Equivalence Classes of Colorings: the Topological Viewpoint

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Abstract

We introduce the notion of equivalence classes of Fox colorings from the topological point of view. We develop the preliminaries that allow the correct formulation of these equivalence classes and enumerate them in a number of instances.

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1 Introduction

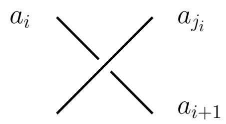
This article has to do with Fox colorings which we discuss below (see also [1], [3], [4]). Moreover it has to do with equivalence classes of these colorings under the natural permutations of these colorings. We enumerate these equivalence classes in a number of instances and offer a conjecture for a more general case.

In Section 2 we discuss preliminaries including the definition of the equivalence classes and in Section 3 we calculate the number of equivalence classes in an infinite number of instances and conjecture the formula for a more general case.

2 Preliminary Material

2.1 Nullity and Arcs of a Diagram

Consider a link, L, along with a diagram D_L for that link. Regarding the arcs of this diagram as algebraic variables we write the homogeneous system of linear equations consisting of the equations read off each crossing as illustrated in Figure 1. We call the matrix of the coefficients of this homogeneous system of linear equations **the coloring matrix of** D_L.



$$a_i + a_{i+1} - 2a_{j_i} = 0$$

Figure 1: Arcs at a crossing and the equation read off it. The coloring system of equations is formed by each of these equations, one per crossing of the diagram under study.

Any coloring matrix is made up of integers. Furthermore, along each row one finds exactly two 1's and one -2, the rest being perhaps 0's. In this way, adding all the columns of a coloring matrix we obtain a column made up of 0's. In particular, a non-trivial linear combination of the columns yields the column made up of 0's. It follows that the determinant of any coloring matrix is 0.

Upon performance of Reidemeister moves on a diagram, the changes on the original coloring matrix are realized by operations that constitute a subset of the following operations on integer matrices. These operations are generated by

1. multiplication of a row (column) by -1;
2. addition to one row (column) of integer linear combinations of other rows (columns);
3. insertion (deletion) of a row and column made up of 0's except for a 1 at the diagonal entry;
4. permutations of rows (columns).

These are the operations which relate equivalent matrices over the integers (see [6], page 50). So the equivalence class of a coloring matrix is a topological invariant of the link under study. For each of these equivalence classes of matrices over the integers there is an outstanding representative which is called the Smith Normal Form (see [9]). A matrix in Smith Normal Form is a diagonal matrix such that the diagonal entries are all non-negative; moreover along the diagonal, the i-th entry divides the $(i+1)$-th entry, up to a certain index l, and after that, the entries are all 0's:

$$\text{diagonal } (d_1, d_2, \dots, d_l, 0, 0, \dots, 0) \qquad \text{with} \qquad d_i | d_{i+1} \quad 1 \leq i \leq l-1$$

The d_i's are called the **invariant factors** of the equivalence class; their name reflects the fact that the set formed by them is an invariant of the equivalence class. It goes without saying that this set is a topological invariant if it originates from a coloring matrix. Moreover, the Smith Normal Form of a coloring matrix is sure to have a 0 at the last entry of the diagonal since we proved above that the determinant of a coloring matrix is 0. The product of the remaining entries of the diagonal of the Smith Normal Form of a coloring matrix is **the determinant of the link** under study. This is also a topological invariant.

We denote the Smith Normal Form of a matrix M by $S(M)$. Being another element of the equivalence class of M, $S(M)$ is obtained by a finite number of the operations listed above. We may then collect all the information concerning the row operations into an invertible matrix called R and likewise for the column operations into an invertible matrix called C to state ([9])

$$S(M) = RMC \tag{1}$$

with the juxtaposition of symbols on the right-hand side of the equation denoting matrix multiplication.

Let us now relate the Smith Normal Form and the invariant factors to the system of linear homogeneous equations of which the coloring matrix is the coefficient matrix. There are always solutions for the aforementioned system of equations namely by assigning the same integer to each arc. This corresponds to the fact that the determinant of the coloring matrix is 0. One of the algebraic variables may take on any value and if there is no other zero entry along the diagonal of the Smith Normal Form, then the remaining variables are uniquely determined once the former variable has been assigned a value. Going back to the original system of equations we obtain the so-called trivial solutions i.e., those solutions that assign the same value to each and every arc of the diagram. (In passing, it is known that for knots i.e., 1-component links, the determinant of the knot is an odd integer, see [8].)

