Locally free sheaves on complex supermanifolds[1]

A.L. Onishchik, E.G. Vishnyakova

1. Introduction

An important part of the classical theory of real or complex manifolds is the theory of (smooth, real analytic or complex analytic) vector bundles. With any vector bundle over a manifold $(M, \mathcal{F})$ the sheaf of its (smooth, real analytic or complex analytic) sections is associated which is a locally free sheaf of $\mathcal{F}$-modules, and in this way all the locally free sheaves of $\mathcal{F}$-modules over $(M, \mathcal{F})$ can be obtained. In the present paper, locally free sheaves of $\mathcal{O}$-modules on a complex analytic supermanifold $(M, \mathcal{O})$ (or equivalently sheaves of sections of vector bundles over $(M, \mathcal{O})$) are studied.

It is well-known that any smooth supermanifold $(M, \mathcal{O})$ is split, i.e. $\mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{G}$, where $\mathcal{G}$ is the sheaf of sections of a certain vector bundle over M. In the complex case this statement is false, see [6]. However, we can assign the split supermanifold $(M, \operatorname{gr} \mathcal{O})$ to any complex analytic supermanifold $(M, \mathcal{O})$, which is called the *retract* of $(M, \mathcal{O})$. Given a locally free sheaf $\mathcal{E}$ of $\mathcal{O}$-modules on a complex analytic supermanifold $(M, \mathcal{O})$, we construct a locally free sheaf $\operatorname{gr} \mathcal{E}$ on the retract $(M, \operatorname{gr} \mathcal{O})$, which is called the *retract* of $\mathcal{E}$. It can be easily shown that $\operatorname{gr} \mathcal{E} \simeq \operatorname{gr} \mathcal{O} \otimes \mathcal{E}_{\mathrm{red}}$, where $\mathcal{E}_{\mathrm{red}}$ is the pullback of $\mathcal{E}$ with respect to the natural embedding of the manifold $(M, \mathcal{F})$ into $(M, \mathcal{O})$. In Section 2 we obtained a classification of locally free sheaves $\mathcal{E}$ of $\mathcal{O}$-modules which have a given retract $\operatorname{gr} \mathcal{E}$ in terms of non-abelian 1-cohomology, Theorem 2. In the special case $\mathcal{O} \simeq \operatorname{gr} \mathcal{O}$ our classification result can be simplified, Theorem 3.

In Section 3 we study locally free sheaves of modules over projective superspaces. In the case of complex projective spaces, the problem of the (indecomposable) bundle classification is far from being solved, see [10]. There are two cases, however, in which all bundles are known to be direct sums of line bundles — over $\mathbb{CP}^1$ by the classical Birkhoff – Grothendieck Theorem and over $\mathbb{CP}^\infty$ by the Barth – Van de Ven – Tyurin theorem. We study similar question in the super context. In the case of $\mathbb{CP}^{1|m}$, $m > 0$, we showed that the Birkhoff – Grothendieck Theorem does not hold true. (The fact that this theorem is false for some $\mathbb{CP}^{1|m}$ was noticed in [9].) Furthermore,

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we achieved the result similar to the Barth – Van de Ven – Tyurin Theorem for projective superspaces.

Section 4 is devoted to the study of the tangent sheaf $\mathcal{T}$ of a split supermanifold $(M, \bigwedge \mathcal{G})$ in more details. The main result is here the equivalence of the triviality of the 1-cohomology class corresponding to $\mathcal{T}$ and the existence of a holomorphic connection of the bundle corresponding to the locally free sheaf of $\mathcal{F}$-modules $\mathcal{G}$.

In Subsection 5 a spectral sequence which connects the cohomology with values in a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ with the cohomology with values in its retract $\operatorname{gr}\mathcal{E}$ is constructed. This spectral sequence permits to compute the cohomology group $H^*(M, \mathcal{E})$ using the cohomology class corresponding to $\mathcal{E}$ by Theorem 3 and the cohomology group $H^*(M, \operatorname{gr}\mathcal{E})$. Note that $\operatorname{gr}\mathcal{E}$ is a sheaf of sections of a certain vector bundle over M. Hence to compute $H^*(M, \operatorname{gr}\mathcal{E})$ we may use the well elaborated tools of complex analytic geometry. We described the first two terms of the spectral sequence and the first non zero differential.

A classification of locally free sheaves of $\mathcal{O}$-modules over a smooth supermanifold $(M, \mathcal{O})$ was obtained in [14], Section 4.3. It was shown that any locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ is isomorphic to $\operatorname{gr}\mathcal{E}$. The similar result for fibre superbundles was proved in [16]. In [4] the split holomorphic case was studied. In particular it was shown there that there exists a holomorphic locally free sheaf of $\mathcal{O}$-modules over a holomorphic supermanifold $(M, \mathcal{O})$, which is not isomorphic to its retract $\operatorname{gr}\mathcal{E}$. There a classification up to isomorphism of locally free sheaves of $\mathcal{O}$-modules over a (holomorphic) split supermanifold $(M, \mathcal{O})$, $\mathcal{O} \simeq \bigwedge \mathcal{G}$, is obtained in terms of cohomology set $H^1(M, \mathrm{GL}(n, \bigwedge \mathcal{G}))$. In the present paper we suggest the different approach to the classification of locally free sheaves of $\mathcal{O}$-modules over a split supermanifold, Theorem 3, and more generally over a non-split supermanifold, Theorem 2. Let us explain the difference in more details. Clearly one has a split homomorphism $T : \mathrm{GL}(n, \bigwedge \mathcal{G}) \to \mathrm{GL}(n, \mathbb{C})$ by taking the degree zero part of $\mathrm{GL}(n, \bigwedge \mathcal{G})$. It induces the map $H^1(T) : H^1(M, \mathrm{GL}(n, \bigwedge \mathcal{G})) \to H^1(M, \mathrm{GL}(n, \mathbb{C}))$. Denote by $a_{\mathcal{E}}$ the element of $H^1(M, \mathrm{GL}(n, \bigwedge \mathcal{G}))$, which corresponds to a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$. Then, in our notations, $\mathcal{E}_{\mathrm{red}}$ corresponds to $H^1(T)(a_{\mathcal{E}})$. In our paper we classify all locally free sheaves $\mathcal{E}$ such that $\mathcal{E}_{\mathrm{red}}$ is fixed. Therefore, instead of computing $H^1(M, \mathrm{GL}(n, \bigwedge \mathcal{G}))$, we suggest to use results concerning classification of holomorphic bundles over a manifold, obtained in classical geometry, and consider locally free sheaves with given retract on a split supermanifold. The idea to classify non-split object, more precisely, supermanifolds, using retrcts appeared firstly in [6].

We would like also to mention that, as in the classical case, the line superbundles can be described using the exp-map, see e.g. [2], Chapter VI, Section 2. The Picard groups of generic super-grassmannians were computed in [13].

Notations.

$(M, \mathcal{O})$	supermanifold
$(M, \operatorname{gr}\mathcal{O})$	the retract of $(M, \mathcal{O})$
$\mathcal{T} = \mathcal{D}er\mathcal{O}$	the tangent sheaf of $(M, \mathcal{O})$
$\mathcal{A}ut\mathcal{O}$	the sheaf of automorphisms of the structure sheaf $\mathcal{O}$
$\mathcal{A}ut_0 \operatorname{gr}\mathcal{O}$	the sheaf of automorphisms of $\operatorname{gr}\mathcal{O}$ preserving the $\mathbb{Z}$-grading of $\operatorname{gr}\mathcal{O}$
$\operatorname{gr}\mathcal{E}$	the retract of a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$
$\mathcal{A}ut^{\mathcal{R}}\mathcal{E}$	the sheaf of automorphisms of a sheaf of $\mathcal{R}$-modules $\mathcal{E}$
$\mathcal{A}ut_0^{\mathcal{R}} \operatorname{gr}\mathcal{E}$	the sheaf of automorphisms of a $\mathbb{Z}$-graded sheaf of $\mathcal{R}$-modules $\operatorname{gr}\mathcal{E}$ preserving the $\mathbb{Z}$-grading of $\operatorname{gr}\mathcal{E}$
$\mathcal{QA}ut\mathcal{E}$	the sheaf of quasi-automorphisms of a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$
$\mathcal{QA}ut_0 \operatorname{gr}\mathcal{E}$	the sheaf of quasi-automorphisms of a $\mathbb{Z}$-graded locally free sheaf $\operatorname{gr}\mathcal{E}$ preserving the $\mathbb{Z}$-grading of $\operatorname{gr}\mathcal{E}$
$\mathcal{A}ut_{\bar{0}}^{\mathcal{F}}\mathcal{S}$	a subsheaf of $\mathcal{A}ut^{\mathcal{F}}\mathcal{S}$ consisting of even automorphisms of a $\mathbb{Z}_2$-graded sheaf S
$\mathcal{E}nd^{\mathcal{O}}\mathcal{E}$	the sheaf of endomorphisms of a sheaf of $\mathcal{O}$-modules $\mathcal{E}$

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2. Main definitions and classification theorems

2.1 Main definitions and classification of complex supermanifolds with a given retract

We consider here complex analytic supermanifolds in the sense of Berezin and Leites (see [3, 8]). Thus, a *supermanifold* $(M, \mathcal{O})$ of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space which is locally isomorphic to a superdomain in $\mathbb{C}^{n|m}$. The underlying complex manifold $(M, \mathcal{F})$ is called the *reduction* of $(M, \mathcal{O})$. Sometime we will denote it by M. A *morphism* $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between two supermanifolds with reductions $(M, \mathcal{F}_M)$ and $(N, \mathcal{F}_N)$ is a morphism between $\mathbb{Z}_2$-graded ringed spaces, i.e., a pair $F = (F_{red}, F^*)$, where $F_{red} : M \to N$ is a continuous mapping and $F^* : \mathcal{O}_N \to (F_{red})_*\mathcal{O}_M$ is a homomorphism of sheaves of $\mathbb{Z}_2$-graded ringed spaces. A morphism F is called an *isomorphism* if F is invertible.

We consider $\mathbb{Z}_2$-graded sheaves of $\mathcal{O}$-modules $\mathcal{S} = \mathcal{S}_{\bar 0} + \mathcal{S}_{\bar 1}$ on $(M,\mathcal{O})$. Denote by $\Pi(\mathcal{S})$ the same sheaf of $\mathcal{O}$-modules $\mathcal{S}$ supplied with the following $\mathbb{Z}_2$-grading:

$$\Pi(\mathcal{S})_{\bar 0} = \mathcal{S}_{\bar 1},\ \Pi(\mathcal{S})_{\bar 1} = \mathcal{S}_{\bar 0}.$$

A $\mathbb{Z}_2$-graded sheaf of $\mathcal{O}$-modules on $(M,\mathcal{O})$ is called *free* (*locally free*) *of rank* $p|q$, $p,q \geq 0$, if it is isomorphic (respectively, locally isomorphic) to the $\mathbb{Z}_2$-graded sheaf of $\mathcal{O}$-modules $\mathcal{O}^p \oplus \Pi(\mathcal{O})^q$. For example, the tangent sheaf $\mathcal{T}$ of a supermanifold $(M,\mathcal{O})$ of dimension $n|m$ is a locally free sheaf of $\mathcal{O}$-modules of rank $n|m$.

The simplest class of supermanifolds constitute the so-called *split supermanifolds*. We recall that a supermanifold $(M,\mathcal{O})$ is called *split* if $\mathcal{O} = \bigwedge_{\mathcal{F}} \mathcal{G}$, where $\mathcal{G}$ is a locally free sheaf of $\mathcal{F}$-modules on M. With any supermanifold $(M,\mathcal{O})$ one can associate a split supermanifold $(M, \operatorname{gr}\mathcal{O})$ of the same dimension which is called the *retract* of $(M,\mathcal{O})$. To construct it, let us consider the $\mathbb{Z}_2$-graded sheaf of ideals $\mathcal{J} = \mathcal{J}_{\bar 0} \oplus \mathcal{J}_{\bar 1} \subset \mathcal{O}$ generated by $\mathcal{O}_{\bar 1}$. The structure sheaf of the retract is defined by

$$\operatorname{gr}\mathcal{O} = \bigoplus_{p\geq 0} \operatorname{gr}\mathcal{O}_p, \text{ where } \operatorname{gr}\mathcal{O}_p = \mathcal{J}^p/\mathcal{J}^{p+1},\ \ \mathcal{J}^0 := \mathcal{O}.$$

It can be easy shown that $\mathcal{F} \simeq \mathcal{O}/\mathcal{J}$, $\operatorname{gr}\mathcal{O}_1$ is a locally free sheaf of $\mathcal{F}$-modules on M and $\operatorname{gr}\mathcal{O}_p = \bigwedge^p_{\mathcal{F}} \operatorname{gr}\mathcal{O}_1$. We will use the following two locally split exact sequences:

$$\begin{array}{c} 0 \to \mathcal{J}_{\bar 0} \to \mathcal{O}_{\bar 0} \to \mathcal{F} \to 0; \\ 0 \to (\mathcal{J}^2)_{\bar 1} \to \mathcal{O}_{\bar 1} \to (\operatorname{gr}\mathcal{O})_1 \to 0. \end{array} \tag{1}$$

Note that a supermanifold is split iff the sequences (1) are globally split.

