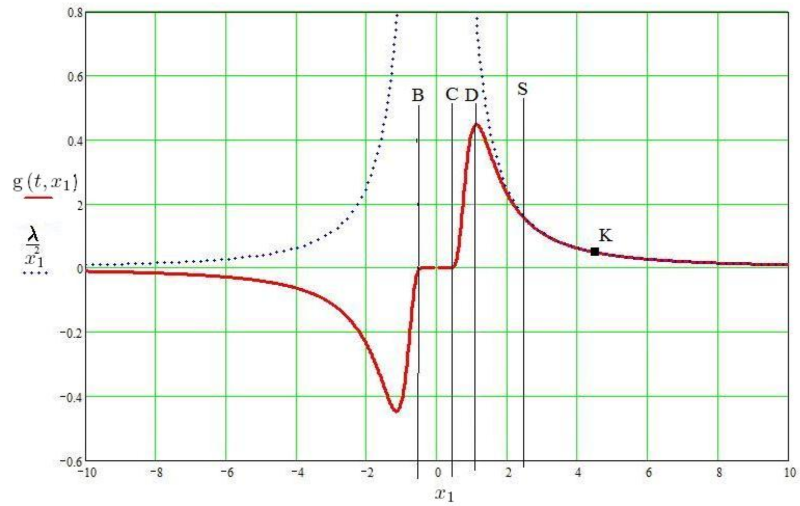

Figure 31:

- interval from B to C the *the Asymptotic Freedom Zone*,
- and interval from C to D *the Confinement Force Zone*.

3.3.1. Dark Energy

In 1998 observations of Type Ia supernovae suggested that the expansion of the universe is accelerating [37]. In the past few years, these observations have been corroborated by several independent sources [38]. This expansion is defined by the Hubble[19] rule [39]:

$$V(r) = Hr, \tag{3.52}$$

here $V(r)$ is the velocity of expansion on the distance r, H is the Hubble's constant ($H \approx 2.3 \times 10^{-18} c^{-1}$ [40]).

Let a black hole be placed in a point O. Then a tremendous number of quarks oscillate in this point. These oscillations bend time-space and if t has some fixed volume, $x > 0$, and $\Lambda := \lambda t$ then

$$v(x) = \mathrm{c} \tanh\left(\frac{\Lambda}{x^2}\right). \tag{3.53}$$

A dependency of $v(x)$ (light years/c) from x (light years) with $\Lambda = 741.907$ is shown in Figure 32.

Let a placed in a point A observer be stationary in the coordinate system $\{t, x\}$. Hence, in the coordinate system $\{t', x'\}$ this observer is flying to the left to the point O with velocity $-v(x_A)$. And point X is flying to the left to the point O with velocity $-v(x)$.

Consequently, the observer A sees that the point X flies away from him to the right with velocity

[19] Edwin Powell Hubble (November 20, 1889 September 28, 1953)[1] was an American astronomer

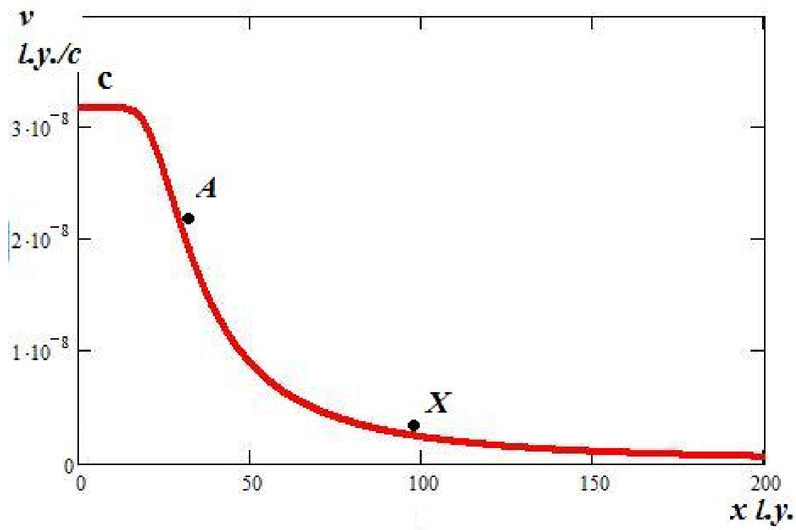

Figure 32: Dependence of v (light year/c) on x (light year) with $\Lambda = 741.907$

