A class of localized solutions of the linear and nonlinear wave equations

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Abstract. Following the tradition from nano and picosecond optics, the basic theoretical studies continue to investigate the processes of propagation of femtosecond and attosecond laser pulses with the corresponding envelope equation for narrow-band laser pulses, working in paraxial approximation. We point, that this approximation is not valid for large band pulses. In air due to small dispersion the wave equation as well as the 3D+1 amplitude equation more accurate describe the pulse dynamics. New exact localized solutions of the linear wave and amplitude equations are presented. The solutions discover non-paraxial semi-spherical diffraction of single-cycle and half-cycle laser pulses and a new class of spherically symmetric solution of the wave equation. The propagation of large band optical pulses in nonlinear vacuum is also investigated in the frame of a system of nonlinear wave vector equations. Exact vector soliton solution with own angular momentum is obtained.

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INTRODUCTION

With the progress of laser innovations it is very important to study the localized waves, especially pulses which admit few cycles under the envelope only and pulses in half-cycle regime. One important experimental result is that even in femtosecond region, the waist (transverse size) of a no modulated initially laser pulse continue to satisfy the Fresnel's law of diffraction. The parabolic diffraction equation governing Fresnel's evolution of a monochromatic wave in continuous regime (CW regime) is suggested for firs time from Leontovich and Fock [1, 2, 3]

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 4ik\frac{\partial w}{\partial z} = 0. \tag{1}$$

These solutions belong circular fundamental Gaussian mode [4, 5], obtained by using axisymmetric complexification of the parabolic equation in the diffraction theory. Higher-order modes, such as Laplace-Gauss [6, 7, 8, 9], Helmholtz-Gauss and Bessel-Gauss [9, 10, 11, 12] beams can be constructed by separation variable method. From other hand the optics of laser pulses, especially in the femtosecond (fs) region operates with strongly polyhromatic waves and their spectral width $\triangle k_z$ can reach values of order of

the main wave number k_0. Additional possibility appear to menage a fs pulse to admits approximately equal duration in x, y and z directions (Light Bullet or LB), or relatively large transverse and small longitudinal size (Light Disk or LD). The evolution of so generated LB and LD in linear or nonlinear regime is quite different from the propagation of light beams and they have drawn the researchers' attention with unexpected dynamical behavior. In [13] we found new exact solution of the wave equation with initial form of Gaussian light bullet with semi-spherical, no paraxial diffraction in air, when the pulse admits few cycles only . We suggest a simple Courant - Hilbert *ansatz* in form

$$E(x,y,z,t) = V(x,y,z,t) \exp\left[ik_0(z - vt)\right], \tag{2}$$

applied to the wave equation

$$\triangle E - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0. \tag{3}$$

The corresponding diffraction equation of the amplitude function V is than

$$-2ik_0\left(\frac{\partial V}{\partial z} + \frac{1}{v}\frac{\partial V}{\partial t}\right) = \Delta V - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2}, \tag{4}$$

where v is phase velocity (or group velocity in air when second order of dispersion is neglected). The developments in spectroscopy of dielectrics, semiconductors call for optical pulses with non-oscillating nature, so-called Half-Cycle Pulses (HCP's) [14, 15]. In term of our *ansatz* (2) this request, that for HCP's the longitudinal spatial $\triangle z$ or temporal $\triangle t$ shape of the amplitude envelope V to be shorter than the main wavelength $\lambda_0 = 2\pi/k_0$ or time period $T_0 = 2\pi/\omega_0$.

In this paper we obtain analytical solutions of the wave (3) and amplitude (4) equations, representing unidirectional propagation of laser pulses and other class of spherically symmetric solutions of these equations. The main result in linear regime of propagation is that all solutions, analytical and numerical, discover enlarging of the pulses and decreasing their amplitude. On this base we claim that there are no stable laser pulse propagation in homogenous media, when the pulse's intensity is small (linear regime). To find stable pulse propagation we investigate mainly large band pulses in two kind homogenous media - air and nonlinear vacuum. In air we obtain scalar stable Lorentz type soliton while in nonlinear vacuum we obtain a vector soliton solution with own angular momentum.

