Equations of hydrodynamic type: exact solutions, reduction of order, transformations, and nonlinear stability/unstability

A. D. Polyanin* S. N. Aristov†

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Abstract

Systems of hydrodynamic type equations derived from the Navier–Stokes equations and the boundary layer equations are considered. A transformation of the Crocco type reducing the equation order for the longitudinal velocity component is described. The issues of nonlinear stability of the obtained solutions are studied. It is found that a specific feature of many solutions of the Navier-Stokes equations is instability. The nonlinear instability of solutions is proved by a new exact method, which may be useful for the analysis of other nonlinear physical models and phenomena.

Self-similar, invariant, partially invariant, generalized separable and some other exact solutions of the Navier–Stokes equations are considered, e.g., in [1–11].

Keywords: Navier–Stokes equations, boundary layer equations, Crocco transformation, exact solutions, nonlinear instability, nonlinear stability, Calogero equation, exact methods.

1 Systems of hydrodynamic type equations

Systems of equations under consideration. The following systems of equation will be considered:

$$\frac{\partial^2 F}{\partial t \partial z} + F\frac{\partial^2 F}{\partial z^2} - m\Big(\frac{\partial F}{\partial z}\Big)^2 = \nu\frac{\partial^3 F}{\partial z^3} + q(t)\frac{\partial F}{\partial z} + p(t), \tag{1}$$

$$\frac{\partial G}{\partial t} + F\frac{\partial G}{\partial z} - kG\frac{\partial F}{\partial z} = \nu\frac{\partial^2 G}{\partial z^2} + s(t), \tag{2}$$

where F is one of the components of the fluid velocity, G is an auxiliary function, and ν is the kinematic viscosity of the fluid. For $m = 1/2$ and $m = 1$, a single equation

*Institute for Problems in Mechanics, Russian Academy of Sciences, 101 Vernadsky Avenue, bldg 1, 119526 Moscow, Russia. E-mail: polyanin@ipmnet.ru

†Institute of Continuous Media Mechanics, Ural Branch, Russian Academy of Sciences, 1 Acad. Koroleva Str., 614013 Perm, Russia. E-mail: asn@icmm.ru

(1) and system of equations (1), (2) with $m = 1$ were treated in [5, 8, 10, 11], where exact solutions of the Navier–Stokes equations and the boundary layer equations were considered. Note that, for $m = 1$, the functions $p = p(t)$ and $q = q(t)$ in equation (1) can be chosen arbitrarily [11]. Below, a new class of exact solutions of the Navier–Stokes equations will be described for system (1), (2) with $m = k = 1/2$ and $q(t) = 0$.

Nonlinear equation (1) can be considered separately and equation (2) is linear with respect to the unknown function G. According to the general property of equation (1) [8, 11], if $\widetilde{F}(z,t)$ is some solution to this equation, then the function

$$F = \widetilde{F}(z + \psi(t), t) - \psi'_t(t) \tag{3}$$

with an arbitrary function $\psi(t)$ will also be a solution to equation (1); in addition, the function $F = -\widetilde{F}(-z,t)$ will also be a solution to this equation.

One class of exact solutions of the Navier–Stokes equations. Three-dimensional nonstationary motions of a viscous incompressible fluid are described by the following system of the Navier-Stokes and continuity equations:

$$\begin{aligned} &\frac{\partial V_n}{\partial t} + V_1\frac{\partial V_n}{\partial x} + V_2\frac{\partial V_n}{\partial y} + V_3\frac{\partial V_n}{\partial z} \\ &\qquad = -\frac{1}{\rho}\nabla_n P + \nu\left(\frac{\partial^2 V_n}{\partial x^2} + \frac{\partial^2 V_n}{\partial y^2} + \frac{\partial^2 V_n}{\partial z^2}\right), \quad n = 1,\,2,\,3, \\ &\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = 0. \end{aligned} \tag{4}$$

Here x, y, and z are the Cartesian coordinates; t is the time; V_1, V_2, and V_3 are the fluid velocity components; P is the pressure; ρ is the fluid density; and $\nabla_1 P = \partial P/\partial x$, $\nabla_2 P = \partial P/\partial y$, $\nabla_3 P = \partial P/\partial z$. In equations (4), it is assumed that the mass forces are potential and included in the pressure.

