

# Five-Dimensional Tangent Vectors in Space-Time

## IV. Generalization of Exterior Calculus

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### Abstract

This part of the series is devoted to the generalization of exterior differential calculus. I give definition to the integral of a five-vector form over a limited space-time volume of appropriate dimension; extend the notion of the exterior derivative to the case of five-vector forms; and formulate the corresponding analogs of the generalized Stokes theorem and of the Poincare theorem about closed forms. I then consider the five-vector generalization of the exterior derivative itself; prove a statement similar to the Poincare theorem; define the corresponding five-vector generalization of flux; and derive the analog of the formula for integration by parts. I illustrate the ideas developed in this paper by reformulating the Lagrange formalism for classical scalar fields in terms of five-vector forms. In conclusion, I briefly discuss the five-vector analog of the Levi-Civita tensor and dual forms.

#### A. Equivalence classes of two-, three- and four-dimensional volumes

Consider a set  $\mathfrak{R}_2$  of all smooth parametrized two-dimensional surfaces going through a fixed space-time point  $Q$ . Let us label these surfaces with calligraphic capital Roman letters:  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ , etc. The two parameters of surface  $\mathcal{A}$ —its two inner coordinates—will be denoted as  $\lambda_{\mathcal{A}}^{(1)}$  and  $\lambda_{\mathcal{A}}^{(2)}$ .

If  $f$  is a real scalar function defined in the vicinity of  $Q$ , one can evaluate its derivatives at  $Q$  relative to the parameters of a given surface  $\mathcal{A}$ :

$$\left[ \frac{df(P(\lambda_{\mathcal{A}}^{(1)}, \lambda_{\mathcal{A}}^{(2)}))}{d\lambda_{\mathcal{A}}^{(k)}} \right]_{\lambda_{\mathcal{A}}^{(1)}=\lambda_{\mathcal{A}}^{(1)}(Q), \lambda_{\mathcal{A}}^{(2)}=\lambda_{\mathcal{A}}^{(2)}(Q)},$$

where  $k = 1$  or  $2$ , and let us denote these derivatives as  $\partial_{\mathcal{A}}^{(k)} f|_Q$ .

Let us focus our attention on the behaviour of two-dimensional surfaces in the infinitesimal vicinity of  $Q$ . From that point of view  $\mathfrak{R}_2$  can be divided into classes of equivalent surfaces that coincide in direction or in direction and parametrization. As in the case of parametrized curves, one can consider several degrees to which two given surfaces,  $\mathcal{A}$  and  $\mathcal{B}$ , may coincide:

1. The two surfaces have the same direction at  $Q$ . A more precise formulation is the following: there exists a real  $2 \times 2$  matrix,  $\|a_i^k\|$ , with a positive determinant and such that for any scalar func-

tion  $f$

$$\partial_{\mathcal{A}}^{(k)} f|_Q = \sum_{l=1,2} a_l^k \partial_{\mathcal{B}}^{(l)} f|_Q, \quad (1)$$

where  $k = 1, 2$ .

2. The two surfaces have the same direction at  $Q$ ; in the vicinity of  $Q$  the direction of the corresponding inner coordinate lines is the same; and along these lines the corresponding parameters change with equal rates. More precisely: for any scalar function  $f$

$$\partial_{\mathcal{A}}^{(k)} f|_Q = \partial_{\mathcal{B}}^{(k)} f|_Q, \quad (2)$$

where  $k = 1, 2$ .

3. In the infinitesimal vicinity of  $Q$ , the two surfaces have the same direction and parametrization. This means that

$$\lambda_{\mathcal{A}}^{(k)}(Q) = \lambda_{\mathcal{B}}^{(k)}(Q) \quad (3a)$$

and for any scalar function  $f$

$$\partial_{\mathcal{A}}^{(k)} f|_Q = \partial_{\mathcal{B}}^{(k)} f|_Q, \quad (3b)$$

where  $k = 1, 2$ .

It is a simple matter to check that relations (1), (2) and (3) are all equivalence relations on  $\mathfrak{R}_2$ , and for each of them one can consider the corresponding

quotient set—the set whose elements are classes of equivalent two-dimensional surfaces.

It is apparent that one can establish a one-to-one correspondence between the equivalence classes of parametrized surfaces associated with relation (2) and ordered pairs of four-vectors,  $(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})$ , such that for any surface  $\mathcal{A}$  from a given class

$$\partial_{\mathcal{A}}^{(k)} f|_Q = \partial_{\mathbf{A}^{(k)}} f|_Q \quad (k = 1, 2)$$

for any scalar function  $f$ . It is evident that the vectors in a pair should be linearly independent: otherwise the pair will not correspond to any nondegenerate surface.

In a similar manner one can establish a one-to-one correspondence between the classes of equivalent two-surfaces corresponding to relation (3) and ordered pairs of five-vectors,  $(\mathbf{a}^{(1)}, \mathbf{a}^{(2)})$ , according to the formulae

$$\partial_{\mathcal{A}}^{(k)} f|_Q = \partial_{\mathbf{a}^{(k)}} f|_Q \quad \text{and} \quad \lambda_{\mathcal{A}}^{(k)} = \lambda_{\mathbf{a}^{(k)}} \quad (k = 1, 2)$$

for any surface  $\mathcal{A}$  from a given class and for any scalar function  $f$ . In this case, besides being linearly independent, the five-vectors in a pair should satisfy one more requirement: neither of them should be an element of  $\mathcal{E}$ . Pairs where one of the five-vectors belongs to  $\mathcal{E}$  can be considered as a special case corresponding to degenerate two-dimensional surfaces, which are lines.

In a similar manner one can consider a set  $\mathfrak{R}_3$  of all smooth parametrized three-dimensional hypersurfaces going through a fixed space-time point  $Q$  and a set  $\mathfrak{R}_4$  of all parametrized four-dimensional volumes containing  $Q$ . If  $f$  is a real scalar function defined in the vicinity of  $Q$ , one can evaluate its derivatives at  $Q$  relative to the parameters  $\lambda^{(k)}$  of a given hypersurface or four-volume  $\mathcal{A}$ , and I will denote these derivatives as  $\partial_{\mathcal{A}}^{(k)} f|_Q$  (here  $k = 1, 2, 3$  for a hypersurface and  $k = 1, 2, 3, 4$  for a four-volume).

One can then focus one's attention on the behaviour of hypersurfaces and four-dimensional volumes in the infinitesimal vicinity of  $Q$  and consider different degrees to which two given hypersurfaces or four-volumes may coincide. The analogs of relations (2) and (3) will have exactly the same form, only  $k$  will now run 1 through 3 or 1 through 4.

Finally, one can consider the quotient sets corresponding to the latter two equivalence relations and observe that it is possible to establish a one-to-one correspondence between their elements and triplets or quadruples of four- and five-vectors, respectively. The vectors in these triplets and quadruples should be linearly independent and, in addition, none of the

five-vectors should belong to  $\mathcal{E}$  unless the corresponding hypersurface or four-volume is degenerate.

## B. Integrals over $m$ -dimensional volumes

Let us first consider integrals over parametrized curves. Each of them can be viewed as a rule that assigns a certain number to every finite continuous parametrized curve within a certain region of space-time. This number is additive and therefore can be presented as an integral

$$\int_{\lambda_a}^{\lambda_b} d\lambda \phi(\lambda), \quad (4)$$

where  $\lambda_a$  and  $\lambda_b$  are the end-point values of the curve parameter and  $\phi(\lambda)$  is a certain numerical function, which may depend on the curve direction in the infinitesimal vicinity of the integration point.

In applications one is usually interested in invariant integrals, whose value is independent of the curve parametrization. This is impossible if  $\phi$  in formula (4) depends only on the integration point (if there exists a scalar function  $f$  such that  $\phi(\lambda) = f(P(\lambda))$  for every curve), so the dependence on the curve direction is quite essential. This dependence can be of different types. Within four-vector exterior calculus one considers integrals where  $\phi(\lambda)$  is a *linear* function of the tangent four-vector  $\mathbf{U}(\lambda)$  and therefore can be presented as a contraction of  $\mathbf{U}(\lambda)$  with some four-vector 1-form,  $\tilde{\mathbf{S}}$ , defined in some region of space-time containing the curve:

$$\phi(\lambda) = \langle \tilde{\mathbf{S}}(P(\lambda)), \mathbf{U}(\lambda) \rangle.$$

Such an integral is denoted simply as  $\int \tilde{\mathbf{S}}$  and is referred to as the integral of 1-form  $\tilde{\mathbf{S}}$  along the specified curve.

In a similar way one can consider integrals over limited two-, three- and four-dimensional volumes. Each of them can be presented in the form

$$\int_{\Lambda} d\lambda^{(1)} \dots d\lambda^{(m)} \phi(\lambda^{(1)}, \dots, \lambda^{(m)}),$$

where  $m = 2, 3$ , or 4;  $\Lambda$  is the range of variation of inner coordinates; and  $\phi(\lambda^{(1)}, \dots, \lambda^{(m)})$  is a certain numerical function, which may depend on the direction of the given  $m$ -dimensional surface in the infinitesimal vicinity of the integration point. Within four-vector exterior calculus one deals with integrals whose integrand is a linear function of each of the tangent four-vectors  $\mathbf{U}^{(k)}$  ( $k = 1, \dots, m$ ) that correspond to the selected surface parameters. It is easy to show that for such an integral to be invariant, the function

$\phi$  has to be antisymmetric with respect to permutations of  $\mathbf{U}^{(1)}, \dots, \mathbf{U}^{(m)}$ . It therefore can be presented as a contraction of the multivector  $\mathbf{U}^{(1)} \wedge \dots \wedge \mathbf{U}^{(m)}$  with some four-vector  $m$ -form  $\tilde{\mathbf{S}}$ :

$$\phi = \langle \tilde{\mathbf{S}}, \mathbf{U}^{(1)} \wedge \dots \wedge \mathbf{U}^{(m)} \rangle.$$

Such an integral is referred to as the integral of  $m$ -form  $\tilde{\mathbf{S}}$  over the specified  $m$ -dimensional surface.

We thus see that in those cases where a given  $m$ -dimensional surface is considered only as an integration volume for the integrals discussed above, it is more adequate to characterize its local direction and parametrization with the multivector  $\mathbf{U}^{(1)} \wedge \dots \wedge \mathbf{U}^{(m)}$  rather than with the set of  $m$  individual tangent four-vectors. One can then consider the corresponding equivalence relation, according to which two nondegenerate parameterized  $m$ -dimensional surfaces going through a given point  $Q$  belong to the same equivalence class if and only if their multivectors coincide. It is not difficult to see that this relation is equivalent to relation (1) (and to its analogs for three- and four-dimensional volumes) with the additional requirement that the matrix  $\|a_i^k\|$  be unimodular. One should also notice that at  $m = 1$  this equivalence relation reproduces relation (2) of part II for parametrized curves.