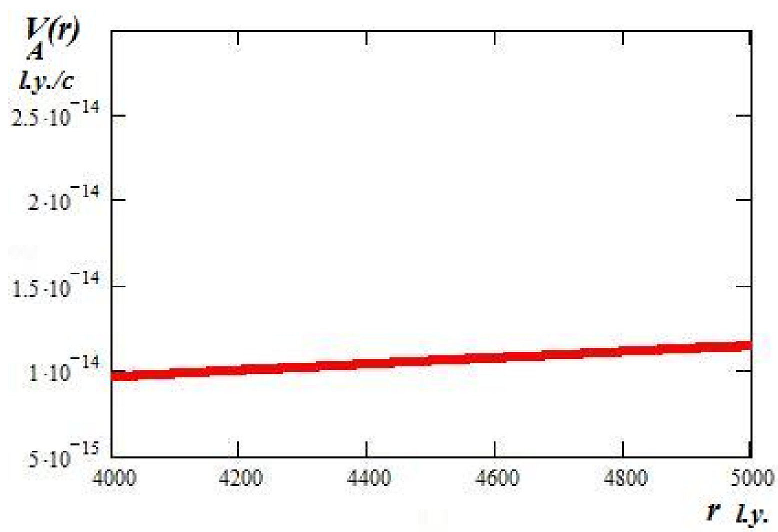

Figure 33: Dependence of $V_A\left(r\right)$ on r with $x_A = 25 \times 10^3$ l.y.

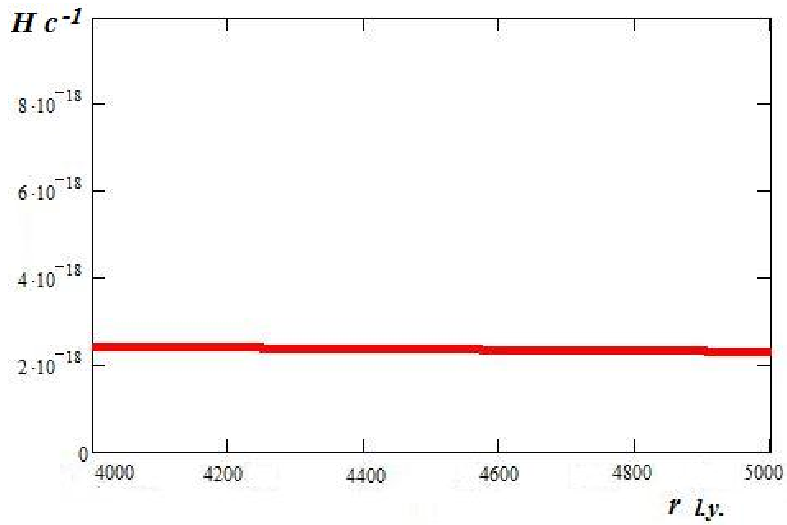

Figure 34: Dependence of H on r

$$V_A(x) = c\tanh\left(\frac{\Lambda}{x_A^2} - \frac{\Lambda}{x^2}\right) \tag{3.54}$$

in accordance with the relativistic rule of addition of velocities.

Let $r := x - x_A$ (i.e. r is distance from A to X), and

$$V_A(r) := c\tanh\left(\frac{\Lambda}{x_A^2} - \frac{\Lambda}{(x_A + r)^2}\right). \tag{3.55}$$

In that case Figure 33 demonstrates the dependence of $V_A(r)$ on r with $x_A = 25 \times 10^3$ l.y.

Hence, X runs from A with almost constant acceleration:

$$\frac{V_A(r)}{r} = H. \tag{3.56}$$

Figure 34 demonstrates the dependence of H on r. (the Hubble constant.).

Therefore, the phenomenon of the accelerated expansion of Universe is explained by oscillations of chromatic states.

