

The Langevin equation with bremsstrahlung

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Abstract

We consider electron as the Brownian particle in the sea of the blackbody photons. The bremsstrahlung force caused by the acceleration of electron is involved in Langevin equation, Then we solve the Langevin equation with the bremsstrahlung force.

The analogue of an electron - muon - is considered as the possible theoretical and experimental activity concerning the Langevin equation. It is not excluded that the theory and experiment will form in future the new deal of the Langevin approach to the Brownian motion of an muon and arbitrary charged particle.

1 Introduction

In 1827 the botanist Brown discovered under his microscope vigorous irregular motion of small particles originating from pollen floating on water (Brown, 1828, 1929). He also observed that very fine particles of

minerals undergo similar incessant motion as if they were living objects. This discovery must have been a great wonder at that time. The idea of combining such a motion Brownian motion with molecular motion became fairly widespread in the latter half of the nineteenth century when atomism had not yet been fully recognized as reality.

Einstein, (1905; 1906) gave the first clear theoretical explanation of such a phenomenon which could be directly verified quantitatively by ex- experiments and thus established the very basic foundation of the atomic theory of matter. This explanation of Brownian motion served as convincing evidence that atoms and molecules exist and was further verified experimentally by Jean Perrin (1908).

Einstein did not know that Brownian motion had actually been observed many years before when he first came upon this idea to verify the reality of the atomic concept. Einstein's theory had a great impact at that time, finally convincing people of the theory of heat as molecular motion. It also greatly influenced pure mathematics, that is, the theory of stochastic processes.

The theory of stochastic processes, called the Wiener process, was initiated by Wiener as a mathematical model of Brownian motion (Blokhintsev, 1966; Pardy, 1973). Some years later this was combined with Feynman's path integral formulation of quantum mechanics (Feynman, 1963, 1964; Caldeira et al., 1983).

The theory of Brownian motion was further developed by Langevin (1908), Smoluchowski (1906, 1916), and many others. The classical theory was excellently reviewed by Wang and Uhlenbeck (Wang et al., 1945). The present treatment follows the Kubo approach (Kubo et al., 1985) which is base on the Wang et al. review. Kubo et al. grasp physical phenomena as stochastic processes which are considered as the very fundamental fundamental ones. Brownian motion is an example of this.

2 Langevin approach with the fluctuating force

Let us consider a particle under the fluctuating force $F(t)$. The Newton equation for the motion of particle in the field of the fluctuating force is as follows:

$$m \frac{dv}{dt} = -\xi v + F(t), \quad (1)$$

where m is mass of a particle and ξ is the coefficient of friction.

We will suppose that $F(t)$ is the stochastic force with the correlation properties given by equation

$$\langle F(t) \rangle = 0; \quad \langle F(t)F(t') \rangle = 2\xi kT\delta(t - t'), \quad (2)$$

where $\langle \rangle$ is the time average of the physical quantity.

The question arises, what is the average velocity of a particle in the stochastic medium with the correlation function (2) and what is the correlation function for velocity $\langle v(t)v(t') \rangle$. The next question is what is the average square displacement of a particle for time $t \gg m/\xi$. Let us find the average and correlation functions v of a particle.

3 Solution of a problem

Putting

$$\tau = \frac{m}{\xi}, \quad (3)$$

We get the solution of equation (1) in the form (Kubo et al., 1985):

$$v(t) = v(t_0)e^{-\frac{t-t_0}{\tau}} + \frac{1}{m} \int_{t_0}^t e^{-\frac{t-s}{\tau}} F(s)ds; \quad t \geq t_0. \quad (4)$$

Because $v(t_0) = \text{const.}$ and $\langle F(s) \rangle = 0$, we see that $\langle v(t) \rangle \rightarrow 0$ for $t_0 \rightarrow -\infty$ and $v_0(t_0) = 0$.

So, we finally have

$$v(t) = \frac{1}{m} \int_{-\infty}^t e^{-\frac{t-s}{\tau}} F(s)ds \quad (5)$$

It follows from eq. (5) that $\langle v(t) \rangle = 0$ because $\langle F(s) \rangle = 0$.

For we $\langle v(t)v(t') \rangle$ have with eq. (2) and with $u = t - s, u' = t' - s$:

$$\begin{aligned} \langle v(t)v(t') \rangle &= \frac{1}{m} \int_{-\infty}^t e^{-\frac{t-s}{\tau}} ds \times \frac{1}{m} \int_{-\infty}^{t'} e^{-\frac{t'-s'}{\tau}} 2\xi kT\delta(s - s')ds' = \\ &= \frac{1}{m^2} \int_0^\infty du e^{-\frac{u}{\tau}} \int_0^\infty du' e^{-\frac{u'}{\tau}} 2\xi kT\delta(t - t' - u + u'). \end{aligned} \quad (6)$$

It holds good for arbitrary $f(x)$

$$\int f(x)\delta(x - a)dx = f(a). \quad (7)$$

Or,

$$\langle v(t)v(t') \rangle = 2\frac{kT}{m}e^{-\frac{t-t'}{\tau}} \int_0^\infty du e^{-\frac{2u}{\tau}} \int_0^\infty du' \delta(t-t'-u+u') \quad (8)$$

