Five-Dimensional Tangent Vectors in Space-Time
I. Introduction and Formal Theory

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Abstract

In this series of papers I examine a special kind of geometric objects that can be defined in space-time — five-dimensional tangent vectors. Similar objects exist in any other differentiable manifold, and their dimension is one unit greater than that of the manifold. Like ordinary tangent vectors, the considered five-dimensional vectors and the tensors constructed out of them can be used for describing certain local quantities and in this capacity find direct application in physics. For example, such familiar physical quantities as the stress-energy and angular momentum tensors prove to be parts of a single five-tensor. In this part of the series five-dimensional tangent vectors are introduced as abstract objects related in a certain way to ordinary four-dimensional tangent vectors. I then make a formal study of their basic algebraic properties and of their differential properties in flat space-time. In conclusion I consider some examples of quantities described by five-vectors and five-tensors.

1. Introduction

Adding a dimension to tangent vectors in space-time is not a new idea in physics. A well-known example is the Kaluza–Klein model [1] and the models that succeeded it, where the extra dimension of tangent vectors results from adding a dimension to the space-time manifold itself. Another example are the theories of gravity formulated as Yang–Mills gauge theories of the de Sitter group [2] and similar models, where the additional dimension is assigned not to the tangent vectors themselves, but to the internal vector space where the vierbein field takes its values. Unlike all these constructions, for introducing the five-dimensional vectors I consider in this paper one does not need to change the space-time manifold in any way nor to endow it with any additional structure. The vectors I am going to discuss here, which I will call *five-dimensional tangent vectors* or simply *five-vectors*, should be viewed as another type of geometric objects that can be defined in space-time and which are more suited for describing certain kinds of geometric and physical quantities than ordinary tangent vectors and tensors.

A hint to the existence of five-dimensional tangent vectors can be found in spinors. For the type of 4-spinors commonly used in physics, the symmetry group of the corresponding Clifford algebra is SO(3,2). Accordingly, there exist five constituents of the Clifford algebra (five matrices) Γ_A, where A runs 0, 1, 2, 3, and 5, that all transform alike under Dirac and charge conjugation:

$$\bar{\Gamma}_A = \Gamma_A \ \text{ and } \ \Gamma_A^c = \Gamma_A, \tag{1}$$

and that satisfy the following anticommutation relations:

$$\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = -\,2\,\eta_{AB}, \tag{2}$$

where $\eta_{AB} \equiv \mathrm{diag}(+1, -1, -1, -1, +1)$. It is evident that one can obtain a new set of five constituents satisfying the same conjugation and anticommutation relations by applying an arbitrary O(3,2) transformation to the original set. Moreover, any two sets of constituents that satisfy relations (1) and (2) prove to be connected by an O(3,2) transformation. For an appropriate choice of the constituent set, the standard γ-matrices (the ones identified with the components of the basis four-vectors) are expressed in terms of Γ_A's as

$$\gamma_\mu = \frac{i}{2}(\Gamma_\mu \Gamma_5 - \Gamma_5 \Gamma_\mu), \tag{3}$$

where $\mu = 0$, 1, 2, or 3.

These observations may give one the idea to consider a new type of vectors that make up a real five-dimensional vector space, V_5, endowed with a symmetric nondegenerate inner product with the signature $(+ - - - +)$ or $(- + + + -)$. Considering the relation that exists between multiplication in a Clifford algebra and exterior multiplication of multivectors and forms, on the grounds of equation (3) one may further suppose that there should exist a certain correspondence between four-dimensional tangent vectors and part of the bivectors constructed from elements of V_5, such that for each orthonormal five-vector basis $\mathbf{e}_A$ from a certain class of such bases, the four-vectors corresponding to the four bivectors $\mathbf{e}_\mu \wedge \mathbf{e}_5$ make up a Lorentz basis in the space V_4 of ordinary tangent vectors.

Basing on these assumptions one can make a formal study of the basic algebraic and differential properties of five-dimensional tangent vectors, as it is done in this paper. This formal analysis may serve as an introduction to the subject and as a guide in developing a more sophisticated theory of five-vectors basing on the principles of differential geometry, which is presented in part II. Within this latter theory five-vectors are introduced first as equivalence classes of parametrized curves and then, more rigorously, as a particular kind of differential-algebraic operators that act upon scalar functions. In part III I consider some applications of five-vectors in flat space-time and of their analogs in three-dimensional Euclidean space. In particular, I show how five-vectors can be used for describing in a coordinate-independent way finite and infinitesimal Poincare transformations and, as an illustration, reformulate the classical mechanics of a perfectly rigit body in terms of the analogs of five-vectors in three-dimensional space. In that part I also introduce the notion of a "bivector" derivative for scalar, four-vector and four-tensor fields in flat space-time and, as an illustration, calculate its analog in three-dimensional Euclidean space for the Lagrange function of a system of several point particles in classical nonrelativistic mechanics.

The fact that five-dimensional tangent vectors and the tensors associated with them enable one to give a coordinate-independent description to finite and infinitesimal Poincare transformations and, as one will see below, to describe as a single local object such quantities as the stress-energy and angular momentum tensors, should be thought of only as a reason for considering five-vectors in the first place and for making an exploratory study of their basic properties. If this were all there is to it, i.e. if five-vectors only enabled one to present certain geometric quantities and the relations between them in a mathematically more attractive form, such vectors would hardly be of particular interest both to physicists, who typically do not care much for fancy mathematics unless it enables them to formulate new physical concepts, and to mathematicians, who would consider five-vectors as merely a particular combination of already known mathematical constructions. A more important reason why the concept of a five-dimensional tangent vector is worth considering is that it enables one to extend the notion of the affine connection on a manifold and of the connections which physicists call gauge fields, and thereby at no cost at all, i.e. without changing the manifold in any way and without introducing new gauge groups, to obtain new geometric properties of space-time in the form of a new kind of torsion and a new kind of gauge fields.

Before discussing these applications of five-vectors, in part IV I develop a five-vector generalization of exterior differential calculus, which is more a technical necessity—a necessity in replacing ordinary tangent vectors with five-vectors in all the formulae related to integration of differential forms and to exterior differentiation of the latter. Apart from allowing one to present certain relations in a more elegant form, for scalar-valued forms this generalization is equivalent to ordinary exterior calculus, which was to be expected since five-vectors in this case are used only for characterizing the infinitesimal elements of integration volumes, and the latter are not changed in any way themselves and are not endowed with any new additional structure.

In part V I discuss the five-vector generalizations of affine connection and gauge fields. I then give definition to the exterior derivative of nonscalar-valued five-vector forms and consider the five-vector analogs of the field strength tensor. In conclusion of that part I briefly comment on the nonspacetime analogs of five-vectors.

In part VI I first define the bivector derivative for four-vector and four-tensor fields in the case of arbitrary Riemannian geometry. I then define this derivative for five-vector and five-tensor fields, examine the bivector analogs of the Riemann tensor, and introduce the notion of a commutator for the fields of five-vector bivectors. After that I examine a more general case of five-vector affine connection, introduce the five-vector analog of the curvature tensor, discuss the canonical stress-energy and angular momentum tensors corresponding to the five-vector generalization of the covariant derivative, and then consider a possible five-vector generalization of the Einstein and Kibble–Sciama equations. In conclusion, I introduce the notion of the bivector derivative for the fields whose values are vectors or tensors not directly related to space-time, and then consider the corresponding gauge fields and discuss some of their properties.

Most of the material presented in parts II, IV, V, and VI can be easily adapted to the case of arbitrary differentiable manifolds with metric. To simplify the presentation, I do not indicate explicitly the smoothness conditions for scalar functions and tensor fields under which the statements formulated are valid. If necessary, these conditions can be retrieved without any difficulty.

2. Invariant formulation of the five-vector hypothesis and notations

For any vector space V (here I will be concerned with real vectors only) one can consider a space of bivectors. A bivector is a wedge product of two vectors:

$$\mathbf{u} \wedge \mathbf{v} \equiv \mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u},$$

or a sum of such products. In the former case the bivector is called *simple*. All bivectors are simple for a three-dimensional V. For V with dimension higher than 3, the sum of two simple bivectors may not be a simple bivector. For example, as one knows from classical electrodynamics, a general antisymmetric four-tensor of rank 2 is not always a wedge product of two four-vectors. There, however, exist such *subsets* of simple bivectors which are closed with respect to addition. Each such subset will be referred to as a *vector space of simple bivectors*. The structure of such spaces is described by the following theorem:

Theorem: If $\mathcal{A}$ is a vector space of simple bivectors constructed from elements of a vector space V and $\dim \mathcal{A} > 3$, then there exists a nonzero vector $\mathbf{w} \in V$ such that each element of $\mathcal{A}$ can be presented in the form

$$\mathbf{u} \wedge \mathbf{w},$$

where $\mathbf{u}$ is some vector from V. For a given $\mathcal{A}$ the vector $\mathbf{w}$ is unique up to a normalization factor.

As one can see, the three-dimensional space is an exception: for it the sum of any two bivectors is a simple bivector, but all its bivectors cannot be presented in the form indicated in the Theorem.

Let us now consider vector spaces of simple bivectors with maximum dimension. Such spaces will be called *maximal*. From the Theorem and the fact that for any vector space V, the set of bivectors $\mathbf{u} \wedge \mathbf{w}$, where $\mathbf{w}$ is fixed and $\mathbf{u}$ runs through V, is a vector space of simple bivectors, it follows that:

1. For an n-dimensional vector space V with $n \geq 4$, the dimension of any maximal vector space of simple bivectors is $n-1$.