LINEAR REGIME OF OPTICAL PULSES

The experiments with ultrashort fs and attosecond pulses as well as the numerical calculations [16] put on the main problem in the diffraction theory: namely, whether it is possible to build such $3D + 1$ diffraction (and dispersion) model that corresponds to

the following experimental results: *a*) at one diffraction length the spot of any spectrally limited laser pulse to satisfy the Fresnel diffraction; *b*) at several diffraction lengths one-two cycle optical pulses diffract semi-spherically. The linear Diffraction - Dispersion Equation (DDE) governing the propagation in an approximation up to second order of dispersion is [13]

$$-2ik_0 \left(\frac{\partial A}{\partial z} + \frac{1}{v} \frac{\partial A}{\partial t} \right) = \Delta A - \frac{1+\beta}{v^2} \frac{\partial^2 A}{\partial t^2}, \tag{5}$$

where $\beta = k''k_0 v^2$ is a number counting the influence of the second order of dispersion. In dispesionless media is obtained also the following Diffraction Equation (DE)

$$-2ik_0 \left(\frac{\partial V}{\partial z} + \frac{1}{v} \frac{\partial V}{\partial t} \right) = \Delta V - \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2}. \tag{6}$$

In air $\beta \simeq 2.1 \times 10^{-5} \simeq 0$, and hence DDE (5) is equal to DE (6), and at hundred diffraction lengths appear only diffraction problems. This means that we can use approximation $\beta \simeq 0$ and investigate Eq. DE (6) only on these distances. As it was mentioned in the Introduction the DE (6) can be obtained directly from the wave equation using the *ansatz* (2). Thus, solving unidirectional amplitude equations (6) and multiplying with the main phase, we can obtain exact unidirectional solution of the wave equation.

DDE (5) and DE (6) are solved by applying spatial Fourier transformation to the components of the amplitude functions A and V. The fundamental solutions of the Fourier images $\hat{A}$ and $\hat{V}$ in (k_x, k_y, k_z, t) space are correspondingly

$$\hat{A}(k_x, k_y, k_z, t) = \hat{A}(k_x, k_y, k_z, 0) \times$$
$$\exp \left\{ i \frac{v}{\beta+1} \left(k_0 \pm \sqrt{k_0^2 + (\beta+1)(k_x^2 + k_y^2 + k_z^2 - 2k_0 k_z)} \right) t \right\}, \tag{7}$$

$$\hat{V}(k_x, k_y, k_z, t) = \hat{V}(k_x, k_y, k_z, 0) \exp \left\{ iv \left(k_0 \pm \sqrt{k_x^2 + k_y^2 + (k_z - k_0)^2} \right) t \right\}. \tag{8}$$

When $\beta \simeq 0$ the fundamental solution (7) is equal to (8) and therefore we will investigate only (8). The exact solution of equation (6) can be obtained by applying the backward Fourier transform

$$V = F^{-1} \left[\hat{V}(k_x, k_y, k_z, 0) \exp \left\{ iv \left(k_0 \pm \sqrt{k_x^2 + k_y^2 + (k_z - k_0)^2} \right) t \right\} \right], \tag{9}$$

or in details

$$V = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{V}(k_x, k_y, k_z, t) \exp \left\{ -i(xk_x + yk_y + zk_z) \right\} dk_x dk_y dk_z =$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{V}(k_x, k_y, k_z, 0) \exp\left\{ iv \left(k_0 \pm \sqrt{k_x^2 + k_y^2 + (k_z - k_0)^2} \right) t \right\} \times$$

$$\exp\left\{ -i(xk_x + yk_y + zk_z) \right\} dk_x dk_y dk_z. \quad (10)$$

Substituting $k_z - k_0 = \hat{k}_z$ in (10) the latter takes the form

$$V = \frac{1}{(2\pi)^3} exp\left\{ ik_0 vt \right\} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{V}(k_x, k_y, \hat{k}_z + k_0, 0) \times$$

$$\exp\left\{ \pm ivt \sqrt{k_x^2 + k_y^2 + \hat{k}_z^2} \right\} \exp\left\{ -i(xk_x + yk_y + z\hat{k}_z) \right\} dk_x dk_y d\hat{k}_z. \quad (11)$$