3.3.2. Dark Matter

”In 1933, the astronomer Fritz Zwicky[20] was studying the motions of distant galaxies. Zwicky estimated the total mass of a group of galaxies by measuring their brightness. When he used a different method to compute the mass of the same cluster of galaxies, he came up with a number that was 400 times his original estimate. This discrepancy in the observed and computed masses is now known as ”the missing mass problem.” Nobody did much with

[20]Fritz Zwicky (February 14, 1898 February 8, 1974) was a Swiss astronomer.

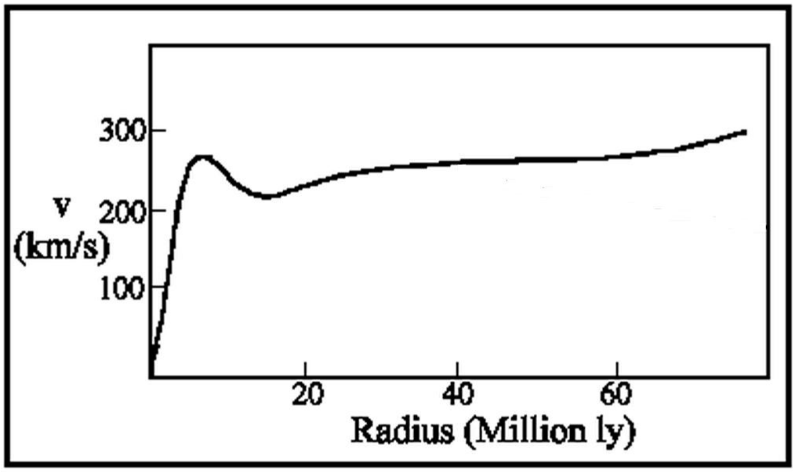

Figure 35: A rotation curve for a typical spiral galaxy. The solid line shows actual measurements (Hawley and Holcomb., 1998, p. 390) [42]

Zwicky's finding until the 1970's, when scientists began to realize that only large amounts of hidden mass could explain many of their observations. Scientists also realize that the existence of some unseen mass would also support theories regarding the structure of the universe. Today, scientists are searching for the mysterious dark matter not only to explain the gravitational motions of galaxies, but also to validate current theories about the origin and the fate of the universe" [41] (Figure 35 [42], Figure 36 [43]).

Some oscillations of chromatic states bend space-time as follows (3.46):

$$\begin{aligned} \frac{\partial}{\partial x'} &= \cos 2\alpha \cdot \frac{\partial}{\partial x} - \sin 2\alpha \cdot \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial y'} &= \cos 2\alpha \cdot \frac{\partial}{\partial y} + \sin 2\alpha \cdot \frac{\partial}{\partial z}. \end{aligned} \tag{3.57}$$

Let

$$\begin{aligned} z &: = x + \mathrm{i}y, \text{ i.e. } z = re^{\mathrm{i}\theta}; \\ z' &: = x' + \mathrm{i}y'. \end{aligned}$$

Because linear velocity of the curved coordinate system $\langle x', y' \rangle$ into the initial system $\langle x, y \rangle$ is the following[21]:

$$v(\theta, r) = \sqrt{\left(\overset{\bullet}{x'}(\theta, r)\right)^2 + \left(\overset{\bullet}{y'}(\theta, r)\right)^2}$$

then in thic case:

[21] $\overset{\bullet}{x'} := \frac{\partial x'}{\partial t}, \overset{\bullet}{y'} := \frac{\partial y'}{\partial t}$.