2. At $n \geq 5$, for each such maximal vector space $\mathcal{A}$ there exists a vector $\mathbf{w} \in V$ such that each element of $\mathcal{A}$ can be presented as $\mathbf{u} \wedge \mathbf{w}$, where $\mathbf{u}$ is some vector from V, and any bivector of such form belongs to $\mathcal{A}$. I will call $\mathbf{w}$ a *directional* vector of $\mathcal{A}$.

3. For a given maximal vector space of simple bivectors, the directional vector is unique up to an arbitrary normalization factor.

We can now reformulate the second part of our assumption about five-vectors as follows: *there exists a certain isomorphism between the space of four-dimensional tangent vectors and one of the maximal vector spaces of simple bivectors over* V_5. It should be emphasized that the meaning of the latter statement is not that the two mentioned vector spaces are isomorphic, which is merely a consequence of the definition of V_5, but that it is supposed that there is given *one specific* isomorphism, by means of which five-vectors are related to space-time and the origin of which will become clear when we turn to the more sophisticated theory of five-vectors presented in part II.

The mentioned isomorphism enables one to make a certain simplification in terminology within the formal theory of five-vectors, which proves to be quite convenient and which I will use in this part only. Namely, basing on this isomorphism one can *identify* four-dimensional tangent vectors with elements of the mentioned maximal vector space of simple bivectors, which in this case will naturally be denoted as V_4, too. Thus, instead of saying that four-vector $\mathbf{U}$ corresponds to bivector $\mathbf{u} \wedge \mathbf{w}$, one can simply write $\mathbf{U} = \mathbf{u} \wedge \mathbf{w}$.

As usual, the inner product of four-vectors will be denoted as g. The nondegenerate inner product on the space of five-vectors will be denoted with the symbol h. Under the above identification, the relation between g and h is given by the following equation:

$$g(\mathbf{u} \wedge \mathbf{w}, \mathbf{v} \wedge \mathbf{w}) = h(\mathbf{u}, \mathbf{v})h(\mathbf{w}, \mathbf{w}) - h(\mathbf{u}, \mathbf{w})h(\mathbf{v}, \mathbf{w}). \tag{4}$$

The overall sign of h is a matter of convention and for purely practical reasons it is convenient to choose it so that h would have the signature $(+---+)$, to make its relation to g simpler.

Let us now determine what kind of a directional vector $\mathbf{w}$ corresponds to V_4. If $\mathbf{w}$ had a negative norm squared, one could always choose its arbitrary normalization factor so that $h(\mathbf{w}, \mathbf{w}) = -1$, and then select an orthonormal basis of five-vectors with $\mathbf{e}_5 = \mathbf{w}$. In that case, for the four-vector basis $\mathbf{E}_\mu = \mathbf{e}_\mu \wedge \mathbf{e}_5$ the inner product matrix would be

$$g_{\mu\nu} \equiv g(\mathbf{E}_\mu, \mathbf{E}_\nu) = \mathrm{diag}(-1, -1, +1, +1),$$

and not of Lorentz type. Thus, the norm squared of $\mathbf{w}$ cannot be negative.

In a similar manner one can check that if $h(\mathbf{w}, \mathbf{w}) = 0$, the inner product induced on the corresponding maximal vector space of simple bivectors would be degenerate, so $\mathbf{w}$ cannot be a null vector either. Thus, one is left with the only possibility that the directional vector of V_4 has a positive norm squared.

Let us now fix our notations:

- Five-vectors will be denoted with lower-case boldface Roman letters: $\mathbf{u}, \mathbf{v}, \mathbf{w}$, etc.

- A typical basis in V_5 will be denoted as $\mathbf{e}_A$, where A (as all capital latin indices) runs 0, 1, 2, 3, and 5. An arbitrary five-vector $\mathbf{u}$ is expressed in terms of its components in a given basis as $\mathbf{u} = u^A \mathbf{e}_A$. One can choose a basis in V_5 arbitrarily, but it is more convenient to select the fifth basis vector coinciding with one of the directional vectors.[1] Such bases will be called *standard* and will be used in all calculations.[2]

- Four-vectors will be denoted with capital boldface Roman letters: $\mathbf{U}, \mathbf{V}, \mathbf{W}$, etc. One can choose a basis in V_4 arbitrarily and independently of the basis in V_5. However, it is more convenient to select it as

$$\mathbf{E}_\mu = \mathbf{e}_\mu \wedge \mathbf{e}_5, \tag{5}$$

where μ (as all lower-case Greek indices) runs 0, 1, 2, and 3. I will refer to this basis as to the one *associated* with the basis $\mathbf{e}_A$ in V_5.

3. Algebraic properties of five-vectors

A. *Transformations from one standard basis to another*

Let $\mathbf{e}_A$ be an arbitrary standard basis in V_5 and let $\mathbf{e}'_A$ be another basis in V_5 such that

$$\mathbf{e}'_A = \mathbf{e}_B L^B_A, \tag{6}$$

where L^B_A is a real nondegenerate 5×5 matrix. The relation between the corresponding associated four-vector bases is

$$\begin{aligned} \mathbf{E}'_\mu &= \mathbf{e}'_\mu \wedge \mathbf{e}'_5 = \mathbf{e}_A \wedge \mathbf{e}_B \, L^A_\mu L^B_5 \\ &= \mathbf{E}_\nu \, (L^\nu_\mu L^5_5 - L^5_\mu L^\nu_5) \\ &\quad + \sum_{\alpha<\beta} \mathbf{e}_\alpha \wedge \mathbf{e}_\beta \, (L^\alpha_\mu L^\beta_5 - L^\beta_\mu L^\alpha_5). \end{aligned} \tag{7}$$

[1] It is not required that $\mathbf{e}_5$ be normalized.

[2] As one can see, the basis vector and vector components related to the fifth dimension are labled with the index 5 rather than 4. This corresponds to the index convention used for γ-matrices, where the notation γ_4 is reserved for the timelike γ-matrix in the Pauli metric: $\gamma_4 = i\gamma_0$. This also better suits the words "fifth dimension", and accentuates the fact that this direction in V_5 is distinguished as being the one that corresponds to the directional vector of V_4.

If the basis $\mathbf{e}'_A$ is also standard, one should have

$$\mathbf{E}'_\mu = \mathbf{E}_\nu \Lambda^\nu_\mu \tag{8}$$

for some real nondegenerate 4×4 matrix Λ^ν_μ. Comparing (7) and (8), one finds that

$$L^\alpha_\mu L^\beta_5 - L^\beta_\mu L^\alpha_5 = 0, \tag{9}$$

$$L^\nu_\mu L^5_5 - L^5_\mu L^\nu_5 = \Lambda^\nu_\mu. \tag{10}$$

Equation (9) is equivalent to the requirement

$$L^\alpha_5 = 0 \ \text{ for all } \ \alpha, \tag{11}$$

which thus is a necessary condition of $\mathbf{e}'_A$ being a standard basis. This is also a sufficient condition, since according to it,

$$\mathbf{e}'_5 = \mathbf{e}_A L^A_5 = \mathbf{e}_5 L^5_5,$$

and L^5_5 cannot be zero because L^A_B is nondegenerate.

¿From (10) and (11) one obtains the formula

$$\Lambda^\nu_\mu = L^5_5 L^\nu_\mu, \tag{12}$$

which relates Λ^ν_μ to L^ν_μ. One should also note that

$$L^5_5 (L^{-1})^5_5 = 1 \quad \text{and} \quad (L^{-1})^\alpha_5 = 0,$$

where $(L^{-1})^A_B$ is the inverse of L^A_B.

It is convenient to distinguish three different types of transformations from one standard basis in V_5 to another:

(*i*) transformations of the form

$$\begin{cases} L^5_5 = a, \ L^\alpha_5 = 0 \\ L^5_\beta = 0, \ L^\alpha_\beta = a^{-1}\,\delta^\alpha_\beta \ \ (a \neq 0), \end{cases}$$

which will be referred to as *U-transformations*;

(*ii*) transformations of the form

$$\begin{cases} L^5_5 = 1, \ L^\alpha_5 = 0 \\ L^5_\beta = a_\beta, \ L^\alpha_\beta = \delta^\alpha_\beta, \end{cases}$$

which will be referred to as *P-transformations*; and

(*iii*) transformations of the form

$$\begin{cases} L^5_5 = 1, \ L^\alpha_5 = 0 \\ L^5_\beta = 0, \ L^\alpha_\beta = t^\alpha_\beta, \end{cases}$$

which will be referred to as *M-transformations* (t^α_β is some nondegenerate 4×4 matrix). An arbitrary transformation from one standard basis to another can be presented as a composition of a U-, a P-, and an M-transformation. It is a simple matter to see that U- and P-transformations have no effect on four-vectors, i.e. that they induce identity transformations in V_4. For M-transformations one evidently has $\Lambda^\alpha_\beta = L^\alpha_\beta = t^\alpha_\beta$.

B. *Symmetries and other special transformations*

If one considers V_5 by itself and takes into account the five-orientation by introducing a Levi-Civita type tensor ϵ_{ABCDE}, the group of isomorphisms of V_5 will be SO(3,2). This symmetry is broken when one of the maximal vector spaces of simple bivectors over V_5 is identified with the space of four-vectors. The symmetry group of the structure as a whole (V_5 plus V_4) is apparently SO(3,1), and the corresponding isomorphisms are M-transformations (which in this case should be interpreted in the active sense) with $t^\alpha_\beta \in$ SO(3,1).

