Five-Dimensional Tangent Vectors in Space-Time
III. Some Applications

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Abstract

In this part of the series I show how five-tensors can be used for describing in a coordinate-independent way finite and infinitesimal Poincare transformations in flat space-time. As an illustration, I reformulate the classical mechanics of a perfectly rigit body in terms of the analogs of five-vectors in three-dimensional Euclidean space. I then introduce the notion of the bivector derivative for scalar, four-vector and four-tensor fields in flat space-time and calculate its analog in three-dimensional space for the Lagrange function of a system of several point particles in classical nonrelativistic mechanics.

A. *Preliminary remarks*

Before developing the mathematical theory of five-vectors further, it will be useful to consider some of their applications in flat space-time. At the same time I will say a few words about the analogs of five-vectors in three-dimensional Euclidean space and will consider some of their applications.

In the case of vectors of the latter type there arises a problem with terminology. Since the analogs of five-vectors in three-dimensional space are four-dimensional, by analogy with five-vectors one should term them as *four*-vectors. This, of course, is unacceptable since the term 'four-vector' is traditionally used for referring to ordinary tangent vectors in space-time. To avoid confusion, in the following the analogs of five-vectors in three-dimensional space will be called (3+1)-vectors. It makes sense to use the same terminology for all other manifolds as well. The only exception from this convention will be made for five-vectors in space-time, which will still be called five-vectors and not (4+1)-vectors.

Let me also say a few words about the notations. Ordinary tangent vectors in three-dimensional Euclidean space will be denoted with capital Roman letters with an arrow: $\vec{A}$, $\vec{B}$, $\vec{C}$, etc. Three-plus-one-vectors will be denoted with lower-case Roman letters with an arrow: $\vec{a}$, $\vec{b}$, $\vec{c}$, etc. It will be taken that lower-case latin indices run 1, 2, and 3 and that capital Greek indices run 1, 2, 3, and 5 (the value "5" corresponds to the additional dimension of the space of (3+1)-vectors). Finally, the Euclidean inner product will be denoted with a dot or with the symbol δ, which should not be confused with the unit tensor of rank $(1,1)$.

In conclusion, let me write out the formulae that express the result of transporting parallelly an active regular five-vector basis associated with some system of Lorentz coordinates, from the origin of the latter to the point with coordinates x^μ:

$$\left\{ \begin{array}{l} [\mathbf{e}_\alpha(0)]^{\text{transported to } x} = \mathbf{e}_\alpha(x) + x_\alpha\, \mathbf{e}_5(x) \\ [\mathbf{e}_5(0)]^{\text{transported to } x} = \mathbf{e}_5(x). \end{array} \right. \tag{1}$$

The analogs of these formulae for (3+1)-vectors are

$$\left\{ \begin{array}{l} [\vec{e}_i(0)]^{\text{transported to } x} = \vec{e}_i(x) + x_i\, \vec{e}_5(x) \\ [\vec{e}_5(0)]^{\text{transported to } x} = \vec{e}_5(x), \end{array} \right. \tag{2}$$

where $x_i \equiv \delta_{ij} x^j$ and x^j is the system of Cartesian coordinates with which the considered active regular basis of (3+1)-vectors is associated.

B. *Active Poincare transformations in the language of five-vectors*

The replacement of any given set of scalar, four-vector and four-tensor fields in flat space-time with an equivalent set of fields, as it was discussed in section 4 of part I, and a similar replacement of five-vector and five-tensor fields, are nothing but active Poincare transformations. Though the procedure for constructing the equivalent fields, described in the same section of part I for four-vector and four-tensor fields and in section 3 of part II for five-vector and

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five-tensor fields, makes use of Lorentz coordinate systems, the latter are only a tool for performing such a construction. In itself, any such active field transformation can be considered without referring to any coordinates and in this sense is an invariant operation. Let us now see how one can give it a coordinate-independent description.

From the very procedure for constructing the equivalent fields it follows that any active Poincare transformation can be characterised by the parameters $L^\alpha_{\ \beta}$ and a^α that specify the transition from the selected initial Lorentz coordinate system x^α to the corresponding final Lorentz coordinate system $x'^{\,\alpha}$:

$$x'^{\,\alpha} = L^\alpha_{\ \beta}\, x^\beta + a^\alpha. \qquad (3)$$

Since parameters $L^\alpha_{\ \beta}$ and a^α explicitly depend on the choice of the initial coordinate system, such a description will naturally be non-invariant. To reduce the dependence on the choice of the coordinate system, instead of $L^\alpha_{\ \beta}$ and a^α one speaks of the displacement vector and rotation tensor. Namely, in the selected initial system of Lorentz coordinates, from the corresponding transformation parameters $L^\alpha_{\ \beta}$ and a^α, basis four-vectors $\mathbf{E}_\alpha$, and dual basis four-vector 1-forms $\widetilde{\mathbf{O}}^\alpha$ one constructs the four-vector $\mathbf{A} \equiv a^\alpha \mathbf{E}_\alpha$ and four-tensor $\mathbf{B} \equiv (L^\alpha_{\ \beta} - \delta^\alpha_{\ \beta})\, \mathbf{E}_\alpha \otimes \widetilde{\mathbf{O}}^\alpha$. By using the transformation formulae for $L^\alpha_{\ \beta}$ and a^α derived in section 5 of part I, one can easily show that $\mathbf{B}$ will be the same at any choice of the initial Lorentz coordinate system and that $\mathbf{A}$ will be the same for any two systems with the same origin. For systems with different origins $\mathbf{A}$ will in general be different, so by using the four-vector quantities $\mathbf{A}$ and $\mathbf{B}$ one cannot get rid of the dependence on the choice of the initial Lorentz coordinate system completely.

To obtain a completely coordinate-independent description of a given active Poincare transformation, let us construct in the selected Lorentz coordinate system a five-tensor field $\boldsymbol{\mathcal{T}}$ which in the P-basis associated with these coordinates has the following constant components:

$$\left\{ \begin{array}{ll} \mathcal{T}^\alpha_{\ \beta} = \Lambda^\alpha_{\ \beta}, & \mathcal{T}^\alpha_{\ 5} = 0 \\ \mathcal{T}^5_{\ \beta} = a_\beta, & \mathcal{T}^5_{\ 5} = 1, \end{array} \right. \qquad (4)$$

where $a_\beta \equiv g_{\beta\sigma} a^\sigma$ and $\Lambda^\alpha_{\ \beta} \equiv (L^{-1})^\alpha_{\ \beta}$ is the inverse of the 4×4 matrix $L^\alpha_{\ \beta}$. As it has been shown in section 5 of part I, the quantities $\mathcal{T}^A_{\ B}$ transform as components of a five-tensor of rank $(1,1)$, and consequently the field $\boldsymbol{\mathcal{T}}$ will be the same at any choice of the initial Lorentz coordinate system.

To understand why the components of $\boldsymbol{\mathcal{T}}$ are constructed from the transformation parameters for covariant coordinates and not from the transformation parameters for the Lorentz coordinates themselves, let us introduce this five-tensor in a slightly different way. Instead of the two coordinate systems x^α and $x'^{\,\alpha}$, let us consider the corresponding P-bases, $\mathbf{p}_A$ and $\mathbf{p}'_A$. Since at each point the vectors $\mathbf{p}'_A$ are expressed linearly in terms of $\mathbf{p}_A$, the former can be obtained by acting on the latter with some linear operator, $\boldsymbol{\mathcal{T}}$. Since all P-bases are self-parallel by definition, $\boldsymbol{\mathcal{T}}$ regraded as a five-tensor field of rank $(1,1)$ will be covariantly constant. If $\widetilde{\mathbf{q}}^A$ and $\widetilde{\mathbf{q}}'^{\,A}$ are the bases of five-vector 1-forms dual to $\mathbf{p}_A$ and $\mathbf{p}'_A$, respectively, one can present the tensor $\boldsymbol{\mathcal{T}}$ as

$$\boldsymbol{\mathcal{T}} = \mathbf{p}'_A \otimes \widetilde{\mathbf{q}}^A,$$

for in this case

$$\boldsymbol{\mathcal{T}}(\mathbf{p}_A) \equiv \mathbf{p}'_B < \widetilde{\mathbf{q}}^B, \mathbf{p}_A > = \mathbf{p}'_A.$$

Furthermore,

$$\boldsymbol{\mathcal{T}}(\widetilde{\mathbf{q}}'^{\,A}) = < \widetilde{\mathbf{q}}'^{\,A}, \mathbf{p}'_B > \widetilde{\mathbf{q}}^B = \widetilde{\mathbf{q}}^A,$$

so when acting on 1-forms, the operator $\boldsymbol{\mathcal{T}}$ performs the reverse transformation. The inverse of $\boldsymbol{\mathcal{T}}$ is apparently $\boldsymbol{\mathcal{T}}^{-1} = \mathbf{p}_A \otimes \widetilde{\mathbf{q}}'^{\,A}$.

It is not difficult to prove that the field $\boldsymbol{\mathcal{T}}$ defined this way will be the same at any choice of the initial P-basis. Indeed, when acting on the fields $\mathbf{p}_A$ of the initial P-basis, the operator $\boldsymbol{\mathcal{T}}$ performs the considered active field transformation, and since this transformation is linear, $\boldsymbol{\mathcal{T}}$ will act the same way on all other covariantly constant five-vector fields, including the basis fields of any other P-basis.