Let $(M,\mathcal{O})$ be a split supermanifold. Then any $\mathbb{Z}_2$-graded locally free sheaf $\mathcal{S} = \mathcal{S}_{\bar 0} \oplus \mathcal{S}_{\bar 1}$ of $\mathcal{F}$-modules on M gives rise to a $\mathbb{Z}_2$-graded locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ on $(M,\mathcal{O})$. It is defined in the following way: $\mathcal{E} := \mathcal{O} \otimes_{\mathcal{F}} \mathcal{S}$. Its $\mathbb{Z}_2$-grading is given by

$$\begin{array}{l} \mathcal{E}_{\bar 0} = \mathcal{O}_{\bar 0} \otimes_{\mathcal{F}} \mathcal{S}_{\bar 0} + \mathcal{O}_{\bar 1} \otimes_{\mathcal{F}} \mathcal{S}_{\bar 1}, \\ \mathcal{E}_{\bar 1} = \mathcal{O}_{\bar 0} \otimes_{\mathcal{F}} \mathcal{S}_{\bar 1} + \mathcal{O}_{\bar 0} \otimes_{\mathcal{F}} \mathcal{S}_{\bar 1}. \end{array} \tag{2}$$

Let now $\mathcal{E} = \mathcal{E}_{\bar 0} \oplus \mathcal{E}_{\bar 1}$ be a locally free sheaf of $\mathcal{O}$-modules of rang $p|q$ on an arbitrary supermanifold $(M,\mathcal{O})$. We are going to construct a locally free sheaf of the same rank on the retract of $(M,\mathcal{O})$. First, we note that

$\mathcal{S} := \mathcal{E}/\mathcal{J}\mathcal{E}$ is a locally free sheaf of $\mathcal{F}$-modules on M. Moreover, $\mathcal{S}$ admits the $\mathbb{Z}_2$-grading

$$\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$$

by two locally free sheaves of $\mathcal{F}$-modules

$$\mathcal{S}_{\bar{0}} := \mathcal{E}_{\bar{0}}/(\mathcal{J}\mathcal{E}) \cap \mathcal{E}_{\bar{0}},\ \mathcal{S}_{\bar{1}} := \mathcal{E}_{\bar{1}}/(\mathcal{J}\mathcal{E}) \cap \mathcal{E}_{\bar{1}}$$

of ranks p and q respectively. We have the following two locally split exact sequences:

$$\begin{aligned} 0 \to \mathcal{J}\mathcal{E} \cap \mathcal{E}_{\bar{0}} \to \mathcal{E}_{(0)\bar{0}} \stackrel{\alpha}{\to} \mathcal{S}_{\bar{0}} \to 0; \\ 0 \to \mathcal{J}\mathcal{E} \cap \mathcal{E}_{\bar{1}} \to \mathcal{E}_{(0)\bar{1}} \stackrel{\beta}{\to} \mathcal{S}_{\bar{1}} \to 0, \end{aligned} \tag{3}$$

where α and β are the natural projection maps. The sheaf $\mathcal{E}$ possesses the filtration:

$$\mathcal{E} = \mathcal{E}_{(0)} \supset \mathcal{E}_{(1)} \supset \mathcal{E}_{(2)} \supset \dots, \tag{4}$$

where

$$\mathcal{E}_{(p)} = \mathcal{J}^p\mathcal{E},\ p \geq 1.$$

Using this filtration, we can construct the following locally free sheaf of $\operatorname{gr}\mathcal{O}$-modules on the retract $(M, \operatorname{gr}\mathcal{O})$:

$$\begin{gathered} \operatorname{gr}\mathcal{E} = \bigoplus\nolimits_p \operatorname{gr}\mathcal{E}_p, \text{ where} \\ \operatorname{gr}\mathcal{E}_p = \mathcal{E}_{(p)}/\mathcal{E}_{(p+1)} \simeq \operatorname{gr}\mathcal{O}_p \otimes_{\mathcal{F}} \mathcal{S}. \end{gathered}$$

From $\operatorname{gr}\mathcal{O} = \bigwedge \operatorname{gr}\mathcal{O}_1$ and $\operatorname{gr}\mathcal{O}_p = \bigwedge^p \operatorname{gr}\mathcal{O}_1$ it follows that

$$\operatorname{gr}\mathcal{E} \simeq \bigwedge \operatorname{gr}\mathcal{O}_1 \otimes_{\mathcal{F}} \mathcal{S}.$$

The sheaf $\operatorname{gr}\mathcal{E}$ we will call the *retract* of $\mathcal{E}$. By definition, the sheaf $\operatorname{gr}\mathcal{E}$ is $\mathbb{Z}$-graded. It possesses also the $\mathbb{Z}_2$-grading given by (2).

Our aim now is to classify locally free sheaves of $\mathcal{O}$-modules on a supermanifold $(M, \mathcal{O})$ which have a fixed retract. First we formulate the well-known theorem of Green (see [4]) which classifies complex supermanifolds $(M, \mathcal{O}_M)$ with a given retract up to isomorphism, inducing the identical isomorphism of reductions. The main tool used in both classification theorems is the 1-cohomology set $H^1(M, \mathcal{Q})$, where $\mathcal{Q}$ is a sheaf of non-abelian groups on M. We denote by ϵ the unit element of $H^1(M, \mathcal{Q})$ which corresponds to the unit 1-cocycle.

In what follows, we denote by $\mathcal{A}ut\mathcal{O}$ the sheaf of automorphisms of the sheaf of superalgebras $\mathcal{O}$ and by $\mathcal{A}ut^{\mathcal{R}}\mathcal{E}$ the sheaf of automorphisms of a sheaf of $\mathcal{R}$-modules $\mathcal{E}$ on M, where $\mathcal{R}$ is a sheaf of (super)algebras on M. The sheaf $\mathcal{A}ut\mathcal{O}$ possesses the filtration

$$\mathcal{A}ut\mathcal{O} = \mathcal{A}ut_{(0)}\mathcal{O} \supset \mathcal{A}ut_{(2)}\mathcal{O} \supset \dots, \tag{5}$$

where

$$\mathcal{A}ut_{(2p)}\mathcal{O} = \{a \in \mathcal{A}ut\mathcal{O} \mid a(u) \equiv u \operatorname{mod} \mathcal{J}^{2p}\}.$$

Furthermore, the group $H^0(M, \mathcal{A}ut_0 \operatorname{gr} \mathcal{O}) \simeq H^0(M, \mathcal{A}ut^{\mathcal{F}} \operatorname{gr} \mathcal{O}_1)$ acts on the sheaf $\mathcal{A}ut \operatorname{gr} \mathcal{O}$ by $\operatorname{Int} : (a, \delta) \mapsto a \circ \delta \circ a^{-1}$, where $\delta \in \mathcal{A}ut \operatorname{gr} \mathcal{O}$ and $a \in H^0(M, \mathcal{A}ut_0 \operatorname{gr} \mathcal{O})$. Clearly, the group $H^0(M, \mathcal{A}ut_0 \operatorname{gr} \mathcal{O})$ leaves invariant the subsheaves of groups $\mathcal{A}ut_{(2p)} \operatorname{gr} \mathcal{O}$. Hence this group acts on the sets $H^1(M, \mathcal{A}ut_{(2p)} \operatorname{gr} \mathcal{O})$, and the unit element ϵ is fixed under this action.

Denote by $[(M, \mathcal{O})]$ the class of supermanifolds which are isomorphic to $(M, \mathcal{O})$. (Here we consider complex supermanifolds up to isomorphisms inducing the identical isomorphism of reductions.)

Theorem 1. [**Green**] *Let* $(M, \mathcal{O}_{\mathrm{gr}})$ *be a split complex supermanifold. Then*

$$\{[(M, \mathcal{O})] \mid \operatorname{gr} \mathcal{O} = \mathcal{O}_{\mathrm{gr}}\} \stackrel{1:1}{\longleftrightarrow} H^1(M, \mathcal{A}ut_{(2)} \operatorname{gr} \mathcal{O})/H^0(M, \mathcal{A}ut_0 \operatorname{gr} \mathcal{O}),$$

where $(M, \mathcal{O}_{\mathrm{gr}})$ *corresponds to* ϵ.

2.2 Classification theorems for locally free sheaves with a given retract

Let $(M, \mathcal{O})$ and $(M, \mathcal{O}')$ be two supermanifolds, $\mathcal{E}_1$ and $\mathcal{E}_2$ be locally free sheaves of $\mathcal{O}$-modules and $\mathcal{O}'$-modules on M respectively. Suppose that $\Psi : \mathcal{O} \to \mathcal{O}'$ is a homomorphism of sheaves of superalgebras. A homomorphism of $\mathbb{Z}_2$-graded sheaves of vector spaces $\Phi : \mathcal{E}_1 \to \mathcal{E}_2$ is called a Ψ-*morphism* if

$$\Phi(fv) = \Psi(f)\Phi(v), \ f \in \mathcal{O}, \ v \in \mathcal{E}_1.$$

In this case we write $\Phi = \Phi_\Psi$. A Ψ-*morphism* $\Phi : \mathcal{E} \to \mathcal{E}$ is called a Ψ-*isomorphism* if Φ is invertible. A Ψ-isomorphism $\Phi : \mathcal{E} \to \mathcal{E}$ we also will call a Ψ-*automorphism* of $\mathcal{E}$. A homomorphism (isomorphism) of $\mathbb{Z}_2$-graded sheaves of vector spaces $\Phi : \mathcal{E}_1 \to \mathcal{E}_2$ will be called a *quasi-morphism* (*quasi-isomorphism*) if it is a Ψ-*morphism* (Ψ-*isomorphism*) for a certain Ψ. The sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ will be called *quasi-isomorphic* if it exists a quasi-isomorphism $\Phi : \mathcal{E}_1 \to \mathcal{E}_2$. A quasi-isomorphism $\mathcal{E} \to \mathcal{E}$ will be called a *quasi-automorphism* of $\mathcal{E}$. We will study the sheaf $\mathcal{QA}ut\mathcal{E}$, where

$$\mathcal{QA}ut\mathcal{E}(U) = \{\Phi \mid \Phi \text{ is a quasi-automorphism of } \mathcal{E}|_U\} \tag{6}$$

for each open subset $U \subset M$. One verifies easily that $\Phi_\Psi \circ \Theta_\Upsilon$, where $\Phi_\Psi, \Theta_\Upsilon \in \mathcal{QA}ut\mathcal{E}$, is a $\Psi \circ \Upsilon$-morphism. It follows that $\mathcal{QA}ut\mathcal{E}$ is a sheaf of groups. It possesses the double filtration by the subsheaves

$$\mathcal{QA}ut_{(p)(q)}\mathcal{E} := \{\Phi_\Psi \in \mathcal{QA}ut\mathcal{E} \mid \Phi_\Psi(v) \equiv v \operatorname{mod} \mathcal{E}_{(p)}, \ \Psi(f) \equiv f \operatorname{mod} \mathcal{J}^q \\ \text{for } v \in \mathcal{E}, f \in \mathcal{O}\}, \ p, q \geq 0.$$

We also define the following subsheaves:

$$\mathcal{QA}ut_0(\operatorname{gr}\mathcal{E}) := \{\Phi_\Psi \mid \Phi_\Psi \in \mathcal{QA}ut(\operatorname{gr}\mathcal{E}),\ \Phi_\Psi \text{ preserves the } \mathbb{Z}\text{-grading of } \operatorname{gr}\mathcal{E}\}. \tag{7}$$

$$\mathcal{A}ut_{\bar 0}^{\mathcal{F}}\mathcal{S} := \{\Phi \mid \Phi \in \mathcal{A}ut^{\mathcal{F}}\mathcal{S},\ \Phi \text{ preserves the } \mathbb{Z}_2\text{-grading of} \,\mathcal{S}\}, \tag{8}$$

where $\mathcal{S}$ is a $\mathbb{Z}_2$-graded sheaf of $\mathcal{F}$-modules.

Lemma 1. *We have an isomorphism of sheaves of groups*

$$\mathcal{QA}ut_0(\operatorname{gr}\mathcal{E}) \simeq \mathcal{A}ut^{\mathcal{F}}(\operatorname{gr}\mathcal{O}_1) \times \mathcal{A}ut_{\bar 0}^{\mathcal{F}}\mathcal{E}_{\mathrm{red}}.$$

Proof. Let us define the mapping

$$\Theta : \mathcal{A}ut^{\mathcal{F}}(\operatorname{gr}\mathcal{O}_1) \times \mathcal{A}ut_{\bar 0}^{\mathcal{F}}\mathcal{E}_{\mathrm{red}} \to \mathcal{QA}ut_0(\operatorname{gr}\mathcal{E})$$

by

$$(\psi, \Phi) \mapsto \Phi_{\wedge\psi},\ \psi \in \mathcal{A}ut^{\mathcal{F}}(\operatorname{gr}\mathcal{O}_1),\ \Phi \in \mathcal{A}ut_{\bar 0}^{\mathcal{F}}\mathcal{E}_{\mathrm{red}},$$

where

$$\Phi_{\wedge\psi}(hv) := \wedge\psi(h)\Phi(v)$$

for $h \in \operatorname{gr}\mathcal{O}$, $v \in \mathcal{E}_{\mathrm{red}}$ and $\wedge\psi$ is the automorphism of the sheaf $\operatorname{gr}\mathcal{O}$ induced by ψ. This is a homomorphism of sheaves of groups. In fact, suppose that another pair (ψ', Φ'), where $\psi' \in \mathcal{A}ut^{\mathcal{F}}(\operatorname{gr}\mathcal{O}_1)$, $\Phi' \in \mathcal{A}ut_{\bar 0}^{\mathcal{F}}\mathcal{E}_{\mathrm{red}}$, is given. Then we have

$$(\Phi_{\wedge\psi} \circ \Phi'_{\wedge\psi'})(hv) = \Phi_{\wedge\psi}(\wedge\psi'(h)\Phi'_{\wedge\psi'}(v)) = \wedge\psi(\wedge\psi'(h))\Phi_{\wedge\psi}(\Phi'_{\wedge\psi'}(v)) = (\Phi \circ \Phi'_{\wedge\psi\circ\wedge\psi'})(hv)$$

for $h \in \operatorname{gr}\mathcal{O}$, $v \in \mathcal{E}_{\mathrm{red}}$.