One may notice that the latter transformations and P-transformations make up a group isomorphic to the Poincare group. This can be easily seen by comparing the formulae for P- and M-transformations at $t^\alpha_\beta \in$ SO(3,1) with the formulae for the Poincare transformation of covariant Lorentz coordinates in the five-dimensional representation (see Appendix) and observing that they are identical in form. This coincidence is not accidental. It turns out that the rules of parallel transport for five-vectors are such that with any Lorentz coordinate system in flat space-time one can associate either an orthonormal set of basis five-vector fields (everywhere $h(\mathbf{e}_A, \mathbf{e}_B) = \eta_{AB}$), which, however, cannot be chosen self-parallel, or a set of self-parallel basis fields (everywhere $\nabla \mathbf{e}_A = \mathbf{0}$), which can be made orthonormal only at one point in space-time, for example, at the origin of the coordinate system. As one will see in section 4, the elements of the self-parallel basis at a given point transform nontrivially under space-time translations, and in the general case the Poincare transformation of such a basis is a composition of a certain M-transformation with $t^\alpha_\beta \in$ SO(3,1) and a certain P-transformation.

C. *Relation between four- and five-vector bases*

For any five-vector basis $\mathbf{e}_A$ one can construct the corresponding associated basis of four-vectors: $\mathbf{E}_\alpha = \mathbf{e}_\alpha \wedge \mathbf{e}_5$. It is evident that this correspondence is not mutually unique: for any basis of five-vectors obtained from $\mathbf{e}_A$ by arbitrary U- and P-transformations the associated basis of four-vectors will be exactly the same. One can distinguish between all these five-vector bases only by imposing additional requirements. One particular way of choosing the five-vector basis for a given basis of four-vectors is based on the following two lemmas:

Lemma 1: For any orthonormal basis of four-vectors $\mathbf{E}_\alpha$, there exists an orthonormal standard basis of five-vectors $\mathbf{e}_A$ such that $\mathbf{e}_\alpha \wedge \mathbf{e}_5 = \mathbf{E}_\alpha$. This five-vector basis is unique up to a common sign of all $\mathbf{e}_A$.

Proof : Since all $\mathbf{E}_\alpha$ are elements of one maximal vector space of simple bivectors, they can be presented as $\mathbf{E}_\alpha = \mathbf{e}'_\alpha \wedge \mathbf{e}'_5$, where $\mathbf{e}'_5$ is a directional vector of this maximal vector space and $\mathbf{e}'_\alpha$ are certain five-vectors. One can easily show that the five vectors $\mathbf{e}'_A$ are linearly independent and therefore form a standard basis in V_5. Let us construct a new basis according to the formulae

$$\begin{aligned} \mathbf{e}_\alpha &= (h_{5'5'})^{1/2}\{\mathbf{e}'_\alpha - (h_{\alpha'5'})/(h_{5'5'})\,\mathbf{e}'_5\} \\ \mathbf{e}_5 &= (h_{5'5'})^{-1/2}\mathbf{e}'_5, \end{aligned}$$

where $h_{A'B'} \equiv h(\mathbf{e}'_A, \mathbf{e}'_B)$. This is also a standard basis, and simple calculations show that $\mathbf{e}_\alpha \wedge \mathbf{e}_5 = \mathbf{E}_\alpha$ and $h(\mathbf{e}_A, \mathbf{e}_B) = \eta_{AB}$, so it has been demonstrated that the required basis exists.

If $\mathbf{e}''_A$ is another basis that satisfies the same requirements as $\mathbf{e}_A$, and $\mathbf{e}''_A = \mathbf{e}_B L^B_A$, then one can easily show that $L^5_5 = \pm 1$, $L^\alpha_5 = L^5_\beta = 0$, and $L^\alpha_\beta = (L^5_5)^{-1}\delta^\alpha_\beta$, so either $\mathbf{e}''_A = \mathbf{e}_A$ or $\mathbf{e}''_A = -\mathbf{e}_A$. ∎

We thus see that for the special case of an *orthonormal* four-vector basis one can fix the corresponding five-vector basis up to a sign by requiring that the latter be orthonormal, too. In a certain sense, this is a natural choice. It is also natural that the orthonormality condition does not fix the overall sign of the basis five-vectors, since this sign has no effect on their inner products.[3] In the general case, the selection of the five-vector basis can be based on the following lemma:

Lemma 2: For an arbitrary basis of four-vectors $\mathbf{E}_\alpha$, there exists a standard basis of five-vectors $\mathbf{e}_A$ such that $h(\mathbf{e}_5, \mathbf{e}_5) = 1$, $h(\mathbf{e}_5, \mathbf{e}_\alpha) = 0$, and $\mathbf{e}_\alpha \wedge \mathbf{e}_5 = \mathbf{E}_\alpha$. This five-vector basis is unique up to a common sign of all $\mathbf{e}_A$.

Proof : It is evident that there exists a matrix Λ^α_β such that $\mathbf{E}'_\alpha = \mathbf{E}_\beta \Lambda^\beta_\alpha$ is an orthonormal basis in V_4. According to Lemma 1, there exists an orthonormal five-vector basis $\mathbf{e}'_A$ such that $\mathbf{e}'_\alpha \wedge \mathbf{e}'_5 = \mathbf{E}'_\alpha$. One can easily check that the basis

$$\mathbf{e}_\alpha = \mathbf{e}'_\beta (\Lambda^{-1})^\beta_\alpha \ \text{ and } \ \mathbf{e}_5 = \mathbf{e}'_5$$

is such that $h(\mathbf{e}_5, \mathbf{e}_5) = 1$, $h(\mathbf{e}_5, \mathbf{e}_\alpha) = 0$, and $\mathbf{e}_\alpha \wedge \mathbf{e}_5 = \mathbf{E}_\alpha$, so it has been demonstrated that the required basis exists.

[3] To fix the five-vector basis unambiguously, one has to impose one more requirement. For example, one may observe that by changing the overall sign of the basis five-vectors one changes the five-orientation of the basis, so one can fix a single basis by requiring that $\epsilon_{01235} = +1$ or that $\epsilon_{01235} = -1$.

If $\mathbf{e}''_A$ is another basis that satisfies the same requirements as $\mathbf{e}_A$, one can construct the basis $\mathbf{e}'''_5 = \mathbf{e}''_5$, $\mathbf{e}'''_\alpha = \mathbf{e}''_\beta \Lambda^\beta_{\ \alpha}$ and check that $\mathbf{e}'''_A$ is orthonormal and that $\mathbf{e}'''_\alpha \wedge \mathbf{e}'''_5 = \mathbf{E}'_\alpha$. Thus, by virtue of Lemma 1, one has $\mathbf{e}'''_A = \pm \mathbf{e}'_A$, so

$$\begin{aligned} \mathbf{e}''_5 &= \mathbf{e}'''_5 = \pm \mathbf{e}'_5 = \pm \mathbf{e}_5 \\ \mathbf{e}''_\alpha &= \mathbf{e}'''_\beta (\Lambda^{-1})^\beta_{\ \alpha} = \pm \mathbf{e}'_\beta (\Lambda^{-1})^\beta_{\ \alpha} = \pm \mathbf{e}_\alpha . \end{aligned} \quad \blacksquare$$

A standard five-vector basis that satisfies the requirements $h(\mathbf{e}_5, \mathbf{e}_5) = 1$ and $h(\mathbf{e}_5, \mathbf{e}_\alpha) = 0$ will be called a *regular* basis. Thus, Lemma 2 states that for a given four-vector basis there exist but two corresponding regular five-vector bases, differing from each other only in the overall sign of the basis five-vectors. A regular basis is very convenient since in it

$$h_{55} = 1, \; h_{\alpha 5} = 0, \text{ and } h_{\alpha\beta} = g_{\alpha\beta},$$

which simplifies algebraic transformations, and (if one chooses the five-vector basis this way at every point)

$$\partial_\mu h_{55} = \partial_\mu h_{\alpha 5} = 0 \;\text{ and }\; \partial_\mu h_{\alpha\beta} = \partial_\mu g_{\alpha\beta},$$

which is convenient when one evaluates the derivatives.

4. Differential properties of five-vectors

A. *Relation between parallel transports of four- and five-vectors*

When considering the differential properties of five-vectors, one should imagine that at each point in space-time there exists a tangent space of five-vectors. As for any other type of vector-like objects considered in space-time, one can speak of parallel transport of five-vectors from one point to another. It seems natural to suppose that the rules of this transport should be related in some way to similar rules for four-dimensional tangent vectors. It is obvious that this relation cannot be derived from algebraic properties of five-vectors, and to obtain it one has to make some additional assumption about five-vectors, which ought to be regarded as part of their definition.