For a viscous incompressible fluid, equations (4) admit the following exact solutions:

$$V_1 = G - \frac{1}{2}x\frac{\partial F}{\partial z}, \quad V_2 = -\frac{1}{2}y\frac{\partial F}{\partial z}, \quad V_3 = F,$$

$$\frac{P}{\rho} = p_0(t) + \frac{1}{4}\alpha(t)(x^2 + y^2) - s(t)x - \frac{1}{2}F^2 + \nu\frac{\partial F}{\partial z} - \int\frac{\partial F}{\partial t}\,dz,$$

where $p_0(t)$, $\alpha(t)$, and $s(t)$ are arbitrary functions of time t and the functions F and G depend on z and t and satisfy the equations

$$\frac{\partial^2 F}{\partial t\partial z} + F\frac{\partial^2 F}{\partial z^2} - \frac{1}{2}\left(\frac{\partial F}{\partial z}\right)^2 = \nu\frac{\partial^3 F}{\partial z^3} + \alpha(t), \tag{5}$$

$$\frac{\partial G}{\partial t} + F\frac{\partial G}{\partial z} - \frac{1}{2}G\frac{\partial F}{\partial z} = \nu\frac{\partial^2 G}{\partial z^2} + s(t). \tag{6}$$

System of equations (5), (6) is a special case of system (1), (2) with $m = k = 1/2$ and $q(t) = 0$.

2 Reduction of order of equation (1) and its generalizations with the use the Crocco type transformation

Reduction of order of equation (1). Let us introduce the notation

$$\eta = \frac{\partial F}{\partial z}, \qquad \Phi = \frac{\partial^2 F}{\partial z^2}. \tag{7}$$

Transferring the term mF_z^2 (here and below, the brief record of derivatives is used) to the right-hand side of equation (1), dividing both the resulting equation by $F_{zz} = \Phi$, differentiating with respect to z, and taking into account formulas (7), we eventually obtain the following equation:

$$\frac{\Phi_t}{\Phi} - \frac{F_{zt}\Phi_z}{\Phi^2} + \eta = \frac{\partial}{\partial z}\frac{\nu\Phi_z + m\eta^2 + q(t)\eta + p(t)}{\Phi}. \tag{8}$$

Now let us pass in equation (8) from the old variables t, x, $F = F(x,t)$ to the new variables t, η, $\Phi = \Phi(t,\eta)$, where η and Φ are defined by formulas (7). The derivatives are transformed as follows:

$$\frac{\partial}{\partial z} = \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} = F_{zz}\frac{\partial}{\partial \eta} = \Phi\frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial t} + F_{zt}\frac{\partial}{\partial \eta}.$$

As a result, equation (8) reduces to the second-order equation

$$\frac{\eta}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left(\frac{m\eta^2 + q\eta + p}{\Phi}\right) + \nu\frac{\partial^2 \Phi}{\partial \eta^2}$$

which can be rewritten as follows:

$$\frac{\partial \Phi}{\partial t} + (m\eta^2 + q\eta + p)\frac{\partial \Phi}{\partial \eta} = [(2m-1)\eta + q]\Phi + \nu\Phi^2\frac{\partial^2 \Phi}{\partial \eta^2}. \tag{9}$$

Here and below, for brevity, the arguments of the functions $p(t)$, $q(t)$, and $s(t)$ will often be omitted.

Note that, in the degenerate case (inviscid fluid, $\nu = 0$), the initial nonlinear second-order equation (1) is reduced to the first-order linear equation (9), which can be completely integrated using the method of characteristics.

If a solution of the initial equation (1) is known, then formulas (7) define a solution to equation (9) in the parametric form.