Let us now find the components of $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{T}}^{-1}$ in the basis $\mathbf{p}_A \otimes \widetilde{\mathbf{q}}^B$. Since

$$\mathbf{p}'_\alpha = \mathbf{p}_\beta \Lambda^\beta_{\ \alpha} + a_\alpha \mathbf{p}_5 \quad \text{and} \quad \mathbf{p}'_5 = \mathbf{p}_5,$$

one has

$$\begin{array}{rcl} \boldsymbol{\mathcal{T}} & = & (\mathbf{p}_\beta \Lambda^\beta_{\ \alpha} + a_\alpha \mathbf{p}_5) \otimes \widetilde{\mathbf{q}}^\alpha + \mathbf{p}_5 \otimes \widetilde{\mathbf{q}}^5 \\ & = & \mathbf{p}_\alpha \otimes \widetilde{\mathbf{q}}^\beta \cdot \Lambda^\alpha_{\ \beta} + \mathbf{p}_5 \otimes \widetilde{\mathbf{q}}^\beta \cdot a_\beta \\ & & + \, \mathbf{p}_\alpha \otimes \widetilde{\mathbf{q}}^5 \cdot 0 + \mathbf{p}_5 \otimes \widetilde{\mathbf{q}}^5 \cdot 1, \end{array}$$

so in this basis the compoents of $\boldsymbol{\mathcal{T}}$ are given by formulae (4). In a similar manner one can find the components of $\boldsymbol{\mathcal{T}}^{-1}$:

$$\left\{ \begin{array}{ll} (\mathcal{T}^{-1})^\alpha_{\ \beta} = L^\alpha_{\ \beta}, & (\mathcal{T}^{-1})^\alpha_{\ 5} = 0 \\ (\mathcal{T}^{-1})^5_{\ \beta} = -\,a_\gamma L^\gamma_{\ \beta}, & (\mathcal{T}^{-1})^5_{\ 5} = 1. \end{array} \right.$$

One should not mix the latter up with the quantities

$$\left\{ \begin{array}{ll} \mathcal{T}^\alpha_{\ \beta} = L^\alpha_{\ \beta}, & \mathcal{T}^\alpha_{\ 5} = a^\beta \\ \mathcal{T}^5_{\ \beta} = 0, & \mathcal{T}^5_{\ 5} = 1 \end{array} \right. \qquad (5)$$

constructed from the transformation parameters for the Lorentz coordinates themselves. What the latter are is explained in Appendix.

Let us now consider an infinitesimal Poincare transformation. In this case the matrix $L^{\alpha}_{\ \beta}$ in formula (3) can be presented as

$$L^{\alpha}_{\ \beta} = \delta^{\alpha}_{\beta} + \omega^{\alpha}_{\ \beta},$$

where $\omega^{\alpha}_{\ \beta}$ are infinitesimals that satisfy the condition

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0, \tag{6}$$

where $\omega_{\alpha\beta} \equiv g_{\alpha\sigma}\omega^{\sigma}_{\ \beta}$. The parameters a^{α} in formula (3) are also infinitesimals and accordingly the field $\boldsymbol{\mathcal{T}}$ that describes this infinitesimal Poincare transformation can be presented as

$$\boldsymbol{\mathcal{T}} = \mathbf{1} - \boldsymbol{\mathcal{S}},$$

where $\mathbf{1}$ is the unity five-tensor and tensor $\boldsymbol{\mathcal{S}}$ has the components

$$\mathcal{S}^{\alpha}_{\ \beta} = \omega^{\alpha}_{\ \beta},\ \mathcal{S}^{5}_{\ \beta} = -\,a_{\beta},\ \mathcal{S}^{A}_{\ 5} = 0, \tag{7}$$

and therefore

$$\boldsymbol{\mathcal{S}} = (\mathbf{p}_{\alpha}\omega^{\alpha}_{\ \beta} - \mathbf{p}_5 a_{\beta}) \otimes \widetilde{\mathbf{q}}^{\beta}. \tag{8}$$

It is evident that the latter expression for $\boldsymbol{\mathcal{S}}$ will be valid in any P-basis.

In view of condition (6), it is more convenient to deal not with $\boldsymbol{\mathcal{S}}$ itself, but with the corresponding five-tensor whose indices are either both lower or both upper. It is essential that this latter five-tensor should be related to $\boldsymbol{\mathcal{S}}$ by the map ϑ_g and not by ϑ_h (see their definition in section 3 of Part II), for the quantities $\omega_{\alpha\beta}$ in the left-hand side of equation (6) are obtained from parameters $\omega^{\alpha}_{\ \beta}$ by contraction with the matrix $g_{\alpha\beta}$, not h_{AB}. (Besides, if one lowers or raises an index of $\boldsymbol{\mathcal{S}}$ with h_{AB}, one will obtain a covariantly *non*constant tensor field.) Since

$$\vartheta_g(\mathbf{p}_{\alpha}) = \widetilde{\mathbf{q}}^{\beta}\eta_{\beta\alpha} \ \text{ and } \ \vartheta_g(\mathbf{p}_5) = \mathbf{0}, \tag{9}$$

the completely covariant tensor related to $\boldsymbol{\mathcal{S}}$ by ϑ_g is

$$\vartheta_g(\mathbf{p}_{\alpha}\omega^{\alpha}_{\ \beta} - \mathbf{p}_5 a_{\beta}) \otimes \widetilde{\mathbf{q}}^{\beta} = \omega_{\alpha\beta}\, \widetilde{\mathbf{q}}^{\alpha} \otimes \widetilde{\mathbf{q}}^{\beta}, \tag{10}$$

and is apparently the five-vector equivalent of the completely covariant four-tensor of infinitesimal rotation: $\mathbf{R} = \omega_{\alpha\beta}\, \widetilde{\mathbf{O}}^{\alpha} \otimes \widetilde{\mathbf{O}}^{\beta}$. However, tensor (10) has only six independent components (which is a consequence of ϑ_g being noninjective) and therefore does not describe the considered infinitesimal Poincare transformation completely.

Let us now observe that according to formulae (8) and (9), $\boldsymbol{\mathcal{S}}$ can be obtained from a completely contravariant five-tensor of rank 2 by lowering one of the indices of the latter with $g_{\alpha\beta}$. Indeed, denoting this contravariant five-tensor as $\boldsymbol{\mathcal{R}}$, by virtue of formulae (9) one has

$$\mathcal{R}^{AB}\, \mathbf{p}_A \otimes \vartheta_g(\mathbf{p}_B) = (\mathbf{p}_{\alpha}\mathcal{R}^{\alpha}_{\ \beta} + \mathbf{p}_5\mathcal{R}^{5}_{\ \beta}) \otimes \widetilde{\mathbf{q}}^{\beta},$$

where $\mathcal{R}^{A}_{\ \beta} \equiv \mathcal{R}^{A\xi} g_{\xi\beta}$, and comparing the right-hand side of the latter equation with formula (8) one finds that

$$\mathcal{R}^{\alpha\beta} = \omega^{\alpha}_{\ \xi}\, g^{\xi\beta} \equiv \omega^{\alpha\beta} \ \text{ and } \ \mathcal{R}^{5\beta} = -\,a^{\beta}. \tag{11}$$

The components $\mathcal{R}^{A5}$ are not fixed by the components of $\boldsymbol{\mathcal{S}}$ and can be selected arbitrarily. A particularly convenient choice is

$$\mathcal{R}^{\alpha 5} = -\,\mathcal{R}^{5\alpha} = a^{\alpha} \ \text{ and } \ \mathcal{R}^{55} = 0. \tag{12}$$

In this case $\boldsymbol{\mathcal{R}}$ becomes completely antisymmetric and consequently has only 10 independent components, i.e. exactly as many as does the five-tensor $\boldsymbol{\mathcal{S}}$. Since antisymmetry is an invariant property, equations (12) will hold in any five-vector basis. Moreover, $\boldsymbol{\mathcal{R}}$ will be a covariantly constant field. By analogy with the formula for generators of Lorentz transformations for four-vectors, the relation between $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{S}}$ can be presented as

$$\mathcal{S}^{A}_{\ B} = -\,\tfrac{1}{2}\,\mathcal{R}^{KL}\,(M_{KL})^{A}_{\ B} = -\mathcal{R}^{|KL|}\,(M_{KL})^{A}_{\ B}, \tag{13}$$

where, as usual, the vertical bars around the indices mean that summation extends only over $K < L$, and the quantities

$$(M_{KL})^{A}_{\ B} \equiv \delta^{A}_{L}\, g_{KB} - \delta^{A}_{K}\, g_{LB} \tag{14}$$

are the analogs of the Lorentz generators $(M_{\mu\nu})^{\alpha}_{\ \beta} \equiv \delta^{\alpha}_{\nu}\, g_{\mu\beta} - \delta^{\alpha}_{\mu}\, g_{\nu\beta}$.

As any other five-vector bivector, $\boldsymbol{\mathcal{R}}$ can be invariantly decomposed into a part made only of five-vectors from $\mathcal{Z}$ and a part which is the wedge product of a five-vector from $\mathcal{E}$ with some other five-vector. In the following these two parts of $\boldsymbol{\mathcal{R}}$ will be called its $\mathcal{Z}$- and $\mathcal{E}$-components, respectively. Since $\mathcal{Z}$ is isomorphic to V_4, at each space-time point to the $\mathcal{Z}$-component of $\boldsymbol{\mathcal{R}}$ one can put into correspondence a certain four-vector bivector, which, as is seen from equation (11), is the completely contravariant form of the infinitesimal rotation four-tensor $\mathbf{R}$ introduced above and which for this reason I will denote with the same letter. It is easy to check that the four-tensor field $\mathbf{R}$ obtained this way is covariantly constant and that if $\mathbf{E}_{\alpha}$ is some four-vector basis and $\mathbf{e}_A$

is the associated active regular five-vector basis, then the components of $\mathbf{R}$ in the basis $\mathbf{E}_\alpha \otimes \mathbf{E}_\beta$ equal the components $\mathcal{R}^{\alpha\beta}$ of $\boldsymbol{\mathcal{R}}$ in the basis $\mathbf{e}_A \otimes \mathbf{e}_B$.