We can solve the backward Fourier transform in (11), if it is possible to present the initial Fourier image $\hat{V}(k_x, k_y, \hat{k}_z + k_0, 0)$ as function of kind of $f(k_0)\hat{V}^*(k_x, k_y, \hat{k}_z, 0)$. Thus (11) becomes

$$V = \frac{1}{(2\pi)^3} exp\left\{ ik_0 vt \right\} f(k_0) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{V}^*(k_x, k_y, \hat{k}_z, 0) \times$$

$$\exp\left\{ \pm ivt \sqrt{k_x^2 + k_y^2 + \hat{k}_z^2} \right\} \exp\left\{ -i(xk_x + yk_y + z\hat{k}_z) \right\} dk_x dk_y d\hat{k}_z, \quad (12)$$

where while integrand in (12) depends from equal spectral variables and can be solved analytically.

Gaussian light bullet

We start with a solution obtained for first time in [13], to illustrate in detail the method for finding of finite energy localized solution of the DE (6) and the wave equation (3). The convolution problem (8) was solved for initial Gaussian light bullet of the kind $V(x, y, z, 0) = \exp\left(-(x^2 + y^2 + z^2)/2r_0^2\right)$. In this case the 3D backward Fourier transform (11) becomes

$$V = \frac{1}{(2\pi)^3} exp\left\{ -\frac{k_0^2 r_0^2}{2} - ik_0(z - vt) \right\} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{ -\frac{\left(k_x^2 + k_y^2 + \hat{k}_z^2\right) r_0^2}{2} \right\} \times$$

$$\exp\left\{ \pm ivt \sqrt{k_x^2 + k_y^2 + \hat{k}_z^2} \right\} \exp\left\{ -i(xk_x + yk_y + (z - ir_0^2 k_0)\hat{k}_z) \right\} dk_x dk_y d\hat{k}_z, (13)$$

which in spherical coordinates can be presented in the following way

$$V = \frac{1}{2\pi^2} exp\left\{ -\frac{k_0^2 r_0^2}{2} - ik_0(z - vt) \right\} \times$$

$$\frac{1}{\hat{r}} \int_0^\infty \hat{k}_r \exp\left\{-\frac{\hat{k}_r^2 r_0^2}{2}\right\} \exp\left\{\pm ivt\hat{k}_r\right\} \sin\left\{\hat{r}\hat{k}_r\right\} d\hat{k}_r, \tag{14}$$

where $\hat{k}_r = \sqrt{k_x^2 + k_y^2 + \hat{k}_z^2}$ and $\hat{r} = \sqrt{x^2 + y^2 + (z - ir_0^2 k_0)^2}$.

The corresponding solution is

$$V(x,y,z,t) = \frac{i}{2\hat{r}} \exp\left[-\frac{k_0^2 r_0^2}{2} - ik_0(z - vt)\right] \times$$

$$\left\{ i(vt + \hat{r}) \exp\left[-\frac{1}{2r_0^2}(vt + \hat{r})^2\right] erfc\left[\frac{i}{\sqrt{2}r_0}(vt + \hat{r})\right] \right. \tag{15}$$

$$\left. -i(vt - \hat{r}) \exp\left[-\frac{1}{2r_0^2}(vt - \hat{r})^2\right] erfc\left[\frac{i}{\sqrt{2}r_0}(vt - \hat{r})\right] \right\}.$$

Multiplying (15) with the main phase, we find actually an exact solution of the wave equation (3)

$$E(x,y,z,t) = \frac{i}{2\hat{r}} \exp\left(-\frac{k_0^2 r_0^2}{2}\right) \times$$

$$\left\{ i(vt + \hat{r}) \exp\left[-\frac{1}{2r_0^2}(vt + \hat{r})^2\right] erfc\left[\frac{i}{\sqrt{2}r_0}(vt + \hat{r})\right] \right. \tag{16}$$

$$\left. -i(vt - \hat{r}) \exp\left[-\frac{1}{2r_0^2}(vt - \hat{r})^2\right] erfc\left[\frac{i}{\sqrt{2}r_0}(vt - \hat{r})\right] \right\}.$$