The simplest and the most natural form of the relation in question can be obtained by postulating that parallel transport preserves the isomorphism between the space of four-vectors and one of the maximal vector spaces of simple bivectors over V_5, which has been discussed above. A more precise formulation of this statement is the following:

If four-vector $\mathbf{U}$ *corresponds to bivector* $\mathbf{u} \wedge \mathbf{w}$*, then the transported* $\mathbf{U}$ *corresponds to the transported* $\mathbf{u} \wedge \mathbf{w}$*.* (13)

This assumption has two consequences, which can be conveniently expressed in terms of connection coefficients. Let us define the latter for five-vectors as

$$\nabla_\mu \mathbf{e}_A = \mathbf{e}_B G^B_{\ A\mu},$$

where $\nabla_\mu \equiv \nabla_{\mathbf{E}_\mu}$ denotes the covariant derivative in the direction of the basis four-vector $\mathbf{E}_\mu$. The connection coefficients for four-vectors will be denoted in the standard way:

$$\nabla_\mu \mathbf{E}_\alpha = \mathbf{E}_\beta \Gamma^\beta_{\ \alpha\mu}. \tag{14}$$

In the usual manner one can obtain the expression for the components of the covariant derivative of an arbitrary five-vector field $\mathbf{u}$:

$$\nabla_\mu \mathbf{u} = \nabla_\mu (u^A \mathbf{e}_A) = (\partial_\mu u^A + G^A_{\ B\mu} u^B)\, \mathbf{e}_A \equiv u^A_{\ ;\mu} \mathbf{e}_A,$$

and the transformation formula for five-vector connection coefficients corresponding to the transformations $\mathbf{E}'_\mu = \mathbf{E}_\nu \Lambda^\nu_{\ \mu}$ and $\mathbf{e}'_A = \mathbf{e}_B L^B_{\ A}$ of the four- and five-vector bases:

$$G'^A_{\ B\mu} = (L^{-1})^A_{\ C} G^C_{\ D\nu} L^D_{\ B} \Lambda^\nu_{\ \mu} + (L^{-1})^A_{\ C} (\partial_\nu L^C_{\ B}) \Lambda^\nu_{\ \mu}.$$

If at each point the five-vector basis $\mathbf{e}_A$ is chosen standard and $\mathbf{E}_\alpha$ is the associated basis of four-vectors, then

$$\begin{aligned} \nabla_\mu \mathbf{E}_\alpha &= \nabla_\mu (\mathbf{e}_\alpha \wedge \mathbf{e}_5) \\ &= (\nabla_\mu \mathbf{e}_\alpha) \wedge \mathbf{e}_5 + \mathbf{e}_\alpha \wedge (\nabla_\mu \mathbf{e}_5) \\ &= \mathbf{e}_\beta \wedge \mathbf{e}_5 \,(G^\beta_{\ \alpha\mu} + \delta^\beta_\alpha G^5_{\ 5\mu}) \\ &\qquad + \mathbf{e}_\alpha \wedge \mathbf{e}_\beta \, G^\beta_{\ 5\mu}. \end{aligned} \tag{15}$$

Comparing (14) and (15) one finds that

$$G^\alpha_{\ 5\mu} = 0 \text{ for all } \alpha, \tag{16}$$

and

$$\Gamma^\alpha_{\ \beta\mu} = G^\alpha_{\ \beta\mu} + \delta^\alpha_\beta G^5_{\ 5\mu}. \tag{17}$$

These equations express the relation between the rules of parallel transport for four- and five-vectors. One should notice that they tell one nothing about the coefficients $G^5_{\ \beta\mu}$, so as far as four-vector parallel transport is concerned, the latter can be absolutely arbitrary.

B. *Five-vectors in flat space-time*

It is a well known fact that owing to its special geometric features, flat space-time possesses a symmetry which in application to scalars, four-vectors and other four-tensors can be formulated as the following principle:

For any set of scalar, four-vector and four-tensor fields in flat space-time, by means of a certain procedure one can construct a new set of fields (which will be called *equivalent*) such that at each point in space-time these new fields satisfy the same algebraic and differential relations that the original fields satisfy at a certain corresponding point.

The procedure by means of which the equivalent fields are constructed can be formulated as follows:

1. Introduce a system of Lorentz coordinates x^α.
   Introduce the corresponding coordinate four-vector basis $\mathbf{E}_\alpha = \partial/\partial x^\alpha$.
   Introduce the corresponding bases for all other four-tensors.

2. Each scalar field f will then determine and be determined by one real coordinate function $f(x)$.
   Each four-vector field $\mathbf{U}$ will determine and be determined by four real coordinate functions $U^\alpha(x)$ (= components of $\mathbf{U}$ in the basis $\mathbf{E}_\alpha$).
   Each four-tensor field $\mathbf{T}$ will determine and be determined by an appropriate number of real coordinate functions $T^{\alpha\beta\ldots\gamma}_{\lambda\nu\ldots\mu}(x)$ (= components of $\mathbf{T}$ in the appropriate tensor basis corresponding to $\mathbf{E}_\alpha$).

3. Introduce a new system of Lorentz coordinates x'^α.
   Introduce the corresponding four-vector basis $\mathbf{E}'_\alpha = \partial/\partial x'^\alpha$.
   Introduce the corresponding bases for all other four-tensors.

4. Then the equivalent scalar, four-vector and four-tensor fields will be determined in the new coordinates and new bases by the *same functions* $f(\cdot)$, $U^\alpha(\cdot)$, $\ldots$, $T^{\alpha\beta\ldots\gamma}_{\lambda\nu\ldots\mu}(\cdot)$ that determine the original fields in the old coordinates and old bases.

The above symmetry principle and the corresponding procedure for constructing equivalent fields follow from the definition of flat space-time and the assumptions that in it ∇ is torsion-free and satisfies the condition of compatibility with metric:

$$\nabla g = 0. \tag{18}$$

The latter two assumptions enable one to find the rules of parallel transport for four-vectors, and knowing these one can *prove* that the above symmetry principle holds. For five-vectors let us reverse the problem: let us *suppose* that only those five-vectors have anything to do with reality for which there holds a symmetry principle similar to the one formulated above for four-vectors, and then use this principle to determine the rules of parallel transport for five-vectors in flat space-time.

Let us introduce a system of Lorentz coordinates, x^α, and consider the following set of fields:

1. Four four-vector basis fields $\mathbf{E}_\alpha = \partial/\partial x^\alpha$.

2. Five *continuous* five-vector fields $\mathbf{e}_A$ such that at each point they make up a *regular* basis corresponding to $\mathbf{E}_\alpha$.[4]

3. $5 \times 5 \times 4 = 100$ scalar fields $H^A_{B\mu}$ such that everywhere
$$\nabla_\mu \mathbf{e}_A = \mathbf{e}_B H^B_{A\mu}.$$

By their definition, $H^A_{B\mu}$ are connection coefficients for the basis fields $\mathbf{e}_A$, and since all these bases are standard, one should have $H^\alpha_{5\mu} = 0$. Furthermore, since space-time is flat and $\mathbf{E}_\alpha$ is a Lorentz basis, the corresponding four-vector connection coefficients are zero, so one should have

$$H^\alpha_{\beta\mu} + \delta^\alpha_\beta H^5_{5\mu} = \Gamma^\alpha_{\beta\mu} = 0. \tag{19}$$

Let us now consider another system of Lorentz coordinates, x'^α, such that

$$x'^\alpha = x^\alpha + a^\alpha, \tag{20}$$

where a^α are four arbitrary constant parameters. The fields equivalent to $\mathbf{E}_\alpha$ are $\mathbf{E}_\alpha$ themselves, since by virtue of the symmetry principle,

$$\mathbf{E}'_\alpha = \partial/\partial x'^\alpha = \partial/\partial x^\alpha = \mathbf{E}_\alpha.$$

In view of this, for the fields equivalent to $\mathbf{e}_A$ one has only two options: either $\mathbf{e}'_A = \mathbf{e}_A$ or $\mathbf{e}'_A = -\,\mathbf{e}_A$. Since coordinate transformation (20) depends continuously on a^α, it is natural to require that the same be true of the corresponding field transformation, which leaves us with only one possibility: $\mathbf{e}'_A = \mathbf{e}_A$.

Finally, by virtue of the symmetry principle, the scalar fields equivalent to $H^A_{B\mu}$ are such that

$$H'^A_{B\mu}(x'^\alpha = y^\alpha) = H^A_{B\mu}(x^\alpha = y^\alpha), \tag{21}$$

at all y^α. Since equivalent fields must satisfy the same relations as the original fields, one should have

$$\mathbf{e}_B H'^B_{A\mu} = \mathbf{e}'_B H'^B_{A\mu} = \nabla_{\mathbf{E}'_\mu} \mathbf{e}'_A = \nabla_{\mathbf{E}_\mu} \mathbf{e}_A = \mathbf{e}_B H^B_{A\mu}.$$

Thus, at any point Q one has

$$H'^A_{B\mu}(Q) = H^A_{B\mu}(Q).$$

[4] There are *two* sets of fields like that (see Lemma 2) and we choose one of them.

Comparing this with equation (21), one finds that at any a^α

$$\begin{aligned} H^A_{B\mu}(x^\alpha=0) &= H'^A_{B\mu}(x^\alpha=0) \\ &= H'^A_{B\mu}(x'^\alpha=a^\alpha) \\ &= H^A_{B\mu}(x^\alpha=a^\alpha), \end{aligned}$$

which means that each $H^A_{B\mu}$ is a constant scalar field.

