Fuzzy Topological Systems*

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Abstract

Dialectica categories are a very versatile categorical model of linear logic. These have been used to model many seemingly different things (e.g., Petri nets and Lambek's calculus). In this note, we expand our previous work on fuzzy petri nets to deal with fuzzy topological systems. One basic idea is to use as the dualizing object in the Dialectica categories construction, the unit real interval $\mathrm{I} = [0, 1]$, which has all the properties of a *lineale*. The second basic idea is to generalize Vickers's notion of a topological system.

1 Introduction

Fuzzy set theory and fuzzy logic have been invented by Lotfi ali Asker Zadeh. This is a theory that started from a generalization of the set concept and the notion of a truth value (for an overview, for example, see [8]). In fuzzy set theory, an element of a fuzzy subset belongs to it to a degree, which is usually a number between 0 and 1. For example, if we have a fuzzy subset of white colors, then all the gray-scale colors are white to a certain degree and, thus, belong to t his set with a degree. The following definition by Zadeh himself explains what fuzzy logic is:[1]

Definition Fuzzy logic is a precise system of reasoning, deduction and computation in which the objects of discourse and analysis are associated with information which is, or is allowed to be, imprecise, uncertain, incomplete, unreliable, partially true or partially possible.

Categories, which were invented by Samuel Eilenberg and Saunders Mac Lane, form a very high-level abstract mathematical theory that unifies all branches of mathematics. Category theory plays a central role in modern mathematics and theoretical computer science, and, in addition, it is used in mathematical physics, in software engineering, etc. Categories have been used to model and study logical systems. In particular, the Dialectica categories of de Paiva [4] are categorical model of linear logic [7]. These categories have been used to model Petri nets [2], the Lambek Calculus [12], state in programming [3], and to define fuzzy petri nets [6]. Using some of the ideas in our previous work on fuzzy petri nets, we wanted to develop the idea of fuzzy topological systems, that is, the fuzzy counterpart of Vickers's [13] topological systems. In this note, we present fuzzy topological systems and discuss some of their properties.

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[1] The definition was posted to the bisc-group mailing list on 22/11/2008.

2 The category $\mathsf{Dial}_{\mathrm{I}}(\mathbf{Set})$

The Dialectica categories construction (see for example [5]) can be instantiated using any lineale and the basic category **Set**. As discussed in [11], the unit interval, since it is a Heyting algebra, has all the properties of a lineale structure. Recall that a lineale is a structure defined as follows:

Definition The quintuple $(L, \leq, \circ, 1, \multimap)$ is a lineale if:

- $(L, \leq)$ is poset,
- $\circ : L \times L \to L$ is an order-preserving multiplication, such that $(L, \circ, 1)$ is a symmetric monoidal structure (i.e., for all $a \in L$, $a \circ 1 = 1 \circ a = a$).
- if for any $a, b \in L$ exists a largest $x \in L$ such that $a \circ x \leq b$, then this element is denoted $a \multimap b$ and is called the pseudo-complement of a with respect to b.

Now, one can prove that the quintuple $(\mathrm{I}, \leq, \wedge, 1, \Rightarrow)$, where I is the unit interval, $a \wedge b = \min\{a, b\}$, and $a \Rightarrow b = \bigvee\{c : c \wedge a \leq b\}$ $(a \vee b = \max\{a, b\})$, is a lineale.

Let U and X be nonempty sets. A binary fuzzy relation R in U and X is a fuzzy subset of $U \times X$, or $U \times X \to \mathrm{I}$. The value of $R(u, x)$ is interpreted as the *degree* of membership of the ordered pair (u, x) in R. Let us now define a category of fuzzy relations.