The invariant factors associated to our coloring matrix via its Smith Normal Form allow us to do something else. Suppose we choose a factor m of one of these invariant factors and decide to work over the integers modulo m. Then our Smith Normal Form in this new setting acquired at least one more 0 along the diagonal. Then, there is at least one more variable which can take on any value, modulo m. Going back to the original system of equations, there are at least two arcs which can take on any value modulo m. Hence, we now have polychromatic colorings i.e., solutions where at least two distinct arcs take on two distinct colors that is, values modulo m. Had we chosen a factor m which does not possess common factors with the invariant factors, then modulo m there would have been only trivial colorings.

Proposition 2.1 *Let p be an odd prime. Let D be link diagram. The number of 0's (modulo p) along the diagonal of the Smith Normal Form of the coloring matrix of D, equals the number of arcs of D that can independently receive colors modulo p, and thus generate all colorings of the knot diagram modulo p i.e. the p-colorings.*

Proof. If the Smith Normal Form exhibits n 0's modulo p, this means that the space of solutions has dimension n; working modulo a prime implies we are doing Linear Algebra over a field so it makes sense to talk about dimensions of spaces and bases. Then matrix C in (1) above operates a change of basis taking us back to algebraic variables equivalent to the arcs of the original diagram. Then n of these arcs have to generate all the colorings (i.e., all the solutions of the indicated system of equations modulo p) in terms of a basis of coloring vectors. ■

Definition 2.1 *The number n in the proof of Proposition 2.1 is called the p-nullity of the coloring matrix of D.*

Corollary 2.1 *We keep the notation of Proposition 2.1. If the p-nullity of a link is n then there are p^n p-colorings of the link, and $p^n - p$ non-trivial p-colorings of this link.*

Proof. Omitted. ■

Corollary 2.2 *Let m be a composite positive integer. Let D be a link diagram. Each zero along the diagonal of the Smith Normal Form of the coloring matrix associated to D contributes with a factor m for the number of solutions. Each zero divisor, z, of m long the diagonal contributes with a factor $\gcd(z, m)$ to the number of solutions. With n_Z for the number of 0's along the diagonal in $S(M)$, and $IZ(M)$ for the set the of invariant factors of M which are zero divisors of m, the formula for the number of m-colorings of D is:*

$$m^{n_Z} \prod_{z \in IZ(M)} \gcd(z, m)$$

Proof. The contribution of the n_Z zero's (modulo m) along the diagonal of the Smith Normal Form to the number of solutions is clear. For the contribution of the zero divisors along the diagonal to the number of solutions see [5], page 40. This concludes the proof. ■

Corollary 2.3 *Let m be a composite positive integer. Let D be a link diagram. Suppose the Smith Normal Form of the coloring matrix associated to D does not exhibit zero divisors (modulo m). If there are n zeros (modulo m) along the diagonal, then there are n independent arcs generating the m colorings. Moreover there are m^n m-colorings and $m^n - m$ non-trivial m-colorings.*

Proof. The argument developed above for prime modulus in the proof of Proposition 2.1 applies again in this case and we can also define nullity in the same way. This concludes the proof. ■

2.2 Equivalence Classes of Colorings

In this section we introduce equivalence classes of colorings as orbits of actions of certain groups of permutations on the set of colorings of a diagram. In order for this notion to be topological we require a special kind of permutation. These are permutations which comply with the $2b - a$ operation. The following material is adapted from [2].

Definition 2.2 (Automorphisms) *Given an integer $m \geq 3$, we define an automorphism of $\mathbf{Z}_m$ to be defined to be a permutation f of $\mathbf{Z}_m$ such that*

$$f(2b - a) = 2f(b) - f(a)$$

for all $a, b \in \mathbf{Z}_m$.