Let us consider a third system of Lorentz coordinates, x''^α, which are related to x^α as

$$x''^\alpha = \Lambda^\alpha_{\ \beta} x^\beta,$$

where $\Lambda^\alpha_{\ \beta}$ is an arbitrary constant matrix from SO(3,1). By virtue of the symmetry principle, the fields equivalent to $\mathbf{E}_\alpha$ are

$$\mathbf{E}''_\alpha = \partial/\partial x''^\alpha = \partial/\partial x^\beta\,(\Lambda^{-1})^\beta_{\ \alpha} = \mathbf{E}_\beta\,(\Lambda^{-1})^\beta_{\ \alpha}.$$

It is a simple matter to check that if one requires the field transformation to depend continuously on parameters $\Lambda^\alpha_{\ \beta}$, the fields equivalent to $\mathbf{e}_A$ will be

$$\mathbf{e}''_\alpha = \mathbf{e}_\beta\,(\Lambda^{-1})^\beta_{\ \alpha} \;\text{ and }\; \mathbf{e}''_5 = \mathbf{e}_5.$$

Finally, since it has been found that each $H^A_{B\mu}$ is a constant scalar field, one should have

$$H''^A_{B\mu} = H^A_{B\mu}.$$

Since equivalent fields satisfy the same relations,

$$\begin{aligned} \mathbf{e}_5 H^5_{5\mu} &= \mathbf{e}''_5 H^5_{5\mu} = \mathbf{e}''_B H^B_{5\mu} = \mathbf{e}''_B H''^B_{5\mu} \\ &= \nabla_{\mathbf{E}''_\mu}\mathbf{e}''_5 = \nabla_{\mathbf{E}_\nu}\mathbf{e}_5\,(\Lambda^{-1})^\nu_{\ \mu} \\ &= \mathbf{e}_B H^B_{5\nu}\,(\Lambda^{-1})^\nu_{\ \mu} = \mathbf{e}_5 H^5_{5\nu}\,(\Lambda^{-1})^\nu_{\ \mu}, \end{aligned}$$

so one should have

$$H^5_{5\mu} = H^5_{5\nu}\,(\Lambda^{-1})^\nu_{\ \mu}$$

for all $\Lambda^\mu_{\ \nu}$ from SO(3,1), which is only possible if $H^5_{5\mu}=0$. From equation (19) it then follows that $H^\alpha_{\beta\mu}=0$ for all α, β, and μ. Finally, one has

$$\begin{aligned} \mathbf{e}_5 H^5_{\alpha\mu} &= \mathbf{e}''_5 H^5_{\alpha\mu} = \mathbf{e}''_B H^B_{\alpha\mu} = \mathbf{e}''_B H''^B_{\alpha\mu} \\ &= \nabla_{\mathbf{E}''_\mu}\mathbf{e}''_\alpha = \nabla_{\mathbf{E}_\nu}\mathbf{e}_\beta\,(\Lambda^{-1})^\beta_{\ \alpha}(\Lambda^{-1})^\nu_{\ \mu} \\ &= \mathbf{e}_A H^A_{\beta\nu}\,(\Lambda^{-1})^\beta_{\ \alpha}\,(\Lambda^{-1})^\nu_{\ \mu} \\ &= \mathbf{e}_5 H^5_{\beta\nu}\,(\Lambda^{-1})^\beta_{\ \alpha}\,(\Lambda^{-1})^\nu_{\ \mu}, \end{aligned}$$

so one should have

$$H^5_{\alpha\mu} = H^5_{\beta\nu}\,(\Lambda^{-1})^\beta_{\ \alpha}\,(\Lambda^{-1})^\nu_{\ \mu}$$

for all $\Lambda^\mu_{\ \nu}$ from SO(3,1). This is only possible if $H^5_{\alpha\mu}$ is proportional to the Minkowski metric tensor, $\eta_{\alpha\mu}$. Denoting the proportionality factor (which should be a constant since $H^5_{\alpha\mu}$ are constant fields) as $-\kappa$, one can summarize our findings about $H^A_{B\mu}$ as follows:

$$H^\alpha_{\beta\mu} = H^\alpha_{5\mu} = H^5_{5\mu} = 0 \;\text{ and }\; H^5_{\beta\mu} = -\,\kappa\eta_{\beta\mu}. \quad (22)$$

Thus, any orthonormal set of continuous five-vector basis fields $\mathbf{e}_A$ associated with a Lorentz four-vector basis in flat space-time satisfy the following differential equations:

$$\nabla_\mu \mathbf{e}_5 = 0 \;\text{ and }\; \nabla_\mu \mathbf{e}_\alpha = -\,\kappa\eta_{\alpha\mu}\,\mathbf{e}_5, \quad (23)$$

where κ is a constant, which cannot be found from symmetry considerations. These equations determine the rules of parallel transport for five-vectors in flat space-time.

C. *Equation for h*

Let us now express the contents of equation (23) in an equivalent form: as an equation for the first covariant derivative of the inner product h regarded as a five-tensor. From equations (22) and the fact that in the orthonormal basis $\mathbf{e}_A$ introduced in the previous subsection $h_{AB}=\eta_{AB}$ at every point, it follows that

$$\begin{aligned} h_{55;\mu} &= \partial_\mu h_{55} - h_{A5}H^A_{5\mu} - h_{5B}H^B_{5\mu} = 0, \\ h_{\alpha5;\mu} &= \partial_\mu h_{\alpha5} - h_{A5}H^A_{\alpha\mu} - h_{\alpha B}H^B_{5\mu} \\ &\qquad = -\,h_{55}H^5_{\alpha\mu} = \kappa\eta_{\alpha\mu}, \\ h_{\alpha\beta;\mu} &= \partial_\mu h_{\alpha\beta} - h_{A\beta}H^A_{\alpha\mu} - h_{\alpha B}H^B_{\beta\mu} = 0. \end{aligned}$$

These equations can be presented in the following covariant form:

$$\begin{aligned} &h_{55;\mu}=0, \quad h_{\alpha5;\mu} = \kappa g_{\alpha\mu} \\ &h_{55}h_{\alpha\beta;\mu} = \kappa(g_{\alpha\mu}h_{\beta5} + g_{\beta\mu}h_{\alpha5}), \end{aligned} \quad (24)$$

which is the same in any standard five-vector basis. It is not difficult to see that equations (24) are components of the following abstract equation:

$$\begin{aligned} h(\mathbf{e},\mathbf{e})\{\nabla_{\mathbf{U}}h\}(\mathbf{v},\mathbf{w}) &\\ = \kappa g(\mathbf{U},\mathbf{v}\wedge\mathbf{e})h(\mathbf{w},\mathbf{e}) &\\ + \kappa g(\mathbf{U},\mathbf{w}\wedge\mathbf{e})h(\mathbf{v},\mathbf{e}), & \end{aligned} \quad (25)$$

where $\{\nabla_{\mathbf{U}}h\}(\mathbf{v},\mathbf{w}) \equiv \partial_{\mathbf{U}}h(\mathbf{v},\mathbf{w}) - h(\nabla_{\mathbf{U}}\mathbf{v},\mathbf{w}) - h(\mathbf{v},\nabla_{\mathbf{U}}\mathbf{w})$ is the covariant derivative of the tensor h; $\mathbf{v}$ and $\mathbf{w}$ are any two five-vector fields; $\mathbf{U}$ is an arbitrary four-vector; and $\mathbf{e}$ is a directional vector of V_4 (which is not required to be normalized).

Equation (25) establishes a relation between the Riemannian geometry of space-time, represented by the inner product h, and the rules of parallel transport for five-vectors. At $\kappa=0$ it acquires a form similar to that of equation (18) for the four-dimensional metric tensor:

$$\nabla h = 0,$$

and can be given a similar simple interpretation: that the inner product of two five-vectors is invariant under parallel transport. Equation (25) at $\kappa \neq 0$ will be discussed further in part II.

Let us now examine more closely the properties of five-vectors in flat space-time.

D. *Self-parallel basis*

Any set of Lorentz basis four-vector fields in flat space-time has two special features: it is orthonormal (everywhere $g_{\alpha\beta} = \eta_{\alpha\beta}$) and self-parallel (everywhere $\Gamma^\alpha_{\beta\mu} = 0$). This fact is closely related to equation (18) for the metric tensor g: if ∇g were nonzero, a basis like that could not exist.

With five-vectors one has a similar situation at $\kappa = 0$: as one can see from formulae (22), the orthonormal basis $\mathbf{e}_A$ is then self-parallel and, accordingly, the first covariant derivative of h is identically zero, as is seen from equation (25).

The situation is different at $\kappa \neq 0$. Since ∇h is nonzero, the requirements of orthonormality and self-parallelism become conflicting in the sense that one can have either orthonormality or self-parallelism but not both at the same time.

The basis $\mathbf{e}_A$ of subsection B is orthonormal by definition but is not self-parallel, as is seen from equations (22) or (23). In the following I will call it an O-basis ('O' stands for '*orthonormal*') associated with a given system of Lorentz coordinates x^μ. Let us now construct a self-parallel basis, $\mathbf{p}_A$, that would coincide with $\mathbf{e}_A$ at the origin of the considered coordinate system. Being a self-parallel basis, $\mathbf{p}_A$ should satisfy the following differential equations:

$$\nabla_\mu \mathbf{p}_A = 0.$$

If $\mathbf{p}_A = \mathbf{e}_B N^B_A$, then

$$\nabla_\mu \mathbf{p}_A = \nabla_\mu(\mathbf{e}_B N^B_A) = \mathbf{e}_B(\partial_\mu N^B_A + H^B_{C\mu} N^C_A),$$

where $H^A_{B\mu}$ are given by equations (22). Considering that $\mathbf{p}_A$ and $\mathbf{e}_A$ should coincide at $x = 0$, one obtains the following system of equations for the 25 scalar coordinate functions $N^A_B(x)$:

$$\partial_\mu N^A_B(x) + H^A_{C\mu} N^C_B(x) = 0 \quad \text{and} \quad N^A_B(0) = \delta^A_B.$$

This system can be easily solved and gives

$$\begin{array}{ll} N^5_5(x) = 1, & N^\alpha_\beta(x) = \delta^\alpha_\beta, \\ N^\alpha_5(x) = 0, & N^5_\alpha(x) = \kappa x_\alpha, \end{array}$$

where $x_\alpha \equiv \eta_{\alpha\beta} x^\beta$ are covariant Lorentz coordinates. We thus see that $\mathbf{p}_A$ are expressed in terms of $\mathbf{e}_A$ as follows:

$$\begin{array}{l} \mathbf{p}_\alpha(x) = \mathbf{e}_\alpha(x) + \kappa x_\alpha \mathbf{e}_5(x) \\ \mathbf{p}_5(x) = \mathbf{e}_5(x). \end{array} \tag{26}$$

I will call $\mathbf{p}_A$ a P-basis ('P' stands for '*parallel*') associated with the given system of Lorentz coordinates. Simple calculations show that

$$\begin{array}{l} h(\mathbf{p}_\alpha, \mathbf{p}_\beta) = \eta_{\alpha\beta} + \kappa^2 x_\alpha x_\beta \\ h(\mathbf{p}_\alpha, \mathbf{p}_5) = \kappa x_\alpha, \;\; h(\mathbf{p}_5, \mathbf{p}_5) = 1, \end{array} \tag{27}$$

so $\mathbf{p}_A$ are orthonormal only at the origin.