Let us now see what will happen if the integration volume is characterized by tangent *five*-vectors. Since we are generalizing exterior calculus, it is natural to consider the case where the integrand is a linear function of each of these five-vectors.

Let us, again, start with parametrized curves. Owing to the invariant decomposition of  $V_5$  into the direct sum of  $\mathcal{Z}$  and  $\mathcal{E}$ , any integral of the considered kind can be presented as

$$\int_{\lambda_a}^{\lambda_b} d\lambda \langle \tilde{\mathbf{s}}, \mathbf{u}^{\mathcal{Z}} \rangle + \int_{\lambda_a}^{\lambda_b} d\lambda \langle \tilde{\mathbf{s}}', \mathbf{u}^{\mathcal{E}} \rangle, \quad (5)$$

where  $\mathbf{u}(\lambda)$  is the tangent five-vector and  $\tilde{\mathbf{s}}$  and  $\tilde{\mathbf{s}}'$  are some five-vector 1-forms. The first term in this formula is an invariant integral, whose value coincides with that of the integral  $\int \tilde{\mathbf{S}}$  along the same curve, where  $\tilde{\mathbf{S}}$  is the four-vector 1-form that corresponds to  $\tilde{\mathbf{s}}^{\mathcal{Z}}$ . The second term in formula (5) is proportional to

$$\int_{\lambda_a}^{\lambda_b} \lambda d\lambda \langle \tilde{\mathbf{s}}', \mathbf{1} \rangle$$

and is not an invariant integral unless  $\langle \tilde{\mathbf{s}}', \mathbf{1} \rangle$  is identically zero. Therefore, any invariant integral of the considered type should have the form of the first term in formula (5). The latter will be referred to as

the integral of five-vector 1-form  $\tilde{\mathbf{s}}$  along the specified curve.

In a similar way one can deal with invariant integrals over two-, three- and four-dimensional volumes. One can show that each of them can be presented in the form

$$\int_{\Lambda} d\lambda^{(1)} \dots d\lambda^{(m)} \langle \tilde{\mathbf{s}}, (\mathbf{u}^{(1)})^{\mathcal{Z}} \wedge \dots \wedge (\mathbf{u}^{(m)})^{\mathcal{Z}} \rangle, \quad (6)$$

where  $m = 2, 3, \text{ or } 4$  and  $\tilde{\mathbf{s}}$  is now some five-vector  $m$ -form. Integral (6) will be denoted as  $\int \tilde{\mathbf{s}}$  and will be referred to as the integral of  $m$ -form  $\tilde{\mathbf{s}}$  over the specified  $m$ -dimensional surface. As one can see, only the  $\mathcal{Z}$ -component of  $\tilde{\mathbf{s}}$  gives contribution to the integral, so for any integration volume the value of the latter coincides with that of the integral  $\int \tilde{\mathbf{S}}$ , where  $\tilde{\mathbf{S}}$  is the four-vector  $m$ -form that corresponds to  $\tilde{\mathbf{s}}^{\mathcal{Z}}$ .

There exists another interesting way of constructing invariant integrals. For that one should consider the given  $m$ -dimensional integration surface (now  $m = 1, 2, 3$  or  $4$ ) as a degenerate volume of dimension  $m + 1$  and characterize it with the multivector  $\mathbf{u}^{(1)} \wedge \dots \wedge \mathbf{u}^{(m)} \wedge \mathbf{e}$ , where  $\mathbf{u}^{(k)}$  are the five-dimensional tangent vectors that correspond to nondegenerate inner coordinates and  $\mathbf{e}$  is a nonzero five-vector from  $\mathcal{E}$ . The integral itself will have the form

$$\int_{\Lambda} d\lambda^{(1)} \dots d\lambda^{(m)} \langle \tilde{\mathbf{t}}, \mathbf{u}^{(1)} \wedge \dots \wedge \mathbf{u}^{(m)} \wedge \mathbf{e} \rangle, \quad (7)$$

where  $\tilde{\mathbf{t}}$  is some five-vector form of rank  $m + 1$ .

One should now select the five-vector  $\mathbf{e}$  from  $\mathcal{E}$ . It is evident that allowing  $\mathbf{e}$  to vary from one point to another is equivalent to introducing a certain weight factor into the integral. As a rule, such factors are not considered, but even if one does introduce one, it is more convenient not to absorb it into the multivector, so that the role of the latter would consist only in specifying the infinitesimal integration volume, as it does in ordinary exterior calculus. Considering this, it will be taken that  $\mathbf{e}$  is a constant vector.

For the same reason one can choose  $\mathbf{e}$  to have any nonzero size. If one wishes the five-vector exterior calculus to be applicable to manifolds without metric and without affine connection (as is its four-vector analog), one should select the size of  $\mathbf{e}$  without any reference to the inner product nor to parallel transport, and the only distinguished choice is then  $\mathbf{e} = \mathbf{1}$ .

Since  $\mathbf{u}^{(1)} \wedge \dots \wedge \mathbf{u}^{(m)} \wedge \mathbf{1} = (\mathbf{u}^{(1)})^{\mathcal{Z}} \wedge \dots \wedge (\mathbf{u}^{(m)})^{\mathcal{Z}} \wedge \mathbf{1}$ , for any integration volume integral (7) will have the same value as integral (6) in which  $\tilde{\mathbf{s}}$  is defined by the condition

$$\begin{aligned} \langle \tilde{\mathbf{s}}, (\mathbf{u}^{(1)})^{\mathcal{Z}} \wedge \dots \wedge (\mathbf{u}^{(m)})^{\mathcal{Z}} \rangle \\ = \langle \tilde{\mathbf{t}}, \mathbf{u}^{(1)} \wedge \dots \wedge \mathbf{u}^{(m)} \wedge \mathbf{1} \rangle. \end{aligned} \quad (8)$$

One should also notice that contrary to the case of integral (6), the contribution to integral (7) is given by the  $\tilde{\mathcal{E}}$ -component of form  $\tilde{\mathbf{t}}$ .

We thus see that a given five-vector  $m$ -form  $\tilde{\mathbf{s}}$  ( $m = 1, 2, 3$  or  $4$ ) can be integrated over volumes of two different dimensions: (i) over an  $m$ -dimensional surface, in which case the integral is determined only by the  $\tilde{\mathcal{Z}}$ -component of  $\tilde{\mathbf{s}}$ , or (ii) over an  $(m-1)$ -dimensional surface, in which case the integral depends only on the  $\tilde{\mathcal{E}}$ -component of  $\tilde{\mathbf{s}}$ . At  $m = 1$  the integration volume in the second case degenerates into an isolated point (or several isolated points), and the integral is replaced by the value of the contraction of the given 1-form with the five-vector  $\mathbf{1}$ . Five-vector 5-forms can be integrated over volumes of only one dimension: four. For an obvious reason, they do not have a  $\tilde{\mathcal{Z}}$ -component.

From our analysis it also follows that each additive rule for assigning numbers to limited  $m$ -dimensional volumes ( $m = 1, 2, 3$  or  $4$ ) produced by the integrals considered in this section can be interpreted in three different ways. It can be regarded as an integral of a four-vector  $m$ -form; or as an integral of a five-vector  $m$ -form; or as an integral of a five-vector  $(m+1)$ -form, over the considered volume. The three mentioned forms—which have been denoted above as  $\tilde{\mathbf{S}}$ ,  $\tilde{\mathbf{s}}$ , and  $\tilde{\mathbf{t}}$ , respectively—are related to one another in the following way:

$$\begin{aligned} \langle \tilde{\mathbf{S}}, \mathbf{U}^{(1)} \wedge \dots \wedge \mathbf{U}^{(m)} \rangle \\ &= \langle \tilde{\mathbf{s}}, (\mathbf{u}^{(1)})^{\mathcal{Z}} \wedge \dots \wedge (\mathbf{u}^{(m)})^{\mathcal{Z}} \rangle \\ &= \langle \tilde{\mathbf{t}}, \mathbf{u}^{(1)} \wedge \dots \wedge \mathbf{u}^{(m)} \wedge \mathbf{1} \rangle, \end{aligned}$$

where  $\mathbf{u}^{(k)} \in \mathbf{U}^{(k)}$  for  $k = 1, \dots, m$ . This invariant relation between forms of different types exists at each  $m$  owing to the isomphicity of the three corresponding vector spaces: (1) the space of multivectors of rank  $m$  made out of four-vectors; (2) the space of multivectors of rank  $m$  made out of five-vectors from  $\mathcal{Z}$ ; and (3) the space of wedge products of  $\mathbf{1}$  with multivectors of rank  $m$  made out of five-vectors. At  $m = 1$  these three vector spaces are respectively  $V_4$ ,  $\mathcal{Z}$ , and the maximal vector space of simple bivectors over  $V_5$  with the directional vector from  $\mathcal{E}$ , the isomorphisms between which have already been discussed in part II. In view of the mentioned isomphicity, the equivalence relation between parametrized  $m$ -dimensional volumes obtained by equating the corresponding multivectors is exactly the same for all three types of multivectors considered above.