In [2] we find the following facts. For a given integer $m \geq 3$, each automorphism is given by:

$$f_{\lambda,\mu}(x) = \lambda x + \mu$$

with $\mu \in \mathbf{Z}_m$ and $\lambda \in \mathbf{Z}_m^*$, the set of units of $\mathbf{Z}_m$. The set of all these automorphisms equipped with composition of functions, constitutes a group isomorphic to the affine group over $\mathbf{Z}_m$ i.e., isomorphic to the semi-direct product $\mathbf{Z}_m \rtimes \mathbf{Z}_m^*$. We denote it $\mathbf{Aut}_m$.

Finally, we will call **inner automorphism** any element of the subgroup of $\mathbf{Aut}_m$ made up of elements of the form

$$f_{\pm,\mu}(x) = \pm x + \mu$$

If m is even this subgroup is isomorphic to the dihedral group of order m and μ can take on only "even" values from $\mathbf{Z}_m$. If m is odd, this subgroup is isomorphic to the dihedral group of order $2m$ and μ can take on any value from $\mathbf{Z}_m$. We denote it $\mathbf{Inn}_m$. This information is contained in the reference [2].

In the sequel, we will use the expression **automorphism** to designate a permutation of the form

$$f_{\lambda,\mu}(x) = \lambda x + \mu$$

with $\mu \in \mathbf{Z}_m$ and $\lambda \in \mathbf{Z}_m^*$, and **inner automorphism** to designate a permutation of the form

$$f_{\pm,\mu}(x) = \pm x + \mu$$

with μ taking on only "even" values from $\mathbf{Z}_m$ if m is even; with μ taking on any value from $\mathbf{Z}_m$ if m is odd.

In the sequel, we will reserve the adjective "inner" to refer to the inner automorphisms.

Definition 2.3 *Let $m > 1$ be an integer. Let L be a link admitting non-trivial m-colorings. Let D be a diagram of L.*

We let $\mathbf{m\mathcal{C}D}$ stand for the set of all m-colorings of D.

Proposition 2.2 *Let $m > 1$ be an integer. Let L be a link admitting non-trivial m-colorings along with D, a diagram of L. Let G be a subgroup of Aut_m.*

Then G acts on $m\mathcal{C}D$ by permuting the colors of m-colorings of D.

Specifically, given $g \in G$ and an m-coloring of D with colors c_i, call it $\mathcal{C}$, $g\mathcal{C}$ is the m-coloring of D obtained by replacing each color c_i of $\mathcal{C}$ by $g(c_i)$.

Proof. We keep the notation of the statement. We regard $\mathcal{C} \in m\mathcal{C}D$ as the map which assigns colors to the arcs of D in such a way that, $\mathcal{C}(a_{i+1}) = 2\mathcal{C}(a_{j_i}) - \mathcal{C}(a_i)$, where j_i designates the index of the over-arc of the crossing where under-arcs with indices i and $i+1$ meet, see Figure 1.

So, given $g \in G$ and $\mathcal{C} \in m\mathcal{C}D$, say then $g\mathcal{C}$ is such that

$$g\big(\mathcal{C}(a_{i+1})\big) = g\big(\mathcal{C}(2a_{j_i} - a_i)\big) = 2g\big(\mathcal{C}(a_{j_i})\big) - g\big(\mathcal{C}(a_i)\big)$$

so $g\mathcal{C}$ is again an m-coloring of D.

Clearly, the identity element $1_G \in G$ is such that $1_G\mathcal{C} = \mathcal{C}$. Furthermore, for any two $g_1, g_2 \in G$, the composition of functions guarantees that $(g_1 g_2)\mathcal{C} = g_1\big(g_2\mathcal{C}\big)$. This concludes the proof.

■

Definition 2.4 (G-Equivalence Classes of m-Colorings of D) *Keeping the notation above, the G-orbits over $m\mathcal{C}D$ are, by definition, the G-Equivalence Classes of m-Colorings of D.*

We will next prove that this notion provides a topological invariant.