Let $\Phi = \Phi(\eta, t)$ be some solution to equation (9). Then, the corresponding solution to the initial equation (1) can also be presented in parametric form as follows:

$$z = \int \frac{ds}{\Phi(s,t)} + \psi(t), \qquad F = \int \frac{s\,ds}{\Phi(s,t)} - \psi_t'(t),$$

where $\psi(t)$ is an arbitrary function (in the integrals, t is treated as a parameter).

Transformation of system (1), (2). Assuming that $F_{zz} \neq 0$, let us pass in system (1), (2) from the old variables t, z, F to the new variables t, η, Φ according to formulas (7). Then, equation (1) is transformed into equation (9) and equation (2) is transformed into the following equation:

$$\frac{\partial G}{\partial t} + (m\eta^2 + q\eta + p)\frac{\partial G}{\partial \eta} - k\eta G = \nu\Phi^2 \frac{\partial^2 G}{\partial \eta^2} + s(t). \tag{10}$$

In deriving this equation, we used a representation for the mixed derivative obtained from equation (1).

At $m = k$, equation (10) has exact solutions of the following form:

$$G = A\eta + B\Phi + C, \tag{11}$$

where $A = A(t)$, $B = B(t)$, and $C = C(t)$ are unknown functions determined by an appropriate system of ordinary differential equations (at $m = k \neq 1$, we have $B = 0$). To prove this fact, one should substitute expression (11) into equation (10) and take into account equation (9). Formula (11) will be used below for the representation of solutions to equation (2) via solutions to equation (1).

Some generalizations. Let us consider the nonlinear nth-order equation

$$\frac{\partial^2 F}{\partial z \partial t} + [a(t)F + b(t)z]\frac{\partial^2 F}{\partial z^2} = H\Big(t, \frac{\partial F}{\partial z}, \frac{\partial^2 F}{\partial z^2}, \frac{\partial^3 F}{\partial z^3}, \dots, \frac{\partial^n F}{\partial z^n}\Big), \tag{12}$$

which generalize equation (1) and allow for order reduction. Passing in equation (12) from t, x, $F = F(x,t)$ to the new variables t, η, $\Phi = \Phi(t,\eta)$, where η and Φ are defined by formulas (7), we obtain the $(n-l)$th-order equation

$$\frac{a(t)\eta + b(t)}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\Big[\frac{1}{\Phi}H\Big(t, \eta, \Phi, \Phi\frac{\partial \Phi}{\partial \eta}, \dots, \frac{\partial^{n-2}\Phi}{\partial z^{n-2}}\Big)\Big], \tag{13}$$

in which the high-order derivatives are calculated using the formulas

$$\frac{\partial^k F}{\partial z^k} = \frac{\partial^{k-2}\Phi}{\partial z^{k-2}} = \Phi\frac{\partial}{\partial \eta}\frac{\partial^{k-3}\Phi}{\partial z^{k-3}}, \quad \frac{\partial}{\partial z} = \Phi\frac{\partial}{\partial \eta}, \quad k = 3, \dots, n.$$

In the special case of a second-order equation with $n = 1$, $a(t) = -1$, $b(t) = 0$, and $H = H(F_z)$, equation (12) reduces to the Calogero equation, which was considered in [8, 12, 13]. It is evident that a more general equation (12) with $n = 1$, $H = H(t, F_z)$, and arbitrary functions $a(t)$ and $b(t)$ (it is logical to call this the generalized Calogero equation), can be reduced using the Crocco type transformation to the first-order equation (13), which becomes linear after the substitution $\Phi = 1/\Psi$.

3 Representation of solutions to equation (2) via solutions to equation (1)

The case of $m = k = 1$. Let $F = F(z,t)$ be a solution to equation (1). Then, by virtue of relations (11) and (7), equation (2) has the solution

$$G = A'_t + Aq + A\frac{\partial F}{\partial z} + B\frac{\partial^2 F}{\partial z^2}, \tag{14}$$

where the functions $A = A(t)$ and $B = B(t)$ satisfy the ordinary differential equations

$$A''_{tt} + qA'_t + (p + q'_t)A = s, \tag{15}$$

$$B'_t + qB = 0. \tag{16}$$

The proof is carried out by eliminating the function G from (2) and (14), followed by the comparison of the obtained expression to both equation (1) and the equation resulting from the differentiation of equation (1) with respect to z.