### C. Generalized Stokes theorem

Let  $\partial V$  be the closed  $m$ -dimensional boundary of an  $(m+1)$ -dimensional surface  $V$  and let  $\tilde{\mathbf{S}}$  be a four-vector  $m$ -form defined throughout  $V$ . The integral of  $\tilde{\mathbf{S}}$  over  $\partial V$ , which will be denoted as

$$\oint_{\partial V} \tilde{\mathbf{S}},$$

is called the flux of form  $\tilde{\mathbf{S}}$  through the closed surface  $\partial V$ . The generalized Stokes theorem states that this flux equals the integral over the interior of  $V$  of a certain  $(m+1)$ -form, which is denoted as  $\mathbf{d}\tilde{\mathbf{S}}$  and is called the exterior derivative of  $\tilde{\mathbf{S}}$ :

$$\oint_{\partial V} \tilde{\mathbf{S}} = \int_V \mathbf{d}\tilde{\mathbf{S}}. \quad (9)$$

One can give several equivalent definitions to the exterior derivative. As a first step, one usually defines it for a scalar function (regarded as a four-vector 0-form):  $\mathbf{d}f$  is such a four-vector 1-form that

$$\langle \mathbf{d}f, \mathbf{U} \rangle = \partial_{\mathbf{U}} f \quad (10)$$

for any four-vector  $\mathbf{U}$ . This enables one to present the basis 1-forms dual to a coordinate four-vector basis associated with coordinates  $x^\alpha$  as exterior derivatives  $\mathbf{d}x^\alpha$ . One can then define the effect of  $\mathbf{d}$  on an arbitrary form in the following way: for the  $m$ -form

$$\tilde{\mathbf{S}} = S_{|\alpha_1 \dots \alpha_m|} \mathbf{d}x^{\alpha_1} \wedge \dots \wedge \mathbf{d}x^{\alpha_m} \quad (11)$$

one has

$$\mathbf{d}\tilde{\mathbf{S}} = \mathbf{d}S_{|\alpha_1 \dots \alpha_m|} \wedge \mathbf{d}x^{\alpha_1} \wedge \dots \wedge \mathbf{d}x^{\alpha_m}. \quad (12)$$

Another, equivalent definition of the exterior derivative can be given by induction: for any  $m$ -form  $\tilde{\mathbf{S}}$  and any  $n$ -form  $\tilde{\mathbf{T}}$

$$\mathbf{d}(\tilde{\mathbf{S}} \wedge \tilde{\mathbf{T}}) = \mathbf{d}\tilde{\mathbf{S}} \wedge \tilde{\mathbf{T}} + (-1)^m \tilde{\mathbf{S}} \wedge \mathbf{d}\tilde{\mathbf{T}}, \quad (13a)$$

and for any form  $\tilde{\mathbf{S}}$

$$\mathbf{d}\mathbf{d}\tilde{\mathbf{S}} = \mathbf{0}. \quad (13b)$$

One can also define the exterior derivative of any four-vector form *à la* equation (10): if  $\tilde{\mathbf{S}}$  is a 1-form, then  $\mathbf{d}\tilde{\mathbf{S}}$  is such a 2-form that

$$\begin{aligned} \langle \mathbf{d}\tilde{\mathbf{S}}, \mathbf{U} \wedge \mathbf{V} \rangle &= \partial_{\mathbf{U}} \langle \tilde{\mathbf{S}}, \mathbf{V} \rangle \\ &\quad - \partial_{\mathbf{V}} \langle \tilde{\mathbf{S}}, \mathbf{U} \rangle - \langle \tilde{\mathbf{S}}, [\mathbf{U}, \mathbf{V}] \rangle \end{aligned} \quad (14a)$$

for any four-vector fields  $\mathbf{U}$  and  $\mathbf{V}$ ; if  $\tilde{\mathbf{S}}$  is a 2-form, then  $\mathbf{d}\tilde{\mathbf{S}}$  is such a 3-form that

$$\begin{aligned} \langle \mathbf{d}\tilde{\mathbf{S}}, \mathbf{U} \wedge \mathbf{V} \wedge \mathbf{W} \rangle \\ &= \partial_{\mathbf{U}} \langle \tilde{\mathbf{S}}, \mathbf{V} \wedge \mathbf{W} \rangle + \partial_{\mathbf{V}} \langle \tilde{\mathbf{S}}, \mathbf{W} \wedge \mathbf{U} \rangle \\ &\quad + \partial_{\mathbf{W}} \langle \tilde{\mathbf{S}}, \mathbf{U} \wedge \mathbf{V} \rangle - \langle \tilde{\mathbf{S}}, [\mathbf{U}, \mathbf{V}] \wedge \mathbf{W} \rangle \\ &\quad - \langle \tilde{\mathbf{S}}, [\mathbf{V}, \mathbf{W}] \wedge \mathbf{U} \rangle - \langle \tilde{\mathbf{S}}, [\mathbf{W}, \mathbf{U}] \wedge \mathbf{V} \rangle \end{aligned} \quad (14b)$$

for any four-vector fields  $\mathbf{U}$ ,  $\mathbf{V}$  and  $\mathbf{W}$ ; etc.

Let us now generalize the concept of flux and the Stokes theorem to the case of five-vector forms.

It is natural to define the flux of a five-vector  $m$ -form  $\tilde{\mathbf{t}}$  through a closed  $m$ -dimensional surface  $\partial V$  as the integral of  $\tilde{\mathbf{t}}$  over  $\partial V$ , and I will denote this particular integral as

$$\oint_{\partial V} \tilde{\mathbf{t}}.$$

According to section B, this flux equals the flux through  $\partial V$  of the four-vector  $m$ -form  $\tilde{\mathbf{T}}$  corresponding to  $\tilde{\mathbf{t}}^{\tilde{\mathcal{Z}}}$ . If  $\partial V$  is the boundary of the  $(m+1)$ -dimensional volume  $V$ , then by virtue of the generalized Stokes theorem for four-vector forms, one has

$$\oint_{\partial V} \tilde{\mathbf{t}} = \oint_{\partial V} \tilde{\mathbf{T}} = \int_V \mathbf{d}\tilde{\mathbf{T}}.$$

Now, using the one-to-one correspondence that exists between four-vector forms and  $\tilde{\mathcal{Z}}$ -components of five-vector forms, one can present the flux of  $\tilde{\mathbf{t}}$  through  $\partial V$  as an integral over the interior of  $V$  of a five-vector  $(m+1)$ -form whose  $\tilde{\mathcal{Z}}$ -component corresponds to  $\mathbf{d}\tilde{\mathbf{T}}$ . It is natural to call this form the exterior derivative of  $\tilde{\mathbf{t}}$  and to denote it as  $\mathbf{d}\tilde{\mathbf{t}}$ . We thus obtain the following variant of the generalized Stokes theorem for five-vector forms:

$$\oint_{\partial V} \tilde{\mathbf{t}} = \int_V \mathbf{d}\tilde{\mathbf{t}}, \quad (15)$$

where in both integrals the rank of the form equals the dimension of the integration volume.

It is not difficult to show that if

$$\tilde{\mathbf{t}} = t_{|A_1 \dots A_m|} \tilde{\mathbf{o}}^{A_1} \wedge \dots \wedge \tilde{\mathbf{o}}^{A_m}, \quad (16)$$

where  $\tilde{\mathbf{o}}^A$  is the basis of five-vector 1-forms dual to a *passive regular coordinate* five-vector basis, then

$$(\mathbf{d}\tilde{\mathbf{t}})^{\tilde{\mathcal{Z}}} = \partial_\alpha t_{|\alpha_1 \dots \alpha_m|} \tilde{\mathbf{o}}^\alpha \wedge \tilde{\mathbf{o}}^{\alpha_1} \wedge \dots \wedge \tilde{\mathbf{o}}^{\alpha_m}. \quad (17)$$

The  $\tilde{\mathcal{E}}$ -component of  $\mathbf{d}\tilde{\mathbf{t}}$  is not fixed by the above correspondence, and to determine it one should consider the flux of  $\tilde{\mathbf{t}}$  through a closed surface of dimension  $m-1$ . Let us denote this surface, again, as  $\partial V$ . The flux in question will then be the integral of  $\tilde{\mathbf{t}}$  over  $\partial V$ . According to section B, it equals the flux through  $\partial V$  of the five-vector  $(m-1)$ -form  $\tilde{\mathbf{s}}$  related to  $\tilde{\mathbf{t}}$  by an equation similar to equation (8). If  $V$  denotes an  $m$ -dimensional volume limited by  $\partial V$ , then according to the variant of the generalized Stokes theorem we have obtained above, one has

$$\oint_{\partial V} \tilde{\mathbf{t}} = \oint_{\partial V} \tilde{\mathbf{s}} = \int_V \mathbf{d}\tilde{\mathbf{s}}.$$

If now one uses the one-to-one correspondence between the  $\tilde{\mathcal{Z}}$ -components of five-vector  $m$ -forms and the  $\tilde{\mathcal{E}}$ -components of five-vector  $(m+1)$ -forms, one can present the flux of  $\tilde{\mathbf{t}}$  through  $\partial V$  as an integral over the interior of  $V$  of a five-vector  $(m+1)$ -form whose  $\tilde{\mathcal{E}}$ -component corresponds to  $(\mathbf{d}\tilde{\mathbf{s}})^{\tilde{\mathcal{Z}}}$ . This  $(m+1)$ -form also has the meaning of an exterior derivative of  $\tilde{\mathbf{t}}$ , and since its  $\tilde{\mathcal{Z}}$ -component is not fixed by the above correspondence, one can take that this is the same form  $\mathbf{d}\tilde{\mathbf{t}}$  we have introduced above. We thus obtain another variant of the generalized Stokes theorem, which has the same form as equation (15), only now in both integrals the rank of the form is one unit greater than the dimension of the integration volume. It is not difficult to show that

$$(\mathbf{d}\tilde{\mathbf{t}})^{\tilde{\mathcal{E}}} = \partial_\alpha t_{|\alpha_1 \dots \alpha_{m-1}|5} \tilde{\mathbf{o}}^\alpha \wedge \tilde{\mathbf{o}}^{\alpha_1} \wedge \dots \wedge \tilde{\mathbf{o}}^{\alpha_{m-1}} \wedge \tilde{\mathbf{o}}^5,$$

which together with equation (17) gives one the following general formula for calculating the exterior derivative of  $m$ -form (16):

$$\mathbf{d}\tilde{\mathbf{t}} = \partial_A t_{|A_1 \dots A_m|} \tilde{\mathbf{o}}^A \wedge \tilde{\mathbf{o}}^{A_1} \wedge \dots \wedge \tilde{\mathbf{o}}^{A_m}. \quad (18)$$

As in four-vector exterior calculus, one can define the exterior derivative of a five-vector form without any reference to the Stokes theorem, i.e. as a certain operator that produces an  $(m+1)$ -form out of an  $m$ -form. One should, again, start with a scalar function  $f$ , which is now regarded as a five-vector 0-form, and define its exterior derivative as such a five-vector 1-form  $\mathbf{d}f$  that

$$\langle \mathbf{d}f, \mathbf{u} \rangle = \partial_{\mathbf{u}} f \quad (19)$$

for any five-vector  $\mathbf{u}$ . One can then present formula (18) in a form similar to equation (12):

$$\mathbf{d}\tilde{\mathbf{t}} = \mathbf{d}t_{|A_1 \dots A_m|} \wedge \tilde{\mathbf{o}}^{A_1} \wedge \dots \wedge \tilde{\mathbf{o}}^{A_m}, \quad (20)$$

and take this to be the definition of  $\mathbf{d}\tilde{\mathbf{t}}$ .

Another way of defining the exterior derivative for five-vector forms is similar to equations (14): if  $\tilde{\mathbf{t}}$  is a 1-form, then  $\mathbf{d}\tilde{\mathbf{t}}$  is such a 2-form that

$$\langle \mathbf{d}\tilde{\mathbf{t}}, \mathbf{u} \wedge \mathbf{v} \rangle = \partial_{\mathbf{u}} \langle \tilde{\mathbf{t}}, \mathbf{v} \rangle - \partial_{\mathbf{v}} \langle \tilde{\mathbf{t}}, \mathbf{u} \rangle - \langle \tilde{\mathbf{t}}, [\mathbf{u}, \mathbf{v}] \rangle \quad (21)$$

for any two five-vector fields  $\mathbf{u}$  and  $\mathbf{v}$ ; etc.