In a similar way, since the maximal vector space of simple bivectors over V_5 with the direction vector from $\mathcal{E}$ is isomorphic to V_4, at each space-time point to the $\mathcal{E}$-component of $\boldsymbol{\mathcal{R}}$ one can put into correspondence a certain four-vector $\mathbf{A}$. For practical reasons, it is more convenient to establish the isomorphism between the above two vector spaces not as it has been done in section 3 of part II, but in a slightly different way: supposing that the space of bivectors is endowed not with the inner product induced by h, but with the inner product differing from the latter by the factor ξ^{-1}. In this case the components of $\mathbf{A}$ in the basis $\mathbf{E}_\alpha$ will equal the components $\mathcal{R}^{\alpha 5}$ of $\boldsymbol{\mathcal{R}}$ in the basis $\mathbf{e}_A \otimes \mathbf{e}_B$, so $\mathbf{A}$ itself will coincide with the infinitesimal displacement four-vector. At $\mathbf{R} \neq \mathbf{0}$ the four-vector field $\mathbf{A}$ will not be covariantly constant, which is in agreement with the fact that the values of $\mathbf{R}$ and $\mathbf{A}$ at any given point Q determine the rotation and translation of a Lorentz coordinate system with the origin at Q that one has to make to perform the considered active Poincare transformation.

C. *Motion of a perfectly rigit body in the language of three-plus-one-vectors*

Everything that has been said above about the description of active Poincare transformations in flat space-time in terms of five-tensors can be applied, with obvious modifications, to the case of flat three-dimensional Euclidean space. Instead of five-vectors one should now speak about (3+1)-vectors and instead of Poincare transformations, about transformations from the group of motions of three-dimensional Euclidean space. I will now show how the formalism developed in the previous section can be applied for describing the motion of a perfectly rigit body in classical nonrelativistic mechanics. In order not to introduce new notations, I will denote the analogs of tensors $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{R}}$ in three-dimensional space with the same symbols.

Owing to the absolute rigidity of the body in question, its motion can be viewed as an active transformation of the fields (discrete or continuous) that describe the distribution of matter inside the body—a transformation that develops in time. Accordingly, the change in the position of the body that occurs over a finite time period t can be described invariantly with a certain (3+1)-tensor, $\boldsymbol{\mathcal{T}}(t)$, and the rate of this change can be described with a certain antisymmetric (3+1)-tensor, $\boldsymbol{\mathcal{W}}$, equal to the ratio of the (3+1)-tensor $\boldsymbol{\mathcal{R}}(dt)$ that describes to the first order the infinitesimal transformation that corresponds to the change in the body position over the time dt to the magnitude of this time interval:

$$\boldsymbol{\mathcal{W}} \equiv \boldsymbol{\mathcal{R}}(dt)/dt. \tag{15}$$

In the following, $\boldsymbol{\mathcal{W}}$ will be referred to as the *velocity bivector* of the body.

As in the case of $\boldsymbol{\mathcal{R}}$, at every point in space one can put into correspondence to $\boldsymbol{\mathcal{W}}$ a pair consisting of a three-vector, which will be denoted as $-\vec{V}$, and of a three-vector bivector. Since the space is three-dimensional, it is more convenient to deal not with the latter bivector itself, but with the three-vector dual to it, which I will denote as $\vec{\Omega}$. If $\vec{e}_\Theta$ is some active regular basis of (3+1)-vectors and $\vec{E}_i$ is the associated three-vector basis, then the components of $\vec{V}$ and $\vec{\Omega}$ in the latter are related to the components of $\boldsymbol{\mathcal{W}}$ in the basis $\vec{e}_\Theta \otimes \vec{e}_\Sigma$ in the following way:

$$\left\{ \begin{array}{lllll} \mathcal{W}^{5i} & = & -\mathcal{W}^{i5} & = & V^i \\ \mathcal{W}^{ij} & = & -\mathcal{W}^{ji} & = & \epsilon^{ij}{}_k\, \Omega^k . \end{array} \right. \tag{16}$$

It is not difficult to show that at each point in space $\vec{V}$ coincides with the translational velocity of a frame rigitly fixed to the body and with the origin at that point. Similarly, $\vec{\Omega}$ can be shown to coincide with the angular velocity of this frame.

Let us introduce in space an arbitrary system of Cartesian coordinates and consider the vectors $\vec{V}$ and $\vec{\Omega}$ at the point with coordinates x^i. Since the field $\boldsymbol{\mathcal{R}}$ is covariantly constant, so is the field $\boldsymbol{\mathcal{W}}$, and consequently the value of $\boldsymbol{\mathcal{W}}$ at the considered point can be obtained by transporting parallelly to this point the value of $\boldsymbol{\mathcal{W}}$ at the origin. By using formulae (2) one can easily find that in the O-basis associated with the selected coordinates,

$$\left\{ \begin{array}{lll} \mathcal{W}^{5j}(x) & = & \mathcal{W}^{5j}(0) + x_i\, \mathcal{W}^{ij}(0) \\ \mathcal{W}^{ij}(x) & = & \mathcal{W}^{ij}(0), \end{array} \right.$$

and substituting the components of $\vec{V}$ and $\vec{\Omega}$ for those of $\boldsymbol{\mathcal{W}}$, one obtains

$$\left\{ \begin{array}{lll} V^i(x) & = & V^i(0) + \epsilon^i{}_{jk}\, \Omega^j(0)\, x^k \\ \Omega^i(x) & = & \Omega^i(0). \end{array} \right.$$

In view of the meaning the vectors $\vec{V}$ and $\vec{\Omega}$ have at a given point, from the latter formulae follows the well-known rule for transformation of translational and angular veclocities as one transfers the origin of the moving frame for which they are defined to another point:

$$\vec{V}' = \vec{V} + \vec{\Omega} \times \vec{X} \quad \text{and} \quad \vec{\Omega}' = \vec{\Omega}, \tag{17}$$

where $\vec{X}$ is the position vector that connects the old origin with the new one.

Let us now suppose that there is a particle of the body at the point with coordinates x^i. Since at any given moment, the velocity of this particle coincides with the translational veclocity of the frame connected to the body with the origin at that point, from equations (17) follows another well-known relation:

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r},$$

where $\vec{v}$ is the particle velocity and $\vec{r}$ is its position vector relative to the frame for which $\vec{V}$ and $\vec{\Omega}$ are defined.

Let us now consider the expression for kinetic energy. As one knows, in the general case the latter can be presented as a sum of three terms: (*i*) a term bilinear in $\vec{V}$ and independent of $\vec{\Omega}$, (*ii*) a term bilinear in $\vec{\Omega}$ and independent of $\vec{V}$, and (*iii*) a term linear both in $\vec{V}$ and in $\vec{\Omega}$. Since with respect to the three-dimensional space the kinetic energy is a scalar, the above means that in terms of (3+1)-tensors it can be presented in the following form:

$$E_{\text{kin.}} = \tfrac{1}{2}\, \mathcal{I}_{\Gamma\Delta\Theta\Sigma}\, \mathcal{W}^{|\Gamma\Delta|} \mathcal{W}^{|\Theta\Sigma|}, \tag{18}$$

where $\mathcal{I}_{\Gamma\Delta\Theta\Sigma}$ are components of some (3+1)-tensor of rank 4, which by definition have the following symmetry properties:

$$\mathcal{I}_{\Gamma\Delta\Theta\Sigma} = \mathcal{I}_{\Theta\Sigma\Gamma\Delta}$$

and

$$\mathcal{I}_{\Gamma\Delta\Theta\Sigma} = -\,\mathcal{I}_{\Delta\Gamma\Theta\Sigma} = -\,\mathcal{I}_{\Gamma\Delta\Sigma\Theta}.$$

It is obvious that the kinetic energy of a single particle can be presented in the same form. By comparing the right-hand side of formula (18) with the usual expression for the kinetic energy of a point particle in classical nonrelativistic mechanics, and considering that at the point where the particle is located, $\vec{V}$ coincides with the particle velocity vector, one finds that in this case

$$\mathcal{I}_{5i5j} = -\,\mathcal{I}_{i55j} = -\,\mathcal{I}_{5ij5} = \mathcal{I}_{i5j5} = m \cdot \delta_{ij}, \tag{19}$$

where m is the particle mass, and all other components of $\boldsymbol{\mathcal{I}}$ are zero (here and below I omit the indices that numerate the particles). To express the kinetic energy of the body as a whole in terms of the velocities $\vec{V}$ and $\vec{\Omega}$ that correspond to some moving frame with the origin at point O, let us make use of the fact that the contraction of (3+1)-tensors is conserved by parallel transport, so the kinetic energy of a given particle of the body equals the contraction of two samples of tensor $\boldsymbol{\mathcal{W}}$ at O with the tensor $\boldsymbol{\mathcal{I}}$ corresponding to this particle, transported from the point where the particle is located to O. Thus, for every particle

$$\begin{aligned} E_{\text{kin.}} = \tfrac{1}{2}\, &(\boldsymbol{\mathcal{I}}^{\,\text{transported}})_{\,5i5j}\, V^i\, V^j \\ &+ (\boldsymbol{\mathcal{I}}^{\,\text{transported}})_{\,5ijk}\, V^i\, \epsilon^{|jk|}{}_{l}\, \Omega^l \\ &+ \tfrac{1}{2}\, (\boldsymbol{\mathcal{I}}^{\,\text{transported}})_{\,ijkl}\, \epsilon^{|ij|}{}_{m}\, \epsilon^{|kl|}{}_{n}\, \Omega^m\, \Omega^n, \end{aligned} \tag{20}$$

and the kinetic energy of the body as a whole is the sum of the expressions in the right-hand side of this formula, taken over all the particles. If the considered particle is located at the point with coordinates x^i, then by using formulae (2) one can easily find that