In [7] the third author showed that, given any two diagrams of the same link, there is a bijection between the two sets of m-colorings of these diagrams. Moreover, this bijection takes trivial m-colorings to trivial m-colorings and non-trivial m-colorings to non-trivial m-colorings. This bijection is materialized via what we like to call the “Colored Reidemeister Moves”. Roughly speaking the “Colored Reidemeister Moves” apply to a diagram endowed with an m-coloring; a Reidemeister move is applied to the diagram and a local adjustment of the coloring is performed. These adjustements are unique and reversible thereby proving the bijection between the two sets of m-colorings of any two diagrams of the same link. Leaning on this notion we prove the next statement.

Proposition 2.3 *Let m be an integer greater than 1. Let L be a link admitting non-trivial m-colorings along with two diagrams of L, D and D'. Let G be a subgroup of $\mathbf{Aut}_m$.*

There is a bijection between the G-Equivalence Classes of m-Colorings of D and the G-Equivalence Classes of m-Colorings of D'.

Proof: We prove that the Colored Reidemeister moves take distinct m-colorings of D along a G-equivalence class of m-colorings of D to distinct m-colorings of D' along a G-equivalence class of m-colorings of D'. Specifically, for $g \in G$ and $\mathcal{C} \in m\mathcal{C}D$, we prove that the “Colored Reidemeister moves” take $\mathcal{C} \in m\mathcal{C}D$ to $\mathcal{C}' \in m\mathcal{C}D'$ and $g\mathcal{C} \in m\mathcal{C}D$ to $g\mathcal{C}' \in m\mathcal{C}D'$. The proofs of these statements for the individual “Colored Reidemeister Moves” f type I, II, and III are displayed in Figures 2, 3, and 4. The $\sim$ relates horizontally colorings on distinct diagrams related by a Colored Reidemeister move.

In Figures 2 and 3 circles with dotted lines were drawn to bring out the local nature of the transformation. This was not done in Figure 4 in order not to overburden the Figure. ■

$$\mathcal{C}(r) = x \qquad \sim \qquad \mathcal{C}'(r'_1) = x = \mathcal{C}'(r'_2)$$

$$g\big(\mathcal{C}(r)\big) = g(x) \qquad \sim \qquad g\big(\mathcal{C}'(r'_1)\big) = g(x) = g\big(\mathcal{C}'(r'_2)\big)$$

Figure 2: Colored Reidemeister move of type I and G-equivalence relation of colorings on the same diagram.

$$\mathcal{C}(r) = x \qquad \mathcal{C}(s) = y \quad \sim \quad \mathcal{C}'(r'_1) = x \qquad \mathcal{C}'(s') = y$$
$$\mathcal{C}'(r'_2) = 2\mathcal{C}'(s') - \mathcal{C}'(r'_1) = 2y - x$$
$$\mathcal{C}'(r'_3) = \cdots = 2y - (2y - x) = x$$

$$g\big(\mathcal{C}(r)\big) = g(x) \qquad g\big(\mathcal{C}(s)\big) = g(y) \quad \sim \quad g\big(\mathcal{C}'(r'_1)\big) = g(x) \qquad g\big(\mathcal{C}'(s')\big) = g(y)$$
$$g\big(\mathcal{C}'(r'_2)\big) = g(2\mathcal{C}'(s') - \mathcal{C}'(r'_1)) = 2g(y) - g(x)$$
$$g\big(\mathcal{C}'(r'_3)\big) = \cdots = g(2y - (2y - x)) = g(x)$$

Figure 3: Colored Reidemeister move of type II and G-equivalence relation of colorings on the same diagram.

$$\mathcal{C}(r_1) = x \qquad \mathcal{C}(s_1) = y \qquad \mathcal{C}(t) = z$$
$$\mathcal{C}(r_2) = \cdots = 2y - x$$
$$\mathcal{C}(s_2) = \cdots = 2z - y$$
$$\mathcal{C}(r_3) = \cdots = 2z - (2y - x) = 2z - 2y + x$$
$$\sim$$
$$\mathcal{C}'(r'_1) = x \qquad \mathcal{C}'(s'_1) = y \qquad \mathcal{C}'(t') = z$$
$$\mathcal{C}'(r'_2) = \cdots = 2z - x$$
$$\mathcal{C}'(s'_2) = \cdots = 2z - y$$
$$\mathcal{C}'(r'_3) = \cdots = 2z - 2y + x$$