Finally, from formula (20) one can derive the analogs of equations (13): for any  $m$ -form  $\tilde{\mathbf{s}}$  and any  $n$ -form  $\tilde{\mathbf{t}}$

$$\mathbf{d}(\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}}) = \mathbf{d}\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} + (-1)^m \tilde{\mathbf{s}} \wedge \mathbf{d}\tilde{\mathbf{t}}, \quad (22a)$$

and for any form  $\tilde{\mathbf{t}}$

$$\mathbf{d}\tilde{\mathbf{t}} = \mathbf{0}. \quad (22b)$$

However, unlike the case of four-vector forms, the latter two equations (together with equation (19)) are not enough to define the exterior derivative completely. To gain a better understanding of this fact, one should recall how things work out in the case of four-vector forms.

From the generalized Stokes theorem one obtains the following formula for the exterior derivative:

$$\mathbf{d}\tilde{\mathbf{S}} = \mathbf{d}S_{|\alpha_1 \dots \alpha_m|} \wedge \tilde{\mathbf{O}}^{\alpha_1} \wedge \dots \wedge \tilde{\mathbf{O}}^{\alpha_m}, \quad (23)$$

where  $\tilde{\mathbf{O}}^\alpha$  is any basis of four-vector 1-forms dual to a coordinate basis. Provided the effect of  $\mathbf{d}$  on a scalar function is known, equation (23) is equivalent to equation (13a) and the requirement

$$\mathbf{d}\tilde{\mathbf{O}}^\alpha = 0 \quad (24)$$

(the latter, by the way, is a necessary and sufficient condition of the corresponding basis of four-vectors being a coordinate basis). From equation (10) one can derive that  $\tilde{\mathbf{O}}^\alpha$  equal  $\mathbf{d}x^\alpha$ , and so equation (24) follows from equation (13b) and can be replaced with it.

In the case of five-vector forms, equation (20) is equivalent to equation (22a) and the requirement

$$\mathbf{d}\tilde{\mathbf{o}}^A = 0, \quad (25)$$

which is a necessary condition of the corresponding basis of five-vectors being a regular coordinate basis. However, from equation (19) one can only derive that  $\tilde{\mathbf{o}}^\alpha = \mathbf{d}x^\alpha$ . The fifth basis 1-form, which in this case equals  $\tilde{\mathbf{j}}$ , cannot be presented as an exterior derivative of any 0-form (this is also true of *any* basis of five-vector 1-forms dual to a *standard* basis). Thus, its exterior derivative is not determined by the rule  $\mathbf{d}\mathbf{d} = 0$ , and to make equations (22) equivalent to equation (20) one should supplement the former with a third equation:

$$\mathbf{d}\tilde{\mathbf{j}} = 0. \quad (26)$$

The fact that basis four-vector 1-forms dual to a coordinate basis can be presented as exterior derivatives of scalar functions, whereas in the case of five-vector forms this is possible only for the first four basis 1-forms, is closely related to the Poincare theorem in application to four-vector forms and to its analog for forms associated with five-vectors. The Poincare theorem states that each four-vector  $m$ -form  $\tilde{\mathbf{S}}$  with  $m \geq 1$ , which in a certain region of space-time<sup>1</sup> satisfies the equation  $\mathbf{d}\tilde{\mathbf{S}} = 0$  (such forms are called

<sup>1</sup>This region should satisfy certain requirements, which can be found e.g. in ref.[1].

*closed*), can be presented in this region as an exterior derivative of some four-vector  $(m-1)$ -form. In agreement with this theorem, the basis four-vector 1-forms  $\tilde{\mathbf{O}}^\alpha$ , which satisfy equation (24), can be presented as exterior derivatives of certain scalar functions (which in this particular case can be chosen to coincide with coordinates). A statement similar to the Poincare theorem can be easily shown to hold for forms corresponding to five-vectors:

Any five-vector  $m$ -form  $\tilde{\mathbf{s}}$  with  $m \geq 2$ , which in a certain region of space-time (subject to the same constraints that are imposed within the Poincare theorem for four-vector forms) satisfies the equation  $\mathbf{d}\tilde{\mathbf{s}} = 0$ , can be presented in this region as an exterior derivative of some five-vector  $(m-1)$ -form.

At  $m = 1$  the theorem works for the  $\tilde{\mathcal{Z}}$ -component only. The  $\tilde{\mathcal{E}}$ -component of any five-vector 1-form, if it is nonzero, is not an exterior derivative of any scalar function; in this case from  $\mathbf{d}(\tilde{\mathbf{s}}^{\tilde{\mathcal{E}}}) = \mathbf{0}$  follows  $\tilde{\mathbf{s}}^{\tilde{\mathcal{E}}} = \text{const} \cdot \tilde{\mathbf{j}}$ . One can easily see how this theorem manifests itself in the case of basis five-vector 1-forms  $\tilde{\mathbf{o}}^A$ , which satisfy equation (25).

The representation of basis four-vector 1-forms  $\tilde{\mathbf{O}}^\alpha$  as exterior derivatives of coordinates is a convenient way of indicating that the selected basis of 1-forms is dual to a coordinate basis. We see that no similar convenient representation exists in the case of five-vector forms, so one should either introduce a special notation for the basis of 1-forms dual to a regular coordinate basis or each time indicate explicitly what kind of a basis is used in the formulae presented.

#### D. Five-vector exterior derivative

In the previous section we have generalized the concept of flux and of exterior derivative to the case of five-vector forms. It turns out that one can go further and consider an operator similar to the exterior derivative but which differs from the latter in that  $\partial$  in the right-hand side of formula (18) is replaced with its five-vector counterpart:  $\partial + \lambda \cdot \varsigma \cdot \mathbf{1}$ . Let us call this operator the *five-vector exterior derivative* and denote its effect on a five-vector form  $\tilde{\mathbf{t}}$  as  $\mathbf{d}\tilde{\mathbf{t}}$ .

Since the operator  $\partial_{\mathbf{u}} + \lambda_{\mathbf{u}} \cdot \varsigma \cdot \mathbf{1}$  will often appear in the following formulae, it is convenient to introduce a special notation for it:  $\mathfrak{D}_{\mathbf{u}}$ . By definition, for any scalar function  $f$ ,

$$\mathfrak{D}_{\mathbf{u}}f \equiv \mathbf{u}[f] = \partial_{\mathbf{u}}f + \varsigma \lambda_{\mathbf{u}}f. \quad (27)$$

From equations (13) of part II it follows that the effect of  $\mathfrak{D}_{\mathbf{u}}$  on the product of two scalar functions is

formally described by the rule

$$\partial_{\mathbf{u}}(fg) = \partial_{\mathbf{u}}f \cdot g + f \cdot \partial_{\mathbf{u}}g - \partial_{\mathbf{u}}1 \cdot fg, \quad (28)$$

where  $1$  is the constant unity function. As in the case of operators  $\partial$  and  $\nabla$ , it is convenient to introduce the notation  $\partial_A \equiv \partial_{\mathbf{e}_A}$ , where  $\mathbf{e}_A$  is the selected basis of five-vectors. Using this notation, one can present the five-vector exterior derivative of  $m$ -form (16) as

$$\mathbf{d}\tilde{\mathbf{t}} = \partial_A t_{|A_1 \dots A_m|} \tilde{\mathbf{o}}^A \wedge \tilde{\mathbf{o}}^{A_1} \wedge \dots \wedge \tilde{\mathbf{o}}^{A_m}, \quad (29)$$

where, let us recall,  $\tilde{\mathbf{o}}^A$  is the basis of five-vector 1-forms dual to a passive regular coordinate basis. One can easily see that there exists a very simple relation between  $\mathbf{d}$  and  $\tilde{\mathbf{d}}$ : for any form  $\tilde{\mathbf{t}}$

$$\mathbf{d}\tilde{\mathbf{t}} = \tilde{\mathbf{d}}\tilde{\mathbf{t}} + \tilde{\mathbf{j}} \wedge \tilde{\mathbf{t}}. \quad (30)$$

From equation (29) it follows that the effect of  $\mathbf{d}$  on a scalar function  $f$  (regarded as a five-vector 0-form) is given by the formula

$$\langle \mathbf{d}f, \mathbf{u} \rangle = \partial_{\mathbf{u}}f, \quad (31)$$

which is the analog of equations (10) and (19) for  $\mathbf{d}$ . One can then present formula (29) in a form similar to equations (20) and (23):

$$\mathbf{d}\tilde{\mathbf{t}} = \mathbf{d}t_{|A_1 \dots A_m|} \wedge \tilde{\mathbf{o}}^{A_1} \wedge \dots \wedge \tilde{\mathbf{o}}^{A_m}, \quad (32)$$

which, together with formula (31), can serve as a definition of the effect of  $\mathbf{d}$  on an arbitrary five-vector form.

Another way of defining the five-vector exterior derivative is similar to equations (14) and (21): if  $\tilde{\mathbf{t}}$  is a five-vector 1-form, then  $\mathbf{d}\tilde{\mathbf{t}}$  is such a 2-form that

$$\langle \mathbf{d}\tilde{\mathbf{t}}, \mathbf{u} \wedge \mathbf{v} \rangle = \partial_{\mathbf{u}} \langle \tilde{\mathbf{t}}, \mathbf{v} \rangle - \partial_{\mathbf{v}} \langle \tilde{\mathbf{t}}, \mathbf{u} \rangle - \langle \tilde{\mathbf{t}}, [\mathbf{u}, \mathbf{v}] \rangle \quad (33)$$

for any two five-vector fields  $\mathbf{u}$  and  $\mathbf{v}$ ; etc.