Let us prove that $\operatorname{Ker}\Theta = (\mathrm{id}, \mathrm{id})$. Suppose that $\Theta(\psi, \Phi) = \mathrm{id}$. Then $\Phi_{\wedge\psi}(hv) = \wedge\psi(h)\Phi(v) = hv$ for all $h \in \operatorname{gr}\mathcal{O}$, $v \in \mathcal{E}_{\mathrm{red}}$. Putting $h = 1$, we see that $\Phi(v) = v$, i.e., $\Phi = \mathrm{id}$. Since $\mathcal{E}_{\mathrm{red}}$ is locally free, this implies that $\wedge\psi(h) = h$, therefore, $\psi = \mathrm{id}$. Thus, the homomorphism Θ is injective.

Let us now prove that it is surjective. Let $\Phi_\Psi \in \mathcal{QA}ut_0(\operatorname{gr}\mathcal{E})$ be given. Let us show that $\Phi_\Psi \in \operatorname{Im}\Theta$. Since $\Phi_\Psi|_{\mathcal{E}_{\mathrm{red}}} : \mathcal{E}_{\mathrm{red}} \to \mathcal{E}_{\mathrm{red}}$ and Φ_Ψ preserves the $\mathbb{Z}_2$-grading of $\operatorname{gr}\mathcal{E}$, we have $\Phi := \Phi_\Psi|_{\mathcal{E}_{\mathrm{red}}} \in \mathcal{A}ut_{\bar 0}^{\mathcal{F}}\mathcal{E}_{\mathrm{red}}$. Furthermore, if $h \in \operatorname{gr}\mathcal{O}_p$ and $v \in \mathcal{E}_{\mathrm{red}}$, then

$$\Phi_\Psi(hv) = \Psi(h)\Phi(v) \in \operatorname{gr}\mathcal{E}_p.$$

It follows that $\Psi(h) \in \operatorname{gr} \mathcal{O}_p$, and hence Ψ preserves the $\mathbb{Z}$-grading of $\operatorname{gr} \mathcal{O}$. We have $\psi = \Psi|_{\operatorname{gr} \mathcal{O}_1} \in \mathcal{A}ut^{\mathcal{F}}(\operatorname{gr} \mathcal{O}_1)$ and $\wedge \psi = \Psi$. The proof is complete.□

We will use the above notation, fixing a split complex supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$ and a $\mathbb{Z}_2$-graded locally free sheaf of $\mathcal{F}$-modules $\mathcal{S}$ on M. Our aim is to classify locally free sheaves $\mathcal{E}$ of $\mathcal{O}$-modules on complex supermanifolds $(M, \mathcal{O})$ with retract $(M, \mathcal{O}_{\mathrm{gr}})$, whose retract $\operatorname{gr} \mathcal{E}$ coincides with $\mathcal{E}_{\mathrm{gr}} = \mathcal{O}_{\mathrm{gr}} \otimes_{\mathcal{F}} \mathcal{S}$.

The group $H^0(M, \mathcal{Q}\mathcal{A}ut_0 \mathcal{E}_{\mathrm{gr}})$ acts on the sheaf $\mathcal{Q}\mathcal{A}ut \mathcal{E}_{\mathrm{gr}}$ by the automorphisms $\delta \mapsto a \circ \delta \circ a^{-1}$, where $a \in H^0(M, \mathcal{Q}\mathcal{A}ut_0 \mathcal{E}_{\mathrm{gr}})$ and $\delta \in \mathcal{Q}\mathcal{A}ut \mathcal{E}_{\mathrm{gr}}$. It is easy to see that this action leaves invariant the subsheaves $\mathcal{Q}\mathcal{A}ut_{(p)(q)} \mathcal{E}_{\mathrm{gr}}$ and hence induces an action of $H^0(M, \mathcal{Q}\mathcal{A}ut_0 \mathcal{E}_{\mathrm{gr}})$ on the cohomology set $H^1(M, \mathcal{Q}\mathcal{A}ut_{(p)(q)} \mathcal{E}_{\mathrm{gr}})$.

If $\phi : M \to N$ is a holomorphic map of manifolds and $p : \mathbb{E} \to N$ is a vector bundle, we may define the pullback bundle $\phi^*(\mathbb{E})$ on N. The corresponding to $\phi^*(\mathbb{E})$ sheaf is $\mathcal{F}_M \otimes_{\phi^*(\mathcal{F}_N)} \phi^*(\mathcal{E})$, where $\mathcal{E}$ is the sheaf of sections corresponding to $\mathbb{E}$, $\mathcal{F}_M$ and $\mathcal{F}_N$ are the sheaves of holomorphic functions on M and N respectively. Let $\pi : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ be a morphism of two supermanifolds and $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_N$-modules on N of rang $p|q$. Similarly, we can define the sheaf $\mathcal{O}_M \otimes_{\pi^*_{\mathrm{red}}(\mathcal{O}_N)} \pi^*_{\mathrm{red}}(\mathcal{E})$. This sheaf is a locally free sheaf of $\mathcal{O}_M$-modules on M of rang $p|q$, since

$$\mathcal{O}_M \otimes_{\pi^*_{\mathrm{red}}(\mathcal{O}_N)} \pi^*_{\mathrm{red}}(\mathcal{O}_N) \simeq \mathcal{O}_M.$$

Sometimes we will denote the sheaf $\mathcal{O}_M \otimes_{\pi^*_{\mathrm{red}}(\mathcal{O}_N)} \pi^*_{\mathrm{red}}(\mathcal{E})$ by $\widetilde{\pi}(\mathcal{E})$.

Let us consider the special case $(M, \mathcal{O}_M) = (N, \mathcal{O}_N)$, $\pi = (\mathrm{id}, \pi^*)$ and $\pi^* \in H^0(M, \mathcal{A}ut \mathcal{O}_M)$. We have

$$\widetilde{\pi}(\mathcal{E}) = \mathcal{O}_M \otimes_{\mathrm{id}^*(\mathcal{O}_N)} \mathrm{id}^*(\mathcal{E}) = \mathcal{O}_M \otimes_{\mathcal{O}_N} \mathcal{E}.$$

The sheaves $\widetilde{\pi}(\mathcal{E})$ and $\mathcal{E}$ are $(\pi^*)^{-1}$-isomorphic, the $(\pi^*)^{-1}$-isomorphism is given by $f \otimes s \mapsto (\pi^*)^{-1}(f)s$, where $f \in \mathcal{O}_M$ and $s \in \mathcal{E}$. Let $\Phi_{\Psi^*} : \mathcal{E} \to \mathcal{E}'$ be an Ψ^*-isomorphism of two locally free sheaves of $\mathcal{O}_M$-modules on M. We put $\Psi := (\mathrm{id}, \Psi^*)$. We see that $\widetilde{\Psi}(\mathcal{E})$ and $\mathcal{E}'$ are id-isomorphic.

Furthermore, let us consider the sheaf $\mathcal{A}ut^{\mathcal{O}} \mathcal{E}$ of automorphisms of the $\mathcal{O}$-modules sheaf $\mathcal{E}$. It possesses the filtration:

$$\mathcal{A}ut^{\mathcal{O}} \mathcal{E} = \mathcal{A}ut^{\mathcal{O}}_{(0)} \mathcal{E} \supset \mathcal{A}ut^{\mathcal{O}}_{(1)} \mathcal{E} \supset \dots,$$

where

$$\mathcal{A}ut^{\mathcal{O}}_{(p)} \mathcal{E} := \{a \in \mathcal{A}ut^{\mathcal{O}} \mathcal{E} \mid a(v) \equiv v \bmod \mathcal{E}_{(p)}\},\ p \geq 0.$$

The group $H^0(M, \mathcal{A}ut^{\mathcal{O}}_0 \operatorname{gr} \mathcal{E}) \simeq H^0(M, \mathcal{A}ut^{\mathcal{F}}_{\bar{0}} \mathcal{E}_{\mathrm{red}})$ acts on the sheaf $\mathcal{A}ut^{\mathcal{O}} \operatorname{gr} \mathcal{E}$ by $\delta \mapsto a \circ \delta \circ a^{-1}$, where $a \in H^0(M, \mathcal{A}ut^{\mathcal{O}}_0 \operatorname{gr} \mathcal{E})$ and $\delta \in \mathcal{A}ut^{\mathcal{O}} \operatorname{gr} \mathcal{E}$. It is easy to see that this action leaves the subsheaves $\mathcal{A}ut^{\mathcal{O}}_{(p)}\operatorname{gr}\mathcal{E}$ invariant and hence induces an action of $H^0(M, \mathcal{A}ut^{\mathcal{O}}_0\operatorname{gr}\mathcal{E})$ on the cohomology set $H^1(M, \mathcal{A}ut^{\mathcal{O}}_{(p)}\operatorname{gr}\mathcal{E})$.

We have the exact sequence of sheaves of groups

$$\mathrm{id} \to \mathcal{A}ut^{\mathcal{O}}\mathcal{E} \to \mathcal{QA}ut\mathcal{E} \to \mathcal{A}ut\mathcal{O} \to \mathrm{id},$$

where the first homomorphism is the natural embedding (an automorphism of $\mathcal{A}ut^{\mathcal{O}}\mathcal{E}$ is regarded as an id-morphism) and the second one, say $F : \mathcal{QA}ut\mathcal{E} \to \mathcal{A}ut\mathcal{O}$, is defined by $\Phi_\Psi \mapsto \Psi$. Note that $F(\mathcal{QA}ut_{(p)(q)}\mathcal{E}) \subset \mathcal{A}ut_{(q)}\mathcal{O}$ and in the case $\mathcal{E} = \operatorname{gr}\mathcal{E}$ the restriction $F|\mathcal{QA}ut_0\operatorname{gr}\mathcal{E}$ coincides with the natural projection

$$\mathcal{QA}ut_0(\mathcal{E}_{\mathrm{gr}}) \simeq \mathcal{A}ut_0\operatorname{gr}\mathcal{O} \times \mathcal{A}ut^{\mathcal{F}}_{\bar{0}}(\mathcal{E}_{\mathrm{red}}) \to \mathcal{A}ut_0\operatorname{gr}\mathcal{O}$$

(see Lemma 1).

The homomorphism F commutes with the actions of $H^0(M, \mathcal{QA}ut_0\operatorname{gr}\mathcal{E})$ and $H^0(M, \mathcal{A}ut_0\operatorname{gr}\mathcal{O})$ on $\mathcal{QA}ut_{(p)(q)}(\operatorname{gr}\mathcal{E})$ and $\mathcal{A}ut_{(q)}(\operatorname{gr}\mathcal{O})$ respectively. More precisely,

$$F(a \circ \delta \circ a^{-1}) = F(a) \circ F(\delta) \circ F(a^{-1}),$$

where $a \in H^0(M, \mathcal{QA}ut_0\operatorname{gr}\mathcal{E})$ and $\delta \in \mathcal{QA}ut\operatorname{gr}\mathcal{E}$. It follows that F induces the map of sets

$$\widetilde{F} : H^1(M, \mathcal{QA}ut_{(1)(2)}\operatorname{gr}\mathcal{E})/H^0(M, \mathcal{QA}ut_0\operatorname{gr}\mathcal{E}) \to H^1(M, \mathcal{A}ut_{(2)}\operatorname{gr}\mathcal{O})/H^0(M, \mathcal{A}ut_0\operatorname{gr}\mathcal{O}).$$

Let $\Phi_\Psi : \mathcal{E}_1 \to \mathcal{E}_2$ be a Ψ-morphism of locally free sheaves of $\mathcal{O}$-modules. Since $\Psi(\mathcal{J}^p) \subset \mathcal{J}^p$, we see that $\Phi_\Psi((\mathcal{E}_1)_{(p)}) \subset (\mathcal{E}_2)_{(p)}$, $p \geq 0$. We denote by $\operatorname{gr}(\Phi_\Psi) : \operatorname{gr}\mathcal{E}_1 \to \operatorname{gr}\mathcal{E}_2$ the induced morphism. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on M. Denote

$$[\mathcal{E}] = \{\mathcal{E}' \mid \mathcal{E}' \text{ is quasi-isomorphic to } \mathcal{E}\}.$$

Theorem 2. *Let $(M, \mathcal{O}_{\mathrm{gr}})$ be a split supermanifold, $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded locally free sheaf of $\mathcal{F}$-modules on M and $\mathcal{E}_{\mathrm{gr}} = \mathcal{O}_{\mathrm{gr}} \otimes_{\mathcal{F}} \mathcal{S}$.*

1) We have a bijection

$$\{[\mathcal{E}] \mid \operatorname{gr}\mathcal{O} = \mathcal{O}_{\mathrm{gr}},\ \operatorname{gr}\mathcal{E} = \mathcal{E}_{\mathrm{gr}}\} \stackrel{1:1}{\longleftrightarrow} H^1(M, \mathcal{QA}ut_{(1)(2)}\mathcal{E}_{\mathrm{gr}})/H^0(M, \mathcal{QA}ut_0\mathcal{E}_{\mathrm{gr}}).$$

The unit $\epsilon \in H^1(M, \mathcal{QA}ut_{(1)(2)}\mathcal{E}_{\mathrm{gr}})$ is fixed with respect to the action of the group $H^0(M, \mathcal{QA}ut_0\mathcal{E}_{\mathrm{gr}})$.