If substitute the time variable $t = 0$ in Eq. (16) the initial solution of the wave equation transforms to the form $E(x,y,z,0) = \exp(ik_0 z) \exp\left(-(x^2 + y^2 + z^2)/2r_0^2\right)$. All analytical and numerical solutions of the wave equation (3) obtained from initial conditions of kind of $E(x,y,z,0) = \exp(ik_0 z)V(x,y,z)$ and for the corresponding amplitude equations (5) and (6) of the kind $V(x,y,z,0)$, where V is three dimensional localized smooth function, produced a translation of the solutions in z direction. The wave equation (3) is hyperbolic type ones, while the amplitude equation (6) is of parabolic type and a initial value problems can be solved (11). And here one method for finding spherically symmetric solutions of the wave equation appear. Let now the initial amplitude function V of the amplitude equation is product of three dimensional localized function, multiplied by a plane wave with opposite direction

$$V(x,y,z,0) = \exp(-ik_0 z)V * (x,y,z), \tag{17}$$

where $V*$ is one spherically symmetric function. The corresponding initial amplitude function of the the wave equation become $E(x,y,z,0) = V * (x,y,z)$, and practically, solving initial value problem of (6) with initial conditions of kind (17), we can found exact spherically symmetric solutions of the wave equation (3).

Spherically symmetric finite energy solutions of the wave equation

One careful analysis on the solution of the amplitude function V presented as integral in Eq.(11) point that all functions under integral depends from translated wave number $\hat{k}_z$, except the Fourier image of the initial function $\hat{V}(k_x, k_y, \hat{k}_z + k_0, 0)$. We can use the Transition theorem from the Fourier optics to present $\hat{V}(k_x, k_y, \hat{k}_z + k_0, 0)$ as a function of translated wave number $\hat{k}_z$ only: $f(k_0)\hat{V}^*(k_x, k_y, \hat{k}_z, 0)$. Formally, this correspond to use the initial conditions in form

$$V(x, y, z, 0) = V^*(x, y, z, 0) \exp\{-ik_0 z\}. \tag{18}$$

Applying the Transition theorem to (18), the next expression of the Fourier image can be written

$$F[V(x, y, z, 0)] = \hat{V}(k_x, k_y, k_z - k_0, 0) = \hat{V}^*(k_x, k_y, \hat{k}_z, 0). \tag{19}$$

Thus, all functions in the backward Fourier transform (11) depends from translated wave number $\hat{k}_z$ only

$$V = \frac{1}{(2\pi)^3} exp\{ik_0(vt - z)\} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{V}^*(k_x, k_y, \hat{k}_z, 0) \times$$
$$\exp\left\{\pm ivt\sqrt{k_x^2 + k_y^2 + \hat{k}_z^2}\right\} \exp\left\{-i(xk_x + yk_y + z\hat{k}_z)\right\} dk_x dk_y d\hat{k}_z. \tag{20}$$

In this paper we will investigate spherically-symmetric functions of kind $V^*(x, y, z, 0) = V^*(r, 0)$, where $r = \sqrt{x^2 + y^2 + z^2}$. This means that the backward radial Fourier transform can be written

$$V = \frac{1}{2\pi^2} exp\{ik_0(vt - z)\} \frac{1}{r} \int_0^{\infty} \hat{k}_r \hat{V}^*(\hat{k}_r, 0) \exp\{\pm ivt\hat{k}_r\} \sin\{r\hat{k}_r\} d\hat{k}_r. \tag{21}$$

Now the method for finding finite energy, spherically symmetric solutions of the wave equation become obvious. Using initial condition in form of localized function $V^*(r) \mapsto 0$ when $r \mapsto \infty$ as one first step we find its Fourier image $\hat{V}^*(\hat{k}_r, 0)$. Solving the the backward radial Fourier transform (21) we can find the time evolution of the corresponding DE (6). Multiplying the solutions of DE (6) with the main phase (Courant -Hilbert *ansatz* (2)) we practically obtain exact solution of the wave equation (3).