Finally, from formula (32) one can derive the analogs of equations (13) and (22): for any  $m$ -form  $\tilde{\mathbf{s}}$  and any  $n$ -form  $\tilde{\mathbf{t}}$

$$\mathbf{d}(\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}}) = \mathbf{d}\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} + (-1)^m \tilde{\mathbf{s}} \wedge \mathbf{d}\tilde{\mathbf{t}} - \mathbf{d}1 \wedge \tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}}, \quad (34a)$$

and for any form  $\tilde{\mathbf{t}}$

$$\mathbf{d}\mathbf{d}\tilde{\mathbf{t}} = \mathbf{0}. \quad (34b)$$

These two equations are equivalent to equation (32) and, together with equation (31), can be used to define the effect of  $\mathbf{d}$  on any five-vector form by induction. Indeed, by using equation (31) one can show that any basis of five-vector 1-forms,  $\tilde{\mathbf{o}}^A$ , dual to a

passive regular coordinate basis associated with coordinates  $x^\alpha$  can be presented as

$$\begin{cases} \tilde{\mathbf{o}}^\alpha = \mathbf{d}x^\alpha - x^\alpha \mathbf{d}1 \\ \tilde{\mathbf{o}}^5 = \mathbf{d}1 \end{cases}. \quad (35)$$

From equations (34) one then obtains that

$$\mathbf{d}\tilde{\mathbf{o}}^A - \mathbf{d}1 \wedge \tilde{\mathbf{o}}^A = \mathbf{0}, \quad (36)$$

which is nothing but equation (25) expressed in terms of  $\mathbf{d}$ . By induction, from the latter equation one can derive that

$$\mathbf{d}(\tilde{\mathbf{o}}^{A_1} \wedge \dots \wedge \tilde{\mathbf{o}}^{A_m}) - \mathbf{d}1 \wedge \tilde{\mathbf{o}}^{A_1} \wedge \dots \wedge \tilde{\mathbf{o}}^{A_m} = \mathbf{0}$$

for any  $m \geq 1$ , which together with equation (34a) is equivalent to equation (32).

It is evident that the analog of the Poincare theorem for five-vector forms presented in the previous section can be reformulated in terms of  $\mathbf{d}$  in the following way: any five-vector  $m$ -form with  $m \geq 2$  or 1-form from  $\tilde{\mathcal{Z}}$ , which in a certain region of space-time (subject to the constraints imposed within the Poincare theorem) satisfies the equation  $\mathbf{d}\tilde{\mathbf{s}} - \tilde{\mathbf{j}} \wedge \tilde{\mathbf{s}} = \mathbf{0}$ , can be presented in this region as  $\tilde{\mathbf{s}} = \mathbf{d}\tilde{\mathbf{t}} - \tilde{\mathbf{j}} \wedge \tilde{\mathbf{t}}$ , where  $\tilde{\mathbf{t}}$  is some  $(m-1)$ -form. There exists another theorem for  $\mathbf{d}$ , which can also be regarded as an analog of the Poincare theorem:

Any five-vector  $m$ -form  $\tilde{\mathbf{s}}$  with  $m \geq 1$ , which in a certain region of space-time (subject to the same constraints that are imposed within the Poincare theorem) satisfies the equation  $\mathbf{d}\tilde{\mathbf{s}} = \mathbf{0}$ , can be presented in this region as a five-vector exterior derivative of some  $(m-1)$ -form. At  $m=0$  from the above equation follows  $\tilde{\mathbf{s}} = \mathbf{0}$ .

*Proof:* At  $m \geq 1$ , from equation (30) one has

$$\mathbf{d}\tilde{\mathbf{s}} = \tilde{\mathbf{d}}\tilde{\mathbf{s}} + \tilde{\mathbf{j}} \wedge \tilde{\mathbf{s}} = \mathbf{d}(\tilde{\mathbf{s}}^{\tilde{\mathcal{Z}}}) + \mathbf{d}(\tilde{\mathbf{s}}^{\tilde{\mathcal{E}}}) + \tilde{\mathbf{j}} \wedge (\tilde{\mathbf{s}}^{\tilde{\mathcal{Z}}}) = \mathbf{0}. \quad (37)$$

Consequently,  $\mathbf{d}(\tilde{\mathbf{s}}^{\tilde{\mathcal{Z}}}) = (\mathbf{d}\tilde{\mathbf{s}})^{\tilde{\mathcal{Z}}} = \mathbf{0}$ , and according to the analog of the Poincare theorem for five-vector forms presented in the previous section, there exists an  $(m-1)$ -form  $\tilde{\mathbf{t}}$  such that  $\tilde{\mathbf{s}}^{\tilde{\mathcal{Z}}} = \mathbf{d}(\tilde{\mathbf{t}}^{\tilde{\mathcal{Z}}})$ . The  $\tilde{\mathcal{E}}$ -component of equation (37) yields  $\mathbf{d}(\tilde{\mathbf{s}}^{\tilde{\mathcal{E}}}) + \tilde{\mathbf{j}} \wedge (\tilde{\mathbf{s}}^{\tilde{\mathcal{Z}}}) = (\mathbf{d}\tilde{\mathbf{s}})^{\tilde{\mathcal{E}}} = \mathbf{0}$ , so

$$\mathbf{d}(\tilde{\mathbf{s}}^{\tilde{\mathcal{E}}}) = -\tilde{\mathbf{j}} \wedge \tilde{\mathbf{s}}^{\tilde{\mathcal{Z}}} = -\tilde{\mathbf{j}} \wedge \mathbf{d}(\tilde{\mathbf{t}}^{\tilde{\mathcal{Z}}}) = \mathbf{d}(\tilde{\mathbf{j}} \wedge \tilde{\mathbf{t}}^{\tilde{\mathcal{Z}}}). \quad (38)$$

At  $m \geq 2$ , there exists an  $(m-1)$ -form  $\tilde{\mathbf{r}}$  such that  $\tilde{\mathbf{s}}^{\tilde{\mathcal{E}}} = \tilde{\mathbf{j}} \wedge \tilde{\mathbf{t}}^{\tilde{\mathcal{Z}}} + \mathbf{d}(\tilde{\mathbf{r}}^{\tilde{\mathcal{E}}})$ , and since  $\tilde{\mathbf{j}} \wedge \tilde{\mathbf{r}}^{\tilde{\mathcal{E}}} = \mathbf{0}$ , one obtains  $\tilde{\mathbf{s}} = \tilde{\mathbf{s}}^{\tilde{\mathcal{Z}}} + \tilde{\mathbf{s}}^{\tilde{\mathcal{E}}} = \mathbf{d}(\tilde{\mathbf{t}}^{\tilde{\mathcal{Z}}} + \tilde{\mathbf{r}}^{\tilde{\mathcal{E}}}) + \tilde{\mathbf{j}} \wedge (\tilde{\mathbf{t}}^{\tilde{\mathcal{Z}}} + \tilde{\mathbf{r}}^{\tilde{\mathcal{E}}}) = \mathbf{d}(\tilde{\mathbf{t}}^{\tilde{\mathcal{Z}}} + \tilde{\mathbf{r}}^{\tilde{\mathcal{E}}})$ .

At  $m = 1$ ,  $\tilde{\mathbf{t}}$  is a scalar function, so  $\tilde{\mathbf{t}}^{\tilde{\mathcal{Z}}} = \tilde{\mathbf{t}} \equiv t$ . From equation (38) it then follows that  $\tilde{\mathbf{s}}^{\tilde{\mathcal{E}}} - t \cdot \tilde{\mathbf{j}} = \text{const} \cdot \tilde{\mathbf{j}}$ , and since  $\mathbf{d}(\text{const}) = \mathbf{0}$ , one obtains

$$\tilde{\mathbf{s}} = \tilde{\mathbf{s}}^{\tilde{\mathcal{Z}}} + \tilde{\mathbf{s}}^{\tilde{\mathcal{E}}} = \mathbf{d}(t + \text{const}) + (t + \text{const}) \cdot \tilde{\mathbf{j}} = \mathbf{d}(t + \text{const}).$$

Finally, at  $m = 0$ , one has  $\tilde{\mathbf{s}}^{\tilde{\mathcal{Z}}} = \tilde{\mathbf{s}}$ , and the  $\tilde{\mathcal{E}}$ -component of equation  $\mathbf{d}\tilde{\mathbf{s}} = \mathbf{0}$  yields  $\tilde{\mathbf{s}} = \mathbf{0}$ . ■

It is not difficult to see that the  $(m-1)$ -form whose five-vector exterior derivative equals  $\tilde{\mathbf{s}}$  is unique only at  $m = 1$ . At  $m \geq 2$  there exists an infinite number of such  $(m-1)$ -forms, but the difference of any two of them is a five-vector exterior derivative of some  $(m-2)$ -form.

### E. Five-vector flux and reflected five-vector exterior derivative

The concept of exterior derivative is directly related to the concept of flux: by definition, for any five-vector form  $\tilde{\mathbf{s}}$ , the exterior derivative  $\mathbf{d}\tilde{\mathbf{s}}$  is such that its integral over the interior of any limited volume of appropriate dimension equals the flux of  $\tilde{\mathbf{s}}$  through the volume boundary. We now have a generalization of  $\mathbf{d}$ : the five-vector exterior derivative  $\mathbf{d}$ , and one can use the above scheme in the opposite direction to obtain the five-vector generalization of the flux, defining the latter as a quantity which for a given form  $\tilde{\mathbf{s}}$  and any limited volume  $V$  of appropriate dimension, equals the integral of  $\mathbf{d}\tilde{\mathbf{s}}$  over  $V$ . Let us call this quantity the *five-vector flux of form  $\tilde{\mathbf{s}}$  through volume  $V$*  and denote it as

$$\oint_V \tilde{\mathbf{s}}.$$

The formal analog of the generalized Stokes theorem for  $\mathbf{d}$  can then be presented as

$$\int_V \mathbf{d}\tilde{\mathbf{s}} = \oint_V \tilde{\mathbf{s}}. \quad (39)$$

Let us now find the expression for the five-vector flux in the form of an integral of  $\tilde{\mathbf{s}}$ . If  $\tilde{\mathbf{s}}$  is an  $m$ -form, then  $\mathbf{d}\tilde{\mathbf{s}}$  can be integrated over volumes of dimension  $m+1$  and  $m$ . In the former case the integral will depend only on the  $\tilde{\mathcal{Z}}$ -component of  $\mathbf{d}\tilde{\mathbf{s}}$ , and since the latter coincides with  $(\mathbf{d}\tilde{\mathbf{s}})^{\tilde{\mathcal{Z}}}$ , the flux corresponding to  $\mathbf{d}$  will be exactly the same as the flux corresponding to  $\mathbf{d}$ .

The integral of  $\mathbf{d}\tilde{\mathbf{s}}$  over an  $m$ -dimensional volume  $V$  can be presented, by using equation (30) and the generalized Stokes theorem (15), as

$$\begin{aligned} \int_V \mathbf{d}\tilde{\mathbf{s}} &= \int_V \mathbf{d}\tilde{\mathbf{s}} + \int_V \tilde{\mathbf{j}} \wedge \tilde{\mathbf{s}} \\ &= \oint_{\partial V} \tilde{\mathbf{s}} + (-1)^m \int_V \tilde{\mathbf{s}} \wedge \tilde{\mathbf{j}}. \end{aligned} \quad (40)$$

One should now notice that the forms  $\tilde{\mathbf{s}}$  and  $\tilde{\mathbf{t}} \equiv \tilde{\mathbf{s}} \wedge \tilde{\mathbf{j}}$  satisfy equation (8) of section B, which means that the second integral in the right-hand side of equation (40) is simply the integral of  $\tilde{\mathbf{s}}$  over  $V$ . One thus obtains the following expression for the five-vector flux of  $m$ -form  $\tilde{\mathbf{s}}$  through the  $m$ -dimensional volume  $V$ :

$$\oint_V \tilde{\mathbf{s}} = \oint_{\partial V} \tilde{\mathbf{s}} + (-1)^m \int_V \tilde{\mathbf{s}}. \quad (41)$$

One should notice that unlike the fluxes of five-vector forms considered in section C, the five-vector flux (41) depends on *both* components of form  $\tilde{\mathbf{s}}$ .