$$\begin{aligned} &(\boldsymbol{\mathcal{I}}^{\,\text{transported}})_{\,5i5j} = m\, \delta_{ij} \\ &(\boldsymbol{\mathcal{I}}^{\,\text{transported}})_{\,5ijk}\, \epsilon^{|jk|}{}_{l} = m\, \epsilon_{il}{}^{j}\, x_j \\ &(\boldsymbol{\mathcal{I}}^{\,\text{transported}})_{\,ijkl}\, \epsilon^{|ij|}{}_{m}\, \epsilon^{|kl|}{}_{n} \\ &\qquad\qquad = m\, (\delta_{mn}\, x_i x^i - x_m x_n), \end{aligned} \tag{21}$$

and substituting these expressions into formula (20), one obtains the usual expression for the kinetic energy of a rigit body in terms of $\vec{V}$ and $\vec{\Omega}$:

$$E_{\text{kin.}} = \tfrac{1}{2}\, (\textstyle\sum m)\, (\vec{V})^2 + \vec{V} \cdot \vec{\Omega} \times (\textstyle\sum m\vec{r}) + \tfrac{1}{2}\, I_{ij}\, \Omega^i\, \Omega^j,$$

where the sum goes over all the particles and

$$I_{ij} \equiv \sum m\, (\delta_{ij}\, x_k x^k - x_i x_j).$$

Thus, the moments of inertia of the body with respect to O can be found by dualizing the $ijkl$-components of the summary (3+1)-tensor $\sum \boldsymbol{\mathcal{I}}$ at O with respect to the indices i and j and with respect to the indices k and l by using the *three*-dimensional ϵ tensor. To find the moments of inertia relative to any other point O', one should simply transport the tensor $\sum \boldsymbol{\mathcal{I}}$ from O to O' according to the rules of parallel transport for (3+1)-tensors and dualize its $ijkl$-components at that point.

Tensor $\boldsymbol{\mathcal{I}}$ can be contracted with only one sample of tensor $\boldsymbol{\mathcal{W}}$. One will then obtain a completely covariant antisymmetric (3+1)-tensor of rank two, which I will denote as $\boldsymbol{\mathcal{M}}$, with the components

$$\mathcal{M}_{\Gamma\Delta} \;=\; \mathcal{I}_{\Gamma\Delta\Theta\Xi}\, \mathcal{W}^{|\Theta\Xi|}. \tag{22}$$

As in the case of $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{W}}$, at each point in space one can put into correspondence to $\boldsymbol{\mathcal{M}}$ a certain pair consisting of a three-vector 1-form and a three-vector 2-form. For practical reasons it is convenient first to replace the 2-form with the 1-form dual to it, thereby obtaining instead of the original pair a pair consisting of two three-vector 1-forms, and then replace these latter 1-forms with the corresponding three-vectors. As a result, to $\boldsymbol{\mathcal{M}}$ there will correspond a pair consisting of two three-vectors, which I will denote as $-\vec{P}$

and $\vec{M}$. If $\widetilde{o}^\Sigma$ is the basis of three-plus-one-vector 1-forms dual to the basis $\vec{e}_\Theta$ introduced above, then the components of $\vec{P}$ and $\vec{M}$ in the associated three-vector basis $\vec{E}_i$ will be related to the components of $\boldsymbol{\mathcal{M}}$ in the basis $\widetilde{o}^\Theta \otimes \widetilde{o}^\Sigma$ as follows:

$$P^i = \delta^{ij} \mathcal{M}_{5j} \quad \text{and} \quad M^i = \tfrac{1}{2}\, \epsilon^{ijk} \mathcal{M}_{jk}.$$

Let us now calculate $\boldsymbol{\mathcal{M}}$ for a single point particle. According to equations (19), at the point where the latter is located

$$\mathcal{M}_{5i} = m\, \delta_{ij} v^j = m v_i \quad \text{and} \quad \mathcal{M}_{ij} = 0,$$

so in this case $\vec{P}$ coincides with the particle momentum three-vector, and $\vec{M} = 0$. Let us now suppose that in some system of Cartesian coordinates the considered particle has the coordinates x^i. Let us transport $\boldsymbol{\mathcal{M}}$ to the origin of this system. Then, according to formulae (2), in the O-basis associated with these coordinates,

$$\begin{aligned} \mathcal{M}_{5j}(0) &= \mathcal{M}_{5j}(x) = m v_j \\ \mathcal{M}_{ij}(0) &= x_i \mathcal{M}_{5j}(x) + x_j \mathcal{M}_{i5}(x) \\ &= m\,(x_i v_j - x_j v_i), \end{aligned}$$

and consequently the pair $(-\vec{P}, \vec{M})$ corresponding to the transported $\boldsymbol{\mathcal{M}}$ is such that $\vec{P}$ coincides with the particle momentum three-vector transported to the origin according to ordinary rules of parallel transport for three-vectors, and $\vec{M}$ coincides with the three-vector of the particle angular momentum relative to the origin. Since the latter can be selected arbitrarily, this correspondence between $\boldsymbol{\mathcal{M}}$ and the particle momentum and angular momentum will exist at every point. Naturally, each of the three-vectors in the pair $(-\vec{P}, \vec{M})$ can be transported to any other point in space according to the rules of parallel transport for three-vectors, however, the pair as a whole will correspond to the (3+1)-tensor $\boldsymbol{\mathcal{M}}$ only at the point with respect to which the angular momentum is defined.

We thus see that the momentum and angular momentum in classical nonrelativistic mechanics can be described by a single geometric object—by the antisymmetric (3+1)-tensor $\boldsymbol{\mathcal{M}}$. It is natural to called the latter the *momentum–angular momentum tensor*. One of the advantages of such a description compared to the description in terms of three-vectors is that $\boldsymbol{\mathcal{M}}$ can be defined in a purely *local* way, with no reference to any other point in space. For example, in the case of a single point particle, at the point where the latter is located the $\widetilde{\mathcal{E}}$-component of $\boldsymbol{\mathcal{M}}$ is expressed in terms of the particle momentum and the $\widetilde{\mathcal{Z}}$-component of $\boldsymbol{\mathcal{M}}$ is zero. Having defined the tensor $\boldsymbol{\mathcal{M}}$ this way, one can then transport it to any other point in space. This transport will result in that $\boldsymbol{\mathcal{M}}$ will acquire a nonzero $\widetilde{\mathcal{Z}}$-component, which will be exactly the angular momentum of the particle relative to the point where $\boldsymbol{\mathcal{M}}$ has been transported to.

In order to calculate the momentum–angular momentum tensor for a *system* of point particles, one should first transport the tensors $\boldsymbol{\mathcal{M}}$ corresponding to all the particles to one point in space. It is at this stage that the tensors $\boldsymbol{\mathcal{M}}$ of individual particles will acquire nonzero $\widetilde{\mathcal{Z}}$-components, which, when summed up, will give the total angular momentum of the system relative to the selected point. In the particular case where the system is a rigit body, its total momentum and total angular momentum can be expressed in terms of the velocities $\vec{V}$ and $\vec{\Omega}$ corresponding to some frame connected to the body, with the origin at point O. To do this, one should follow the same procedure that has been used above for calculating the kinetic energy: one should transport the tensors $\boldsymbol{\mathcal{I}}$ corresponding to individual particles of the body to O, sum them up there, and then contract the sum $\sum \boldsymbol{\mathcal{I}}$ with the velocity bivector of the body at that point. As a result, one will obtain the usual expressions for momentum and angular momentum of the rigit body in terms of $\vec{V}$ and $\vec{\Omega}$, which I will not present here.

Let us now discuss the equations of motion. In the case of a single point particle one has:

$$d\vec{P}/dt = \vec{F}, \tag{23}$$

where $\vec{P}$ is the particle momentum and $\vec{F}$ is the acting force. Since $\vec{P}$ corresponds to the $\widetilde{\mathcal{E}}$-component of the (3+1)-tensor $\boldsymbol{\mathcal{M}}$, one may suppose that the above equation of motion corresponds to the $\widetilde{\mathcal{E}}$-component of some three-plus-one-tensor equation. One should expect that the left-hand side of this latter equation is the time derivative of $\boldsymbol{\mathcal{M}}$ and that its right-hand side is some antisymmetric (3+1)-tensor of rank 2, which I will denote as $\boldsymbol{\mathcal{K}}$. Thus, the three-plus-one-tensor equation will have the form

$$d\boldsymbol{\mathcal{M}}/dt = \boldsymbol{\mathcal{K}}, \tag{24}$$

and now we should determine how $\boldsymbol{\mathcal{K}}$ is related to the known three-vector quantities.

Let us introduce in space some system of Cartesian coordinates and let $x(t)$ denote the trajectory of the particle. To evaluate the time derivative in the left-hand side of equation (24), one should take the tensor $\boldsymbol{\mathcal{M}}(t+dt)$ at the point $x(t+dt)$, transport it according to the rules of parallel transport for (3+1)-tensors to the point $x(t)$, subtract from it the tensor $\boldsymbol{\mathcal{M}}(t)$ at that point, and divide the difference by dt. Following this procedure, one will find that in the O-basis associated with the selected coordinates,

$$\mathcal{K}_{5i}(t) = \frac{mv_i(t+dt) - mv_i(t)}{dt} = \frac{d(mv_i)}{dt} = \delta_{ij}F^j(t).$$

Similarly, since at any t the $\widetilde{\mathcal{Z}}$-component of $\boldsymbol{\mathcal{M}}(t)$ at the point $x(t)$ is zero, one will have

$$\begin{aligned}\mathcal{K}_{ij}(t) &= v_i(t)\,\mathcal{M}_{5j}(t) + v_j(t)\,\mathcal{M}_{i5}(t) \\ &= mv_i(t)v_j(t) - mv_j(t)v_i(t) = 0.\end{aligned}$$

As $\boldsymbol{\mathcal{M}}$, $\boldsymbol{\mathcal{K}}$ can be represented by a pair of three-vectors. The results we have just obtained mean that at the point where the particle is located, the pair corresponding to $\boldsymbol{\mathcal{K}}$ is $(-\vec{F}, \vec{0})$. In the particular case where no forces act on the particle, one obtains

$$d\boldsymbol{\mathcal{M}}/dt = 0,$$

which is nothing but the conservation law for momentum and angular momentum of a free particle, written down in the language of (3+1)-tensors.