Definition The category $\mathsf{Dial}_{\mathrm{I}}(\mathbf{Set})$ has as objects triples $A = (U, X, \alpha)$, where U and X are sets and α is a map $U \times X \to \mathrm{I}$. Thus, each object is a fuzzy relation. A map from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ is a pair of **Set** maps (f, g), $f : U \to V$, $g : Y \to X$ such that

$$\alpha(u, g(y)) \leq \beta(f(u), y),$$

or in pictorial form:

$$\begin{array}{ccc} U \times Y & \xrightarrow{\mathrm{id}_U \times g} & U \times X \\ {\scriptstyle f \times \mathrm{id}_Y}\downarrow & \geq & \downarrow{\scriptstyle \alpha} \\ V \times Y & \xrightarrow[\beta]{} & \mathrm{I} \end{array}$$

Assume that (f, g) and (f', g') are the following arrows:

$$(U, X, \alpha) \overset{(f,g)}{\longrightarrow} (V, Y, \beta) \overset{(f',g')}{\longrightarrow} (W, Z, \gamma).$$

Then $(f, g) \circ (f', g') = (f \circ f', g' \circ g)$ such that

$$\alpha\Big(u, (g' \circ g)(z)\Big) \leq \gamma\Big((f \circ f')(u), z\Big).$$

Tensor products and the internal-hom in $\mathsf{Dial}_{\mathrm{I}}(\mathbf{Set})$ are given as in the Girard-variant of the Dialectica construction [4]. Given objects $A = (U, X, \alpha)$ and $B = (V, Y, \beta)$, the tensor product $A \otimes B$ is $(U \times V, X^V \times Y^U, \alpha \times \beta)$, where the $\alpha \times \beta$ is the relation that, using the lineale structure of I, takes the minimum of the membership degrees. The linear function-space or internal-hom is given by $A \to B = (V^U \times Y^X, U \times X, \alpha \to \beta)$, where again the relation $\alpha \to \beta$ is given by the implication in the lineale. With this structure we obtain:

Theorem 2.1 *The category* $\mathsf{Dial}_{\mathrm{I}}(\mathbf{Sets})$ *is a monoidal closed category with products and coproducts.*

Products and coproducts are given by $A \times B = (U \times V, X + Y, \gamma)$ and $A \oplus B = (U + V, X \times Y, \delta)$, where $\gamma : U \times V \times (X + Y) \to \mathrm{I}$ is the fuzzy relation that is defined as follows

$$\gamma\big((u,v),z\big) = \begin{cases} \alpha(u,x), & \text{if } z = (x,0) \\ \beta(v,y), & \text{if } z = (y,1) \end{cases}$$

Similarly for the coproduct $A \oplus B$.

3 Fuzzy Topological Systems

Let $A = (U, X, \alpha)$ be an object of $\mathsf{Dial}_{\mathrm{I}}(\mathbf{Set})$, where X is a frame, that is, a poset $(X, \leq)$ where

1. every subset S of X has a join
2. every finite subset S of X has a meet
3. binary meets distribute over joins, if Y is a subset of X:

$$x \wedge \bigvee Y = \bigvee \Big\{ x \wedge y : y \in Y \Big\}.$$

Given such a triple, we can view A as a *fuzzy topological system*, that is, the fuzzy counterpart of Vickers's [13] *topological systems*.

A *topological system* in Vicker's monograph[13] is a triple $(U, \models, X)$, where X is a frame whose elements are called *opens* and U is a set whose elements are called *points*. Also, the relation $\models$ is a subset of $U \times X$, and when $u \models x$, we say that u *satisfies* x. In addition, the following must hold

- if S is a finite subset of X, then

$$u \models \bigwedge S \Longleftrightarrow u \models x \text{ for all } x \in S.$$

- if S is any subset of X, then

$$u \models \bigvee S \Longleftrightarrow u \models x \text{ for some } x \in S.$$

Given two topological systems (U, X) and (V, Y), a map from (U, X) to (V, Y) consists of a function $f : U \to V$ and a frame homomorphism $\phi : Y \to X$, if $u \models \phi(y) \Leftrightarrow f(u) \models y$. Topological systems and continuous maps between them form a category, which we write as **TopSystems**.