For $t - t' > 0$, and $u < t - t'$ it is $\int du'[\dots] = 0$ and we have

$$\langle v(t)v(t') \rangle = \frac{kT}{m} \frac{2}{\tau} e^{-\frac{t-t'}{\tau}} \int_{\max(0, t-t')}^\infty e^{-\frac{2u}{\tau}} du \quad (9)$$

Or,

$$\langle v(t)v(t') \rangle = \frac{kT}{m} e^{-\frac{t-t'}{\tau}}. \quad (10)$$

4 The average displacement of the Brownian particle

According to the definition of average displacement of the Brownian particle, we have

$$\begin{aligned} \langle [x(t) - x(0)]^2 \rangle &= \int_0^t dt_1 \int_0^t dt_2 \langle v(t_1)v(t_2) \rangle = \\ &= \frac{kT}{m} \int_0^t dt_1 \int_0^t dt_2 e^{-\frac{t-t'}{\tau}} \end{aligned} \quad (11)$$

Integral in (11) is equal to $2 \times$ contribution of the integration the triangle $(0, t, t)$

Putting $t_1 - t_2 = t'$, $0 < t_2 < t - t'$, we get

$$\begin{aligned} 2 \int_0^t (t - t') e^{-\frac{t'}{\tau}} dt' &= 2 \left\{ \frac{t}{\alpha} (1 - e^{-\frac{\alpha t}{\tau}}) + \frac{d}{d\alpha} \frac{1}{\alpha} (1 - e^{-\frac{\alpha t}{\tau}}) \right\} \\ &= 2 \left\{ \tau t - \tau^2 (1 - e^{-\frac{t}{\tau}}) \right\}; \left(\alpha = \frac{1}{\tau} \right) \end{aligned} \quad (12)$$

For $t \gg \tau$, we have

$$\langle [x(t) - x(0)]^2 \rangle \rightarrow 2 \frac{kT}{m} \tau t = 2 \frac{kT}{\xi} t \quad (13)$$

Or,

$$\langle (\Delta x)^2 \rangle = 2Dt; \quad D = \frac{kT}{\xi} \quad (14)$$

5 Lorentz-Dirac equation with bremsstrahlung

Lorentz-Dirac equation describes motion of a charged particle in electromagnetic field where also the bremsstrahlung force is involved in the equation (Parady, 2012).

The bremsstrahlung force is in the non-relativistic limit expressed in the following form:

$$\mathbf{f} = \frac{2e^2}{3c^3} \ddot{\mathbf{v}}. \quad (15)$$

The force $\mathbf{f}$ is not active force and it means it cannot cause motion of electron. In other words the equation $\mathbf{f} = m\dot{\mathbf{v}}$ has no physical meaning. The bremsstrahlung force is meaningful only with addition of the active force. Then the bremsstrahlung force acts as the so called light friction.

The relativistic equation which involves the bremsstrahlung force is so called Lorentz-Dirac equation and it can be evidently written in the form (Landau et al., 1988):

$$mc \frac{dv_\mu}{d\tau} = \frac{e}{c} F_{\mu\nu} v^\nu + g_\mu, \quad (16)$$

where g^μ can be expressed according to Landau et al. (1988) in the form:

$$g_\mu = \frac{2e^2}{3c} \left(\frac{d^2 v_\mu}{d\tau^2} - v_\mu v^\nu \frac{d^2 v_\nu}{d\tau^2} \right), \quad (17)$$

where the form of the bremsstrahlung term leads in the non-relativistic limit to eq. (15), where u_μ is the four-velocity and the radiative term was approximated by Landau et al. in the form (Landau et al., 1988):

$$g_\mu = \frac{2e^3}{3mc^3} \frac{\partial F_{\mu\nu}}{\partial x^\alpha} v^\nu v^\alpha - \frac{2e^4}{3m^2 c^5} F_{\mu\alpha} F^{\beta\alpha} v_\beta + \frac{2e^4}{3m^2 c^5} (F_{\alpha\beta} v^\beta) (F^{\alpha\gamma} v_\gamma) v_\mu. \quad (18)$$

It is possible to show that the space components of the 4-vector force g_μ is of the form (Landau et al., 1988)

$$\begin{aligned} \mathbf{f} = & \frac{2e^3}{3mc^3} \left(1 - \frac{v^2}{c^2} \right)^{-1/2} \left\{ \left(\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right) \mathbf{E} + \frac{1}{c} \left[\mathbf{v} \left(\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right) \mathbf{H} \right] \right\} + \\ & + \frac{2e^4}{3m^2 c^4} \left\{ \mathbf{E} \times \mathbf{H} + \frac{1}{c} \mathbf{H} \times (\mathbf{H} \times \mathbf{v}) + \frac{1}{c} \mathbf{E} (\mathbf{v} \cdot \mathbf{E}) \right\} - \end{aligned}$$

$$-\frac{2e^4}{3m^2c^5\left(1-\frac{v^2}{c^2}\right)}\mathbf{v}\left\{\left(\mathbf{E}+\frac{1}{c}(\mathbf{v}\times\mathbf{H})\right)^2-\frac{1}{c^2}(\mathbf{E}\mathbf{v})^2\right\}. \quad (19)$$