Thus, with any system of Lorentz coordinates in flat space-time one can associate two special sets of five-vector basis fields: an O-basis, which is orthonormal everywhere but is not self-parallel, or a P-basis, which is self-parallel but is not orthonormal anywhere except for the origin. At $\kappa = 0$ the two bases coincide.

E. *Poincare transformation of five-tensor components*

Let us now derive the formulae for transformation of five-vector components and of components of other five-tensors as one passes from one system of Lorentz coordinates to another.

In the general case, with transformation of the five-vector basis according to the formula

$$\mathbf{e}_A \to \mathbf{e}'_A = \mathbf{e}_B L^B_A,$$

the components of an arbitrary five-vector $\mathbf{v}$ transform as

$$v^A \to v'^A = (L^{-1})^A_B \, v^B. \tag{28}$$

If $\tilde{\mathbf{o}}^A$ is the basis of five-vector 1-forms dual to $\mathbf{e}_A$, one should have

$$\tilde{\mathbf{o}}^A \to \tilde{\mathbf{o}}'^A = (L^{-1})^A_B \, \tilde{\mathbf{o}}^B,$$

and, accordingly, the components of an arbitrary five-vector 1-form $\widetilde{\mathbf{w}}$ in this dual basis transform as

$$w_A \to w'_A = w_B L^B_A. \tag{29}$$

Consider now an arbitrary Poincare transformation of Lorentz coordinates:

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu. \tag{30}$$

The same reasoning as in subsection B shows that the corresponding O-basis transforms as

$$\mathbf{e}'_\alpha = \mathbf{e}_\beta \, (\Lambda^{-1})^\beta_\alpha \quad \text{and} \quad \mathbf{e}'_5 = \mathbf{e}_5, \tag{31}$$

and from formulae (28) and (29) one obtains the following transformation laws for components of five-vectors and forms:

$$\left\{ \begin{array}{l} v'^\alpha = \Lambda^\alpha_\beta \, v^\beta \\ v'^5 = v^5 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} w'_\alpha = w_\beta (\Lambda^{-1})^\beta_\alpha \\ w'_5 = w_5. \end{array} \right. \tag{32}$$

Thus, the first four components of any five-vector or five-vector 1-form in the O-basis transform exactly as components of a four-vector or a four-vector 1-form, while the fifth component behaves as a scalar.

Let us now see what happens in the P-basis. According to formulae (26) and (31) and to formula (44) of Appendix, one has

$$\mathbf{p}'_5 = \mathbf{e}'_5 = \mathbf{e}_5 = \mathbf{p}_5$$

and

$$\begin{aligned} \mathbf{p}'_\alpha &= \mathbf{e}'_\alpha + \kappa x'_\alpha \mathbf{e}'_5 \\ &= (\mathbf{e}_\beta + \kappa x_\beta \mathbf{e}_5)\,(\Lambda^{-1})^\beta_{\ \alpha} + \kappa a_\alpha \mathbf{e}_5 \\ &= \mathbf{p}_\beta\,(\Lambda^{-1})^\beta_{\ \alpha} + \kappa a_\alpha \mathbf{p}_5, \end{aligned}$$

where $a_\alpha = \eta_{\alpha\beta} a^\beta$. Formulae (28) and (29) now give

$$\left\{ \begin{array}{l} v'^\alpha = \Lambda^\alpha_{\ \beta} v^\beta \\ v'^5 = v^5 - \kappa a_\alpha \Lambda^\alpha_{\ \beta}\, v^\beta \end{array} \right. \tag{33a}$$

and

$$\left\{ \begin{array}{l} w'_\alpha = w_\beta (\Lambda^{-1})^\beta_{\ \alpha} + \kappa a_\alpha w_5 \\ w'_5 = w_5. \end{array} \right. \tag{33b}$$

We thus see that at $\kappa \neq 0$ the transformation laws for five-tensor components in the P-basis are essentially different from what one has in the O-basis. In particular, these components transform nontrivially under space-time translations, and now one is able to understand why.

A global P-basis can exist only in flat space-time, where the parallel transport of five-vectors is independent of the path along which it is made. A P-basis can be constructed by choosing an orthonormal five-vector basis at one point and transporting it parallelly to all other points in space-time. Since (at $\kappa \neq 0$) the inner product of five-vectors is not conserved by parallel transport, the P-basis cannot be orthonormal at every point. Actually, the rules of parallel transport for five-vectors are such that $\mathbf{p}_A$ are orthonormal only at the origin. Moreover, as one can see from formulae (27), at each point the inner product matrix $h_{AB} \equiv h(\mathbf{p}_A, \mathbf{p}_B)$ has its own value, different from the values it has at all other points. This means that having a P-basis, one is able to distinguish points without using any coordinates. In fact, if need be, one can recover the relevant Lorentz coordinates by simply calculating the inner product of $\mathbf{p}_\alpha$ and $\mathbf{p}_5$ and using the formula

$$x_\alpha(Q) = \kappa^{-1} h(\mathbf{p}_\alpha(Q), \mathbf{p}_5(Q)).$$

Thus, the P-basis is a structure which is rigitly connected to space-time points and to one of the Lorentz coordinate systems. When the latter is changed, the P-basis changes too.

5. Examples of five-tensors

A. *How to find a five-vector or a five-tensor*

In the previous two sections we have examined the basic algebraic and differential properties of five-vectors. There now arises a natural question: are there any physical or purely geometric quantities that are described by five-vectors or by other nontrivial five-tensors (by the ones not reducible to a four-tensor)? This brings us to another question: how can one discover a five-vector or a five-tensor? One possible answer to this question is the same as to a similar question for four-vectors: one has to find several quantities that under Lorentz transformations and translations in flat space-time transform as components of a five-vector or of some other five-tensor. Since one is talking about components, one has to specify the basis in which they are evaluated. This is a simple matter if the definition of the quantities one considers involves only scalars and components of four-tensors in a Lorentz basis: since in either case $\nabla_\mu = \partial_\mu$, the same should be true for the quantities defined, and considering that in this basis $g_{\mu\nu} = \eta_{\mu\nu}$, one concludes that the five-tensor components should correspond to a P-basis and consequently should transform according to formulae (33).

It is apparent that this method of searching for five-tensors fails if $\kappa = 0$, since in this case the transformation formulae do not enable one to distinguish the components of a five-tensor from components of several four-tensors. At $\kappa \neq 0$ the method works, but it does not allow one to determine the precise value of κ. Indeed, if one has, say, five quantities, v^A, that transform according to formulae (33) at a certain value of κ, one can always construct five other quantities:

$$u^\alpha = v^\alpha \text{ and } u^5 = \lambda v^5,$$

where λ is an arbitrary nonzero constant, which will transform as

$$u'^5 = u^5 - (\lambda\kappa) a_\alpha \Lambda^\alpha_{\ \beta}\, u^\beta \quad \text{and} \quad u'^\alpha = \Lambda^\alpha_{\ \beta} u^\beta.$$

So as far as transformation laws are concerned, this quintuple may correspond to a five-vector at any nonzero κ.

In the following I will suppose that $\kappa \neq 0$. In this case it is convenient to slightly modify the definitions of the O- and P-bases by taking that in both cases the fifth basis vector is normalized to $|\kappa|$ rather than to unity. In other words, it will be taken that $\mathbf{e}_5 = \mathbf{p}_5 = \kappa\mathbf{n}$, where $\mathbf{n}$ is one of the two normalized directional vectors of V_4. Such a change in the definitions results in that the constant κ disappears from formulae (26) and (33) and the latter acquire a simpler form:

$$\begin{aligned} \mathbf{p}_\alpha(x) &= \mathbf{e}_\alpha(x) + x_\alpha \mathbf{e}_5(x) \\ \mathbf{p}_5(x) &= \mathbf{e}_5(x). \end{aligned} \tag{34}$$

and

$$\begin{cases} v'^\alpha = \Lambda^\alpha_\beta v^\beta \\ v'^5 = v^5 - a_\alpha \Lambda^\alpha_\beta v^\beta \end{cases} \tag{35a}$$

and

$$\begin{cases} w'_\alpha = w_\beta (\Lambda^{-1})^\beta_\alpha + a_\alpha w_5 \\ w'_5 = w_5. \end{cases} \tag{35b}$$

B. *Covariant Lorentz coordinates and parameters of Poincare transformations*

The simplest example of quantities that transform as components of a nontrivial five-tensor are covariant Lorentz coordinates. Comparing formula (44) of Appendix with formulae (35), we see that under Lorentz transformations and translations the five quantities x_A, where $x_5 \equiv 1$, transform as components of a five-vector 1-form. Consequently, if $\widetilde{\mathbf{q}}^A$ is the basis of five-vector 1-forms dual to the P-basis associated with the selected Lorentz coordinate system, the 1-form $\widetilde{\mathbf{x}}$ constructed according to the formula

$$\widetilde{\mathbf{x}}(x) \equiv x_\alpha \widetilde{\mathbf{q}}^\alpha(x) + \widetilde{\mathbf{q}}^5(x), \tag{36}$$

will be the same no matter which system of Lorentz coordinates is used.[5]

¿From equations (34) one can easily obtain the formulae that relate the basis $\widetilde{\mathbf{q}}^A$ to the basis of five-vector 1-forms $\widetilde{\mathbf{o}}^A$ dual to the O-basis corresponding to the same coordinates:

$$\widetilde{\mathbf{q}}^\alpha(x) = \widetilde{\mathbf{o}}^\alpha(x) \;\text{ and }\; \widetilde{\mathbf{q}}^5(x) = \widetilde{\mathbf{o}}^5(x) - x_\alpha \widetilde{\mathbf{o}}^\alpha(x). \tag{37}$$

Substituting these relations into definition (36), one obtains the following expression for the 1-form $\widetilde{\mathbf{x}}$ in the basis $\widetilde{\mathbf{o}}^A$:

$$\widetilde{\mathbf{x}}(x) = x_\alpha \widetilde{\mathbf{o}}^\alpha(x) + \widetilde{\mathbf{o}}^5(x) - x_\alpha \widetilde{\mathbf{o}}^\alpha(x) = \widetilde{\mathbf{o}}^5(x),$$

from which one can clearly see that $\widetilde{\mathbf{x}}$ is indeed independent of the choice of the coordinate system.