Suppose now that $\vec{F} \neq 0$. Let us transport the tensors in both sides of equation (24) to some other point O. As we know, the pair corresponding to the transported $\boldsymbol{\mathcal{M}}$ will consist of the particle momentum three-vector with the minus sign and of the three-vector of particle angular momentum relative to O. Similarly, one can find that the pair corresponding to the transported $\boldsymbol{\mathcal{K}}$ will consist of the three-vector $-\vec{F}$ transported to O according to the rules of parallel transport for three-vectors, and of the three-vector $\vec{K}$ of the force moment relative to O. Thus, when transported to the indicated point, the three-plus-one-tensor equation (24) is equivalent to the following two three-vector equations: equation (23) and the equation

$$d\vec{M}/dt = \vec{K}.$$

In the case of a system of point particles, one can sum up equations (24) corresponding to all the particles in the system, provided one first transports them all to some point O, and obtain the three-plus-one-tensor equation

$$d\boldsymbol{\mathcal{M}}^{tot}/dt = \boldsymbol{\mathcal{K}}^{tot}, \tag{25}$$

which is apparently equivalent to two three-vector equations that equate the time derivatives of the total momentum three-vector and of the three-vector of total angular momentum relative to O respectively to the three-vector of total force evaluated in the usual way and to the three-vector of the total force moment relative to O.

D. *Bivector derivative*

Let us consider the group of active Poincare tranformations of scalar, four-vector and four-tensor fields in flat space-time. Let us distinguish in it some one-parameter family $\mathcal{H}$ that includes the identity transformation. Let us denote the parameter of this family as s and the image of an arbitrary field $\mathcal{G}$ under a transformation from $\mathcal{H}$ as $\mathbf{\Pi}_s\{\mathcal{G}\}$. It is convenient to take that the identity transformation corresponds to $s = 0$.

For the selected one-parameter family $\mathcal{H}$ and for any sufficiently smooth field $\mathcal{G}$ from the indicated class of fields, one can define the derivative

$$\mathsf{D}_{\mathcal{H}}\mathcal{G} \equiv (d/ds)\mathbf{\Pi}_s\{\mathcal{G}\}|_{s=0}, \tag{26}$$

which is a field of the same type as $\mathcal{G}$. It is apparent that for every type of fields, the operators $\mathsf{D}_{\mathcal{H}}$ corresponding to all possible one-parameter families $\mathcal{H}$ make up a 10-dimensional real vector space, which is nothing but the representation of the Lie algebra of the Poincare group that corresponds to the considered type of fields.

Let us introduce in space-time some system of Lorentz coordinates x^α and select a basis in the space of operators $\mathsf{D}_{\mathcal{H}}$ consisting of the six operators $\mathsf{M}_{\mu\nu}$ that correspond to rotations in the planes $x^\mu x^\nu$ ($\mu < \nu$) and of the four operators P_μ that correspond to translations along the coordinate axes. If one parametrizes the indicated transformations with the parameters $\omega^{\alpha\beta}$ and a^α introduced in section B, then for an arbitrary scalar function f one will have

$$\begin{aligned}\mathsf{P}_\mu f(x) &= \partial_\mu f(x) \\ \mathsf{M}_{\mu\nu} f(x) &= x_\nu \partial_\mu f(x) - x_\mu \partial_\nu f(x),\end{aligned} \tag{27}$$

for an arbitrary four-vector field $\mathbf{U}$ one will have

$$\begin{aligned}(\mathsf{P}_\mu \mathbf{U})^\alpha(x) &= \partial_\mu U^\alpha(x) \\ (\mathsf{M}_{\mu\nu}\mathbf{U})^\alpha(x) &= x_\nu \partial_\mu U^\alpha(x) - x_\mu \partial_\nu U^\alpha(x) \\ &\quad + (M_{\mu\nu})^\alpha_{\ \beta}\, U^\beta(x),\end{aligned} \tag{28}$$

where the components correspond to the Lorentz four-vector basis associated with the selected coordinates; and so on.

With transition to some other system of Lorentz coordinates with the origin at the *same* point, the derivatives $\mathsf{M}_{\mu\nu}\mathcal{G}$ transform with respect to the indices μ and ν as components of a four-vector 2-form, and the derivatives $\mathsf{P}_\mu\mathcal{G}$ transform with respect to μ as components of a four-vector 1-form. Consequently, if one constructs out of these quantities the fields

$$\mathsf{P}\mathcal{G} \equiv \mathsf{P}_\mu\mathcal{G} \cdot \widetilde{\mathbf{O}}^\mu \quad \text{and} \quad \mathsf{M}\mathcal{G} \equiv \mathsf{M}_{|\mu\nu|}\mathcal{G} \cdot \widetilde{\mathbf{O}}^\mu \wedge \widetilde{\mathbf{O}}^\nu, \tag{29}$$

where $\widetilde{\mathbf{O}}^\mu$ is the basis of four-vector 1-forms associated with the selected coordinate system, these fields will be the same at any choice of the latter. From definition (29) it follows that at every point in space-time

$$\begin{array}{c} \mathsf{P}_\mu \mathcal{G} = < \mathsf{P}\mathcal{G}\, ,\, \mathbf{E}_\mu > \\ \mathsf{M}_{\mu\nu} \mathcal{G} = < \mathsf{M}\mathcal{G}\, ,\, \mathbf{E}_\mu \wedge \mathbf{E}_\nu >, \end{array} \tag{30}$$

where $\mathbf{E}_\mu$ is the four-vector basis corresponding to the Lorentz coordinate system with respect to which the operators $\mathsf{P}_\mu \mathcal{G}$ and $\mathsf{M}_{\mu\nu} \mathcal{G}$ are defined. If the fields $\mathsf{P}\mathcal{G}$ and $\mathsf{M}\mathcal{G}$ were completely independent of the choice of the Lorentz coordinate system, then basing on relations (30) one could regard $\mathsf{M}_{\mu\nu} \mathcal{G}$ as a special kind of derivative whose argument is a four-vector bivector:

$$\mathsf{M}_{\mu\nu} \mathcal{G} \equiv \mathsf{D}_{\mathbf{E}_\mu \wedge \mathbf{E}_\nu} \mathcal{G},$$

and $\mathsf{P}_\mu \mathcal{G}$ as a derivative whose argument is a four-vector:

$$\mathsf{P}_\mu \mathcal{G} \equiv \mathsf{D}_{\mathbf{E}_\mu} \mathcal{G},$$

and then at any point in space-time, the family $\mathcal{H}$ for which the derivative $\mathsf{D}_\mathcal{H}$ is evaluated could be identified in a coordinate-free way by indicating the four-vector and the four-vector bivector that correspond to this family. The field $\mathsf{P}\mathcal{G}$ is indeed independent of the choice of the coordinate system, and it is easy to see that for all types of fields from the considered class of fields the operator $\mathsf{D}_{\mathbf{E}_\mu}$ coincides with the operator of the covariant derivative in the direction of the four-vector $\mathbf{E}_\mu$ (I am talking about flat space-time only). However, the field $\mathsf{M}\mathcal{G}$ does depend on the choice of the origin, since under the translation $x^\alpha \to x^\alpha + a^\alpha$ it transforms as

$$\mathsf{M}\mathcal{G} \to \mathsf{M}\mathcal{G} + \mathsf{P}\mathcal{G} \wedge \widetilde{\mathbf{A}},$$

where $\widetilde{\mathbf{A}} \equiv a_\alpha \widetilde{\mathbf{O}}^\alpha$, so one cannot regard $\mathsf{M}_{\mu\nu} \mathcal{G}$ as a derivative whose argument is a four-vector bivector irrespective of the choice of the coordinate system.

Since derivative (26) is associated with active Poincare transformations, basing on the results of section B one may expect that at any point in space-time the derivatives $\mathsf{D}_\mathcal{H}$ corresponding to various one-parameter families $\mathcal{H}$ can be parametrized invariantly with five-vector bivectors. To see that this is indeed so, one may observe that with transition from one Lorentz coordinate system to another, the quantities $\mathsf{M}_{\mu\nu} \mathcal{G}$ and $\mathsf{P}_\mu \mathcal{G}$ transform respectively as the $\mu\nu$- and $\mu 5$-components of a five-vector 2-form in the P-basis. Consequently, the field

$$\mathsf{D}\mathcal{G} \equiv \mathsf{M}_{|\mu\nu|} \mathcal{G} \cdot \widetilde{\mathbf{q}}^\mu \wedge \widetilde{\mathbf{q}}^\nu + \mathsf{P}_\mu \mathcal{G} \cdot \widetilde{\mathbf{q}}^\mu \wedge \widetilde{\mathbf{q}}^5,$$

where $\widetilde{\mathbf{q}}^A$ is the basis of five-vector 1-forms dual to the P-basis, $\mathbf{p}_A$, associated with the selected Lorentz coordinate system, will be the same at any choice of the latter. Similar to equations (30), one will have the relations

$$\begin{array}{c} \mathsf{P}_\mu \mathcal{G} = < \mathsf{D}\mathcal{G}\, ,\, \mathbf{p}_\mu \wedge \mathbf{p}_5 > \\ \mathsf{M}_{\mu\nu} \mathcal{G} = < \mathsf{D}\mathcal{G}\, ,\, \mathbf{p}_\mu \wedge \mathbf{p}_\nu >, \end{array}$$

basing on which one can regard $\mathsf{P}_\mu \mathcal{G}$ and $\mathsf{M}_{\mu\nu} \mathcal{G}$ as particular values of the derivative whose argument is a five-vector bivector, and which in view of this I will call the *bivector* derivative. For any Lorentz coordinate system one will apparently have

$$\mathsf{P}_\mu \mathcal{G} = \mathsf{D}_{\mathbf{p}_\mu \wedge \mathbf{p}_5} \mathcal{G} \quad \text{and} \quad \mathsf{M}_{\mu\nu} \mathcal{G} = \mathsf{D}_{\mathbf{p}_\mu \wedge \mathbf{p}_\nu} \mathcal{G}, \tag{31}$$

where $\mathbf{p}_A$ is the P-basis associated with these coordinates. Comparing the latter formulae with formulae (27) at the origin, one can see that for any active regular basis $\mathbf{e}_A$ and any scalar function f,

$$\mathsf{D}_{\mathbf{e}_\mu \wedge \mathbf{e}_5} f = \partial_{\mathbf{e}_\mu} f \quad \text{and} \quad \mathsf{D}_{\mathbf{e}_\mu \wedge \mathbf{e}_\nu} f = 0. \tag{32}$$