In order to fuzzify topological systems, we need to fuzzify the relation "$\models$." However, the requirement imposed on the relation of satisfaction is too severe when dealing with fuzzy structures. Indeed, in some reasonable categorical models of fuzzy structures (see, for example [1, 11]), the authors use a weaker condition where the e quivalence operator is replaced by an implication operator. Thus we suggest that the corresponding condition for morphisms of fuzzy topological systems should become $u \models \phi(y) \Rightarrow f(u) \models y$.

Definition A *fuzzy topological system* is a triple (U, α, X), where U is a set, X is a frame and $\alpha : U \times X \to \mathrm{I}$ a binary fuzzy relation such that:

(i) If S is a finite subset of X, then

$$\alpha(u, \bigwedge S) \leq \alpha(u, x) \text{ for all } x \in S.$$

(ii) If S is any subset of X, then

$$\alpha(u, \bigvee S) \leq \alpha(u, x) \text{ for some } x \in S.$$

(iii) $\alpha(u, \top) = 1$ and $\alpha(u, \bot) = 0$ for all $u \in U$.

To see that fuzzy topological systems also form a category we need to show that given morphisms $(f, F) : (U, X) \to (V, Y)$ and $(g, G) : (V, Y) \to (W, Z)$, the obvious composition $(g \circ f, F \circ G) : (U, X) \to (W, Z)$ is also a morphism of fuzzy topological systems. But we know $\mathsf{Dial}_{\mathrm{I}}(\mathbf{Set})$ is a category and conditions (i), (ii) and (iii) do not apply to morphisms. Identities are given by $(id_U, id_X) : (U, X) \to (U, X)$.

The collection of objects of $\mathsf{Dial}_{\mathrm{I}}(\mathbf{Set})$ that are fuzzy topological systems and the arrows between them, form the category **FTopSystems**, which is a subcategory of $\mathsf{Dial}_{\mathrm{I}}(\mathbf{Set})$.

Proposition 3.1 *Any topological system (U, X) is a fuzzy topological system (U, ι, X), where*

$$\iota(u, x) = \begin{cases} 1, & \text{when } u \models x \\ 0, & \text{otherwise} \end{cases}$$

Proof Consider the first property of the relation "$\models$"

$$u \models \bigwedge S \Longleftrightarrow u \models x \text{ for all } x \in S.$$

This will be translated to

$$\iota(u, \bigwedge S) \leq \iota(u, x) \text{ for all } x \in S.$$

The inequality is in fact an equality since whenever $u \models x$, $\iota(u, x) = 1$. Therefore, we can transform this condition into the following one

$$\iota(u, \bigwedge S) = \iota(u, x) \text{ for all } x \in S.$$

A similar argument holds true for the second property.

The following result is based on the previous one:

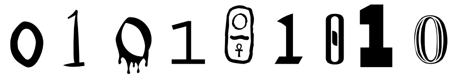
Theorem 3.2 *The category of topological systems is a full subcategory of* $\mathsf{Dial}_{\mathrm{I}}(\mathbf{Set})$.

Obviously, it is not enough to provide generalization of structures—one needs to demonstrate that these new structures have some usefulness. The following example gives an interpretation of these structures in a "real-life" situation.

Example Vickers [13, p. 53] gives an interesting physical interpretation of topological systems. In particular, he considers the set U to be a set of programs that generate bit streams and the opens to be assertions about bit streams. For exanple, if u is a program that generates the infinite bit stream 010101010101... and "**starts** 01010" is an assertion that is satified if a bit stream starts with the digits "01010", then this is expressed as follows:

$$x \models \textbf{starts } 01010.$$

Assume now that x' is a program that produces bit streams that look like the following one



The individuals bits are not identical to either "1" or "0," but rather similar to these. One can speculate that these bits are the result of some interaction of x' with its environment and this is the reason they are not identical. Then, we can say that x' satisfies the assertion "**starts** 01010" to some degree, since the elements that make up the stream produced by x' are not identical, but rather similar.