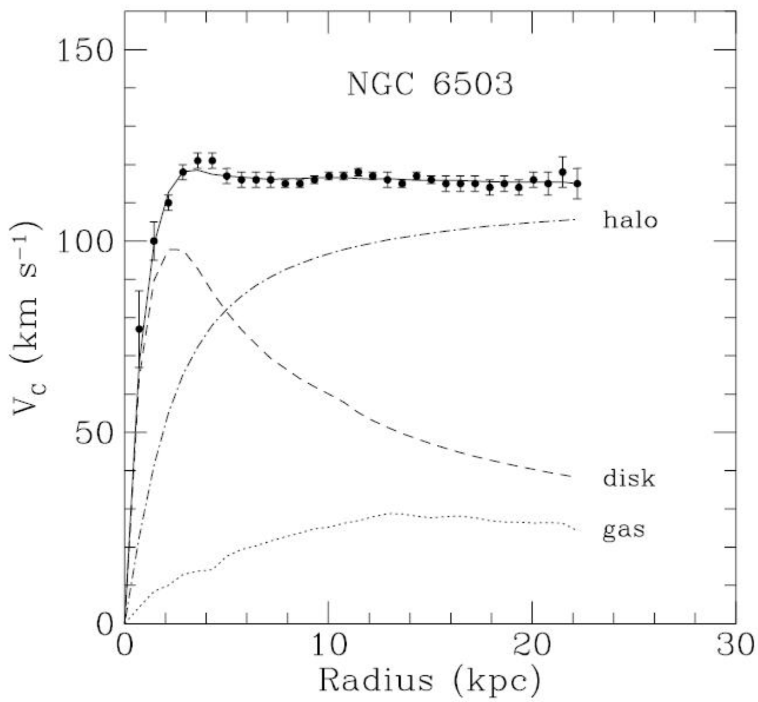

Figure 36: Rotation curve of NGC 6503. The dotted, dashed and dash-dotted lines are the contributions of gas, disk and dark matter, respectively.

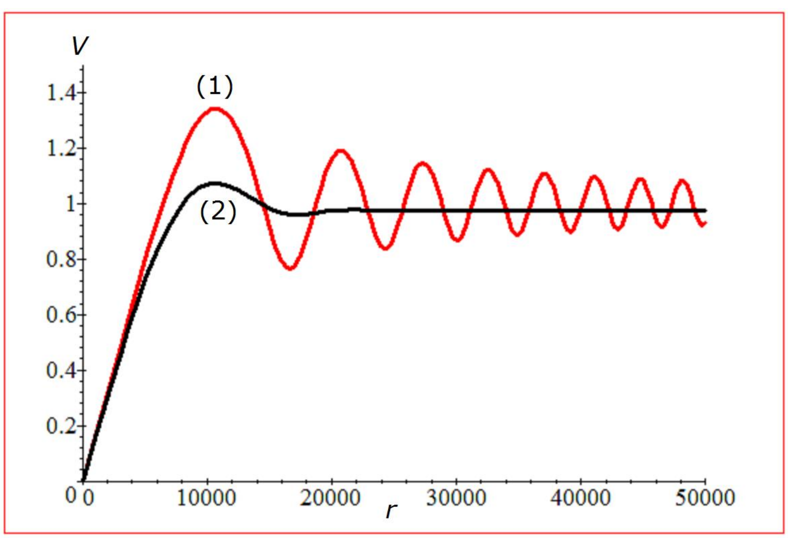

Figure 37:

$$v(\theta, r) = \left|\overset{\bullet}{x'}\right|.$$

Let function z' be a holomorphic function. Hence, in accordance with the Cauchy-Riemann conditions the following equations are fulfilled:

$$\begin{aligned}\frac{\partial x'}{\partial x} &= \frac{\partial y'}{\partial y}, \\ \frac{\partial x'}{\partial y} &= -\frac{\partial y'}{\partial x}.\end{aligned}$$

Therefore, in accordance with (3.57):

$$dz' = e^{-\mathrm{i}(2\alpha)}dz$$

where 2α is an holomorphic function, too. For example, let

$$2\alpha := \frac{1}{\mathrm{i}}\left((x+y)+\mathrm{i}(y-x)\right).$$

In this case:

$$\overset{\bullet}{z'} = -\frac{1}{16}(1-\mathrm{i})\left(A(t,r,\theta)+B(t,r,\theta)+C(t,r,\theta)\right)$$

where

$$\begin{aligned}A(t,r,\theta) &: = \frac{4r\cos\theta}{\sqrt{t}}\exp\left(\frac{2r^2}{t}(\sin 2\theta)-\mathrm{i}\cos 2\theta\right), \\ B(t,r,\theta) &: = -\frac{2\sqrt{\pi}}{\sqrt{t}}\operatorname{erf}\left(Q(\theta)\frac{r}{\sqrt{t}}\right), \\ C(t,r,\theta) &: = -\sqrt{\pi}\cdot\cos\theta\cdot Q^*(\theta)\cdot\operatorname{erf}\left(Q(\theta)\frac{r}{\sqrt{t}}\right)\end{aligned}$$

where

$$Q(\theta) := (\cos\theta-\sin\theta)+\mathrm{i}(\cos\theta+\sin\theta).$$

Figure 37 shows the dependence of velocity v on the radius r at large $t \sim 10^8$ and line (1) at $\theta = \pi$, and line (2) at $\theta = 13\pi/14$. Compare it with Figure 35 and Figure 36.