The general solution to equation (16) is as follows: $B = C\exp\Bigl(-\int q\,dt\Bigr)$, where C is an arbitrary constant.

The case of $m = k \neq 1$. In this case, by virtue of relations (11) and (7), equation (2) has the solution

$$G = \frac{1}{m}(A'_t + Aq) + A\frac{\partial F}{\partial z}, \tag{17}$$

where function $A = A(t)$ satisfies the ordinary differential equation

$$A''_{tt} + qA'_t + (mp + q'_t)A = ms. \tag{18}$$

The exact solutions to equation (1) can be found in [5, 8, 10, 11]. Formulas (14)-(18) are used to obtain the corresponding exact solutions to equation (2).

4 Use of linear transformations for constructing exact solutions to equations (1)

Using the following linear transformation with respect to the unknown function,

$$\begin{aligned} F &= a(t)f(\tau,\xi) + b(t)z + c(t),\\ \xi &= \lambda(t)z + \sigma(t), \quad \tau = \int \lambda^2(t)\,dt + C_0, \end{aligned} \tag{19}$$

where $a = a(t)$, $b = b(t)$, $c = c(t)$, $\lambda = \lambda(t)$, and $\sigma = \sigma(t)$ are arbitrary functions, we write equation (1) in the following form:

$$\frac{\partial^2 f}{\partial\tau\partial\xi} + [\widetilde{a}(\tau)f + \widetilde{b}(\tau)\xi + \widetilde{c}(\tau)]\frac{\partial^2 f}{\partial\xi^2} - m\widetilde{a}(\tau)\Bigl(\frac{\partial f}{\partial\xi}\Bigr)^2 = \nu\frac{\partial^3 f}{\partial\xi^3} + \widetilde{q}(\tau)\frac{\partial f}{\partial\xi} + \widetilde{p}(\tau), \tag{20}$$

where

$$\begin{aligned} &\widetilde{a} = \frac{a}{\lambda}, \quad \widetilde{b} = \frac{1}{\lambda^3}(b\lambda + \lambda'_t), \quad \widetilde{c} = \frac{1}{\lambda^3}(c\lambda^2 - b\lambda\sigma + \lambda\sigma'_t - \sigma\lambda'_t),\\ &\widetilde{q} = \frac{1}{a\lambda^3}[aq\lambda + 2mab\lambda - (a\lambda)'_t], \quad \widetilde{p} = \frac{1}{a\lambda^3}(p + bq + mb^2 - b'_t). \end{aligned} \tag{21}$$

The argument of the functions on the left-hand side of the equations is τ and that on the right-hand side is t; the variables τ and t are linked by the last relation in (19).

The presence of a large number (from five to seven) of arbitrary functions in equations (1) and (19) allows us to construct various exact solutions to equation (1).

Example 1. Assuming sequentially that (20) $f = \xi^{-1}$, $f = Ae^{\xi} + Be^{-\xi}$, $f = A\sin\xi + B\cos\xi$, and $f = \tanh\xi$ at $m = 1$ in transformation (19) and equation (20), we can obtain the solutions given in [8, 11] by selecting appropriate arbitrary functions. Equation (20) is also satisfied by the functions $f = (1 \pm e^{\xi})^{-1}$ and $f = \tan(\xi + A)$, which give new solutions.