From these equations it follows that at the point with coordinates x^α,

$$\mathsf{D}_{\mathbf{p}_\mu \wedge \mathbf{p}_5} f = \mathsf{D}_{\mathbf{p}^z_\mu \wedge \mathbf{p}_5} f = \mathsf{D}_{\mathbf{e}_\mu \wedge \mathbf{e}_5} f = \partial_{\mathbf{e}_\mu} f = \partial_\mu f$$

and

$$\begin{array}{rcl} \mathsf{D}_{\mathbf{p}_\mu \wedge \mathbf{p}_\nu} f & = & \mathsf{D}_{\mathbf{e}_\mu \wedge \mathbf{e}_\nu} f + x_\nu \mathsf{D}_{\mathbf{e}_\mu \wedge \mathbf{e}_5} f + x_\mu \mathsf{D}_{\mathbf{e}_5 \wedge \mathbf{e}_\nu} f \\ & = & x_\nu \partial_\mu f - x_\mu \partial_\nu f, \end{array}$$

which is in agreement with formulae (27) in the general case (in the latter two chains of equations and in equations (33) and (34) that follow, $\mathbf{e}_A$ denotes the O-basis associated with the considered coordinates). Comparing formulae (31) with formulae (28) at the origin, one can see that for any Lorentz four-vector basis $\mathbf{E}_\alpha$,

$$\mathsf{D}_{\mathbf{e}_\mu \wedge \mathbf{e}_5} \mathbf{E}_\alpha = \mathbf{0} \quad \text{and} \quad \mathsf{D}_{\mathbf{e}_\mu \wedge \mathbf{e}_\nu} \mathbf{E}_\alpha = \mathbf{E}_\beta \left(M_{\mu\nu} \right)^\beta{}_\alpha, \tag{33}$$

so for any such basis

$$\mathsf{D}_{\mathbf{p}_A \wedge \mathbf{p}_B} \mathbf{E}_\alpha = \mathsf{D}_{\mathbf{e}_A \wedge \mathbf{e}_B} \mathbf{E}_\alpha \tag{34}$$

at all A and B. From the properties of Poincare transformations and from definition (26) it follows that for any scalar function f and any four-vector field $\mathbf{V}$,

$$\mathsf{D}_{\mathbf{p}_A \wedge \mathbf{p}_B} (f \mathbf{V}) = \mathsf{D}_{\mathbf{p}_A \wedge \mathbf{p}_B} f \cdot \mathbf{V} + f \cdot \mathsf{D}_{\mathbf{p}_A \wedge \mathbf{p}_B} \mathbf{V}, \tag{35}$$

which together with equations (33) and (34) gives formulae (28) for an arbitrary four-vector field $\mathbf{U}$. Similar formulae can be obtained for all other four-tensor fields.

One can now consider a more general derivative than $\mathsf{D}_\mathcal{H}$ by allowing the one-parameter family $\mathcal{H}$ to vary from point to point. For any field $\mathcal{G}$ from the considered class of fields, such a derivative is a field whose value at each space-time point coincides with the value of one of the fields $\mathsf{D}_{\mathcal{H}}\mathcal{G}$ for some family $\mathcal{H}$, which at different points may be different. Everywhere below, when speaking of the bivector derivative I will refer to this more general type of differentiation.

According to the results obtained above, any such derivative can be uniquely fixed by specifying a certain field of five-vector bivectors. Therefore, by analogy with the covariant derivative, for any type of fields D can be formally regarded as a map that puts into correspondence to every pair consisting of a bivector field and a field of the considered type another field of that type. For example, the bivector derivative for four-vector fields can be viewed as a map

$$\boldsymbol{\mathcal{F}}\wedge\boldsymbol{\mathcal{F}}\times\boldsymbol{\mathcal{D}}\to\boldsymbol{\mathcal{D}}, \tag{36}$$

where $\boldsymbol{\mathcal{F}}\wedge\boldsymbol{\mathcal{F}}$ is the set of all fields of five-vector bivectors and $\boldsymbol{\mathcal{D}}$ is the set of all four-vector fields. From the definition of the bivector derivative it follows that map (36) has the following formal properties: for any scalar functions f and g, any four-vector fields $\mathbf{U}$ and $\mathbf{V}$, and any bivector fields $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$,

$$\mathsf{D}_{(f\mathcal{A}+g\mathcal{B})}\mathbf{U} = f\cdot\mathsf{D}_{\mathcal{A}}\mathbf{U} + g\cdot\mathsf{D}_{\mathcal{B}}\mathbf{U} \tag{37a}$$
$$\mathsf{D}_{\mathcal{A}}(\mathbf{U}+\mathbf{V}) = \mathsf{D}_{\mathcal{A}}\mathbf{U} + \mathsf{D}_{\mathcal{A}}\mathbf{V} \tag{37b}$$
$$\mathsf{D}_{\mathcal{A}}(f\mathbf{U}) = \mathsf{D}_{\mathcal{A}}f\cdot\mathbf{U} + f\cdot\mathsf{D}_{\mathcal{A}}\mathbf{U}. \tag{37c}$$

In the third equation, the action of D on the function f is determined by the rules:

$$\begin{cases} \mathsf{D}_{(\mathcal{A}+\mathcal{B})}f = \mathsf{D}_{\mathcal{A}}f + \mathsf{D}_{\mathcal{B}}f, \\ \mathsf{D}_{\mathcal{A}^{\mathcal{Z}}}f = 0, \quad \mathsf{D}_{\mathcal{A}^{\mathcal{E}}}f = \partial_{\mathbf{A}}f, \end{cases} \tag{38}$$

where $\mathbf{A}$ denotes the four-vector field that corresponds to the $\mathcal{E}$-component of $\boldsymbol{\mathcal{A}}$.

The properties of D presented above are similar to the three main properties of the covariant derivative that are used for defining the latter formally. Using properties (37) for the same purpose is not very convenient, since to define the bivector derivative completely one has to supplement them with the formulae that determine the relation of D to space-time metric, and usually from such relations one is already able to derive part of the properties expressed by equations (37). As an example, let us consider the formulae that express the operator D in terms of the operator ∇ of the torsion-free g-conserving covariant derivative and of the linear local operator $\widehat{\mathbf{M}}$ defined below, both of which are completely determined by the metric. For an arbitrary four-vector field $\mathbf{U}$ one has:

$$\mathsf{D}_{\mathcal{A}^{\mathcal{E}}}\mathbf{U} = \nabla_{\mathbf{A}}\mathbf{U} \quad\text{and}\quad \mathsf{D}_{\mathcal{A}^{\mathcal{Z}}}\mathbf{U} = \widehat{\mathbf{M}}_{\mathbf{B}}\mathbf{U}, \tag{39}$$

where $\mathbf{A}$, as in definition (38), denotes the four-vector field corresponding to the $\mathcal{E}$-component of $\boldsymbol{\mathcal{A}}$, $\mathbf{B}$ denotes the field of four-vector bivectors corresponding to the $\mathcal{Z}$-component of $\boldsymbol{\mathcal{A}}$, and the operator $\widehat{\mathbf{M}}$, which depends linearly on its argument, has the following components in an arbitrary four-vector basis $\mathbf{E}_\alpha$:

$$\widehat{\mathbf{M}}_{\mathbf{E}_\alpha\wedge\mathbf{E}_\beta}\mathbf{E}_\mu = \mathbf{E}_\nu(M_{\alpha\beta})^\nu{}_\mu.$$

It is easy to see that properties (37b) and (37c) follow from formulae (39) and property (37a), and property (37a) itself follows from equations (39) and the following simpler property:

$$\mathsf{D}_{(\mathcal{A}+\mathcal{B})}\mathbf{U} = \mathsf{D}_{\mathcal{A}}\mathbf{U} + \mathsf{D}_{\mathcal{B}}\mathbf{U},$$

which is similar to the first equation in definition (38) and which, together with equations (39), can serve as a definition of the bivector derivative for four-vector fields.

The action of operator D on all other four-tensor fields can be defined either independently—according to formula (26), or as in the case of the covariant derivative—according to the equations that express the Leibniz rule in application to the contraction of a four-vector 1-form with a four-vector field and to the tensor product of any two four-tensor fields. The corresponding formulae are quite obvious and will not be presented here.