2) Let $a \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})/H^0(M, \mathcal{A}ut_0\mathcal{O}_{\mathrm{gr}})$. Then there is a bijection between elements of the set $\widetilde{F}^{-1}(a)$ and classes of isomorphic locally free sheaves on supermanifolds which are contained in $[(M, \mathcal{O})]$.

Proof. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on $(M, \mathcal{O})$ and $\mathcal{U} = \{U_i\}$ be an open covering of M such that (1) and (3) are split over U_i and $\mathcal{E}|_{U_i}$ are free. In this case $(\mathrm{gr}\,\mathcal{E})|_{U_i}$ are free sheaves of $(\mathrm{gr}\,\mathcal{O})$-modules, too. We fix local bases $(\hat{e}^i_j)$ and $(\hat{f}^i_k)$ of the sheaves of $\mathcal{F}$-modules $(\mathcal{E}_{\mathrm{red}})_{\bar{0}}|_{U_i}$ and $(\mathcal{E}_{\mathrm{red}})_{\bar{1}}|_{U_i}$, $U_i \in \mathcal{U}$, respectively.

We are going to define an isomorphism $\delta_i : \mathcal{E}|_{U_i} \to (\mathrm{gr}\,\mathcal{E})|_{U_i}$. Let $e^i_j \in \mathcal{E}_{(0)\bar{0}}$ such that $\alpha(e^i_j) = \hat{e}^i_j$ and $f^i_k \in \mathcal{E}_{(0)\bar{1}}$ such that $\beta(f^i_k) = \hat{f}^i_k$. Then (e^i_j, f^i_k) is a local basis of $\mathcal{E}|_{U_i}$. A splitting of (1) determines local isomorphisms $\sigma_i : \mathcal{O}|_{U_i} \to \mathrm{gr}\,\mathcal{O}|_{U_i}$. We put

$$\delta_i(\sum h_j e^i_j + \sum g_k f^i_k) = \sum \sigma_i(h_j)\hat{e}^i_j + \sum \sigma_i(g_k)\hat{f}^i_k, \;\; h_j, g_k \in \mathcal{O}.$$

Obviously, δ_i is an isomorphism. We put $\gamma_{ij} := \sigma_i \circ \sigma_j^{-1}$ and $(g_{ij})_{\gamma_{ij}} := \delta_i \circ \delta_j^{-1}$. It is clear that $(\gamma_{ij}) \in Z^1(\mathcal{U}, \mathcal{A}ut_{(2)}(\mathrm{gr}\,\mathcal{O}))$ and

$$((g_{ij})_{\gamma_{ij}}) \in Z^1(\mathcal{U}, \mathcal{QA}ut_{(1)(2)}(\mathrm{gr}\,\mathcal{E})).$$

Conversely, if $((g_{ij})_{\gamma_{ij}}) \in Z^1(\mathcal{U}, \mathcal{QA}ut_{(1)(2)}(\mathrm{gr}\,\mathcal{E}))$, we can construct a locally free sheaf of $\mathcal{O}$-modules on $(M, \mathcal{O}(\gamma_{ij}))$, where $(M, \mathcal{O}(\gamma_{ij}))$ is the supermanifold corresponding to the cocycle $(\gamma_{ij}) \in Z^1(\mathcal{U}, \mathcal{A}ut_{(2)}\,\mathrm{gr}\,\mathcal{O})$ by the Green Theorem. Indeed, we have to identify $\mathrm{gr}\,\mathcal{E}|_{U_i}$ with $\mathrm{gr}\,\mathcal{E}|_{U_j}$ over $U_i \cap U_j$ using $(g_{ij})_{\gamma_{ij}}$.

The standard calculation shows that if two cocycles $((g_{ij})_{\gamma_{ij}})$ and $((g'_{ij})_{\gamma'_{ij}})$ are cohomological, then the corresponding locally free sheaves of $\mathcal{O}$-modules are quasi-isomorphic and this quasi-isomorphism denoted by Φ_Ψ has the property $\mathrm{gr}(\Phi_\Psi) = \mathrm{id}_{\mathrm{id}}$. Conversely, if $\Phi_\Psi : \mathcal{E} \to \mathcal{E}'$ is a quasi-isomorphism of locally free sheaves of $\mathcal{O}$-modules such that $\mathrm{gr}(\Phi_\Psi) = \mathrm{id}_{\mathrm{id}}$, then the corresponding cocycles are cohomological.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two locally free sheaves of $\mathcal{O}$-modules on $(M, \mathcal{O})$ such that $\mathrm{gr}\,\mathcal{E} = \mathrm{gr}\,\mathcal{E}' = \mathcal{E}_{\mathrm{gr}}$. Assume that $\Phi_\Psi : \mathcal{E} \to \mathcal{E}'$ is an isomorphism. Then $\mathrm{gr}(\Phi_\Psi) \in H^0(M, \mathcal{QA}ut_0\,\mathrm{gr}\,\mathcal{E})$. Suppose that $\mathcal{E}$ corresponds to $(g_{ij})_{\gamma_{ij}} = \delta_i \circ \delta_j^{-1}$, where $\gamma_{ij} = \sigma_i \circ \sigma_j^{-1}$, and $\mathcal{E}'$ corresponds to $(g'_{ij})_{\gamma'_{ij}} = \delta'_i \circ (\delta'_j)^{-1}$, where $\gamma'_{ij} = \sigma'_i \circ (\sigma'_j)^{-1}$. There exist isomorphisms $(\widetilde{\Phi}_i)_{\widetilde{\Psi}_i} : \mathrm{gr}\,\mathcal{E}|_{U_i} \to \mathrm{gr}\,\mathcal{E}|_{U_i}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{gr}\,\mathcal{E}|_{U_i} & \xrightarrow{(\widetilde{\Phi}_i)_{\widetilde{\Psi}_i}} & \mathrm{gr}\,\mathcal{E}|_{U_i} \\ {\scriptstyle\delta_i}\uparrow & & \uparrow{\scriptstyle\delta'_i} \\ \mathcal{E}|_{U_i} & \xrightarrow{\Phi_\Psi} & \mathcal{E}|_{U_i} \end{array}.$$

Since $\operatorname{gr}\delta_i = \operatorname{gr}\delta'_i$, it follows that $\operatorname{gr}((\widetilde{\Phi}_i)_{\widetilde{\Psi}_i}) = \operatorname{gr}(\Phi_\Psi)$ and hence

$$(\Theta_i)_{\Omega_i} := \operatorname{gr}(\Phi_\Psi)^{-1} \circ (\widetilde{\Phi}_i)_{\widetilde{\Psi}_i} \in \mathcal{Q}\mathcal{A}ut_{(1)(2)} \operatorname{gr}\mathcal{E}.$$

Further, we have

$$(g'_{ij})_{\gamma'_{ij}} = \delta'_i \circ (\delta'_j)^{-1} = (\widetilde{\Phi}_i)_{\widetilde{\Psi}_i} \circ \delta_i \circ (\Phi_\Psi)^{-1} \circ \Phi_\Psi \circ \delta_j^{-1} \circ ((\widetilde{\Phi}_j)_{\widetilde{\Psi}_j})^{-1} =$$
$$(\widetilde{\Phi}_i)_{\widetilde{\Psi}_i} \circ (g_{ij})_{\gamma_{ij}} \circ ((\widetilde{\Phi}_j)_{\widetilde{\Psi}_j})^{-1} = \operatorname{gr}(\Phi_\Psi) \circ (\Theta_i)_{\Omega_i} \circ (g_{ij})_{\gamma_{ij}} \circ (\Theta_j^{-1})_{\Omega_j^{-1}} \circ \operatorname{gr}(\Phi_\Psi)^{-1}.$$

Hence, the cohomology classes corresponding to $(g_{ij})_{\gamma_{ij}}$ and $(g'_{ij})_{\gamma'_{ij}}$ belong to the same orbit of the group $H^0(M, \mathcal{Q}\mathcal{A}ut_0\mathcal{E}_{\mathrm{gr}})$.

Conversely, assume that $b \in H^0(M, \mathcal{Q}\mathcal{A}ut_0\mathcal{E}_{\mathrm{gr}})$ and $(g'_{ij})_{\gamma'_{ij}} = b \circ (g_{ij})_{\gamma_{ij}} \circ b^{-1}$. Then $\delta'_i \circ (\delta'_j)^{-1} = b \circ \delta_i \circ \delta_j^{-1} \circ b^{-1}$ and we can define the isomorphism $\Gamma : \mathcal{E} \to \mathcal{E}'$ by $\Gamma|_{U_i} := (\delta'_i)^{-1} \circ b \circ \delta_i$, where $\mathcal{E}$ and $\mathcal{E}'$ correspond to $(g_{ij})_{\gamma_{ij}}$ and $(g'_{ij})_{\gamma'_{ij}}$ respectively.

Let $a \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})/H^0(M, \mathcal{A}ut_0\mathcal{O}_{\mathrm{gr}})$. By Theorem 1 we may assign to each a the class of isomorphic supermanifolds $[(M, \mathcal{O})]$. From the proof of Theorem 2 it follows that there is a bijection between elements of the set $\widetilde{F}^{-1}(a)$ and classes of isomorphic locally free sheaves on supermanifolds which are contained in $[(M, \mathcal{O})]$.□

2.3 A classification theorem for locally free sheaves on a split supermanifold

Denote by $[\mathcal{E}]_{\mathrm{id}}$ the class of id-isomorphic (i.e., isomorphic) to $\mathcal{E}$ locally free sheaves of $\mathcal{O}$-modules on a split complex supermanifold $(M, \mathcal{O})$.

Theorem 3. *Let $(M, \mathcal{O})$ be a split supermanifold, $\mathcal{S} = \mathcal{S}_{\bar 0} \oplus \mathcal{S}_{\bar 1}$ be a $\mathbb{Z}_2$-graded locally free sheaf of $\mathcal{F}$-modules on M and $\mathcal{E}_{\mathrm{gr}} = \mathcal{O} \otimes_{\mathcal{F}} \mathcal{E}_{\mathrm{red}}$. Then*

$$\{[\mathcal{E}]_{\mathrm{id}} \mid \operatorname{gr}\mathcal{E} = \mathcal{E}_{\mathrm{gr}}\} \stackrel{1:1}{\longleftrightarrow} H^1(M, \mathcal{A}ut^{\mathcal{O}}_{(1)}\mathcal{E}_{\mathrm{gr}})/H^0(M, \mathcal{A}ut^{\mathcal{O}}_0\mathcal{E}_{\mathrm{gr}}).$$

Moreover, the unit $\epsilon \in H^1(M, \mathcal{A}ut^{\mathcal{O}}_{(1)}\mathcal{E}_{\mathrm{gr}})$ is a fixed point with respect to the action of the group $H^0(M, \mathcal{A}ut^{\mathcal{O}}_0\mathcal{E}_{\mathrm{gr}})$.

Proof. Let us use the notations from the proof of Theorem 2. Since $(M, \mathcal{O})$ is split, we may assume that $\sigma_i = \sigma|_{U_i}$, where σ is determined by a global splitting of (1). It follows that the cocycle (g_{ij}) lies in $Z^1(\mathcal{U}, \mathcal{A}ut^{\mathcal{O}}_{(1)}\mathcal{E}_{\mathrm{gr}})$. The further proof is similar to the proof the Theorem 2. □

3. Locally free sheaves of modules on projective superspaces

In this subsection we will discuss two remarkable theorems about locally free sheaves on projective spaces, proved by Barth – Van de Ven – Tyurin and Birkhoff – Grothendieck, in the super-context.