1. Localized algebraic function of the kind $V^* = 1/[1 + r^2/r_0^2]$

Here we demonstrate how to obtain with this method the finite energy solution for initial algebraic localized function of the kind

$$V^* (x, y, z, t = 0) = 1/[1 + r^2/r_0^2]. \tag{22}$$

The 3D Fourier expression of (22) in spherical variables is

$$\hat{V}^* (k_r, t = 0) = \frac{\pi}{2k_r} \exp(-r_0 k_r). \tag{23}$$

Hence, the corresponding solution of the amplitude equation after solving the spectral kernels (21) of (6) is

$$V (x, y, z, t) = \exp[-ik_0(z - vt)] / \left[\frac{r^2}{r_0^2} + \left(1 + \frac{ivt}{r_0}\right)^2 \right]. \tag{24}$$

Now again by multiplying with the main phase, the corresponding finite energy solution of the wave equation (3) becomes

$$E (x, y, z, t) = 1 / \left[\frac{r^2}{r_0^2} + \left(1 + \frac{ivt}{r_0}\right)^2 \right]. \tag{25}$$

2. Localized algebraic function of the kind $V^* = 1/[(1 + r^2/r_0^2)^2]$

The function $V^*(r) = 1/(1 + r^2/r_0^2)^2$ has a Fourier image $\hat{V}^* (k_r) = \frac{1}{4}\pi r_0 \exp\{-r_0 k_r\}$. Hence, the corresponding solution of the amplitude equation is

$$V(x, y, z, t) = \exp[-ik_0(z - vt)] \left\{ \frac{2(r_0 + itv)}{(\tilde{r}^2 + (r_0 + itv)^2)^2} \right\}, \tag{26}$$

and the corresponding wave solution is

$$E(x, y, z, t) = \frac{2(r_0 + itv)}{(\tilde{r}^2 + (r_0 + itv)^2)^2}. \tag{27}$$

3. Localized algebraic function of the kind $V^* = 1/\left[(1 + r^2)^4\right]$

The Fourier images of the function

$$V^*(r) = 1/\left[(1 + r^2)^4\right] \tag{28}$$

is the following

$$\hat{V}^* = \frac{1}{96}\pi\exp\{-k_r\}\left(3 + k_r + 3k_r\right). \tag{29}$$

The solution V of DE (6) become

$$V(x,y,z,t) = 6\exp\left[-ik_0(z-vt)\right] \times$$
$$\frac{[8 + 29itv] + tv[-r^2 + t^2v^2][-ir^2 + tv(8 + itv)] - 2tv[-3ir^2 + tv(20 + 13itv)]}{[r^2 + (1 + itv)^2]^4}, \tag{30}$$

and the corresponding wave solution is

$$E(x,y,z,t) = 6\times$$
$$\frac{[8 + 29itv] + tv[-r^2 + t^2v^2][-ir^2 + tv(+itv)] - 2tv[-3ir^2 + tv(20 + 13itv)]}{[r^2 + (1 + itv)^2]^4}. \tag{31}$$

4. Localized algebraic function of the kind $V^* = 24(1 - r^2)/(1 + r^2)^4$

The function $V^*(r) = 24(1 - r^2)/(1 + r^2)^4$ has a Fourier image $\frac{1}{2}\pi\exp\{-k_r\}k_r^2$. Hence, the solution V of DE (6) is

$$V(x,y,z,t) = \frac{3i}{4\pi\tilde{r}}\exp\left[-ik_0(z-vt)\right]\left\{\frac{1}{(vt + \tilde{r} - i)^4} - \frac{1}{(-vt + \tilde{r} + i)^4}\right\}, \tag{32}$$

and the corresponding wave solution is

$$E(x,y,z,t) = \frac{3i}{4\pi\tilde{r}}\left\{\frac{1}{(vt + \tilde{r} - i)^4} - \frac{1}{(-vt + \tilde{r} + i)^4}\right\}. \tag{33}$$

5. Localized algebraic function of the kind $V^* = 2r(3 - r^2)/(1 + r^2)^3$

The function $V^*(r) = 2r(3 - r^2)/(1 + r^2)^3$ has a Fourier image $\frac{1}{2}\pi\exp\{-k_r\}k_r$. Hence, the solution V of DE (6) is

$$V(x,y,z,t) = \frac{1}{\tilde{r}}\exp\left[-ik_0(vt - z)\right]\left\{\frac{1}{(vt - \tilde{r} - i)^3} - \frac{1}{(vt + \tilde{r} - i)^3}\right\}, \tag{34}$$

and the corresponding wave solution is

$$E(x,y,z,t) = \frac{1}{2\pi^2\tilde{r}}\left\{\frac{1}{(vt - \tilde{r} - i)^3} - \frac{1}{(vt + \tilde{r} - i)^3}\right\}. \tag{35}$$