For the bivector derivative one can define the analogs of connection coefficients. Namely, for any set of basis four-vector fields $\mathbf{E}_\alpha$ and any set of basis five-vector fields $\mathbf{e}_A$ one can take

$$\mathsf{D}_{AB}\mathbf{E}_\mu = \mathbf{E}_\nu\Gamma^\nu{}_{\mu AB}, \tag{40}$$

where $\mathsf{D}_{AB} \equiv \mathsf{D}_{\mathbf{e}_A\wedge\mathbf{e}_B}$. According to equations (33), for any Lorentz four-vector basis and any standard five-vector basis associated with it, one has

$$\begin{aligned} \Gamma^\mu{}_{\nu\alpha5} &= -\Gamma^\mu{}_{\nu5\alpha} = 0 \\ \Gamma^\mu{}_{\nu\alpha\beta} &= -\Gamma^\mu{}_{\nu\beta\alpha} = (M_{\alpha\beta})^\mu{}_\nu. \end{aligned} \tag{41}$$

The bivector connection coefficients for any other choice of the basis fields can be found either by using the following transformation formula:

$$\begin{aligned} \Gamma'^\mu{}_{\nu AB} &= (\Lambda^{-1})^\mu_\sigma\Gamma^\sigma{}_{\tau ST}\Lambda^\tau_\nu L^S_A L^T_B \\ &\quad + (\Lambda^{-1})^\mu_\sigma(\mathsf{D}_{ST}\Lambda^\sigma_\nu)L^S_A L^T_B, \end{aligned} \tag{42}$$

which corresponds to the transformations $\mathbf{E}'_\alpha = \mathbf{E}_\beta\Lambda^\beta_\alpha$ and $\mathbf{e}'_A = \mathbf{e}_B L^B_A$ of the four- and five-vector basis fields, or by using formulae (39). In particular, for an arbitrary four-vector basis and the corresponding active regular five-vector basis one has

$$\Gamma^\mu{}_{\nu\alpha5} = \Gamma^\mu{}_{\nu\alpha} \quad\text{and}\quad \Gamma^\mu{}_{\nu\alpha\beta} = (M_{\alpha\beta})^\mu{}_\nu, \tag{43}$$

where $\Gamma^{\mu}_{\ \nu\alpha}$ are ordinary four-vector connection coefficients associated with ∇.

Let me also observe that the quantities $\mathsf{D}_{AB}\mathcal{G}$ for an arbitrary field $\mathcal{G}$ can be presented as derivatives with respect to the components of the bivector $\boldsymbol{\mathcal{R}}$ introduced in section B. Indeed, according to the definition of the operators P_{μ} and $\mathsf{M}_{\mu\nu}$, one has

$$\begin{array}{l} \mathsf{D}_{5\mu}\mathcal{G} = -(\partial/\partial a^{\mu})\mathbf{\Pi}\{\mathcal{G}\}|_{\omega^{\alpha\beta}=a^{\alpha}=0} \\ \mathsf{D}_{\mu\nu}\mathcal{G} = (\partial/\partial \omega^{\mu\nu})\mathbf{\Pi}\{\mathcal{G}\}|_{\omega^{\alpha\beta}=a^{\alpha}=0} \quad (\mu<\nu), \end{array} \tag{44}$$

where it is assumed that the image $\mathbf{\Pi}\{\mathcal{G}\}$ of field $\mathcal{G}$ under an infinitesimal Poincare transformation is a function of parameters $\omega^{\alpha\beta}$ and a^{α}. In view of equations (11), the latter formulae can be rewritten as

$$\begin{array}{l} \mathsf{D}_{5\mu}\mathcal{G} = (\partial/\partial \mathcal{R}^{5\mu})\mathbf{\Pi}\{\mathcal{G}\}|_{\mathcal{R}^{\alpha\beta}=\mathcal{R}^{5\alpha}=0} \\ \mathsf{D}_{\mu\nu}\mathcal{G} = (\partial/\partial \mathcal{R}^{\mu\nu})\mathbf{\Pi}\{\mathcal{G}\}|_{\mathcal{R}^{\alpha\beta}=\mathcal{R}^{5\alpha}=0}, \end{array} \tag{45}$$

and since $\mathsf{D}_{\mu\nu} = -\mathsf{D}_{\nu\mu}$ and $\mathcal{R}^{\mu\nu} = -\mathcal{R}^{\nu\mu}$, the second equation will be valid at $\mu > \nu$ as well. Since $\mathsf{D}_{\mu 5} = -\mathsf{D}_{5\mu}$, one can write that

$$\mathsf{D}_{\mu 5}\mathcal{G} = (\partial/\partial(-\mathcal{R}^{5\mu}))\mathbf{\Pi}\{\mathcal{G}\}|_{\mathcal{R}^{\alpha\beta}=\mathcal{R}^{5\alpha}=0},$$

so if one takes $\mathcal{R}^{\mu 5} = -\mathcal{R}^{5\mu}$, as it has been done in section B, one will have

$$\mathsf{D}_{AB}\mathcal{G} = (\partial/\partial \mathcal{R}^{AB})\mathbf{\Pi}\{\mathcal{G}\}|_{\mathcal{R}=0}, \tag{46}$$

for all $A \neq B$, which is one more argument in favour of the choice (12).

E. *Bivector derivative of the Lagrange function*

In the previous section I have defined the bivector derivative for scalar, four-vector and four-tensor fields in flat space-time. In a similar manner the bivector derivative can be defined for more complicated objects. As an example, I will now consider the definition of the analog of the bivector derivative in three-dimensional Euclidean space for the Lagrange function of a system of several point particles, in classical nonrelativistic mechanics.

As is known, the state of motion of such a system at every moment of time can be fixed by specifying the position of each particle in space and its velocity. The Lagrange function $\mathbf{L}$ for this system can be viewed as a map that puts into correspondence to each allowed state of motion $\mathcal{C}$ a real number, $\mathbf{L}(\mathcal{C})$. In the general case this map may be explicitly time-dependent.

The bivector derivative of the Lagrange function can be defined according to an equation similar to formula (26). For that one apparently has to define first how $\mathbf{L}$ changes under active transformations from the group of motions of three-dimensional Euclidean space. This can be done in a standard way if one knows how such transformations affect the states of motion of the system. Namely, for any transformation from the indicated group the image $\mathbf{\Pi}\{\mathbf{L}\}$ of the Lagrange function is such a function of the state of motion that for any $\mathcal{C}$

$$\mathbf{\Pi}\{\mathbf{L}\}(\mathbf{\Pi}\{\mathcal{C}\}) = \mathbf{L}(\mathcal{C}), \tag{47}$$

where $\mathbf{\Pi}\{\mathcal{C}\}$ is the image of state $\mathcal{C}$ under the considered active transformation. It is apparent that equation (47) can be presented in the following equivalent way:

$$\mathbf{\Pi}\{\mathbf{L}\}(\mathcal{C}) = \mathbf{L}(\mathbf{\Pi^{-1}}\{\mathcal{C}\}), \tag{48}$$

where $\mathbf{\Pi^{-1}}$ is the transformation inverse to $\mathbf{\Pi}$. Basing on definition (47), one can define the derivative of $\mathbf{L}$ relative to some one-parameter family of transformations $\mathcal{H}$ as follows: $\mathsf{D}_{\mathcal{H}}\mathbf{L}$ is such a real-valued function of the state of motion that for any $\mathcal{C}$

$$\mathsf{D}_{\mathcal{H}}\mathbf{L}(\mathcal{C}) = (d/ds)\mathbf{\Pi}_s\{\mathbf{L}\}(\mathcal{C})|_{s=0}, \tag{49}$$

where s is the parameter of the considered family. If one proceeds from definition (48), then instead of (49) one will have the following equivalent definition:

$$\mathsf{D}_{\mathcal{H}}\mathbf{L}(\mathcal{C}) = (d/ds)\mathbf{L}(\mathbf{\Pi}_s^{-1}\{\mathcal{C}\})|_{s=0}. \tag{50}$$

By using some particular set of variables for characterizing the state of motion of the system, for example, the coordinates of all the particles in some Cartesian coordinate system and the components of their velocities in the corresponding three-vector basis, it is not difficult to show that derivatives (49) for all one-parameter families $\mathcal{H}$ are correlated with the derivatives of scalar, three-vector and three-tensor fields relative to all these families in the following sense: if families $\mathcal{H}$, $\mathcal{H}'$, and $\mathcal{H}''$ are such that for any field $\mathcal{G}$ from the indicated class of fields,

$$\mathsf{D}_{\mathcal{H}}\mathcal{G} = a \cdot \mathsf{D}_{\mathcal{H}'}\mathcal{G} + b \cdot \mathsf{D}_{\mathcal{H}''}\mathcal{G},$$

where a and b are some real numbers, then for any state of motion $\mathcal{C}$,

$$\mathsf{D}_{\mathcal{H}}\mathbf{L}(\mathcal{C}) = a \cdot \mathsf{D}_{\mathcal{H}'}\mathbf{L}(\mathcal{C}) + b \cdot \mathsf{D}_{\mathcal{H}''}\mathbf{L}(\mathcal{C}).$$

This fact enables one to construct another function of the state of motion, which, as $\mathbf{L}$, may explicitly depend on time, but whose values will be covariantly constant fields of antisymmetric (3+1)-tensors of rank two rather than numbers. To this end, let us observe that as in the case of space-time, to every one-parameter family $\mathcal{H}$ one can put into correspondence a certain covariantly constant field $\boldsymbol{\mathcal{A}}$ of three-plus-one-vector bivectors, such that the derivative $\mathsf{D}_{\mathcal{H}}\mathcal{G}$ of any scalar, three-vector or three-tensor field $\mathcal{G}$ can be presented as the contraction $< \mathsf{D}\mathcal{G}, \boldsymbol{\mathcal{A}} >$, where $\mathsf{D}\mathcal{G}$ is a three-plus-one-vector 2-form independent of $\mathcal{H}$, whose values are quantities of the same kind as those of $\mathcal{G}$. In a similar way one can define the scalar-valued three-plus-one-vector 2-form $\mathsf{D}\mathbf{L}$, which will be a function of the state of motion of the system (and of time), by taking that for any $\mathcal{C}$

$$< \mathsf{D}\mathbf{L}(\mathcal{C}), \boldsymbol{\mathcal{A}} > = \mathsf{D}_{\mathcal{H}}\mathbf{L}(\mathcal{C}),$$

where $\boldsymbol{\mathcal{A}}$ is an arbitrary covariantly constant field of three-plus-one-vector bivectors and $\mathcal{H}$ is the one-parameter family that corresponds to it. Since $\mathsf{D}_{\mathcal{H}}\mathbf{L}(\mathcal{C})$ is simply a number, from the fact that $\boldsymbol{\mathcal{A}}$ is covariantly constant follows that at any $\mathcal{C}$ the field $\mathsf{D}\mathbf{L}(\mathcal{C})$ will be covariantly constant, too.