3.1 Exact sequences corresponding to $\mathcal{A}ut^{\mathcal{O}}\mathcal{E}$

Let $(M, \mathcal{O})$ be a split complex supermanifold and $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on M. Denote by $\mathcal{E}nd^{\mathcal{O}}\mathcal{E}$ the sheaf of $\mathcal{O}$-endomorphisms of $\mathcal{E}$. This sheaf possesses the filtration

$$\mathcal{E}nd^{\mathcal{O}}\mathcal{E} = \mathcal{E}nd^{\mathcal{O}}_{(0)}\mathcal{E} \supset \mathcal{E}nd^{\mathcal{O}}_{(1)}\mathcal{E} \supset \dots,$$

$$\mathcal{E}nd^{\mathcal{O}}_{(p)}\mathcal{E} := \{A \in \mathcal{E}nd^{\mathcal{O}}\mathcal{E} \mid A(\mathcal{E}_{(q)}) \subset \mathcal{E}_{(q+p)} \text{ for all } q \geq 0\}.$$

The map

$$\exp : \mathcal{E}nd^{\mathcal{O}}_{(p)}\mathcal{E} \to \mathcal{A}ut^{\mathcal{O}}_{(p)}\mathcal{E},$$

given by the usual exp-series is a bijection of sheaves of sets for all $p \geq 1$ due to the fact that $\log = (\exp)^{-1}$ is well defined. In general it is not a homomorphism of sheaves of groups. We may define the map

$$\lambda_p : \mathcal{A}ut^{\mathcal{O}}_{(p)}\mathcal{E} \to \mathcal{E}nd^{\mathcal{O}}_{(p)}\mathcal{E}/\mathcal{E}nd^{\mathcal{O}}_{(p+1)}\mathcal{E},\ p \geq 1,$$

given by

$$a \mapsto A + \mathcal{E}nd^{\mathcal{O}}_{(p+1)},\ \text{where } a = \exp(A).$$

This map is surjective and $\operatorname{Ker} \lambda_p = \mathcal{A}ut^{\mathcal{O}}_{(p+1)}\mathcal{E}$. Clearly, it is a homomorphism of sheaves of groups. We will also consider the subsheaves of $\mathcal{E}nd^{\mathcal{O}} \operatorname{gr} \mathcal{E}$

$$\mathcal{E}nd^{\mathcal{O}}_{p} \operatorname{gr} \mathcal{E} := \{A \in \mathcal{E}nd^{\mathcal{O}} \operatorname{gr} \mathcal{E} \mid A(\operatorname{gr} \mathcal{E}_q) \subset \operatorname{gr} \mathcal{E}_{p+q}\},\ p \geq 0.$$

Then

$$\mathcal{E}nd^{\mathcal{O}}_{(p)} \operatorname{gr} \mathcal{E} = \bigoplus_{q \geq p} \mathcal{E}nd^{\mathcal{O}}_{q} \operatorname{gr} \mathcal{E}.$$

It follows that

$$\mathcal{E}nd^{\mathcal{O}}_{(p)} \operatorname{gr} \mathcal{E}/\mathcal{E}nd^{\mathcal{O}}_{(p+1)} \operatorname{gr} \mathcal{E} \simeq \mathcal{E}nd^{\mathcal{O}}_{p} \operatorname{gr} \mathcal{E}.$$

Hence, we get the exact sequence

$$0 \to \mathcal{A}ut^{\mathcal{O}}_{(p+1)} \operatorname{gr} \mathcal{E} \to \mathcal{A}ut^{\mathcal{O}}_{(p)} \operatorname{gr} \mathcal{E} \xrightarrow{\lambda_p} \mathcal{E}nd^{\mathcal{O}}_{p} \operatorname{gr} \mathcal{E} \to 0,\ p \geq 1. \tag{9}$$

The following lemma gives a description of the sheaf $\mathcal{E}nd_p \operatorname{gr} \mathcal{E}$, $p \geq 1$, in terms of the sheaves $\mathcal{O}$ and $\mathcal{E}_{\mathrm{red}}$.

Lemma 2. *We have*

$$\mathcal{E}nd^{\mathcal{O}}_{p} \operatorname{gr} \mathcal{E} \simeq \begin{cases} \mathcal{O}_p \otimes ((\mathcal{E}_{\mathrm{red}})_{\bar{0}} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar{1}} \oplus (\mathcal{E}_{\mathrm{red}})_{\bar{1}} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar{0}}), & p \text{ is odd;} \\ \mathcal{O}_p \otimes ((\mathcal{E}_{\mathrm{red}})_{\bar{0}} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar{0}} \oplus (\mathcal{E}_{\mathrm{red}})_{\bar{1}} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar{1}}), & p \text{ is even.} \end{cases}$$

Proof. Firstly, note that an endomorphism $A \in \mathcal{E}nd_p(\operatorname{gr}\mathcal{E})$ is determined by its restriction $A|_{\operatorname{gr}\mathcal{E}_0}$. Secondly, $A|_{\operatorname{gr}\mathcal{E}_0} : \operatorname{gr}\mathcal{E}_0 \to \operatorname{gr}\mathcal{E}_p$ is an $\mathcal{F}$-linear map preserving parity (2). The result follows from the relation $\operatorname{gr}\mathcal{E}_q \simeq \operatorname{gr}\mathcal{O}_q \otimes \mathcal{E}_{\mathrm{red}}$. □

Now we can recover the following well-known result, see [9, 14]:

Proposition 1. *Let $(M, \mathcal{O})$ be a smooth supermanifold and $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on M. Then $\mathcal{E} \simeq \mathcal{O} \otimes_{\mathcal{F}} \mathcal{E}_{\mathrm{red}}$.*

Proof. Indeed, $(M, \mathcal{O})$ is split by the Batchelor Theorem. In this case

$$H^1(M, \mathcal{E}nd_p^{\mathcal{O}} \operatorname{gr}\mathcal{E}) = \{0\}$$

by Lemma 2. Hence

$$H^1(M, \mathcal{A}ut_{(1)}^{\mathcal{O}} \operatorname{gr}\mathcal{E}) = \{\epsilon\},$$

and our assertion follows from the Theorem 3.□

3.2 The Barth – Van de Ven – Tyurin Theorem for supermanifolds

Let us briefly recall the classical Barth – Van de Ven – Tyurin Theorem. Consider the sequence of complex projective spaces

$$\mathbb{CP}^1 \xrightarrow{\varphi_1} \mathbb{CP}^2 \xrightarrow{\varphi_2} \ldots,$$

where φ_i are standard embeddings. (The inductive limit of this sequence is also called the *complex projective ind-space* $\mathbb{CP}^\infty$ (see [5, 17] and more detailed [7].) We consider collections $E = \{E_N\}_{N\geq 1}$ of holomorphic vector bundles E_N of a finite rank over $\mathbb{CP}^N$, $N \geq 1$, such that $\widetilde{\varphi}_N(E_{N+1}) = E_N$. (Such collections are also called *vector bundles over* $\mathbb{CP}^\infty$.) If $E = \{E_N\}_{N\geq 1}$ and $E' = \{E'_N\}_{N\geq 1}$ are two such collections, then the collection $E \oplus E' := \{E_N \oplus E'_N\}_{N\geq 1}$ is called the *direct sum* of E and E'. A *morphism of collections* $f : E \to E'$ is a set $\{f_N : E_N \to E'_N\}_{N\geq 1}$ of morphisms of vector bundles such that $\widetilde{\varphi}_N \circ f_{N+1} = f_N \circ \widetilde{\varphi}_N$. A morphism of two collections $f : E \to E'$ is called an *isomorphism* if it possesses the inverse morphism.

Theorem 4. **[Barth – Van de Ven – Tyurin]** *Any collection $E = \{E_N\}_{N\geq 1}$ of holomorphic vector bundles E_N of a finite rank over $\mathbb{CP}^N$ is isomorphic to a direct sum of collections $E^i = \{E^i_N\}_{N\geq 1}$ of vector bundles E^i_N of rank* 1.

For collections of rank 2 this result was proved by W. Barth and A. Van de Ven in [1], and for collections of an arbitrary finite rank by A. Tyuirin in [17].

The similar question may be considered in the case of complex supermanifolds. Recall that the *projective superspace* $(M, \mathcal{O}) = \mathbb{CP}^{n|m}$ of dimension

$n|m$ is a complex supermanifold with the reduction $M = \mathbb{CP}^n$ and the structure sheaf $\mathcal{O} = \bigwedge \mathcal{L}(-1)^m$, where $\mathcal{L}(-1)$ is the sheaf of $\mathcal{F}$-modules inverse to the sheaf $\mathcal{L}(1)$, which corresponds to a hyperplane in $\mathbb{CP}^n$. The classical homogeneous coordinates $z_0, ..., z_n$ on $\mathbb{CP}^n$ can be supplemented by odd homogeneous coordinates $\zeta_1, ..., \zeta_m$, giving rise to the system of homogeneous coordinates on $\mathbb{CP}^{n|m}$.

Let us consider the sequence of projective superspaces

$$\mathbb{CP}^{1|k_1} \xrightarrow{\varphi_1} \mathbb{CP}^{2|k_2} \xrightarrow{\varphi_2} \dots,$$

where $k_i \leq k_{i+1}$ and φ_i are standard embeddings, i.e any map $\varphi_i : \mathbb{CP}^{i|k_i} \to \mathbb{CP}^{i+1|k_{i+1}}$ is given in homogeneous coordinates (z_j, ζ_r) and (z'_s, ζ'_t) on $\mathbb{CP}^{i|k_i}$ and $\mathbb{CP}^{i+1|k_{i+1}}$ respectively by

$$z'_s = z_s,\ s = 1, \cdots, i,\ z_{i+1} = 0;$$
$$\zeta'_t = \zeta_t,\ t = 1, \cdots, k_i,\ \zeta'_t = 0,\ t = k_i + 1, \cdots, k_{i+1}.$$

We study collections $\mathcal{E} = \{\mathcal{E}_n\}_{n\geq 1}$ of locally free sheaves $\mathcal{E}_n$ of a finite rank over $\mathbb{CP}^{n|k_n}$, $n \geq 1$, such that $\widetilde{\varphi}_n(\mathcal{E}_{n+1}) = \mathcal{E}_n$. A morphism of two collections and their direct sum are defined similarly to the classical case. We are going to prove the following theorem:

Theorem 5. *Any collection $\mathcal{E} = \{\mathcal{E}_n\}_{n\geq 1}$ of locally free sheaves $\mathcal{E}_n$ of a finite rank over $\mathbb{CP}^{n|k_n}$ is isomorphic to a direct sum of collections $\mathcal{E}^i = \{\mathcal{E}^i_n\}_{n\geq 1}$ of locally free sheaves $\mathcal{E}^i_n$ of rank $1|0$ or $0|1$.*

Proof. Note that $\mathcal{E}_{\mathrm{red}} = \{(\mathcal{E}_n)_{\mathrm{red}}\}$ is the collection of locally free sheaves such that $\widetilde{(\varphi_i)}_{\mathrm{red}}((\mathcal{E}_{i+1})_{\mathrm{red}}) = (\mathcal{E}_i)_{\mathrm{red}}$ and $(\varphi_i)_{\mathrm{red}} : \mathbb{CP}^i \to \mathbb{CP}^{i+1}$ are standard embeddings. By Theorem 4 we have $\mathcal{E}_{\mathrm{red}} \simeq \bigoplus_r \mathcal{S}^r$, where $\mathcal{S}^r = \{\mathcal{S}^r_n\}$ is a collection of locally free sheaves of rang 1 (and of super-rank $1|0$ or $0|1$). Hence the collection $\operatorname{gr}\mathcal{E} = \{\operatorname{gr}\mathcal{E}_n\}$, where we identify $\operatorname{gr}\mathcal{E}_n = \mathcal{O}_{\mathbb{CP}^n} \otimes (\mathcal{E}_n)_{\mathrm{red}}$, is isomorphic to the collection $\{\mathcal{O}_{\mathbb{CP}^n} \otimes \bigoplus_r \mathcal{S}^r_n\}$.

Our aim is to show that $\mathcal{E} \simeq \operatorname{gr}\mathcal{E}$. Using Lemma 2 and the well-known fact: $H^1(\mathbb{CP}^n, \mathcal{L}(r)) = \{0\}$ for $n > 1$ and any $r \in \mathbb{Z}$, we conclude that $H^1(\mathbb{CP}^n, \mathcal{E}nd^{\mathcal{O}}_p(\operatorname{gr}\mathcal{E}_n)) = \{0\}$ for $p \geq 1$ and $n > 1$. Hence, by the sequence (9) we get

$$H^1(\mathbb{CP}^n, \mathcal{A}ut^{\mathcal{O}}_{(1)}(\operatorname{gr}\mathcal{E}_n)) = \{\epsilon\} \text{ for } n > 1.$$

It follows by Theorem 3 that the following isomorphisms

$$f_n : \mathcal{E}_n \xrightarrow{\sim} \operatorname{gr}\mathcal{E}_n = \sum_r \mathcal{O}_{\mathbb{CP}^n} \otimes \mathcal{S}^r_n.$$

exist. Let us show that we can choose the isomorphisms f_n such that they commute with pullbacks of the bundles. Fix an isomorphism f_n. Let us construct an isomorphism

$$f'_{n+1} : \mathcal{E}_{n+1} \xrightarrow{\sim} \mathcal{O}_{\mathbb{CP}^{n+1}} \otimes (\mathcal{E}_{n+1})_{\mathrm{red}}$$

such that $\widetilde{\varphi}_n \circ f'_{n+1} = f_n \circ \widetilde{\varphi}_n$. Denote by $\mathcal{I}_n$ the sheaf of ideals corresponding to the subsupermanifold $\varphi_n : \mathbb{CP}^{n|k_n} \to \mathbb{CP}^{n+1|k_{n+1}}$. By definition we have

$$\mathcal{E}_n = \widetilde{\varphi}_n(\mathcal{E}_{n+1}) = \varphi^*_{\mathrm{red}}(\mathcal{E}_{n+1}/\mathcal{I}_n\mathcal{E}_{n+1}),$$
$$\mathrm{gr}\,\mathcal{E}_n = \widetilde{\varphi}_n(\mathrm{gr}\,\mathcal{E}_{n+1}) = \varphi^*_{\mathrm{red}}(\mathrm{gr}\,\mathcal{E}_{n+1}/\mathcal{I}_n\,\mathrm{gr}\,\mathcal{E}_{n+1}).$$

Denote by $\mathcal{B}_n$ the sheaf of automorphisms of the sheaf of $\mathcal{O}_{\mathbb{CP}^{n+1}}/\mathcal{I}_n\mathcal{O}_{\mathbb{CP}^{n+1}}$-modules $\mathrm{gr}\,\mathcal{E}_{n+1}/\mathcal{I}_n\,\mathrm{gr}\,\mathcal{E}_{n+1}$ and by $(\mathcal{B}_n)_{(1)}$ the subsheaf of $\mathcal{B}_n$:

$$(\mathcal{B}_n)_{(1)} := \{a \in \mathcal{B}_n \mid a(v) = v \bmod (\mathrm{gr}\,\mathcal{E}_{n+1}/\mathcal{I}_n\,\mathrm{gr}\,\mathcal{E}_{n+1})_{(1)}\},$$

where $(\mathrm{gr}\,\mathcal{E}_{n+1}/\mathcal{I}_n\,\mathrm{gr}\,\mathcal{E}_{n+1})_{(1)}$ is the image of $(\mathrm{gr}\,\mathcal{E}_{n+1})_{(1)}$ by the natural homomorphism. Note that we have $\sup((\mathcal{B}_n)_{(1)}) = \varphi_{\mathrm{red}}(\mathbb{CP}^n)$ and $\varphi^*_{\mathrm{red}}((\mathcal{B}_n)_{(1)}) = \mathcal{A}ut^{\mathcal{O}_{\mathbb{CP}^n}}_{(1)}(\mathrm{gr}\,\mathcal{E}_n)$.