Hence, Dark Matter and Dark Energy can be mirages in the space-time, which is curved by oscillations of chromatic states.

,

Conclusion

Fundamental Theoretical Physics contains sequences of theories, each of which is explained of previous ones by rules of the classical logic. For example, optics is absorbed by theory of electromagnetism, classical mechanics - by special theory of relativity and quantum theory, the theory of electromagnetism and weak interactions - by theory of electroweak interactions of Sheldon Glashow and so on. That means that basic notions and statements of every subsequent theory are more logical than basic notions and axioms of the preceding one.

When these basic elements of the theory become absolutely logical, i.e. when they become notions and rules of classical logic, theoretical physics will come to an end, it will rather be logic than physics.

———————————————-

Any subjects, connected with an information is called informational objects. For example, it can be a physics device, or computer disks and gramophone records, or people, carrying memory on events of their lifes, or trees, on cuts which annual rings tell on past climatic and ecological changes, or stones with imprints of long ago extincted plants and bestials, or minerals, telling on geological cataclysms, or celestial bodies, carrying an information on a remote distant past Universe, etc., etc.

It is clearly that an information, received from such information object, can be expressed by a text which made of sentences.

A set of sentences, expressing an information of some informational object, is called recorder of this object (p.15).

Obviously, the following conditions are satisfied:

I. A recorder does not kept logically hereafter refers to the classical propositional logic inconsistent sentence.

II. If a recorder contains some sentence then one contains all propositional consequences of that sentence.

+III. If recorder a contains sentence "recorder b contains sentence A" then recorder a contains sentence A.

For example, if recorder a contains sentence "recorder b contains sentence "Big Theorem is proved" " then recorder a contains sentence "Big Theorem is proved".

Some recorders systems form structures like clocks. The following results come from the logical properties of a set of recorders (p.16)

First, all such clocks have the same direction, i.e. if an event expressed by sentence A precedes an event expressed by sentence B according to one of such clocks then the same for others as well (p.18).

Secondly, time, according to this clock, is irreversible, i.e. there's no recorde which can receive information about an event that has happened until this event really happens. Thus, nobody can come back in past or receive information from future (p.32).

Thirdly, a set of recorders are naturally embedded into a metrical space, i.e. all four axioms of metrical space are received from logical properties of the set of recorders (p.23).

Fourthly, if this metrical space is Euclidean, then the corresponding ”space and time” of recorders obeys to transformations of the complete Poincare group. In this case Special Theory of Relativity follows the logical properties of information. If this metric space is not Euclidean then suitable non-linear geometry may be built on this space. And an appropriate version of the General Relativity Theory can be implemented in that space-time (pp.29–48).

Therefore, basic properties of time - unidirectionality and irreversibility, metrical properties of space and principles of the theory of relativity derive from logical properties of the set of recorders. Thus, if you have some set of objects, dealing with information, then ”time” and ”space” are inevitable. And it doesn’t matter whether this set is included in our world or some other worlds, which don’t have a space-time structure initially.

Such ”Time–Space” is called ”Informational Time–Space”.

Because we receive our time with our informational system then all other our times’ notions (thermodynamical time, cosmological time, psychological time, quantum time etc.) should be defined by that Informational Time.

As it is well known, classical propositional logic can be formulated on the basis of the properties of Boolean function. If the range of this function will be extended to the interval [0, 1] of the real number axes then we shall obtain the function which has all properties of the function of probability. Logical analogue of Law of Large Numbers in form of Bernoulli is derived for this function. So, probability theory is a generalization of classical propositional logic and, therefore, it is also propositional logic (pp.48–56).

I consider the events, each of which can bound to a certain point in space-time. Such events are called dot events[45]. Combinations (sums, products, supplements) of such events are events, called physical events.