NONLINEAR REGIME OF BROAD-BAND OPTICAL PULSES

Nonlinear propagation of broad-band pulses in air. Lorentz type soliton

As it was pointed in [18], after neglecting two small perturbation terms, the corresponding nonlinear amplitude equation for broad band femtosecond pulses can be reduced to

$$\Delta C - \frac{1}{v^2} \frac{\partial^2 C}{\partial t^2} + \gamma C^3 = 0, \tag{36}$$

where $\gamma = C_0^2 k_0^2 n_2$ is the nonlinear coefficient. The Eq. (36) admits soliton solution propagating in forward direction only. The soliton solution of Eq. (36) is

$$C = \frac{sech(ln(\tilde{r}))}{\tilde{r}} = \frac{2}{1 + \tilde{r}^2}, \tag{37}$$

where $\gamma = 2$ and $\tilde{r} = \sqrt{x^2 + y^2 + (z + ia)^2 - v^2(t + ia/v)^2}$. The main difference from the solutions of linear wave (3) and amplitude (6) equations, which are not stable, is that the soliton solution of nonlinear wave equation (36) preserve its spatial and spectral shape in time. Thus, for first time photon-like propagation is obtained as exact solution of the corresponding $3D + 1$ nonlinear wave equation Eq. (36). While the stable soliton solution of the one-dimmensional nonlinear Schrödinger equation admits *sech* form, the 3D+1 soliton solution of the nonlinear wave equation (36) has Lorentz shape with asymmetric k_z spectrum.

Propagation of broad-band pulses in nonlinear vacuum. Nonlinear spherical wave

As one natural next step is our intention to find nonlinear solutions with angular momentum. In 1935 Euler and Kockel [19] predict one intrinsic nonlinearity to the electromagnetic vacuum due to electron-positron nonlinear polarization. This leads to field-dependent dielectric tensor in form

$$\varepsilon_{ik} = \delta_{ik} + \frac{7e^4\hbar}{45\pi m^4 c^7} \left[2 \left(|E|^2 - |B|^2 \right) + 7B_i B_k \right], \tag{38}$$

where complex form of presentation to the electrical E_i and magnetic B_i components is used. Note that the term containing $B_i B_k$ vanishes, when localized electromagnetic wave with only one magnetic component B_l is investigated. The dielectric response relevant to such optical pulse is thus

$$\varepsilon_{ik} = \delta_{ik} + \frac{14e^4\hbar}{45\pi m^4 c^7} \left(|E|^2 - |B|^2 \right). \tag{39}$$

The magnetic field, rather than the electrical field, appears in the expression for the dielectric response (39) and the nonlinear addition to the intensity profile (effective mass density) of one electromagnetic wave in nonlinear vacuum can be expressed in electromagnetic units as $I_{nl} = (|E|^2 - |B|^2)$. When the spectral width of one pulse $\triangle k_z$ exceeds the values of the main wave-vector $\triangle k_z \simeq k_0$ the system of amplitude equations in nonlinear vacuum become

$$\Delta \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \gamma(|\vec{E}|^2 - |\vec{B}|^2)|\vec{E} = 0$$

$$\Delta \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} + \gamma(|\vec{E}|^2 - |\vec{B}|^2)|\vec{B} = 0, \tag{40}$$

where $\gamma = \frac{7k_0^2 e^4 \hbar}{90\pi m^4 c^7}$ and E, B are amplitude functions. Initially, we present the components of of the electrical and magnetic fields as a vector sum of circular and linear components

$$\begin{aligned} &E_z \\ E_c &= iE_x - E_y \\ B_l &= -B_z. \end{aligned} \tag{41}$$

Thus (40) is transformed to the next scalar system of equations

$$\Delta E_z - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} + \gamma(|E_z|^2 + |E_c|^2 - |B_l|^2)E_z = 0$$

$$\Delta E_c - \frac{1}{c^2}\frac{\partial^2 E_c}{\partial t^2} + \gamma(|E_z|^2 + |E_c|^2 - |B_l|^2)E_c = 0 \tag{42}$$

$$\Delta B_l - \frac{1}{c^2}\frac{\partial^2 B_l}{\partial t^2} + \gamma(|E_z|^2 + |E_c|^2 - |B_l|^2)B_l = 0.$$