$$g\big(\mathcal{C}(r_1)\big) = g(x) \qquad g\big(\mathcal{C}(s_1)\big) = g(y) \qquad g\big(\mathcal{C}(t)\big) = g(z)$$
$$g\big(\mathcal{C}(r_2)\big) = \cdots = 2g(y) - g(x)$$
$$g\big(\mathcal{C}(s_2)\big) = \cdots = 2g(z) - g(y)$$
$$g\big(\mathcal{C}(r_3)\big) = \cdots = 2g(z) - 2g(y) + g(x)$$
$$\sim$$
$$g\big(\mathcal{C}'(r'_1)\big) = g(x) \quad g\big(\mathcal{C}(s_1)\big) = g(y) \quad g\big(\mathcal{C}(t)\big) = g(z)$$
$$g\big(\mathcal{C}'(r'_2)\big) = \cdots = 2g(y) - g(x)$$
$$g\big(\mathcal{C}'(s'_2)\big) = \cdots = 2g(z) - g(y)$$
$$g\big(\mathcal{C}'(r'_3)\big) = \cdots = 2g(z) - 2g(y) + g(x)$$

Figure 4: Colored Reidemeister move of type III and G-equivalence relation of colorings on the same diagram.

Theorem 2.1 *Let L be a link and D one of its diagrams. Let $m > 1$ be an integer.*

The number of G-equivalence classes of m-colorings of D is a topological invariant. The multi-set whose elements are the number of m-colorings per G-equivalence class of m-colorings of D is a topological invariant.

Proof. Omitted. ∎

We remark that in the sequel G will be either $\mathbf{Aut}_m$ or $\mathbf{Inn}_m$.

3 Applications

In this Section we apply the theory developed above to specific situations.

3.1 p-nullity 2

Lemma 3.1 (Main Lemma) *Let p be a prime.*

Let L be a link with p-nullity 2.

Then L only has one equivalence class of p-colorings.

Proof: Since the p-nullity of L is 2 then on any given diagram of L there are two arcs which generate any p-coloring of L (see Section 2). We will thus represent p-colorings of L by specifying ordered pairs of colors modulo p assigned to these arcs. Let then (a, b) and (a', b') stand for two distinct non-trivial p-colorings of L; in particular, $a' \neq b'$ and $a \neq b$.

Let

$$\lambda = (a' - b')(a - b)^{-1} \qquad \text{and} \qquad \mu = (b'a - a'b)(a - b)^{-1}$$

Then

$$f(x) = \lambda x + \mu$$

is an automorphism and

$$a' = f(a) \qquad \text{and} \qquad b' = f(b)$$

i.e., the two generic non-trivial p-colorings are related. Therefore there is only one equivalence class of p-colorings. ■

Corollary 3.1 *Let L be a link with the following property.*

The Smith Normal Form of a(ny) coloring matrix of L has only one 0 and only one $0 \neq d \neq 1$.

Then for any $m|d$, there is only one equivalence class of m-colorings. In particular, rational links satisfy this property.

Proof. Omitted. ■

Corollary 3.2 *The following links have only one class of p-colorings for each prime p for which they admit non-trivial p-colorings.*

1. *Links whose determinant is prime.*
2. *Links of non-zero determinant whose Smith Normal Form of the coloring matrix displays different primes on different diagonal entries (besides the 0 entry and possible 1's).*
3. *Knots whose knot group can be presented with two generators, especially torus knots.*

Proof. Omitted. ■

3.1.1 p-nullity 2: the inner equivalence relation instance.

Lemma 3.2 *Let p be a prime.*

Let L be a link with p-nullity 2.

Then L has $\frac{p-1}{2}$ inner equivalence classes of p-colorings.

Proof: We keep the notaton and terminology we have been using concerning nullities and representations of colorings.