The probability density of dot events in space-time is invariant under Lorentz transformations. But probability density of such events in space at a certain instant of time is not invariant under these transformations. I consider the dot events for which density of probability in space at some instant of time is the null component of a 3+1-vector function which is transformed by the Lorentz formulas (pp.58–59).

I call these probabilities the traceable probabilities.

It is known that Dirac’s equation contains four anticommutive complex 4X4 matrices. And this equation is not invariant under electroweak transformations. But it turns out that there is another such matrix anticommutive with all these four matrices. If additional mass term with this matrix will be added to Dirac’s equation then the resulting equation shall be invariant under these transformations I call these five of anticommutive complex 4X4 matrices Clifford pentade. There exist only six Clifford pentads I call one of them the light pentad, three - the chromatic pentads, and two - the gustatory pentads.

The light pentad contains three matrices corresponding to the coordinates of 3-dimensional space, and two matrices relevant to mass terms - one for the lepton and one for the neutrino of this lepton.

Each chromatic pentad also contains three matrices corresponding to three coordinates and two mass matrices - one for top quark and another - for bottom quark.

Each gustatory pentad contains one coordinate matrix and two pairs of mass matrices - these pentads are not needed yet (pp.59–60).

It is proven (pp.65–68, 80–82) that any square-integrable 4x1-matrix function with bounded domain (Planck's function) obeys some generalization of Dirac's equation with additional gauge members. This generalization is the sum of products of the coordinate matrices of the light pentad and covariant derivatives of the corresponding coordinates plus product of all the eight mass matrices (two of light and six of chromatic) and the corresponding mass numbers.

If this equation does not contain chromatic mass numbers then we obtain Dirac's equation for leptons with gauge members which are similar to electroweak fields obtained for gauge fields W and Z (pp.83–89, 106–139).

If this equation does not contain lepton's and neutrino's mass terms then we obtain the Dirac's equation with gauge members similar to eight gluon's fields (pp.141 –155). And oscillations of chromatic states of this equation bend space-time. This bend gives rise to the effects of redshift, confinement and asymptotic freedom, and Newtonian gravity turns out to be a continuation of subnucleonic forces (pp.155–157).

And it turns out that these oscillations bend space-time so that at large distance space expands with acceleration according to Hubble's law. And these oscillations bend space-time so that here appears the discrepancy between q uantity of the luminous matter in space structures and the traditional picture of gravitational interaction of stars in these structures (pp.157–162)

Thus, concepts and statements of Quantum Theory are concepts and statements of the probability of dot events and their ensembles.

Elementary physical particles in vacuum behave as these probabilities. For example, in accordance with doubleslit experiment.

Thus, if between event of the creating of a particle and event of the detecting of ones here events do not occur then at this period of time this particle does not exist - here only probability of this particle detecting in some point. But this probability, as we have seen, obeys the equations of quantum theory, and we get the interference. But in a cloud chamber events of condensation form a chain, meaning the trajectory of this particle. In this case the interference disappears. But this trajectory is not continuous - each point of this line has a neighbour point. And the effect of this particle moving arises from the fact that a wave of probability propagates between these points.

Consequently, the elementary physical particle represents an ensemble of dot events associated probabilities. And charge, mass, energy, momentum, spins, etc. represent parameters of distribution of these probabilities. It explains all paradoxes of quantum physics. Schrodinger's cat lives easy without any superposition of states until the microevent awaited by all occures. And the wave function disappears without any collapse in the moment when an event probability disappears after the event occurs (pp.71–79).

Thus, the fundamental essence of nature are not particles and fields, but dot events and connecting them probability.

Hence, the fundamental theoretical physics is one among of extensions of classical propositional logic.

Epilogue

"... They sawed dumb-bells ...

"What's the matter?" Balaganov said suddenly, stopping work. "I've been sawing away for three hours, and still it isn't gold!" Panikovsky did not reply. He had made the discovery a half hour before, and had continued to move the saw only for the sake of appearance. "Well, let's saw some more," redhaired Shura said gallantly. "Of course we must saw," remarked Panikovsky, trying to defer the moment of reckoning as long as possible. ... "I can't make it out," said Shura, when he had sawed the dumbbell into two halves. "This is not gold!" "Go on sawing! Go on!" gabbled Panikovsky..."

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