An important consequence of the generalized Stokes theorem is the formula for integration by parts. In the case of five-vector forms, this formula can be easily obtained by integrating both sides of equation (22a) over a given volume  $V$  and then using equation (15) to convert the integral of  $\mathbf{d}(\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}})$  over  $V$  into the integral of  $\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}}$  over  $\partial V$ . After rearranging the terms, one has

$$\int_V \mathbf{d}\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} = \oint_{\partial V} \tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} - (-1)^m \int_V \tilde{\mathbf{s}} \wedge \mathbf{d}\tilde{\mathbf{t}}, \quad (42)$$

where  $m$  is the rank of  $\tilde{\mathbf{s}}$ . The generalized Stokes theorem itself can be regarded as a particular case of this formula where  $\tilde{\mathbf{t}}$  is the constant unity 0-form. Let us now derive the analog of this formula for  $\mathbf{d}$ .

Following the same procedure, one integrates both sides of equation (34a) over a given volume  $V$  and then uses equation (39) to convert the integral of  $\mathbf{d}(\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}})$  over  $V$  into the five-vector flux of  $\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}}$ . After rearranging the terms, one obtains

$$\begin{aligned} \int_V \mathbf{d}\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} &= \oint_V \tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} \\ &\quad - (-1)^m \int_V \tilde{\mathbf{s}} \wedge \mathbf{d}\tilde{\mathbf{t}} + \int_V \tilde{\mathbf{j}} \wedge \tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}}. \end{aligned} \quad (43)$$

It is a simple matter to see that for volume  $V$  with dimension one unit greater than the rank of  $\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}}$ , this equation reproduces formula (42). For volume  $V$  with dimension equal to the rank of  $\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}}$ , it gives one a new formula, which, however, is not very useful since the first term in its right-hand side includes an integral over the interior of  $V$  and, in addition, there exists a third term, which is also an integral over  $V$ . A more useful formula can be obtained if in the right-hand side of equation (43) one isolates the integral over the boundary of  $V$  and combines everything else into a single second term. After simple transformations one obtains:

$$\begin{aligned} \int_V \mathbf{d}\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} &= \oint_{\partial V} \tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} \\ &\quad - (-1)^m \int_V \tilde{\mathbf{s}} \wedge (\mathbf{d}\tilde{\mathbf{t}} - 2\tilde{\mathbf{j}} \wedge \tilde{\mathbf{t}}). \end{aligned} \quad (44)$$



We thus see that it makes sense to consider one more operator, which will be denoted as  $\mathbf{d}^*$  and will be called the *reflected five-vector exterior derivative*. By definition, for any form  $\tilde{\mathbf{t}}$ ,

$$\mathbf{d}^*\tilde{\mathbf{t}} = \tilde{\mathbf{d}}\tilde{\mathbf{t}} - \tilde{\mathbf{j}} \wedge \tilde{\mathbf{t}}. \quad (45)$$

Formula (44) can now be rewritten as

$$\int_V \mathbf{d}\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} = \oint_{\partial V} \tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} - (-1)^m \int_V \tilde{\mathbf{s}} \wedge \mathbf{d}^*\tilde{\mathbf{t}}. \quad (46)$$

It is easy to see that the latter equation can also be presented as

$$\int_V \mathbf{d}^*\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} = \oint_{\partial V} \tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} - (-1)^m \int_V \tilde{\mathbf{s}} \wedge \mathbf{d}\tilde{\mathbf{t}}. \quad (47)$$

Formulae (46) and (47) can be derived directly from the following equations:

$$\begin{aligned} \mathbf{d}(\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}}) &= \mathbf{d}\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} + (-1)^m \tilde{\mathbf{s}} \wedge \mathbf{d}\tilde{\mathbf{t}} \\ &= \mathbf{d}^*\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}} + (-1)^m \tilde{\mathbf{s}} \wedge \mathbf{d}\tilde{\mathbf{t}}, \end{aligned} \quad (48)$$

which are analogs of equations (22a) and (34a).

In order to express  $\mathbf{d}^*\tilde{\mathbf{t}}$  in terms of its components, it is convenient to introduce the corresponding analog of the operator  $\mathbf{\partial}_{\mathbf{u}}$ , which I will denote as  $\mathbf{\partial}_{\mathbf{u}}^*$ . By definition, for any scalar function  $f$

$$\mathbf{\partial}_{\mathbf{u}}^* f \equiv \partial_{\mathbf{u}} f - \varsigma \lambda_{\mathbf{u}} f.$$

From equation (28) one can then derive that

$$\partial_{\mathbf{u}}(fg) = \mathbf{\partial}_{\mathbf{u}} f \cdot g + f \cdot \mathbf{\partial}_{\mathbf{u}}^* g = \mathbf{\partial}_{\mathbf{u}}^* f \cdot g + f \cdot \mathbf{\partial}_{\mathbf{u}} g,$$

which is similar to equations (48) for  $\mathbf{d}$ ,  $\mathbf{d}$  and  $\mathbf{d}^*$ . Using this new notation, one can present the reflected five-vector exterior derivative of  $m$ -form (16) as

$$\mathbf{d}^*\tilde{\mathbf{t}} = \mathbf{\partial}_{A_1}^* t_{|A_1 \dots A_m|} \tilde{\mathbf{o}}^A \wedge \tilde{\mathbf{o}}^{A_1} \wedge \dots \wedge \tilde{\mathbf{o}}^{A_m}. \quad (49)$$

One can then derive the analogs of equations (31) through (34), which will differ from the latter in that everywhere  $\mathbf{d}$  will be replaced with  $\mathbf{d}^*$  and  $\mathbf{\partial}_{\mathbf{u}}$  will be replaced with  $\mathbf{\partial}_{\mathbf{u}}^*$ .

#### F. Euler-Lagrange equations for classical scalar fields

A good illustration to the ideas developed in this paper can be found in the Lagrange formalism for classical scalar fields. Let us suppose that we have  $N$  such fields and let us denote them as  $\phi_\ell$  ( $\ell$  runs 1 through  $N$  and labels the fields, not components). For simplicity let us confine ourselves to the case where all  $\phi_\ell$  are real. Mathematically, the action  $S$  corresponding to these fields is an invariant four-dimensional integral

of the type considered in section B. In local field theory, its integrand depends on the values of the fields and of their first derivatives at the integration point. Thus, for an arbitrary four-dimensional volume  $V$  one has

$$S(V) = \int_V \mathcal{L},$$

where  $\mathcal{L} = \mathcal{L}(\mathbf{d}\phi_\ell, \phi_\ell)$ . According to section B, the Lagrangian density  $\mathcal{L}$  can be regarded as a four-vector 4-form, or as a five-vector 4-form, or as a five-vector 5-form. Let us first recall the traditional interpretation. In this case

$$\mathcal{L} = \frac{1}{4!} \mathcal{L}_{\alpha\beta\gamma\delta} \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta \wedge \mathbf{d}x^\gamma \wedge \mathbf{d}x^\delta,$$

and for a given volume  $V$ ,

$$\int_V d^4x \langle \mathcal{L}, \mathbf{E}_0 \wedge \mathbf{E}_1 \wedge \mathbf{E}_2 \wedge \mathbf{E}_3 \rangle = \int_V d^4x \mathcal{L}_{0123},$$

where  $\mathbf{E}_\alpha$  is the corresponding coordinate four-vector basis. The equations of motion for the fields  $\phi_\ell$  are such that their variation inside a given volume  $V$  with the boundary condition  $\delta\phi_\ell|_{\partial V} = 0$  yields a zero first variation of the action corresponding to  $V$ . In the standard way, from this principle one can derive the corresponding Euler-Lagrange equations:

$$\partial_\mu \left[ \frac{\partial \mathcal{L}_{0123}}{\partial (\partial_\mu \phi_\ell)} \right] = \frac{\partial \mathcal{L}_{0123}}{\partial \phi_\ell},$$

which, equivalently, can be presented as

$$\begin{aligned} \frac{1}{4!} \partial_\mu \left\{ \frac{\partial \mathcal{L}_{\alpha\beta\gamma\delta}}{\partial (\partial_\mu \phi_\ell)} \right\} \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta \wedge \mathbf{d}x^\gamma \wedge \mathbf{d}x^\delta \\ = \frac{1}{4!} \left\{ \frac{\partial \mathcal{L}_{\alpha\beta\gamma\delta}}{\partial \phi_\ell} \right\} \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta \wedge \mathbf{d}x^\gamma \wedge \mathbf{d}x^\delta. \end{aligned} \quad (50)$$

One should now recall the invariant definition of the derivatives of  $\mathcal{L}$  with respect to  $\phi_\ell$  and  $\mathbf{d}\phi_\ell$  and find that

$$\left[ \frac{\partial \mathcal{L}}{\partial \phi_\ell} \right]_{\alpha\beta\gamma\delta} = \frac{\partial \mathcal{L}_{\alpha\beta\gamma\delta}}{\partial \phi_\ell},$$

so the four-vector 4-form in the right-hand side of equation (50) is exactly  $\partial \mathcal{L} / \partial \phi_\ell$ . In a similar manner one can find that the quantities in the curly brackets in the left-hand side of equation (50) are components of the four-vector-valued 4-form  $\partial \mathcal{L} / \partial (\mathbf{d}\phi_\ell)$ . One can then use formula (63) of Appendix to present this side of equation (50) as

$$\begin{aligned} \partial_\alpha \left\{ \frac{\partial \mathcal{L}_{\mu|\beta\gamma\delta|}}{\partial (\partial_\mu \phi_\ell)} \right\} \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta \wedge \mathbf{d}x^\gamma \wedge \mathbf{d}x^\delta \\ = \mathbf{d} \left\{ \frac{\partial \mathcal{L}}{\partial (\mathbf{d}\phi_\ell)} \right\}_{\mu|\beta\gamma\delta|}^\mu \wedge \mathbf{d}x^\beta \wedge \mathbf{d}x^\gamma \wedge \mathbf{d}x^\delta \equiv \mathbf{d}\tilde{\mathbf{J}}^\ell, \end{aligned}$$

where  $\tilde{\mathbf{J}}^\ell$  is the scalar-valued 3-form obtained from  $\partial\mathcal{L}/\partial(\mathbf{d}\phi_\ell)$  by contracting its upper and its first lower four-vector indices. We thus see that the Euler-Lagrange equations (50) can be cast into the following abstract form:

$$\mathbf{d}\tilde{\mathbf{J}}^\ell = \tilde{\mathbf{K}}^\ell, \quad (51)$$

where  $\tilde{\mathbf{K}}^\ell \equiv \partial\mathcal{L}/\partial\phi_\ell$ . The integral formulation of this relation is the following: the flux of the four-vector 3-form  $\tilde{\mathbf{J}}^\ell$  through the boundary of any limited four-dimensional volume  $V$  equals the integral of the four-vector 4-form  $\tilde{\mathbf{K}}^\ell$  over the interior of  $V$ .