Example 2. Assuming that

$$\widetilde{a} = C_1, \quad \widetilde{b} = C_2, \quad \widetilde{c} = C_3, \quad \widetilde{q} = C_4, \quad \widetilde{p} = C_5, \tag{22}$$

where C_n are arbitrary constants, we obtain an ordinary differential equation for the function $f = f(\xi)$ from equation (20). In this case, relations (21) under conditions (22) represent a system of ordinary differential equations for the functional coefficients of transformation (19). In equation (1) with $m = 1$ and in transformation (19) with constraints (21) and (22), two functions can be set arbitrarily (it should be recalled that $p = p(t)$ and $q = q(t)$ are arbitrary functions). Stationary solutions $f = f(\xi)$ of equation (20) generate nonstationary traveling-wave solutions (19) of the initial equation (1).

Example 3. Now we assume that

$$\begin{aligned} &\widetilde{a} = C_1\tau^{-\frac{k+1}{2}}, \quad \widetilde{b} = C_2\tau^{-1}, \quad \widetilde{c} = C_3\tau^{-\frac{1}{2}}, \\ &\widetilde{q} = C_4\tau^{-1}, \quad \widetilde{p} = C_5\tau^{\frac{k-3}{2}}, \quad \tau = \int \lambda^2(t)\,dt + C_0, \end{aligned} \tag{23}$$

where k and C_n are arbitrary constants. In this case, equation (20) admits self-similar solutions of the following form:

$$f = \tau^{k/2}h(\zeta), \quad \zeta = \xi\tau^{-1/2},$$

where the function $h = h(\zeta)$ satisfies the ordinary differential equation

$$\Big(\frac{k-1}{2} - C_4\Big)h'_\zeta + \Big[C_1 h + \Big(C_2 - \frac{1}{2}\Big)\zeta + C_3\Big]h''_{\zeta\zeta} - mC_1(h'_\zeta)^2 = \nu h'''_{\zeta\zeta\zeta} + C_5. \tag{24}$$

Substituting expressions (23) into (21), we obtain a system of integro-differential equations for the functional parameters of the initial equation (1) and transformation (19). Note that, using the substitution $\lambda = \sqrt{\varphi'_t}$, and taking into account the relation $\tau = \varphi(t) + C_0$, we arrive at a standard system of ordinary differential equations. As a result, a non-self-similar solution of the form (19) is obtained.

5 Nonlinear analysis of stability/instability of solutions

Analysis of stability/instability of solutions based on equation (2). Consider system (1), (2) with $m = k = 1$ and $s = 0$ obtained in [11]. To analyze stability/instability of solutions, we use formula (14) and equations (15) and (16) relating the solutions of system (1), (2). It is important to note that, in many cases, there is no necessity to know the explicit form of the function F.

First, let us study problems with a stationary longitudinal velocity, which correspond to the case of $F = F(z)$, $p = \mathrm{const}$, and $q = \mathrm{const}$. In this case, the solution to equation (15) depends on the sign of the discriminant $\Delta = q^2 - 4p$:

$$A(t) = \begin{cases} \exp\Bigl(-\dfrac{qt}{2}\Bigr)\Bigl[C_1 \exp\Bigl(\dfrac{t\sqrt{\Delta}}{2}\Bigr) + C_2 \exp\Bigl(-\dfrac{t\sqrt{\Delta}}{2}\Bigr)\Bigr] & \text{if } \Delta > 0, \\ \exp\Bigl(-\dfrac{qt}{2}\Bigr)\Bigl[C_1 \sin\Bigl(\dfrac{t\sqrt{|\Delta|}}{2}\Bigr) + C_2 \cos\Bigl(\dfrac{t\sqrt{|\Delta|}}{2}\Bigr)\Bigr] & \text{if } \Delta < 0, \\ \exp\Bigl(-\dfrac{qt}{2}\Bigr)(C_1 t + C_2) & \text{if } \Delta = 0, \end{cases} \tag{25}$$

where C_1 and C_2 are arbitrary constants. Further, for the sake of simplicity, we assume $B = 0$ in equations (14) and (16). In the analysis, we consider the following two cases.