Characterizing the state of motion of the system with coordinates x^i_ℓ of all the particles in some Cartesian coordinate system and with the components $v^i_\ell \equiv dx^i_\ell/dt$ of their velocities in the corresponding three-vector basis (the index ℓ numerates the particles), it is not difficult to evaluate the components of $\mathsf{D}\mathbf{L}$ in the basis of three-plus-one-vector 2-forms corresponding to the P-basis of (3+1)-vectors associated with the selected coordinates. One obtains:

$$\mathsf{D}_{i5}\mathbf{L} = \sum_\ell \partial\mathbf{L}/\partial x^i_\ell \tag{51}$$

and

$$\mathsf{D}_{ij}\mathbf{L} = \sum_\ell \left(x_{j\,\ell} \cdot \partial\mathbf{L}/\partial x^i_\ell - x_{i\,\ell} \cdot \partial\mathbf{L}/\partial x^j_\ell \right), \tag{52}$$

where $x_{i\,\ell} \equiv \delta_{ij} x^j_\ell$. Since the quantities $\partial\mathbf{L}/\partial x^i_\ell$ are covariant components of the force that acts on the ℓth particle relative to the basis of three-vector 1-forms associated with the selected coordinate system, equations (51) and (52) mean that at the origin O of this system the $\widetilde{\mathcal{E}}$-component of $\mathsf{D}\mathbf{L}$ corresponds to the three-vector 1-form of the total force that acts on the system and the $\widetilde{\mathcal{Z}}$-component of $\mathsf{D}\mathbf{L}$ corresponds to the three-vector 2-form of the total force moment relative to O taken with the opposite sign. Therefore, $-\mathsf{D}\mathbf{L}$ is exactly the three-plus-one-vector 2-form $\boldsymbol{\mathcal{K}}^{tot}$ introduced in section C.

Among other thing, from the latter fact follows the result we have obtained earlier: that momentum and angular momentum of a system of particles in classical nonrelativistic mechanics can be described by a single local object—by a three-plus-one-vector 2-form. Indeed, according to the equations of motion, the force that acts on the particle and the moment of this force relative to an arbitrary point O are total time derivatives respectively of the particle momentum and of its angular moment relative to O. Consequently, the three-plus-one-vector 2-form $\boldsymbol{\mathcal{K}}$ is also a total time derivative of some (3+1)-tensor. Since in the nonrelativistic case time is an external parameter, the rank of this (3+1)-tensor should be the same as that of $\boldsymbol{\mathcal{K}}$, and according to what has been said above, at an arbitrary point in space the $\widetilde{\mathcal{E}}$-component of this (3+1)-tensor will correspond, with the opposite sign, to the three-vector 1-form of the particle momentum, and its $\widetilde{\mathcal{Z}}$-component will correspond to the three-vector 2-form of the particle angular momentum relative to that point.

Appendix: Contravariant basis

As in the case of any other vector space endowed with a nondegenerate inner product, to any basis $\mathbf{E}_\alpha$ in V_4 one can put into correspondence the basis

$$\mathbf{E}^\alpha \equiv g^{\alpha\beta}\mathbf{E}_\beta, \tag{53}$$

which will be called *contravariant*. Here, as usual, $g^{\alpha\beta}$ denote the matrix inverse to $g_{\alpha\beta} \equiv g(\mathbf{E}_\alpha, \mathbf{E}_\beta)$. Definition (53) is equivalent to the following relation:

$$< \widetilde{\mathbf{O}}^\alpha, \mathbf{V} > = g(\mathbf{E}^\alpha, \mathbf{V}) \text{ for any four-vector } \mathbf{V},$$

where $\widetilde{\mathbf{O}}^\alpha$ is the basis of four-vector 1-forms dual to $\mathbf{E}_\alpha$. The latter relation means that the four-vectors $\mathbf{E}^\alpha$ are inverse images of the basis 1-forms $\widetilde{\mathbf{O}}^\alpha$ under the map $V_4 \to \widetilde{V}_4$ defined by the inner product g. From definition (53) it also follows that $g(\mathbf{E}_\alpha, \mathbf{E}^\beta) = \delta^\beta_\alpha$, so the four-vector 1-forms $\widetilde{\mathbf{O}}_\alpha$ that make up the basis dual to $\mathbf{E}^\alpha$ are images of the basis four-vectors $\mathbf{E}_\alpha$ under the indicated map. It is evident that when $\mathbf{E}_\alpha$ is a Lorentz basis in flat spacetime, associated with some Lorentz coordinate system x^α, one has $\mathbf{E}^\alpha = \partial/\partial x_\alpha$, where x_α are the corresponding covariant coordinates. It is also evident that at any affine connection relative to which the metric tensor is covariantly constant, the relation between the bases $\mathbf{E}_\alpha$ and $\mathbf{E}^\alpha$ is preserved by parallel transport, and it is easy to see that in this case

$$\nabla_\mu \mathbf{E}^\alpha = -\,\Gamma^\alpha_{\ \beta\mu}\mathbf{E}^\beta,$$

where $\Gamma^\alpha_{\ \beta\mu}$ are the connection coefficients corresponding to the basis fields $\mathbf{E}_\alpha$.

In a similar manner, by using the nondegenerate inner product h, one can define the contravariant basis $\mathbf{e}^A$ corresponding to an arbitrary five-vector basis $\mathbf{e}_A$:

$$< \widetilde{\mathbf{o}}^A, \mathbf{v} > = h(\mathbf{e}^A, \mathbf{v}) \text{ for any five-vector } \mathbf{v},$$

where $\widetilde{\mathbf{o}}^A$ is the basis of five-vector 1-forms dual to $\mathbf{e}_A$. However, such a definition of the contravariant basis is inconvenient in two ways. First of all, the contravariant basis corresponding to an arbitrary standard basis $\mathbf{e}_A$ will in general not be a standard basis itself (this will be the case only if $\mathbf{e}_A$ is a *regular* basis). Secondly, since the inner product h is not conserved by parallel transport, the latter will not preserve the correspondence between $\mathbf{e}_A$ and $\mathbf{e}^A$ either. In view of this, in the case of five-vectors it is more convenient to define the contravariant basis in another way. Namely, for any standard basis $\mathbf{e}_A$ one takes that

$$\mathbf{e}^\alpha = g^{\alpha\beta}\mathbf{e}_\beta \text{ and } \mathbf{e}^5 = \mathbf{e}_5. \tag{54}$$

It is a simple matter to see that the first four vectors of the contravariant basis defined this way satisfy the relation

$$< \widetilde{\mathbf{o}}^\alpha, \mathbf{v} > = g(\mathbf{e}^\alpha, \mathbf{v}) \text{ for any five-vector } \mathbf{v},$$

which, however, is not equivalent to the first equation in definition (54) since it does not fix the $\mathcal{E}$-components of the vectors $\mathbf{e}^\alpha$. It is not difficult to see that the basis of five-vector 1-forms dual to $\mathbf{e}^A$, which will be denoted as $\widetilde{\mathbf{o}}_A$, is expressed in terms of the basis 1-forms $\widetilde{\mathbf{o}}^A$ as

$$\widetilde{\mathbf{o}}_\alpha = \widetilde{\mathbf{o}}^\beta g_{\beta\alpha} \text{ and } \widetilde{\mathbf{o}}_5 = \widetilde{\mathbf{o}}^5.$$

The first of these equations is equivalent to the relation

$$< \widetilde{\mathbf{o}}_\alpha, \mathbf{v} > = g(\mathbf{e}_\alpha, \mathbf{v}) \text{ for any five-vector } \mathbf{v},$$

which means that $\widetilde{\mathbf{o}}_\alpha$ are images of the basis five-vectors $\mathbf{e}_\alpha$ with respect to the map $\vartheta_g : V_5 \to \widetilde{V}_5$ defined in section 3 of part II. It is also evident that from $\mathbf{e}_\alpha \in \mathbf{E}_\alpha$ follows $\mathbf{e}^\alpha \in \mathbf{E}^\alpha$.

According to definition (54), the contravariant P-basis associated with some system of Lorentz coordinates x^α in flat space-time is expressed in terms of the corresponding contravariant O-basis as

$$\mathbf{p}^\alpha = \mathbf{e}^\beta + x^\alpha \mathbf{e}^5 \text{ and } \mathbf{p}^5 = \mathbf{e}^5. \tag{55}$$

Among other things, from the latter equation it follows that in such a P-basis, the five-vector 1-form $\widetilde{\mathbf{x}}$ introduced in section 5 of part I has the components $(x^\alpha, 1)$. Since the correspondence between $\mathbf{p}_A$ and $\mathbf{p}^A$ is preserved by parallel transport, from formula (55) one finds that for the contravariant O-basis

$$\nabla_\mu \mathbf{e}^\alpha = -\,\delta^\alpha_\mu \mathbf{e}^5 \text{ and } \nabla_\mu \mathbf{e}^5 = 0.$$

Finally, it is a simple matter to show that in the basis of five-tensors of rank $(1,1)$ associated with the P-basis (55), the components of the tensor $\boldsymbol{\mathcal{T}}$ that describes the active Poincare transformation corresponding to transformation (3) of Lorentz coordinates, are given by formula (5).