Let us now turn to the other two possible interpretations of  $\mathcal{L}$ .

In the case where the Lagrangian density is regarded as a five-vector 4-form one obtains practically the same results except that now in all the formulae all Greek indices are five-vector ones and everywhere  $\mathcal{L}$  is replaced with  $\mathcal{L}^{\tilde{\mathcal{Z}}}$ . Nothing is said about the  $\tilde{\mathcal{E}}$ -component of  $\mathcal{L}$  and unless some additional ideas are invoked, it has no relation to the Lagrange formalism.

More interesting results are obtained if  $\mathcal{L}$  is regarded as a five-vector 5-form. In this case

$$\mathcal{L} = \frac{1}{5!} \mathcal{L}_{ABCDE} \tilde{\mathbf{o}}^A \wedge \tilde{\mathbf{o}}^B \wedge \tilde{\mathbf{o}}^C \wedge \tilde{\mathbf{o}}^D \wedge \tilde{\mathbf{o}}^E,$$

and for a given volume  $V$ ,

$$\int_V d^4x \langle \mathcal{L}, \mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_5 \rangle = \int_V d^4x \mathcal{L}_{01235},$$

where  $\mathbf{e}_A$  is the corresponding passive regular coordinate basis. Following the standard procedure, one can obtain the Euler-Lagrange equations

$$\partial_\mu \left[ \frac{\partial \mathcal{L}_{01235}}{\partial(\partial_\mu \phi_\ell)} \right] - \frac{\partial \mathcal{L}_{01235}}{\partial \phi_\ell} = 0,$$

and considering that in a passive regular basis one has  $\partial_\mu = \partial_\mu$  and  $\partial_5 = 1$ , one can rewrite them as

$$\partial_\mu \left[ \frac{\partial \mathcal{L}_{01235}}{\partial(\partial_\mu \phi_\ell)} \right] - \partial_5 \left[ \frac{\partial \mathcal{L}_{01235}}{\partial(\partial_5 \phi_\ell)} \right] = 0. \quad (52)$$

One should then change the sign of the second term by replacing one of the  $\partial_A$  with  $\partial_A^*$ . This enables one to present equation (52) as

$$\frac{1}{5!} \partial_H^* \left\{ \frac{\partial \mathcal{L}_{ABCDE}}{\partial(\partial_H^* \phi_\ell)} \right\} \tilde{\mathbf{o}}^A \wedge \tilde{\mathbf{o}}^B \wedge \tilde{\mathbf{o}}^C \wedge \tilde{\mathbf{o}}^D \wedge \tilde{\mathbf{o}}^E = \mathbf{0} \quad (53a)$$

or as

$$\frac{1}{5!} \partial_H^* \left\{ \frac{\partial \mathcal{L}_{ABCDE}}{\partial(\partial_H^* \phi_\ell)} \right\} \tilde{\mathbf{o}}^A \wedge \tilde{\mathbf{o}}^B \wedge \tilde{\mathbf{o}}^C \wedge \tilde{\mathbf{o}}^D \wedge \tilde{\mathbf{o}}^E = \mathbf{0}. \quad (53b)$$

One should now recall the invariant definition of the derivatives of  $\mathcal{L}$  and show that the quantities in the

curly brackets in equations (53a) and (53b) are exactly the components of the derivatives  $\partial\mathcal{L}/\partial(\mathbf{d}^*\phi_\ell)$  and  $\partial\mathcal{L}/\partial(\mathbf{d}\phi_\ell)$ , respectively. Finally, one can use formula (64) of Appendix to present equation (53a) as

$$\mathbf{d} \left\{ \frac{\partial \mathcal{L}}{\partial(\mathbf{d}^* \phi_\ell)} \right\}_{H|ABCD}^H \wedge \tilde{\mathbf{o}}^A \wedge \tilde{\mathbf{o}}^B \wedge \tilde{\mathbf{o}}^C \wedge \tilde{\mathbf{o}}^D = \mathbf{0} \quad (54a)$$

and equation (53b) as

$$\mathbf{d}^* \left\{ \frac{\partial \mathcal{L}}{\partial(\mathbf{d} \phi_\ell)} \right\}_{H|ABCD}^H \wedge \tilde{\mathbf{o}}^A \wedge \tilde{\mathbf{o}}^B \wedge \tilde{\mathbf{o}}^C \wedge \tilde{\mathbf{o}}^D = \mathbf{0}, \quad (54b)$$

and it is a simple matter to show that equation (54b) can be obtained from equation (54a) by changing the sign of the  $\tilde{\mathcal{Z}}$ -components of all the quantities involved. We thus see that the Euler-Lagrange equations in this case can be presented as

$$\mathbf{d}\mathbf{\Lambda}^\ell = \mathbf{0}, \quad (55)$$

where  $\mathbf{\Lambda}^\ell$  is the scalar-valued five-vector 4-form obtained from  $\partial\mathcal{L}/\partial(\mathbf{d}^*\phi_\ell)$  by contracting its upper and its first lower five-vector indices:

$$(\mathbf{\Lambda}^\ell)_{ABCD} = [\partial\mathcal{L}/\partial(\mathbf{d}^*\phi_\ell)]_{HABCD}^H. \quad (56)$$

In the language of integrals, equation (55) means that the five-vector flux of the 4-form  $\mathbf{\Lambda}^\ell$  through any limited four-dimensional volume  $V$  is zero:

$$\oint_V \mathbf{\Lambda}^\ell = 0. \quad (57)$$

### G. Five-vector Levi-Civita tensor and dual forms

As on any other vector space endowed with a nondegenerate inner product, one can define on  $V_5$  a completely antisymmetric tensor,  $\epsilon$ , whose rank in this case is five. The magnitude of its only independent component, say, of  $\epsilon_{01235}$ , is fixed by the condition:

$$\epsilon_{01235} = +1 \text{ in any orthonormal basis with positive orientation}^2,$$

and then the components of  $\epsilon$  in an arbitrary five-vector basis  $\mathbf{e}_A$  will be

$$\epsilon_{ABCDE} = \eta \cdot |h|^{1/2} \cdot |_{ABCDE}|,$$

where  $h$  denotes the determinant of the matrix  $h_{AB} \equiv h(\mathbf{e}_A, \mathbf{e}_B)$  (at some particular choice of the constant

<sup>2</sup>The notion of orientation for five-vector bases is defined in the usual way. In the following I will consider positive the orientation of a normalized regular basis for which the associated four-vector basis has positive *four-dimensional* orientation.

$\xi$ ); parameter  $\eta$  equals  $+1$  if the basis  $\mathbf{e}_A$  has positive orientation and  $-1$  otherwise; and the symbol  $|ABCDE|$  is defined in the usual way:

$$|ABCDE| = \begin{cases} +1, & \text{if } (ABCDE) \text{ is an even} \\ & \text{permutation of } (01235), \\ -1, & \text{if } (ABCDE) \text{ is an odd} \\ & \text{permutation of } (01235), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show that in any standard basis one has  $h = h_{55} \cdot g$ , where  $g$  denotes the determinant of the  $4 \times 4$  matrix  $g_{\alpha\beta} \equiv g(\mathbf{e}_\alpha, \mathbf{e}_\beta)$ , and therefore in any *active* regular basis one has

$$\epsilon_{01235} = \eta \cdot |g|^{1/2} \cdot \kappa,$$

where  $\kappa \equiv |\xi|^{1/2}$ , and in any *passive* regular basis one has

$$\epsilon_{01235} = \eta \cdot |g|^{1/2} \cdot \varpi,$$

where  $\varpi \equiv |\xi|^{1/2} \cdot \zeta^{-1}$  and  $\zeta$  is the dimensional constant introduced in section 3 of part II.

As in the case of the four-vector Levi-Civita tensor, it is convenient to introduce the completely contravariant tensor corresponding to  $\epsilon$ , whose components are

$$\epsilon^{ABCDE} = h^{AA'} h^{BB'} h^{CC'} h^{DD'} h^{EE'} \epsilon_{A'B'C'D'E'},$$

where matrix  $h^{AB}$  is the inverse of  $h_{AB}$ . It is not difficult to demonstrate that in any standard basis

$$\epsilon^{ABCDE} = -\text{sign } \xi \cdot \eta \cdot |h|^{-1/2} \cdot |ABCDE|,$$

and therefore

$$\epsilon^{ABCDE} \epsilon_{ABCDE} = -5! \cdot \text{sign } \xi.$$

The latter equation is a particular case of the general relation that expresses the contraction of  $\epsilon$  with its completely contravariant counterpart over a certain number of indices, in terms of the so-called permutation tensors:

$$\begin{aligned} \epsilon^{A_1 \dots A_m C_1 \dots C_{5-m}} \epsilon_{B_1 \dots B_m C_1 \dots C_{5-m}} \\ = -(5-m)! \cdot \text{sign } \xi \cdot \delta^{A_1 A_2 \dots A_m}_{B_1 B_2 \dots B_m}, \end{aligned}$$

where  $m$  can be any integer from 0 to 5 and

$$\delta^{A_1 A_2 \dots A_m}_{B_1 B_2 \dots B_m} \equiv m! \delta_{B_1}^{[A_1} \delta_{B_2}^{A_2} \dots \delta_{B_m}^{A_m]}.$$

Let me say a few words about the differential properties of  $\epsilon$ . Since the normalization of the latter is determined by the inner product  $h$  and since any non-degenerate  $h$  is not conserved by parallel transport, it is *a priori* not clear whether or not  $\epsilon$  is a covariantly

constant tensor. One can gain an understanding of the situation from the following general reasoning.