Suppose (a, b) and (a', b') are inner-related. Then there exists a

$$f_{\pm,\mu}(x) = \pm x + \mu$$

such that

$$a' = f_{\pm,\mu}(a) = \pm a + \mu$$
$$b' = f_{\pm,\mu}(b) = \pm b + \mu$$

This is equivalent to saying that $|a' - b'| = |a - b|$, modulo p. Since we do not want trivial colorings, these absolute values can take on the values $1, 2, \ldots, p - 1$. But $|a - b| = k$ and $|a - b| = p - k$ lead to the same modulo p. There are then $\frac{p-1}{2}$ inner equivalence classes of p-colorings. ■

The analogues of Corollaries 3.1 and 3.2 are also valid.

Corollary 3.3 *Let L be a link with the following property.*

The Smith Normal Form of a(ny) coloring matrix of L has only one 0 and only one $0 \neq d \neq 1$.

Then for any $m|d$, there are $\frac{m-1}{2}$ equivalence class of m-colorings. In particular, rational links satisfy this property.

Proof. Omitted. ■

Corollary 3.4 *The following links have $\frac{p-1}{2}$ classes of p-colorings for each prime p for which they admit non-trivial p-colorings.*

1. *Links whose determinant is prime.*
2. *Links of non-zero determinant whose Smith Normal Form of the coloring matrix displays different primes on different diagonal entries (besides the 0 entry and possible 1's).*
3. *Knots whose knot group can be presented with two generators, especially torus knots.*

Proof. Omitted. ■

3.2 p-nullity 3

Proposition 3.1 *Let p be an odd prime. Let L be a link with p-nullity 3. Then L admits $p+1$ equivalence classes of p-colorings.*

Proof. We keep the notation and terminology of the statement. The indicated link admits $p^3 - p = p(p^2 - 1) = p(p-1)(p+1)$ non-trivial p-colorings. We will next produce a number of different equivalence classes of p-colorings along with counting the number of elements of each class. Eventually we will realize we already have the $p^3 - p$ non-trivial p-colorings, thus obtaining all equivalence classes of p-colorings.

As before we will represent each non-trivial p-coloring by a triple (a, b, c) corresponding to the colors to be assigned to three generating arcs of a given diagram of link L.

We will first look at $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$. We note that if two entries are equal for a representative of the equivalence class, then any other representative has the same entries equal (although not necessarily with the same element on both representatives). We also note that each of these representatives is related to another one with 0 and 1 replaced by a and b (distinct, modulo p). In fact, $f(x) = \lambda x + \mu$ with $\mu = a$ and $\lambda = b - a$ realizes $a = f(0)$, $b = f(1)$. So, so far we know that $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$ are representative of three distinct equivalence classes, and that each of these equivalence classes contains $p(p-1)$ elements corresponding to λ being able to take on $p-1$ elements and μ p elements from $\mathbf{Z}_p$. At this point we have $3p(p-1)$ non-trivial p-colorings accounted for.

For distinct $2 \leq k, k' \leq p-1$ (modulo p), let us consider the representatives $(0,1,k)$ and $(0,1,k')$. If they were related then there would exist an $f(x) = \lambda x + \mu$ such that $0 = f(0) = \mu$, $1 = f(1) = \lambda$, and $k' = f(k) = k$ which is absurd since we assumed k, k' distinct. Each of these equivalence classes contains $p(p-1)$ elements and there are $p-2$ of them since $p-2$ is the number of distinct k's form 2 to $p-1$ mod p. Therefore we are here accounting for $(p-2)p(p-1)$ non-trivial p-colorings.

In total we have so far $3p(p-1) + (p-2)p(p-1) = (p+1)p(p-1)$ non-trivial p-colorings. But this is precisely the total number of non-trivial p-colorings. Hence we found all the equivalence classes of the p-colorings. A set of admissible representatives is $\{(0,0,1),(0,1,0),(1,0,0),(0,1,2),(0,1,3),\ldots,(0,1,p-1)\}$. We have then $3+p-2 = p+1$ equivalence classes of p-colorings. ■

3.3 p-nullity n: conjecture.

Conjecture 3.1 *Given an odd prime p and an integer $n > 1$, a link with p-nullity n has the following number of equivalence classes of p-colorings:*

$$\frac{p^{n-1}-1}{p-1}$$

The reader can easily verify that this formula holds for the cases studied in the preceding subsections.

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