4 From Fuzzy Topological Systems to Fuzzy Topological Spaces

It is not difficult to map fuzzy topological systems to fuzzy topological spaces (for an overview of the theory of fuzzy topologies see [14]). The following definition shows how to map an open to fuzzy set:

Definition Assume that $a \in A$, where (U, α, X) is a fuzzy topological space. Then the *extent* of an open x is a function whose graph is given below:

$$\Big\{ \big(u, \alpha(u, x)\big) : u \in U \Big\}.$$

Proposition 4.1 *The collection of all fuzzy sets created by the extents of the members of A correspond to a fuzzy topology on X.*

Proof Assume that a and b are opens and let $\mathbf{a}(x) = \alpha(x, a)$, $\mathbf{b}(x) = \alpha(x, b)$, and $\psi(x) = \alpha(x, a \wedge b)$. Then $\alpha(x, a \wedge b) \leq \alpha(x, a)$ and $\alpha(x, a \wedge b) \leq \alpha(x, b)$. In different words, $\psi(x) \leq \mathbf{a}(x)$ and $\psi(x) \leq \mathbf{b}(x)$, which implies that $\psi(x) \leq \min\{\mathbf{a}(x), \mathbf{b}(x)\}$ that is $\psi = a \cap b$. Similarly, assume that $\{a_i\}$ is a collection of opens such that $\mathbf{a}_i(x) = \alpha(x, a_i)$ and $\phi(x) = \alpha(x, \bigvee_i a_i)$. The fact that there is one $\phi(x) \leq \mathbf{a}_j(x)$, while for all other $\mathbf{a}_i$ it holds that $\phi(x) \geq \mathbf{a}_j(x)$, implies that $\phi(x) = \sup_i \mathbf{a}_i(x)$, that is, $\phi = \bigcup_i \mathbf{a}_i(x)$. Finally, the last conditions generate the sets $\mathbf{1}(x) = 1$ and $\mathbf{0}(x) = 0$. So, the opens form a fuzzy topology on X.

5 Products and Sums of Fuzzy Topological Systems

In section 2 we described the categorical products and coproducts of any two objects of $\mathsf{Dial}_{\mathrm{I}}(\mathbf{Set})$. Given two fuzzy topological systems $A = (U, X, \alpha)$ and $B = (V, Y, \beta)$, their topological product is the space $A \times B = (U \times V, X + Y, \gamma)$. Since X and Y are frames it is necessary to modify the definition of $X + Y$ and, consequently, the definition of γ.

Definition Assume that $A = (U, X, \alpha)$ and $B = (V, Y, \beta)$ are two fuzzy topological systems. Then their topological product $A \times B$ is the system $(U \times V, \gamma, X \otimes Y)$, where $X \otimes Y$ is the tensor product of the two frames X and Y (see [13, pp. 80–85] for details) and γ is defined as follows:

$$\gamma\big((u, v), \bigvee_i x_i \otimes y_i\big) = \max\big\{\alpha(u, x), \beta(v, y)\big\}.$$

Obviously, the topological product is not the same as the categorical product. The topological sum is more straigthtforward:

Definition Assume that $A = (U, X, \alpha)$ and $B = (V, Y, \beta)$ are two fuzzy topological systems. Then their topological sum $A + B$ is the system $(U + V, \gamma, X \times Y)$, where γ is defined as follows:

$$\gamma\big(z, (x, y)\big) = \begin{cases} \alpha(u, x), & \text{if } z = (u, 0) \\ \beta(v, y), & \text{if } z = (v, 1) \end{cases}$$

Comparing the topological sum with the categorical sum reveals that they are identical.

6 Conclusions

We have simply started thinking about the possibilities of using Dialectica-like models in the context of fuzzy topological structures. Much remains to be done, in particular we would like to see if a framework based on an implicational notion of morphism like ours can cope with embedding several of the other notions of fuzzy sets considered by Rodabaugh [9]. Also seems likely that we could extend the work of Solovyov [10] on variable-basis topological spaces using similar ideas.

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