Further, any automorphism from $\mathcal{A}ut^{\mathcal{O}_{\mathbb{CP}^n}}_{(1)}(\mathrm{gr}\,\mathcal{E}_{n+1})$ preserves $\mathcal{I}_n\,\mathrm{gr}\,\mathcal{E}_{n+1}$. Hence, we have the map

$$F_n : \mathcal{A}ut^{\mathcal{O}_{\mathbb{CP}^n}}_{(1)}(\mathrm{gr}\,\mathcal{E}_{n+1}) \to (\mathcal{B}_n)_{(1)},$$

which is surjective as a sheaf homomorphism because we always can find locally preimage of elements from $(\mathcal{B}_n)_{(1)}$. Denote by $\mathcal{A}_n$ the kernel of F_n. Let us choose a Stein cover $\mathcal{U} = \{U_i\}$ of $\mathbb{CP}^{n+1}$ such that

$$0 \to \mathcal{A}_n(U_i) \to \mathcal{A}ut^{\mathcal{O}_{\mathbb{CP}^{n+1}}}_{(1)}(\mathrm{gr}\,\mathcal{E}_{n+1})(U_i) \to (\mathcal{B}_n)_{(1)}(U_i) \to 0.$$

is exact for any i. Assume also that $\mathcal{U}$ satisfies conditions of the proof of Theorem 2. Denote by

$$(g^n_{ij}) \in H^1(\mathcal{U}, (\mathcal{B}_n)_{(1)}) \text{ and } (g^{n+1}_{ij}) \in H^1(\mathcal{U}, \mathcal{A}ut^{\mathcal{O}_{\mathbb{CP}^n}}_{(1)}(\mathrm{gr}\,\mathcal{E}_{n+1}))$$

the cocycles corresponding to $\mathcal{E}_n$ and $\mathcal{E}_{n+1}$ by Theorem 3. Recall that $g^n_{ij} = \delta^n_i \circ (\delta^n_j)^{-1}$, where $\delta^n_i : \mathcal{E}_n|_{U_i} \to \mathrm{gr}\,\mathcal{E}_n|_{U_i}$ is the isomorphism from Theorem 2 assuming is addition $\sigma_i = \mathrm{id}$ for any i. Similarly, $g^{n+1}_{ij} = \delta^{n+1}_i \circ (\delta^{n+1}_j)^{-1}$. Since $\widetilde{\varphi}(\mathcal{E}_{n+1}) = \mathcal{E}_n$, we may assume that $\widetilde{\varphi}^n \circ \delta^{n+1}_i|_{U_i} = \delta^n_i \circ \widetilde{\varphi}^n|_{U_i}$. Therefore, $F_n(g^{n+1}_{ij}) = g^n_{ij}$.

We have shown that $(g^n_{ij}) \sim \epsilon$ hence there are $\alpha^n_i \in \mathcal{B}_{(1)}(U_i)$ such that $(\alpha^n_i)^{-1} \circ g^n_{ij} \circ \alpha^n_j = \mathrm{id}$. Using the surjectivity of $F_n|_{U_i}$, we may choose $\alpha^{n+1}_i \in$

$F_n^{-1}(\alpha_i^n)$. Then $(h_{ij}) \in H^1(\mathcal{U}, \mathcal{A}_n)$, where $h_{ij} = (\alpha_i^{n+1})^{-1} \circ g_{ij}^{n+1} \circ \alpha_j^{n+1}$. It is easy to see that

$$\begin{aligned}\mathcal{A}_n = \exp(\quad & (\mathcal{I}_n)_{\bar 0} \otimes ((\mathcal{E}_{\mathrm{red}})_{\bar 0} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar 1} \oplus (\mathcal{E}_{\mathrm{red}})_{\bar 1} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar 0}) \oplus \\ & (\mathcal{I}_n)_{\bar 1} \otimes ((\mathcal{E}_{\mathrm{red}})_{\bar 0} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar 0} \oplus (\mathcal{E}_{\mathrm{red}})_{\bar 1} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar 1}).\end{aligned}$$

Therefore, we get as for $\mathcal{A}ut_{(1)}^{\mathcal{O}_{\mathbb{CP}^n}}(\operatorname{gr}\mathcal{E}_n)$ that $H^1(\mathbb{CP}^{n+1}, \mathcal{A}_n) = \{\epsilon\}$. Therefore, there are $\beta_i \in \mathcal{A}_n(U_i)$ such that $h_{ij} = \beta_i \circ \beta_j^{-1}$. Denote

$$f'_{n+1}|_{U_i} := \beta_i^{-1} \circ (\alpha_i^{n+1})^{-1} \circ \delta_i^{n+1}.$$

By construction, we have $\widetilde{\varphi}_n \circ f'_{n+1} = f_n \circ \widetilde{\varphi}_n$. The proof is complete.□

3.3 About the Birkhoff – Grothendieck Theorem for supermanifolds.

In this subsection we will show that the Birkhoff – Grothendieck Theorem: *Any finite rank vector bundle on the complex projective space* $\mathbb{CP}^1$ *is isomorphic to a direct sum of line bundles*,

does not hold true for the projective superspace $\mathbb{CP}^{1|n}$, where $n \geq 1$. Denote by $\mathcal{O}_n$ the structure sheaf of $\mathbb{CP}^{1|n}$ and by i_n the standard embedding $\mathbb{CP}^{1|1} \to \mathbb{CP}^{1|n}$, $n \geq 1$. Clearly, there is a map $j_n : \mathbb{CP}^{1|n} \to \mathbb{CP}^{1|1}$, $n \geq 1$, such that $j_n^* : \mathcal{O}_1 \to \mathcal{O}_n$ is injective and $j_n \circ i_n = \mathrm{id}$. Let $\mathcal{E}_1$ be a locally free sheaf of $\mathcal{O}_1$-modules. Denote

$$\mathcal{E}_n := \mathcal{O}_n \otimes_{j_n^*(\mathcal{O}_1)} \mathcal{E}_1.$$

Then $\mathcal{E}_n$ is also locally free and $\mathcal{E}_n$ is an extension of $\mathcal{E}_1$. In other words, we have proved that any locally free sheaf on $\mathbb{CP}^{1|1}$ admits an extension to $\mathbb{CP}^{1|n}$. It follows that to prove our assertion it is enough to show that there exists a locally free sheaf of $\mathcal{O}_1$-modules of rank ≥ 2, which is not a direct sum of two lines bundles.

Let us study firstly line bundles on $\mathbb{CP}^{1|1}$. By (9) we get that $\mathcal{A}ut_{(1)}^{\mathcal{O}_1}\operatorname{gr}\mathcal{E} \simeq \mathcal{E}nd_1^{\mathcal{O}}\operatorname{gr}\mathcal{E}$ for any rank and from Lemma 2 it follows that $\mathcal{E}nd_1^{\mathcal{O}}\operatorname{gr}\mathcal{E} = \{0\}$ if $\operatorname{rank}\operatorname{gr}\mathcal{E} = 1|0$ or $0|1$. Therefore, by Theorem 3 any line bundle $\mathcal{E}$ is isomorphic to $\operatorname{gr}\mathcal{E}$.

Further, let $(\mathcal{E}_{\mathrm{red}})_{\bar 0} = \mathcal{L}(0)$, $(\mathcal{E}_{\mathrm{red}})_{\bar 1} = \mathcal{L}(-1)$ and $\mathcal{E}_{\mathrm{gr}} = \mathcal{O}_1 \otimes ((\mathcal{E}_{\mathrm{red}})_{\bar 0} \oplus (\mathcal{E}_{\mathrm{red}})_{\bar 1})$. Then

$$H^1(\mathbb{CP}^1, \mathcal{E}nd_1^{\mathcal{O}}\mathcal{E}_{\mathrm{gr}}) \simeq H^1(\mathbb{CP}^1, \mathcal{L}(-2)) \simeq \mathbb{C}.$$

Using the fact that the unit 1-cohomology class is a fixed point for the action of $H^0(\mathbb{CP}^1, \mathcal{A}ut_0^{\mathcal{O}_1}\mathcal{E}_{\mathrm{gr}})$ on $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(1)}^{\mathcal{O}_1}\mathcal{E}_{\mathrm{gr}})$, we see that there is a locally free sheaf of $\mathcal{O}_1$-modules $\mathcal{E}$ such that $\operatorname{gr}\mathcal{E} = \mathcal{E}_{\mathrm{gr}}$ but $\mathcal{E}$ is not isomorphic to $\mathcal{E}_{\mathrm{gr}}$.

4. The tangent sheaf of a split supermanifold.

Let us recall some well-known facts about the tangent sheaf $\mathcal{T}$ of a split supermanifold $(M, \mathcal{O}) \simeq (M, \bigwedge \mathcal{G})$. First, the sheaf $\mathcal{T}$ is $\mathbb{Z}$-graded (not only $\mathbb{Z}_2$-graded):

$$\mathcal{T} = \bigoplus_{p \geq -1} \mathcal{T}_p,$$

where

$$\mathcal{T}_p := \{v \in \mathcal{T} \mid v(\mathcal{O}_q) \subset \mathcal{O}_{p+q} \text{ for all } q \geq 0\},\ p \geq -1.$$

Second, the following sequence

$$0 \to \bigwedge^{p+1} \mathcal{G} \otimes \mathcal{G}^* \xrightarrow{\delta} \mathcal{T}_p \xrightarrow{\gamma} \bigwedge^{p} \mathcal{G} \otimes \Theta \to 0,\ p \geq -1, \tag{10}$$

where Θ is the tangent sheaf of M, is exact (see [12]). The mapping γ is the restriction of a derivation of degree p onto the subsheaf $\mathcal{F} \subset \mathcal{O}$ and δ identifies any sheaf homomorphism $\mathcal{G} \to \bigwedge^{p+1} \mathcal{G}$ with a derivation of degree p that is zero on $\mathcal{F}$.

Denote by $\mathbb{G}$ the vector bundle corresponding to $\mathcal{G}$. As usual by a *(holomorphic) connection* in a vector bundle $\mathbb{G} \to M$ over a complex manifold M, we mean a bilinear map

$$\nabla : \Theta \times \mathcal{G} \to \mathcal{G}$$

satisfying the following conditions:

- $\nabla_{fX} s = f \nabla_X s$,
- $\nabla_X (fs) = f \nabla_X s + X(f) s$,

where $f \in \mathcal{F}$, $X \in \Theta$ and $s \in \mathcal{G}$. If ∇ and ∇' are connections in $\mathbb{G} \to M$ and $\mathbb{G}' \to M$ respectively, the *tensor product connection* $\nabla \otimes \nabla'$ in $\mathbb{G} \otimes \mathbb{G}'$ is well defined. Recall that

$$(\nabla \otimes \nabla'_X)(s \otimes s') = \nabla_X(s) \otimes s' + s \otimes \nabla'_X(s').$$

It is easy to see that the tensor product connection $\nabla \otimes \cdots \otimes \nabla$ in $\mathbb{G} \otimes \cdots \otimes \mathbb{G}$ (p-times) induces the *wedge product connection* $\wedge^p \nabla$ in $\bigwedge^p \mathbb{G}$, $p > 0$.

Let ∇ be a connection on $\mathbb{G}$. Then to each $X \in \Theta$ we may assign a vector field Y_X on $(M, \mathcal{O}) \simeq (M, \bigwedge \mathcal{G})$ of degree 0 defined by

$$Y_X(f) = X(f),\ f \in \mathcal{F}, \quad Y_X(f) = \wedge^p \nabla(f),\ f \in \bigwedge^p \mathcal{G},$$

The Leibniz rule for Y_X follows from the definitions of a connection and a wedge product connection. Consider the sequence (10) for $p = 0$

$$0 \to \mathcal{G} \otimes \mathcal{G}^* \xrightarrow{\delta} \mathcal{T}_0 \xrightarrow{\gamma} \Theta \to 0. \tag{11}$$

We have just shown that the connection ∇ defines the splitting of (11) by $X \mapsto Y_X$. The converse statement is also true: if we have a splitting i of (11), we may define the connection ∇_i by

$$(\nabla_i)_X(s) := i(X)(s),\ \ s \in \mathcal{G}.$$

Note that the curvature tensor of $\nabla = \nabla_i$

$$R(X,Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]} = ([i(X), i(Y)] - i([X,Y]))|_{\mathcal{G}}$$

measures the defection of i to be a homomorphism of sheaves of Lie algebras.