Using the Lorentz equation we can express the second derivative in the form:

$$\frac{d^2v_\mu}{d\tau^2}=\frac{e}{mc^2}\partial_\alpha(F_{\mu\nu})v^\alpha v^\nu+\frac{e^2}{mc^4}F_{\mu\nu}F^{\nu\alpha}v_\alpha. \quad (20)$$

After insertion of the last equation in the Lorentz-Dirac equation, we get the Lorentz-Dirac equation in the final form:

$$mc\frac{dv_\mu}{d\tau}=\frac{e}{c}F_{\mu\nu}v^\nu+\frac{2e^3}{3mc^3}\times\left\{\partial_\alpha F_{\mu\nu}v^\nu v^\alpha-\frac{e}{mc^2}F_{\mu\alpha}F^{\nu\alpha}v_\nu+\frac{e}{mc^2}v_\mu F_{\alpha\beta}v^\beta F^{\alpha\gamma}v_\gamma\right\}, \quad (21)$$

6 Langevin equation involving bremsstrahlung

In case if we consider the existence of the back reaction of the the charged accelerated particle with one-dimensional force

$$f=\frac{2e^2}{3c^3}\ddot{v}, \quad (22)$$

we get the generalized Langevin equation involving bremsstrahlung. In a one-dimensional case we get for an electron in the stochastic field of the blackboddy photons:

$$m\frac{dv}{dt}=-\xi v-\frac{2e^2}{3c^3}\ddot{v}+F(t), \quad (23)$$

Now, let us transform eq. (23) into the harmonic oscillator equation with the damping force (Parady, 2003; Arfken, et al., 2013) - we get:

$$m\frac{d^2v(t)}{dt^2}+b\dot{v}(t)+kv(t)=F(t). \quad (24)$$

After application of the Laplace transform

$$\int_0^\infty e^{-st}v(t)dt=v(s), \quad (25)$$

and with regard to the initial conditions, $\dot{v}(0)=0, v(0)=0$, we get the following algebraic equation:

$$Ms^2v(s) + Bsv(s) + Kv(s) = F(s), \quad (26)$$

or,

$$v(s) = \frac{F(s)}{M\omega_1} \frac{\omega_1}{(s + B/2M)^2 + \omega_1^2} \quad (27)$$

with $\omega_1^2 = K/M - B^2/4M^2$.

Using inverse Laplace transform denoted by symbol $\mathcal{L}^{-1}$ applied to multiplication of functions $f_1(s)f_2(s)$

$$\mathcal{L}^{-1}(f_1(s)f_2(s)) = \int_0^t d\tau F_1(t - \tau)F_2(\tau), \quad (28)$$

we obtain with $f_1(s) = F(s)/M\omega_1$, $f_2(s) = \omega_1/((s + B/2M)^2 + \omega_1^2)$, $F_1(t) = F(t)/M\omega_1$, $F_2(t) = \exp(-Bt/2M) \sin \omega_1 t$.

$$v(t) = \frac{1}{M\omega_1} \int_0^t F(t - \tau) e^{-\frac{B}{2M}\tau} \sin(\omega_1 \tau) d\tau. \quad (29)$$

Let us remark that for impulsive force $F(t) = P\delta(t)$ we have from the last formula

$$v(t) = \frac{P}{M\omega_1} e^{-\frac{B}{2M}t} \sin \omega_1 t. \quad (30)$$

Using eq. (29), we can determine $\langle v(t)v(t') \rangle$ and

$$\langle [x(t) - x(0)]^2 \rangle = \int_0^t dt_1 \int_0^t dt_2 \langle v(t_1)v(t_2) \rangle \quad (31)$$

7 Discussion

The description of Brownian motion by the Langevin equation is simple and instructive. However, since it is phenomenological, it is necessary to develop a molecular theory. This is not a simple problem, and there is realized only a somewhat simplified version of the theoretical treatments (Isihara, 1971).

At present time we are sure that there is no evidence of transitions of a muon into an electron and a photon, excluding the possibility that the muon is an excited state of the electron. This apparent mystery was resolved by postulating the existence of two neutrino flavors (and incidentally lepton flavor conservation), experimentally confirmed by the discovery of the muon neutrino a few years later. In parallel, experiments

on parity non-conservation in muon decays yielded the first measurements of its magnetic moment. The results clearly established that the muon is a structureless spin 1/2 Dirac particle, further solidifying its role as a fundamental tool in the study of the weak interaction (Echenard et al., 2026).

Now, muon is the possible theoretical and experimental goal concerning the Langevin equation. It is not excluded that the theory and experiment with muon will form the new deal of the Langevin approach to the Brownian motion of the charged particles.

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