[5] As it has already been noted, for each Lorentz coordinate system there exist *two* associated O-bases differing from each other only in the overall sign of the basis five-vectors. By virtue of equations (34), the same is true of the P-bases: there are two of them, and when constructing the 1-form $\widetilde{\mathbf{x}}$ corresponding to the quintuple x_A one may use either of them. The 1-forms obtained with these two bases will apparently differ in the sign. However, this ambiguity is of no significance to us, since the results obtained below will be the same no matter which of the two P-bases is selected.

Let us also evaluate the covariant derivative of the field $\widetilde{\mathbf{x}}$. Since in the P-basis all five-vector connection coefficients are zero, one has

$$\nabla_\mu \widetilde{\mathbf{x}} = \partial_\mu x_\alpha \cdot \widetilde{\mathbf{q}}^\alpha = \eta_{\mu\alpha} \widetilde{\mathbf{q}}^\alpha. \tag{38}$$

The same result can be obtained in the O-basis, if one considers that in this case the only nonzero connection coefficients are $G^5_{\ \alpha\mu} = -\,\eta_{\alpha\mu}$, and so

$$\nabla_\mu \widetilde{\mathbf{x}} = \nabla_\mu \widetilde{\mathbf{o}}^5 = -\,G^5_{\ A\mu} \widetilde{\mathbf{o}}^A = -\,G^5_{\ \alpha\mu} \widetilde{\mathbf{o}}^\alpha = \eta_{\mu\alpha} \widetilde{\mathbf{o}}^\alpha,$$

which on account of the first of equations (37), coincides with result (38).

Another example of purely geometric quantities that transform as components of a nontrivial five-tensor are parameters of Poincare transformations. When formulating the symmetry properties of flat space-time in section 4, I have used Lorentz coordinates only as a tool for constructing the equivalent fields. By itself, the replacement of a given set of fields with an equivalent set, which is nothing but an active field transformation, is an invariant procedure and can be considered without referring to any coordinates. However, depending on how the latter are selected, a given field transformation will correspond to different coordinate transformations. Let us now find how the parameters of these coordinate transformations change as one passes from one system of Lorentz coordinates to another.

The idea of the following calculation is very simple. One selects some set of fields and a system of Lorentz coordinates, and by means of an arbitrary Poincare transformation constructs the equivalent set of fields. One then considers another system of Lorentz coordinates and determines the precise Poincare transformation that one has to make in these new coordinates to obtain the same set of equivalent fields. Finally, one expresses the parameters of this second Poincare transformation in terms of the parameters of the first one.

As a set of fields it is convenient to choose the covariant coordinates associated with the selected Lorentz coordinate system x^α, i.e. four scalar fields $\varphi_{(\alpha)}$ $(\alpha = 0, 1, 2, 3)$ such that

$$\varphi_{(\alpha)}(Q) = \eta_{\alpha\beta} x^\beta(Q)$$

at every point Q. Let us consider an arbitrary Poincare transformation that corresponds to the coordinate transformation

$$x_\alpha \to y_\alpha = x_\beta L^\beta_{\ \alpha} + b_\alpha. \tag{39}$$

By virtue of the symmetry principle, the equivalent fields obtained by this transformation are

$$\varphi^{\text{equiv}}_{(\alpha)} = y_\alpha = x_\beta L^\beta_{\ \alpha} + b_\alpha.$$

Let us now consider another system of Lorentz coordinates:

$$x'^{\alpha} = \Lambda^{\alpha}_{\ \beta} x^{\beta} + a^{\alpha}.$$

In these coordinates the original fields acquire the form

$$\varphi_{(\alpha)} = (x'_{\beta} - a_{\beta})\Lambda^{\beta}_{\ \alpha},$$

and the equivalent fields are

$$\varphi^{\text{equiv}}_{(\alpha)} = (x'_{\gamma} - a_{\gamma})\Lambda^{\gamma}_{\ \beta} L^{\beta}_{\ \alpha} + b_{\alpha}.$$

One should now present the right-hand side of the latter equation as

$$\varphi^{\text{equiv}}_{(\alpha)} = (y'_{\beta} - a_{\beta})\Lambda^{\beta}_{\ \alpha},$$

where

$$y'_{\alpha} \equiv x'_{\beta} L'^{\beta}_{\ \alpha} + b'_{\alpha},$$

and then express $L'^{\beta}_{\ \alpha}$ and b'_{α} in terms of $L^{\beta}_{\ \alpha}$ and b_{α}. Straightforward calculations give

$$\begin{array}{l} L'^{\alpha}_{\ \beta} = \Lambda^{\alpha}_{\ \sigma} L^{\sigma}_{\ \tau} (\Lambda^{-1})^{\tau}_{\ \beta} \\ b'_{\beta} = b_{\tau}(\Lambda^{-1})^{\tau}_{\ \beta} + a_{\beta} - a_{\rho}\Lambda^{\rho}_{\ \sigma} L^{\sigma}_{\ \tau}(\Lambda^{-1})^{\tau}_{\ \beta}, \end{array} \qquad (40)$$

which shows that the quantities $\mathcal{T}^{A}_{B}$ defined as

$$\mathcal{T}^{\alpha}_{\beta} = L^{\alpha}_{\beta},\ \mathcal{T}^{5}_{\beta} = b_{\beta},\ \mathcal{T}^{\alpha}_{5} = 0, \ \text{ and } \ \mathcal{T}^{5}_{5} = 1,$$

transform as components of a five-tensor of rank $(1, 1)$.

It is also interesting to find the transformation formulae for the parameters of infinitesimal Poicare transformations. In this case the matrix $L^{\alpha}_{\ \beta}$ in equation (39) can be presented as

$$L^{\alpha}_{\ \beta} = \delta^{\alpha}_{\ \beta} + \tfrac{1}{2}(\delta^{\alpha}_{\nu}\eta_{\beta\mu} - \delta^{\alpha}_{\mu}\eta_{\beta\nu})\,\omega^{\mu\nu},$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$, and both $\omega^{\mu\nu}$ and b_{α} are infinitesimals. From formulae (40) one obtains

$$\omega'^{\mu\nu} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\omega^{\alpha\beta} \quad \text{and} \quad b'^{\mu} = \Lambda^{\mu}_{\ \nu}(b^{\nu} - a_{\alpha}\Lambda^{\alpha}_{\ \beta}\,\omega^{\nu\beta}),$$

which shows that the quantities $\mathcal{R}^{AB}$ defined as

$$\mathcal{R}^{\mu\nu} = \omega^{\mu\nu},\ \mathcal{R}^{\mu 5} = -\mathcal{R}^{5\mu} = b^{\mu}, \ \text{ and } \ \mathcal{R}^{55} = 0,$$

transform as components of an antisymmetric five-tensor of rank $(2, 0)$.

Further discussion of tensors $\mathcal{T}^{A}_{B}$ and $\mathcal{R}^{AB}$ will be made in part III.

C. *Stress-energy and angular momentum tensors*

Let us now consider an example of physical quantities that transform as components of a five-tensor: the canonical stress-energy and angular momentum tensors, Θ^{μ}_{α} and $M^{\mu}_{\alpha\beta}$.

Let us begin by writing out the formulae that express the components of these two tensors in one Lorentz coordinate system in terms of their components in another Lorentz coordinate system. If the two coordinate systems are related as in equation (30), then

$$\begin{array}{rcl} \Theta'^{\mu}_{\ \alpha} &=& \Lambda^{\mu}_{\ \nu}\,\Theta^{\nu}_{\ \beta}\,(\Lambda^{-1})^{\beta}_{\ \alpha}, \\ M'^{\mu}_{\ \alpha\beta} &=& x'_{\alpha}\Theta'^{\mu}_{\ \beta} - x'_{\beta}\Theta'^{\mu}_{\ \alpha} + \Sigma'^{\mu}_{\ \alpha\beta} \\ &=& \Lambda^{\mu}_{\ \nu} M^{\nu}_{\ \sigma\tau}(\Lambda^{-1})^{\sigma}_{\ \alpha}(\Lambda^{-1})^{\tau}_{\ \beta} \\ && \quad + a_{\alpha}\Lambda^{\mu}_{\ \nu}\Theta^{\nu}_{\ \tau}\,(\Lambda^{-1})^{\tau}_{\ \beta} \\ && \qquad - a_{\beta}\Lambda^{\mu}_{\ \nu}\Theta^{\nu}_{\ \sigma}\,(\Lambda^{-1})^{\sigma}_{\ \alpha}, \end{array} \qquad (41)$$

where $\Sigma^{\mu}_{\alpha\beta}$ is the spin angular momentum tensor.