Let us consider an arbitrary standard orthonormal basis  $\mathbf{e}_A$  at some space-time point  $Q$  and let us parallel transport it to a neighbouring point  $Q'$  along some continuous curve  $\mathcal{C}$  connecting  $Q$  with  $Q'$ . Let us denote the transported vector  $\mathbf{e}_A$  as  $\mathbf{e}'_A$ .

Owing to the general properties of five-vector parallel transport discussed in section 3 of part II,  $\mathbf{e}'_5$  is, again, a vector from  $\mathcal{E}$  and the  $\mathcal{Z}$ -components of  $\mathbf{e}'_\alpha$  are orthogonal one to another and are normalized. Therefore,

$$\begin{aligned} \epsilon(\mathbf{e}'_0, \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3, \mathbf{e}'_5) \\ = \epsilon(\mathbf{e}'_0^{\mathcal{Z}}, \mathbf{e}'_1^{\mathcal{Z}}, \mathbf{e}'_2^{\mathcal{Z}}, \mathbf{e}'_3^{\mathcal{Z}}, \mathbf{e}'_5) \\ = (\lambda_{\mathbf{e}'_5} / \lambda_{\mathbf{e}_5}) \epsilon(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5), \end{aligned}$$

and so  $\epsilon$  is covariantly constant or not depending on whether or not parallel transport conserves the length of the vectors from  $\mathcal{E}$ . The same result can be obtained by computing the covariant derivative of  $\epsilon$  in components and finding that in any standard five-vector basis where the length of the fifth basis vector is constant

$$\epsilon_{ABCDE; \mu} = -G_{5\mu}^5 \epsilon_{ABCDE}. \quad (58)$$

Thus, in the case where the connection for five-vectors possesses the local symmetry described in section 3 of part II, tensor  $\epsilon$  is covariantly constant.

As its four-vector analog, tensor  $\epsilon$  can be used for converting multivectors into forms and vice versa. Namely, if  $\mathbf{w}$  is a multivector on  $V_5$  of rank  $m$  ( $0 \leq m \leq 5$ ) with components  $w^{A_1 \dots A_m}$ , one can construct from it a five-vector  $(5-m)$ -form with the components

$$(m!)^{-1} w^{B_1 \dots B_m} \epsilon_{B_1 \dots B_m A_1 \dots A_{5-m}}. \quad (59)$$

To formulate this correspondence in invariant form, one should regard  $\epsilon$  as an inner product defined for any two multivectors on  $V_5$  whose ranks total up to 5, assuming that

$$\begin{aligned} \epsilon(\mathbf{e}_{A_1} \wedge \dots \wedge \mathbf{e}_{A_m}, \mathbf{e}_{B_1} \wedge \dots \wedge \mathbf{e}_{B_{5-m}}) \\ \equiv \epsilon(\mathbf{e}_{A_1}, \dots, \mathbf{e}_{A_m}, \mathbf{e}_{B_1}, \dots, \mathbf{e}_{B_{5-m}}). \end{aligned}$$

Regarding multivectors of rank  $m$  as elements of a vector space with dimension  $5!/m!(5-m)!$  and identifying the linear forms on this space with five-vector  $m$ -forms, one can employ the same method that has been used in subsection 3.E of part II to define the maps  $\vartheta_g$  and  $\vartheta_h$  and put into correspondence to each multivector  $\mathbf{w}$  of rank  $m$  a certain form  $\vartheta_\epsilon(\mathbf{w})$  of rank  $5-m$  such that

$$\langle \vartheta_\epsilon(\mathbf{w}), \mathbf{v} \rangle = \epsilon(\mathbf{w}, \mathbf{v})$$

for any multivector  $\mathbf{v}$  of rank  $5 - m$ . It is a simple matter to check that the relation between the components of  $\mathbf{w}$  and those of the form  $\vartheta_\epsilon(\mathbf{w})$  is indeed given by formula (59).

There exists another correspondence between multivectors and forms, which is determined by the non-degenerate inner product  $h$  on  $V_5$  or, more precisely, by the inner product of multivectors induced by  $h$ . In this case to any multivector  $\mathbf{w}$  of rank  $m$  with components  $w^{A_1 \dots A_m}$  one puts into correspondence a form of *same* rank with the components

$$w^{B_1 \dots B_m} h_{B_1 A_1} \dots h_{B_m A_m}.$$

This latter form, which I will denote as  $\vartheta_h(\mathbf{w})$ , can be defined invariantly by requiring that for any multivector  $\mathbf{v}$  of rank  $m$

$$\langle \vartheta_h(\mathbf{w}), \mathbf{v} \rangle = h(\mathbf{w}, \mathbf{v}),$$

where  $h$  in this case denotes the inner product of multivectors of rank  $m$  induced by the nondegenerate inner product on  $V_5$ .

One can now combine the maps  $\vartheta_\epsilon$  and  $\vartheta_h$  and define a one-to-one correspondence between forms or multivectors of rank  $m$  and forms or multivectors of rank  $5 - m$ . The image of an  $m$ -form  $\tilde{\mathbf{w}}$  with respect to this map will be called a form *dual* to  $\tilde{\mathbf{w}}$  and will be denoted as  $\tilde{\mathbf{w}}^{\text{dual}}$ . It is evident that for any form  $\tilde{\mathbf{w}}$

$$\tilde{\mathbf{w}}^{\text{dual}} = \vartheta_\epsilon \circ \vartheta_h^{-1}(\tilde{\mathbf{w}}),$$

and the components of  $\tilde{\mathbf{w}}^{\text{dual}}$  are expressed in terms of those of  $\tilde{\mathbf{w}}$  according to the well-known formula:

$$\begin{aligned} (\tilde{\mathbf{w}}^{\text{dual}})_{A_1 \dots A_{5-m}} &= (m!)^{-1} w_{C_1 \dots C_m} \\ &\times h^{C_1 B_1} \dots h^{C_m B_m} \epsilon_{B_1 \dots B_m A_1 \dots A_{5-m}}. \end{aligned}$$

It is easy to prove that

$$(\tilde{\mathbf{w}}^{\text{dual}})^{\text{dual}} = -\text{sign } \xi \cdot \tilde{\mathbf{w}}$$

and that the duality operation transforms the  $\tilde{\mathcal{E}}$ - and  $\tilde{\mathcal{Z}}$ -components of any form  $\tilde{\mathbf{w}}$  respectively into the  $\mathcal{E}$ - and  $\mathcal{Z}$ -components of  $\tilde{\mathbf{w}}^{\text{dual}}$ . Let me also note that in the case of five-vector forms, as in the case of forms associated with any other vector space of odd dimension, the rank of the dual form never equals that of the initial form, so in the general case one is not able to define the operation of dual rotation. The only exception are the 2-forms with the zero  $\tilde{\mathcal{E}}$ -component, which the duality operation transforms into 3-forms with the zero  $\tilde{\mathcal{Z}}$ -component. Since there exists another correspondence between these two types of forms, given by equation (8) at  $m = 2$ , one is able to define a map similar to the duality

operation, which transforms a 2-form with the zero  $\tilde{\mathcal{E}}$ -component into a 2-form of the same type. It is easy to see that such an operation corresponds to the duality transformation of four-vector 2-forms.

In conclusion, let me mention one useful identity that involves dual forms: if  $\tilde{\mathbf{s}}$  and  $\tilde{\mathbf{t}}$  are any two five-vector forms of same rank, then

$$\tilde{\mathbf{s}} \wedge \tilde{\mathbf{t}}^{\text{dual}} = \tilde{\mathbf{s}}^{\text{dual}} \wedge \tilde{\mathbf{t}} = h(\tilde{\mathbf{s}}, \tilde{\mathbf{t}}) \cdot \epsilon, \quad (60)$$

where the inner product of forms  $\tilde{\mathbf{s}}$  and  $\tilde{\mathbf{t}}$  is defined in the usual way:

$$h(\tilde{\mathbf{s}}, \tilde{\mathbf{t}}) = s_{|A_1 \dots A_m|} t^{A_1 \dots A_m}.$$

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## Appendix: Index transposition identity

There exists a very useful identity of purely combinatorial nature, which enables one to transpose a single index with a group of antisymmetrized indices if the number of the latter equals the number of values the indices run through. In the general case, this identity can be formulated as follows:

If the array  $S_{i j_1 \dots j_m}$  ( $m \geq 2$ ) is completely antisymmetric in  $j_1, \dots, j_m$  and all indices run through the same  $m$  values, then

$$S_{i j_1 \dots j_m} = m (-1)^{m+1} S_{[j_1 \dots j_m] i}. \quad (61)$$

*Proof:* One has

$$\begin{aligned} S_{i j_1 \dots j_m} &= S_{[j_1 \dots j_m] i} \\ &= m S_{[i j_1 \dots j_m]} + S_{[j_1 | i | j_2 \dots j_m]} \\ &\quad - \dots + (-1)^{m+1} S_{[j_1 \dots j_m] i}, \end{aligned} \quad (62)$$

where, as usual, the notation  $[j_1 \dots j_k | i | j_{k+1} \dots j_m]$  means antisymmetrization with respect to all the indices inside the square brackets except for  $i$ . Since all indices run through  $m$  values only, the first term

in the right-hand side of equation (62) is identically zero. Since  $S_{i j_1 \dots j_m}$  is antisymmetric in its last  $m$  indices, one has

$$\begin{aligned} (-1)^{k+1} S_{[j_1 \dots j_k | i | j_{k+1} \dots j_m]} \\ &= (-1)^{k+1} (-1)^{m-k} S_{[j_1 \dots j_m | i]} \\ &= (-1)^{m+1} S_{[j_1 \dots j_m] i} \end{aligned}$$

for all  $k$  from 1 to  $m - 1$ , so the remaining  $m$  terms in the right-hand side of equation (62) are all equal, and equation (62) acquires the form of equation (61).  $\blacksquare$

We will need the following two particular cases of identity (61):

- If  $S_{\alpha\beta\gamma\delta}^\mu = S_{[\alpha\beta\gamma\delta]}^\mu$  are components of a four-vector-valued four-vector 4-form, then

$$\frac{1}{4!} \partial_\mu S_{\alpha\beta\gamma\delta}^\mu = \frac{1}{3!} \partial_{[\alpha} T_{\beta\gamma\delta]}, \quad (63)$$

where  $T_{\beta\gamma\delta} \equiv S_{\mu\beta\gamma\delta}^\mu$ .

- If  $S_{ABCDE}^H = S_{[ABCDE]}^H$  are components of a five-vector-valued five-vector 5-form, then

$$\frac{1}{5!} \partial_H S_{ABCDE}^H = \frac{1}{4!} \partial_{[A} T_{BCDE]}. \quad (64)$$

where  $T_{BCDE} \equiv S_{HBCDE}^H$ .

## Reference

1. L.Schwartz, *Analyse Mathématique*, vol.II, Hermann, 1967.