1°. *Nondegenerate case* $F_z \not\equiv 0$. For $q < 0$ (and arbitrary p) or $p < 0$ (and arbitrary q), solutions (14) and (25) (with $C_1 \neq 0$) increase exponentially as $t \to \infty$. Therefore, the specified values of the parameters p and q determine the domain of nonlinear instability of system (1), (2) for any limited stationary profile of the longitudinal component of the velocity $F(z)$ (other than a constant). The point $p = q = 0$ also belongs to the domain of instability of system (1), (2).

Indeed, by choosing suitable values of the constants C_1 and C_2 and using equations (14) and (25), we can make the initial value of $|G|_{t=0}$ (interpreted as the initial perturbation relative to the trivial solution $G = 0$ of equation (2)) smaller than any preset ε. However, for $q < 0$ (and arbitrary p) or $p < 0$ (and arbitrary q), we have $|G| \to \infty$ as $t \to \infty$. This means that arbitrary small perturbations of the solutions to system (1), (2) exhibit unbounded growth with time.

Remark. If $F \to F_1$ at $z \to -\infty$ and $F \to F_2$ as $z \to +\infty$ ($F_1, F_2 = \mathrm{const}$), then solution (14) at $A = 0$, $B \neq 0$ tends to zero as $z \to \pm\infty$.

For $q = 0$ and $p > 0$, solution (25) and hence solution (14) are periodic. The inequalities $q \geq 0$, $p \geq 0$ ($|p| + |q| \neq 0$) determine the domain of conditional stability of the solutions under consideration.

It is important to emphasize that (i) here we are speaking of the *nonlinear instability*; (ii) all the results and solutions obtained above are exact (rather than linearized, which is the case in the theory of linear stability); and (iii) various assumptions, expansions, and approximations inherent in many nonlinear theories are not used either [2, 14, 15]).

Example 1. The stationary spatially periodic solution

$$F = a\sin(\sigma z + b), \quad G = 0 \quad (p = -a^2\sigma^2, \quad q = \nu\sigma^2)$$

of system (1), (2) is unstable for any values of a, b, and σ ($a \neq 0$, $\sigma \neq 0$).

Example 2. The stationary monotonic restricted solution

$$F = -6\nu\sigma\tanh(\sigma z + b), \quad G = 0 \quad (p = 0, \quad q = 8\nu\sigma^2)$$

of system (1), (2) is stable.

All conclusions concerning the stability/instability of solutions presented above, as well as formulas (14) and (25), remain valid for any nonstationary solutions $F = F(z,t)$, $G = G(z,t)$ (under the condition that the derivative $F_z \not\equiv 0$ is bounded) of system (1), (2) with $p = \mathrm{const}$ and $q = \mathrm{const}$.

According to the above considerations, three quarters of the plane of parameters p, q correspond to non-stationary solutions. It is important to note that the flow instability described above is not associated with a specific velocity profile and is realized due to equation (2) governing the transverse components of the fluid velocity. Since the fluid viscosity ν does not enter equations (15) and formulas (25), the above results do not depend on the Reynolds number; i.e., the instability of solutions takes place not only at large but also at small Reynolds numbers ($0 < \mathrm{Re} < \infty$).

Remark. Likewise, we can use equations (14) and (15) to study instability of the nonstationary solutions of system (1), (2) with variables $p = p(t)$ and $q = q(t)$.

2°. *Degenerate case* $F_z \equiv 0$. Let

$$F = a = \mathrm{const} \qquad (\text{if } \ p = 0). \tag{26}$$

Then any solution to equation (2) that, when passing from t, z to the new variables t, $\xi = z - at$, reduces to the classical heat equation, is stable for any values of the parameters a and q.

Analysis of stability of solution $F = \mathrm{const}$ of equation (l) for $p = 0$.

1°. Let us study stability of the trivial solution $F = 0$ of equation (1) for $p = 0$ and various values of the parameter $q = \mathrm{const}$. Equation (1) admits the following exact solution:

$$F = \varepsilon e^{ikz+\lambda t}, \quad \lambda = q - \nu k^2, \tag{27}$$

where ε, k, and λ are real quantities. This solution is also a solution to the linearized equation (1) with the quadratic terms discarded. The absolute value of the difference between solution (27) and the trivial solution at the initial moment of time is $|\varepsilon|$ (this difference can be made arbitrarily small by choosing a suitable ε value).