With respect to their lower indices, Θ^{μ}_{α} and $M^{\mu}_{\alpha\beta}$ are traditionally regarded as components of four-tensors, and the fact that under space-time translations $M^{\mu}_{\alpha\beta}$ acquires additional terms proportional to Θ^{μ}_{α} is interpreted as a consequence of one actually making a switch from one quantity—the angular momentum relative to the point $x^{\mu} = 0$, to another quantity—the angular momentum relative to the point $x'^{\mu} = 0$. Five-vectors enable one to give this fact a different interpretation, which in several ways is more attractive.

One should notice that equations (41) coincide exactly with the transformation formulae for components in the P-basis of a tensor—let us denote it as $\mathcal{M}$—that has one (upper) four-vector index and two (lower) five-vector indices and whose components are related to Θ^{μ}_{α} and $M^{\mu}_{\alpha\beta}$ as follows:

$$\begin{array}{l} \mathcal{M}^{\mu}_{\alpha\beta} = M^{\mu}_{\alpha\beta},\ \ \mathcal{M}^{\mu}_{5\alpha} = \Theta^{\mu}_{\alpha} \\ \mathcal{M}^{\mu}_{\alpha 5} = -\Theta^{\mu}_{\alpha},\ \ \mathcal{M}^{\mu}_{55} = 0. \end{array} \qquad (42)$$

This coincidence means that Θ^{μ}_{α} and $M^{\mu}_{\alpha\beta}$ can be regarded *as components of a single five-tensor*. Since by definition $M^{\mu}_{\alpha\beta} = -M^{\mu}_{\beta\alpha}$, this tensor is antisymmetric in its lower (five-vector) indices.[6]

Such an interpretation of Θ^{μ}_{α} and $M^{\mu}_{\alpha\beta}$ implies that there exists a single local physical quantity: the stress–energy–angular momentum tensor $\mathcal{M}$. The belief that there are many different angular momenta should now be regarded as merely a wrong impression created by interpreting Θ^{μ}_{α} and $M^{\mu}_{\alpha\beta}$ as four-tensors: in reality, all these angular momenta are components of $\mathcal{M}$ in different five-vector bases.

[6] More precisely, here $\mathcal{M}$ is regarded as a dual of a four-vector 3-form whose values are covariant antisymmetric five-tensors of rank 2.

There is now no difficulty in defining the angular momentum density in curved space-time. To see how this can be done, let us evaluate the components of $\mathcal{M}$ in the O-basis. Using relations (37), one has

$$\begin{aligned}\mathcal{M} &= (x_\alpha \Theta^\mu_\beta - x_\beta \Theta^\mu_\alpha + \Sigma^\mu_{\alpha\beta})\, \tilde{\mathbf{q}}^\alpha \otimes \tilde{\mathbf{q}}^\beta \otimes \mathbf{E}_\mu \\ &\qquad + (\Theta^\mu_\beta)\, \tilde{\mathbf{q}}^5 \otimes \tilde{\mathbf{q}}^\beta \otimes \mathbf{E}_\mu \\ &\qquad\qquad + (-\Theta^\mu_\alpha)\, \tilde{\mathbf{q}}^\alpha \otimes \tilde{\mathbf{q}}^5 \otimes \mathbf{E}_\mu \\ &= \Sigma^\mu_{\alpha\beta}\, \tilde{\mathbf{o}}^\alpha \otimes \tilde{\mathbf{o}}^\beta \otimes \mathbf{E}_\mu \\ &\qquad + (\Theta^\mu_\beta)\, \tilde{\mathbf{o}}^5 \otimes \tilde{\mathbf{o}}^\beta \otimes \mathbf{E}_\mu \\ &\qquad\qquad + (-\Theta^\mu_\alpha)\, \tilde{\mathbf{o}}^\alpha \otimes \tilde{\mathbf{o}}^5 \otimes \mathbf{E}_\mu .\end{aligned}$$

Thus, in the O-basis $\mathcal{M}^\mu_{\alpha\beta}$ coincide with the components of the spin angular momentum tensor. In the case of flat space-time one gives preference to the P-basis, since in it $\nabla_\mu = \partial_\mu$, and, accordingly, the $\mathcal{M}^\mu_{\alpha\beta}$ components acquire additional terms proportional to covariant Lorentz coordinates and to the components $\mathcal{M}^\mu_{\alpha 5}$ and $\mathcal{M}^\mu_{5\beta}$. In the case of curved space-time, where a global self-parallel basis does not exist, it is more convenient to use a regular basis and have $\mathcal{M}^\mu_{\alpha\beta} = \Sigma^\mu_{\alpha\beta}$.

Let us now recall that canonical Θ^μ_α and $M^\mu_{\alpha\beta}$ are defined as Noether currents corresponding to Poincare transformations and as such satisfy the following "conservation laws":

$$\begin{aligned}&\partial_\mu \Theta^\mu_\alpha = 0 \\ &\partial_\mu M^\mu_{\alpha\beta} = \eta_{\alpha\mu}\Theta^\mu_\beta - \eta_{\beta\mu}\Theta^\mu_\alpha + \partial_\mu \Sigma^\mu_{\alpha\beta} = 0.\end{aligned}$$

One can now replace these two four-tensor equations with a single covariant five-tensor equation:

$$\mathcal{M}^\mu_{\alpha\beta;\mu} = 0, \tag{43}$$

where it has been taken into account that in the P-basis all five-vector connection coefficients are zero. It is interesting to see how equation (43) works in the O-basis. One has

$$\begin{aligned}\mathcal{M}^\mu_{5\alpha;\,\mu} &= \partial_\mu \mathcal{M}^\mu_{5\alpha} - \mathcal{M}^\mu_{A\alpha} G^A_{\ 5\mu} - \mathcal{M}^\mu_{5A} G^A_{\ \alpha\mu} \\ &= \partial_\mu \Theta^\mu_\alpha - \mathcal{M}^\mu_{55} G^5_{\ \alpha\mu} \ = \ \partial_\mu \Theta^\mu_\alpha \ = \ 0\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}^\mu_{\alpha\beta;\,\mu} &= \partial_\mu \mathcal{M}^\mu_{\alpha\beta} - \mathcal{M}^\mu_{A\beta} G^A_{\ \alpha\mu} - \mathcal{M}^\mu_{\alpha A} G^A_{\ \beta\mu} \\ &= \partial_\mu \Sigma^\mu_{\alpha\beta} - \Theta^\mu_\beta G^5_{\ \alpha\mu} + \Theta^\mu_\alpha G^5_{\ \beta\mu} \\ &= \partial_\mu \Sigma^\mu_{\alpha\beta} + \Theta^\mu_\beta \eta_{\alpha\mu} - \Theta^\mu_\alpha \eta_{\beta\mu} \ = \ 0.\end{aligned}$$

Thus, one obtains the same conservation laws for Θ^μ_α and $\Sigma^\mu_{\alpha\beta}$, only now the terms proportional to Θ^μ_α in the second equation come from connection coefficients.

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Appendix: Poincare transformation of covariant Lorentz coordinates in the five-dimensional representation

With any system of Lorentz coordinates, x^α, in flat space-time one can associate a system of covariant Lorentz coordinates defined as $x_\alpha \equiv \eta_{\alpha\beta} x^\beta$, where $\eta_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$ is the Minkowski metric tensor. Under the Poincare transformation

$$x'^\alpha = \Lambda^\alpha_{\ \beta} x^\beta + a^\alpha,$$

the covariant coordinates transform as

$$x'_\alpha = x_\beta (\Lambda^{-1})^\beta_{\ \alpha} + a_\alpha, \tag{44}$$

where $(\Lambda^{-1})^\beta_{\ \alpha}$ is the inverse of $\Lambda^\alpha_{\ \beta}$ and $a_\alpha \equiv \eta_{\alpha\beta} a^\beta$. Formally, one can present this inhomogeneous transformation as a homogeneous transformation by introducing a *fifth* coordinate, x_5, which is assigned a constant nonzero value, for example, $x_5 = 1$.[7] Transformation (44) can then be presented as

$$x'_A = x_B L^B_{\ A},$$

where A and B run 0, 1, 2, 3, and 5 and where

$$\begin{cases} L^5_{\ 5} = 1, \ L^\alpha_{\ 5} = 0, \\ L^5_{\ \beta} = a_\beta, \ L^\alpha_{\ \beta} = (\Lambda^{-1})^\alpha_{\ \beta}. \end{cases}$$

If x_5 is assigned some other nonzero value, say, $x_5 = \kappa^{-1}$, then instead of the latter formulae one will have

$$\begin{cases} L^5_{\ 5} = 1, \ L^\alpha_{\ 5} = 0, \\ L^5_{\ \beta} = \kappa a_\beta, \ L^\alpha_{\ \beta} = (\Lambda^{-1})^\alpha_{\ \beta}. \end{cases}$$

References

1. Th. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Berlin, Math.–Phys. K1.* (1921) 966; O. Klein, *Z. Phys.*, **46** (1927) 188.

2. See e.g. K. S. Stelle and P. C. West, *Phys. Rev.* **D21** (1980) 1466.

[7] A representation of this kind is used e.g. in the theory of crystallographic groups.