Theorem 6. *Let $(M, \mathcal{O}_M) \simeq (M, \bigwedge \mathcal{G})$ be a (holomorphic) split supermanifold and $\mathcal{T}$ the tangent sheaf. The following conditions are equivalent:*

1. *the sheaf $\mathcal{T}$ corresponds to the unit 1-cohomology class with values in $\mathcal{A}ut^{\mathcal{O}}_{(1)} \operatorname{gr} \mathcal{T}$ by the Theorem 3;*
2. *the sequence (11) splits;*
3. *$\mathcal{G}$ possesses a (holomorphic) connection.*

Proof. By the discussion above we have to prove only that $\mathcal{T}$ corresponds to the trivial 1-cocycles of $H^1(M, \mathcal{A}ut^{\mathcal{O}}_{(1)} \operatorname{gr} \mathcal{T})$ if and only if the sequence (11) splits. Let $\theta_0 : \Theta \to \mathcal{T}_0$ be a splitting of (11). Then the sequence (10) splits for all $p \geq 0$, we may define the splitting $\theta_p : \bigwedge^p \mathcal{G} \otimes \Theta \to \mathcal{T}_p$ by $\theta_p(f \otimes v) = f\theta_0(v)$. It follows that

$$\mathcal{T}_p \simeq \bigwedge^{p} \mathcal{G} \otimes \Theta + \bigwedge^{p+1} \mathcal{G} \otimes \mathcal{G}^*.$$

Hence,

$$\mathcal{T} \simeq \bigwedge \mathcal{G} \otimes (\mathcal{G}^* + \Theta) \simeq \bigwedge \mathcal{G} \otimes (\mathcal{T}_{\mathrm{red}}) = \operatorname{gr} \mathcal{T}.$$

Conversely, since the unit cocycle of $H^1(M, \mathcal{A}ut^{\mathcal{O}}_{(1)} \operatorname{gr} \mathcal{T})$ is a fixed point with respect to the action of $H^0(M, \mathcal{A}ut^{\mathcal{O}}_0 \operatorname{gr} \mathcal{T})$, there is an isomorphism

$\Phi : \mathcal{T} \to \operatorname{gr} \mathcal{T}$ such that $\operatorname{gr} \Phi = \mathrm{id}$ (see proof of Theorem 2). It follows that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{T}_{\bar 0} & \xrightarrow{\Phi|_{\mathcal{T}_{\bar 0}}} & (\operatorname{gr} \mathcal{T})_{\bar 0} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathcal{T}_{\bar 0}/(\mathcal{J}\mathcal{T})_{\bar 0} & = & \mathcal{T}_{\bar 0}/(\mathcal{J}\mathcal{T})_{\bar 0} \end{array},$$

where pr is the projection of

$$\operatorname{gr} \mathcal{T} = \bigoplus_{p \geq 0} (\mathcal{J}^p \mathcal{T})_{\bar 0}/(\mathcal{J}^{p+1}\mathcal{T})_{\bar 0} + \bigoplus_{p \geq 0} (\mathcal{J}^p \mathcal{T})_{\bar 1}/(\mathcal{J}^{p+1}\mathcal{T})_{\bar 1}$$

onto $\mathcal{T}_{\bar 0}/(\mathcal{J}\mathcal{T})_{\bar 0}$ and π is the natural projection. Further, by the definitions of all morphisms the following diagram is also commutative

$$\begin{array}{ccc} \mathcal{T}_{\bar 0} & \xrightarrow{\pi} & \mathcal{T}_{\bar 0}/(\mathcal{J}\mathcal{T})_{\bar 0} \\ {\scriptstyle \mathrm{pr}_{\mathcal{T}_0}}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathcal{T}_0 & \xrightarrow{\gamma} & \Theta \end{array},$$

where τ is an isomorphism defined by $v + (\mathcal{J}\mathcal{T})_{\bar 0} \mapsto \mathrm{pr}_{\mathcal{F}} \circ v|_{\mathcal{F}}$. Denote by i the natural embedding $\mathcal{T}_{\bar 0}/(\mathcal{J}\mathcal{T})_{\bar 0} \hookrightarrow (\operatorname{gr} \mathcal{T})_{\bar 0}$. We may define a splitting of (11) by $\mathrm{pr}_{\mathcal{T}_0} \circ (\Phi|_{\mathcal{T}_{\bar 0}})^{-1} \circ i \circ \tau^{-1}$. The proof is complete.□

5. A spectral sequence

An important problem is to calculate the cohomology group $H^*(M, \mathcal{E})$ of a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ on a supermanifold $(M, \mathcal{O})$. If $(M, \mathcal{O})$ is split, then $\mathcal{E}$ is a locally free sheaf of $\mathcal{F}$-modules on M, and its cohomology group can be calculated in many cases using the well elaborated tools of complex analytic geometry. In non-split case these methods cannot be applied directly, but we can use the associated split supermanifold $(M, \operatorname{gr} \mathcal{O})$ and the sheaf $\operatorname{gr} \mathcal{E}$.

5.1 Quasi-derivations.

Let $(M, \mathcal{O})$ be an arbitrary supermanifold and $\mathcal{E}$ a locally free sheaf on $(M, \mathcal{O})$. Let us take an even vector field $\Gamma \in \mathcal{T}_{\bar 0}(U)$ on a superdomain $(U, \mathcal{O}|U) \subset (M, \mathcal{O})$. A $\mathbb{Z}_2$-graded vector spaces sheaf homomorphism $A_\Gamma : \mathcal{E}|U \to \mathcal{E}|U$ is called a Γ-*derivation* if $A_\Gamma(fv) = \Gamma(f)v + fA_\Gamma(v)$, $f \in \mathcal{O}|U$ and $v \in \mathcal{E}|U$. A homomorphism of $\mathbb{Z}_2$-graded sheaf of vector spaces $B : \mathcal{E} \to \mathcal{E}$ will be called a *quasi-derivation* if it is a Γ-derivation for a certain

Γ. Denote by $\mathcal{QD}er\mathcal{E}$ the sheaf of quasi-derivations. It is a sheaf of Lie algebras with respect to the commutator $[A_\Gamma, B_\Upsilon] := A_\Gamma \circ B_\Upsilon - B_\Upsilon \circ A_\Gamma$. The sheaf $\mathcal{QD}er\mathcal{E}$ possesses the double filtration:

$$\mathcal{QD}er_{(p)(q)}\mathcal{E} := \{A_\Gamma \in \mathcal{QD}er\mathcal{E} \mid \begin{array}{l} A_\Gamma(\mathcal{E}_{(r)}) \subset \mathcal{E}_{(r+p)},\ \Gamma(\mathcal{J}^s) \subset \mathcal{J}^{s+q} \\ \text{for all } r, s \in \mathbb{Z}\}. \end{array}$$

The map

$$\exp : \mathcal{QD}er_{(1)(2)}\mathcal{E} \to \mathcal{QA}ut_{(1)(2)}\mathcal{E}$$

is an isomorphism of sheaves of sets. Let us consider the subsheaf $\mathcal{QD}er_{k,k}\operatorname{gr}\mathcal{E}$ of $\mathcal{QD}er_{(k)(k)}\operatorname{gr}\mathcal{E}$ defined by

$$\begin{array}{rl} \mathcal{QD}er_{k,k}\operatorname{gr}\mathcal{E} := & \{A_\Gamma \in \mathcal{QD}er_{(k)(k)}\operatorname{gr}\mathcal{E} \mid A_\Gamma(\operatorname{gr}\mathcal{E}_r) \subset \operatorname{gr}\mathcal{E}_{r+k}, \\ & \Gamma(\operatorname{gr}\mathcal{O}_s) \subset \operatorname{gr}\mathcal{O}_{s+k} \text{ for all } r, s \in \mathbb{Z}\}. \end{array}$$

Note that $\mathcal{QD}er_{k,k}\operatorname{gr}\mathcal{E} = \mathcal{E}nd_k^{\operatorname{gr}\mathcal{O}}\operatorname{gr}\mathcal{E}$ if k is odd.

Denote by μ_k, $k \geq 1$, the following mapping:

$$\mu_k : \mathcal{QA}ut_{(k)(2)}\operatorname{gr}\mathcal{E} \to \mathcal{QD}er_{k,k}\operatorname{gr}\mathcal{E},$$

$$\mu_k(a_\gamma) = \bigoplus_q \operatorname{pr}_{q+k} \circ A_\Gamma \circ \operatorname{pr}_q,$$

where $a_\gamma = \exp(A_\Gamma)$ and $\operatorname{pr}_k : \operatorname{gr}\mathcal{E} \to \operatorname{gr}\mathcal{E}_k$ is the natural projection. The kernel of this map is $\mathcal{QA}ut_{(k+1)(2)}\operatorname{gr}\mathcal{E}$. Moreover, the following sequence

$$0 \to \mathcal{QA}ut_{(k+1)(2)}\operatorname{gr}\mathcal{E} \longrightarrow \mathcal{QA}ut_{(k)(2)}\operatorname{gr}\mathcal{E} \xrightarrow{\mu_k} \mathcal{QD}er_{k,k}\operatorname{gr}\mathcal{E} \to 0$$

is exact. Denoting by $H_{(k)}(\operatorname{gr}\mathcal{E})$ the image of the natural mapping

$$H^1(M, \mathcal{QA}ut_{(k)(2)}\operatorname{gr}\mathcal{E}) \to H^1(\mathcal{QA}ut_{(1)(2)}\operatorname{gr}\mathcal{E}),$$

we get the filtration:

$$H^1(M, \mathcal{QA}ut_{(1)(2)}\operatorname{gr}\mathcal{E}) = H_{(1)}(\operatorname{gr}\mathcal{E}) \supset H_{(2)}(\operatorname{gr}\mathcal{E}) \supset \ldots.$$

Take $a_\gamma \in H_{(1)}(\operatorname{gr}\mathcal{E})$. We define the order of a_γ the maximal one of the numbers k such that $a_\gamma \in H_{(k)}(\operatorname{gr}\mathcal{E})$. The *order* of a locally free sheaf $\mathcal{E}$ of $\mathcal{O}$-modules on a supermanifold $(M, \mathcal{O}_M)$ is by definition the order of the corresponding cohomology class.

5.2 The spectral sequence.

Let $\mathcal{E}$ be a vector superbundle on a supermanifold $(M, \mathcal{O})$ of dimension $n|m$. Now we will construct a spectral sequence for the cohomology of the sheaf $\mathcal{E}$. We fix an open Stein cover $\mathfrak{U} = (U_i)_{i\in I}$ of M and consider the corresponding Čech cochain complex $C^*(\mathfrak{U}, \mathcal{E}) = \bigoplus_{p\geq 0} C^p(\mathfrak{U}, \mathcal{E})$.

The $\mathbb{Z}_2$-grading of $\mathcal{E}$ gives rise to the $\mathbb{Z}_2$-gradings in $C^*(\mathfrak{U}, \mathcal{E})$ and $H^*(M, \mathcal{E})$ given by

$$\begin{aligned} C_{\bar 0}(\mathfrak{U}, \mathcal{E}) &= \bigoplus_{q\geq 0} C^{2q}(\mathfrak{U}, \mathcal{E}_{\bar 0}) \oplus \bigoplus_{q\geq 0} C^{2q+1}(\mathfrak{U}, \mathcal{E}_{\bar 1}), \\ C_{\bar 1}(\mathfrak{U}, \mathcal{E}) &= \bigoplus_{q\geq 0} C^{2q}(\mathfrak{U}, \mathcal{E}_{\bar 1}) \oplus \bigoplus_{q\geq 0} C^{2q+1}(\mathfrak{U}, \mathcal{E}_{\bar 0}). \\ H_{\bar 0}(M, \mathcal{E}) &= \bigoplus_{q\geq 0} H^{2q}(M, \mathcal{E}_{\bar 0}) \oplus \bigoplus_{q\geq 0} H^{2q+1}(M, \mathcal{E}_{\bar 1}), \\ H_{\bar 1}(M, \mathcal{E}) &= \bigoplus_{q\geq 0} H^{2q}(M, \mathcal{E}_{\bar 1}) \oplus \bigoplus_{q\geq 0} H^{2q+1}(M, \mathcal{E}_{\bar 0}). \end{aligned} \tag{12}$$

The filtration (4) for $\mathcal{E}$ gives rise to the filtration

$$C^*(\mathfrak{U}, \mathcal{E}) = C_{(0)} \supset \ldots \supset C_{(p)} \supset \ldots \supset C_{(m+1)} = 0 \tag{13}$$

of this complex by the subcomplexes

$$C_{(p)} = C^*(\mathfrak{U}, \mathcal{E}_{(p)}).$$

Denoting by $H(M, \mathcal{E})_{(p)}$ the image of the natural mapping $H^*(M, \mathcal{E}_{(p)}) \to H^*(M, \mathcal{E})$, we get the filtration

$$H^*(M, \mathcal{E}) = H(M, \mathcal{E})_{(0)} \supset \ldots \supset H(M, \mathcal{E})_{(p)} \supset \ldots . \tag{14}$$

Denote by $\operatorname{gr} H^*(M, \mathcal{E})$ the bigraded group associated with the filtration (14); its bigrading is given by

$$\operatorname{gr} H^*(M, \mathcal{E}) = \bigoplus_{p,q\geq 0} \operatorname{gr}_p H^q(M, \mathcal{E}).$$

By the general procedure, invented by Leray, the filtration (13) gives rise to a spectral sequence of bigraded groups E_r converging to $E_\infty \simeq \operatorname{gr} H^*(M, \mathcal{E})$. It is constructed in the following way.