For $q - \nu k^2 > 0$, the trivial solution will be unstable, whereas it will be stable for $q - \nu k^2 < 0$. The stability boundary is the parabola $q = \nu k^2$ on the k, q plane. For decreasing fluid viscosity, $\nu \to 0$ (which corresponds to increasing Reynolds numbers), the branches of this parabola tend to the line $q = 0$ and the domain of instability expands to become, in the limiting case, the entire upper half-plane $q > 0$. Increasing ν or k results in the expansion of the stability domain. Since the parameter k can be set arbitrarily, then, for any $q > 0$ we can achieve instability of the trivial solution by choosing an appropriate k value.

2°. Consider an arbitrary stationary solution of the form (26). Instead of solution (27), we use the following function:

$$F = \varepsilon e^{ik(z-at)+\lambda t} + a, \quad \lambda = q - \nu k^2, \tag{28}$$

Owing to property (3) with $\psi(t) = -at$, this function is also a solution to equation (1). The absolute value of the difference between equations (26) and (28) at the initial time can be made arbitrarily small by choosing a suitable ε value. All criteria of stability and instability of solution (26) depending on the parameters k and q remain the same as those for the trivial solution.

Remark. It follows from the above results that, for $p = 0$ and $q \le 0$, only a constant profile of the longitudinal velocity component ($F = \mathrm{const}$) is stable.

6 Transformation and exact solutions of the boundary layer equations

Nonstationary equations of the plane boundary layer in terms of the stream function w are reduced to a single third-order equation [8]:

$$\frac{\partial^2 w}{\partial t \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \nu \frac{\partial^3 w}{\partial y^3} + f(x,t).$$

Let us pass from t, x, y, and w to the new variables t, x, η, and Φ, where η, Φ are the generalized Crocco variables defined as follows:

$$\eta = \frac{\partial w}{\partial y}, \qquad \Phi(x,t,\eta) = \frac{\partial^2 w}{\partial y^2}.$$

As a result, we obtain the following second-order equation,

$$\frac{\partial \Phi}{\partial t} + \eta \frac{\partial \Phi}{\partial x} + f(x,t)\frac{\partial \Phi}{\partial \eta} = \nu \Phi^2 \frac{\partial^2 \Phi}{\partial \eta^2},$$

which is reduced by the substitution $\Phi = 1/\Psi$ to the nonlinear heat equation:

$$\frac{\partial \Psi}{\partial t} + \eta \frac{\partial \Psi}{\partial x} + f(x,t)\frac{\partial \Psi}{\partial \eta} = \nu \frac{\partial}{\partial \eta}\Big(\frac{1}{\Psi^2}\frac{\partial \Psi}{\partial \eta}\Big).$$

1°. Let us first consider the special case of $f(x,t) = f(t)$ and for special solutions of the following form:

$$\Psi = Z(\xi,\tau), \quad \xi = x - \eta t + \int t f(t)\,dt, \quad \tau = \frac{1}{3}t^3.$$

Here, we have the solvable equation

$$\frac{\partial Z}{\partial \tau} = \nu \frac{\partial}{\partial \xi}\Big(\frac{1}{Z^2}\frac{\partial Z}{\partial \xi}\Big), \tag{29}$$

which can be reduced to the linear heat equation [8].

2°. Consider a more general case of $f(x,t) = f(t)x + g(t)$ and seek the following special solutions:

$$\Psi = Z(\xi,\tau), \quad \xi = \varphi(t)x + \psi(t)\eta + \theta(t), \quad \tau = \int \psi^2(t)\,dt,$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\theta = \theta(t)$ are determined by the following linear system of ordinary differential equations

$$\varphi'_t + f\psi = 0, \quad \psi'_t + \varphi = 0, \quad \theta'_t + g\psi = 0.$$

As a result, we also arrive at the solvable equation (29).

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