For any $p, r \geq 0$, define the vector spaces

$$C^p_r = \{c \in C_{(p)} \mid dc \in C_{(p+r)}\}.$$

Then, for a fixed p, we have

$$C_{(p)} = C^p_0 \supset \ldots \supset C^p_r \supset C^p_{r+1} \supset \ldots .$$

The r-th term of the spectral sequence is defined by

$$E_r = \bigoplus_{p=0}^{m} E_r^p, \ r \geq 0,$$

where

$$E_r^p = C_r^p / C_{r-1}^{p+1} + dC_{r-1}^{p-r+1}.$$

Since $d(C_r^p) \subset C_r^{p+r}$, d induces a derivation d_r of E_r of degree r such that $d_r^2 = 0$. Then E_{r+1} is naturally isomorphic to the homology algebra $H(E_r, d_r)$. Denoting $Z_r = \operatorname{Ker} d_r$, we have the natural mapping $\kappa_{r+1}^r : Z_r \to E^{r+1}$. For any $s > r$, denote $\kappa_s^r = \kappa_s^{s-1} \circ \ldots \circ \kappa_{r+1}^r$ (this composition is not defined on the entire Z_r).

The $\mathbb{Z}_2$-grading (12) in $C^*(\mathfrak{U}, \mathcal{E})$ gives rise to certain $\mathbb{Z}_2$-gradings in C_r^p and E_r^p, turning E_r into a superspace. Clearly, the coboundary operator d in $C^*(\mathfrak{U}, \mathcal{E})$ is odd. It follows that the coboundary d_r is odd for any $r \geq 0$.

The superspaces E_r are also endowed with a second $\mathbb{Z}$-grading. Namely, for any $q \in \mathbb{Z}$, set

$$\begin{aligned} C_r^{p,q} &= C_r^p \cap C^{p+q}(\mathfrak{U}, \mathcal{E}), \\ E_r^{p,q} &= C_r^{p,q} / C_{r-1}^{p+1,q-1} + dC_{r-1}^{p-r+1,q+r-2}. \end{aligned}$$

Then

$$E_r = \bigoplus_{p,q} E_r^{p,q}.$$

Clearly,

$$d_r(E_r^{p,q}) \subset E_r^{p+r,q-r+1} \tag{15}$$

for any r, p, q.

One sees easily that $C_r^{p,q} = 0$ for all p and r if $q \leq -(m+1)$. Therefore, for a fixed q, we have $d(C_r^{p,q}) = 0$ for all $r \geq q + m + 2$. This implies that $\kappa_{r+1}^r : E_r^{p,q} \to E_{r+1}^{p,q}$ is an isomorphism for all p and $r \geq r_0(q) = q + m + 2$. Setting $E_\infty^{p,q} = E_{r_0(q)}^{p,q}$, we get the bigraded superspace

$$E_\infty = \bigoplus_{p,q} E_\infty^{p,q}.$$

Now we prove certain properties of the spectral sequence (E_r). Some of them are well known and are valid in a more general situation.

Proposition 2. *The first two terms of the spectral sequence* (E_r) *can be identified with the following bigraded spaces:*

$$\begin{aligned} E_0 &= C^*(\mathfrak{U}, \operatorname{gr} \mathcal{E}), \\ E_1 &= H^*(M, \operatorname{gr} \mathcal{E}). \end{aligned}$$

Here

$$E_0^{p,q} = C^{p+q}(\mathfrak{U}, (\operatorname{gr}\mathcal{E})_p),$$
$$E_1^{p,q} = H^{p+q}(M, (\operatorname{gr}\mathcal{E})_p).$$

Proof. By definition, we have

$$E_0^p = C_{(p)}/C_{(p+1)},\ p \geq 0,$$

where the coboundary operator d_0 of degree 0 is induced by $d : C_{(p)} \to C_{(p)}$. On the other hand, the exact sequence

$$0 \to \mathcal{E}_{(p+1)} \to \mathcal{E}_{(p)} \to \operatorname{gr}\mathcal{E}_p \to 0$$

and Theorem B for Stein supermanifolds imply the exact sequence

$$0 \to \mathcal{E}_{(p+1)}(U) \to \mathcal{E}_{(p)}(U) \to \operatorname{gr}\mathcal{E}_p(U) \to 0$$

for any Stein open subset $U \subset M$. Therefore

$$C^*(\mathfrak{U}, (\operatorname{gr}\mathcal{E})_p) \simeq C_{(p)}/C_{(p+1)} = E_0^p,\ p \geq 0.$$

One sees easily that this is an isomorphism of complexes and that the resulting isomorphism $C^*(\mathfrak{U}, \operatorname{gr}\mathcal{E}) \simeq E_0$ is an isomorphism of bigraded spaces. It follows that

$$E_1 \simeq H(E_0, d_0) \simeq H^*(\mathfrak{U}, \operatorname{gr}\mathcal{E}) \simeq H^*(M, \operatorname{gr}\mathcal{E}).\Box$$

Proposition 3. *There is the following identification of bigraded algebras:*

$$E_\infty = \operatorname{gr} H^*(M, \mathcal{E}),$$

where

$$E_\infty^{p,q} = \operatorname{gr}_p H^{p+q}(M, \mathcal{E}).$$

Proof. Clearly, for $r \geq r_0(q)$ we have $C_r^{p,q} = Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p)})$. It follows that

$$\begin{aligned} E_\infty^{p,q} &= Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p)})/Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p+1)}) + dC^{p+q-1}(\mathfrak{U}, \mathcal{E}) \cap Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p)}) \\ &= H^{p+q}(M, \mathcal{E})_{(p)}/(Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p+1)})/dC^{p+q-1}(\mathfrak{U}, \mathcal{E}) \cap Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p+1)}) \\ &= H^{p+q}(M, \mathcal{E})_{(p)}/H^{p+q}(M, \mathcal{E})_{(p+1)} = \operatorname{gr}_p H^{p+q}(M, \mathcal{E}).\Box \end{aligned}$$

Corollary. *If M is compact, then*

$$\dim H^k(M, \mathcal{E}) = \sum_{p+q=k} \dim E_\infty^{p,q}.$$

Proof. In fact, if M is compact, then all cohomology groups with values in a coherent analytic sheaf on $(M,\mathcal{O})$ or M are of finite dimension.□

Now we prove our main result concerning the first non-zero coboundary operators among $d_1,\ d_2,\ldots$. We may suppose that for each $i\in I$ there exists an isomorphism of sheaves $\sigma_i:\mathcal{O}|U_i\to \operatorname{gr}\mathcal{O}|U_i$, inducing the identity isomorphism $\operatorname{gr}\mathcal{O}|U_i\to\operatorname{gr}\mathcal{O}|U_i$.

By Theorem 2, a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}\to(M,\mathcal{O})$ corresponds to the cohomology class a_γ of the 1-cocycle $((a_\gamma)_{ij})\in Z^1(\mathfrak{U},\mathcal{QA}ut_{(1)(2)}\operatorname{gr}\mathcal{E})$, where $(a_\gamma)_{ij}=\delta_i\circ\delta_j^{-1}$. If the order of $(a_\gamma)_{ij}$ is equal to k, then we may choose $\delta_i,\ i\in I$, in such a way that $((a_\gamma)_{ij})\in Z^1(\mathfrak{U},\mathcal{QA}ut_{(k)(2)}\operatorname{gr}\mathcal{E})$. We can write $a_\gamma=\exp A_\Gamma$, where $A_\Gamma\in C^1(\mathfrak{U},\mathcal{QD}er_{(1)(2)}\operatorname{gr}\mathcal{E})$.

We will identify the differential spaces (E_0,d_0) and $(C^*(\mathfrak{U},\operatorname{gr}\mathcal{E}),d)$ via the isomorphism of Proposition 2. Clearly, $\delta_i:\mathcal{E}_{(p)}|U_i\to\operatorname{gr}\mathcal{E}_{(p)}|U_i=\sum_{r\geq p}\operatorname{gr}\mathcal{E}_r|U_i$ is an isomorphism of sheaves for any $i\in I,\ p\geq 0$. These local sheaf isomorphisms permit us to define an isomorphism of graded cochain groups

$$\psi: C^*(\mathfrak{U},\mathcal{E})\to C^*(\mathfrak{U},\operatorname{gr}\mathcal{E})$$

such that

$$\psi: C^*(\mathfrak{U},\mathcal{E}_{(p)})\to C^*(\mathfrak{U},\operatorname{gr}\mathcal{E}_{(p)}),\ p\geq 0.$$

We give it by

$$\psi(c)_{i_0\ldots i_q}=\delta_{i_0}(c_{i_0\ldots i_q})$$

for any $(i_0,\ldots,i_q)$ such that $U_{i_0}\cap\ldots\cap U_{i_q}\neq\emptyset$. In general, ψ is not an isomorphism of complexes. Nevertheless, we can express explicitly the coboundary d of the complex $C^*(\mathfrak{U},\mathcal{E})$ by means of d_0 and a_γ.

Proposition 4. *For any $c\in C^q(\mathfrak{U},\operatorname{gr}\mathcal{E})=\oplus_p E_0^{q-p,p}$, we have*

$$(\psi(d\psi^{-1}(c)))_{i_0\ldots i_{q+1}}=(d_0c)_{i_0\ldots i_{q+1}}+((a_\gamma)_{i_0i_1}-\mathrm{id})(c_{i_1\ldots i_{q+1}}).$$

Proof. We can write

$$\begin{aligned}(d\psi^{-1}(c))_{i_0\ldots i_{q+1}}&=\sum_{\alpha=0}^{q+1}(-1)^\alpha\psi^{-1}(c)_{i_0\ldots\hat{i}_\alpha\ldots i_{q+1}}\\&=\sum_{\alpha=1}^{q+1}(-1)^\alpha\psi^{-1}(c)_{i_0\ldots\hat{i}_\alpha\ldots i_{q+1}}+\psi^{-1}(c)_{i_1\ldots i_{q+1}}\\&=\delta_{i_0}^{-1}(\sum_{\alpha=1}^{q+1}(-1)^\alpha c_{i_0\ldots\hat{i}_\alpha\ldots i_{q+1}})+\delta_{i_1}^{-1}(c_{i_1\ldots i_{q+1}})\\&=\delta_{i_0}^{-1}((d_0c)_{i_0\ldots i_{q+1}}-c_{i_1\ldots i_{q+1}})+\delta_{i_1}^{-1}(c_{i_1\ldots i_{q+1}}).\end{aligned}$$

Therefore

$$
\begin{aligned}
(\psi(d\psi^{-1}(c)))_{i_0\dots i_{q+1}} &= \delta_{i_0}(d\psi^{-1}(c))_{i_0\dots i_{q+1}} \\
&= (d_0c)_{i_0\dots i_{q+1}} - c_{i_1\dots i_{q+1}} + (a_\gamma)_{i_0i_1}(c_{i_1\dots i_{q+1}}) \\
&= (d_0c)_{i_0\dots i_{q+1}} + ((a_\gamma)_{i_0i_1} - \mathrm{id})(c_{i_1\dots i_{q+1}}).
\end{aligned}
$$

This implies our assertion.□

This proposition makes it possible to calculate the spectral sequence (E_r) whenever d_0 and the cochain a_γ are known. Now we find the explicit form of certain coboundary operators $d_r,\ r \geq 1$.

Theorem 7. *Suppose that the locally free sheaf of $\mathcal{O}$-modules $\mathcal{E} \to (M, \mathcal{O}_M)$ has order k and denote by a_γ the cohomology class corresponding to $\mathcal{E}$ by Theorem 2. Then $d_r = 0$ for $r = 1, \dots, k-1$, and $d_k = \mu_k(a_\gamma)$.*

Proof. Take a cocycle $c \in E_0^{p,q-p}$, $d_0c = 0$, and denote by c^* its cohomology class in $E_1^{p,q-p}$. Clearly, c and c^* are represented by the cochain $\psi^{-1}(c) \in C_0^p$. By Proposition 4,

$$(\psi(d\psi^{-1}(c)))_{i_0\dots i_{q+1}} = ((a_\gamma)_{i_0i_1} - \mathrm{id})(c_{i_1\dots i_{q+1}}).$$

Now we see that

$$(\psi(d\psi^{-1}(c)))_{i_0\dots i_{q+1}} = \mu_k(a_\gamma)_{i_0i_1}(c_{i_1\dots i_{q+1}}) + u_{i_0\dots i_{q+1}},$$

where $u \in C_{(p+k+1)}$. This means that

$$\psi(d\psi^{-1}(c)) = \mu_k(a_\gamma)(c) + u,$$

whence $d_1 = d_2 = \ldots = d_{(k-1)} = 0$. Identifying E_k with E_1, we also see that d_kc^* is represented by the cochain $\psi^{-1}(\mu_k(a_\gamma)(c))$. It follows that

$$d_{2k}c^* = \mu_k(a_\gamma)(c^*).\square$$

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Arkady Onishchik
Yaroslavl University
150 000 Yaroslavl, Russia
E-mail address: aonishch@aha.ru,

Elizaveta Vishnyakova
Max-Planck-Institut für Mathematik
P.O.Box: 7280
53072 Bonn
Germany
E-mail address: VishnyakovaE@googlemail.com, Liza@mpim-bonn.mpg.de