Let now parameterize the space-time by the pseudospherical coordinates $(r, \tau, \theta, \varphi)$ in the following way

$$\begin{aligned} ct &= r\sinh(\tau) \\ z &= r\cosh(\tau)\cos(\theta) \\ y &= r\cosh(\tau)\sin(\theta)\sin(\varphi) \\ x &= r\cosh(\tau)\sin(\theta)\cos(\varphi), \end{aligned} \tag{43}$$

where $r = \sqrt{x^2 + y^2 + z^2 - c^2 t^2}$. After calculations the corresponding d'Alambert operator in pseudospherical coordinates we get [17]

$$\triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2}{\partial \tau^2} - 2\frac{\tanh\tau}{r^2}\frac{\partial}{\partial \tau} + \frac{1}{r^2\cosh^2\tau}\triangle_{\theta,\varphi}, \tag{44}$$

where with $\triangle_{\theta,\varphi}$ the angular part of the usual Laplace operator is written

$$\triangle_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{45}$$

The system of equations (42) in pseudo-spherical coordinates becomes

$$\frac{3}{r}\frac{\partial E_z}{\partial r} + \frac{\partial^2 E_z}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 E_z}{\partial\tau^2} - 2\frac{\tanh\tau}{r^2}\frac{\partial E_z}{\partial\tau} + \frac{1}{r^2\cosh^2\tau}\triangle_{\theta,\varphi}E_z +$$
$$\gamma(|E_z|^2 + |E_c|^2 - |B_l|^2)E_z = 0$$

$$\frac{3}{r}\frac{\partial E_c}{\partial r} + \frac{\partial^2 E_c}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 E_c}{\partial\tau^2} - 2\frac{\tanh\tau}{r^2}\frac{\partial E_c}{\partial\tau} + \frac{1}{r^2\cosh^2\tau}\triangle_{\theta,\varphi}E_c + \tag{46}$$
$$\gamma(|E_z|^2 + |E_c|^2 - |B_l|^2)E_c = 0$$

$$\frac{3}{r}\frac{\partial B_l}{\partial r} + \frac{\partial^2 B_l}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 B_l}{\partial\tau^2} - 2\frac{\tanh\tau}{r^2}\frac{\partial B_l}{\partial\tau} + \frac{1}{r^2\cosh^2\tau}\triangle_{\theta,\varphi}B_l +$$
$$\gamma(|E_z|^2 + |E_c|^2 - |B_l|^2)B_l = 0.$$

Eq's (46) are solved using the method of separation of the variables.

$$E_i(r,\tau,\theta,\varphi) = R(r)T_i(\tau)Y_i(\theta,\varphi) \tag{47}$$
$$B_l(r,\tau,\theta,\varphi) = R(r)T_l(\tau)Y_l(\theta,\varphi),$$

where $i = z,c$. We use additional constrains on the angular and "spherical" time parts

$$|T_z|^2|Y_z(\theta,\varphi)|^2 + |T_c|^2|Y_c(\theta,\varphi)|^2 - |T_l|^2|Y_l(\theta,\varphi)|^2 = const. \tag{48}$$

The condition (48) separate the variables in the nonlinear terms to dependance from radial part only. Thus the radial parts obey the equation

$$\frac{3}{r}\frac{\partial R}{\partial r} + \frac{\partial^2 R}{\partial r^2} - \frac{A_i}{r^2}R + \gamma|R|^2R = 0, \tag{49}$$

where A_i are separation constants. We look for soliton solutions which have more strong localization than scalar soliton solutions

$$R = \frac{sech(ln(r^\alpha))}{r}, \tag{50}$$

where α, γ and the separation constants A_i satisfy the relation

$$\alpha^2 - 1 = A_i \; ; 2\alpha^2 = \gamma. \tag{51}$$

The corresponding τ - dependent part of the equation is linear

$$\cosh^2 \tau \frac{d^2 T_i}{d\tau^2} + 2\sinh \tau \cosh \tau \frac{dT_i}{d\tau} + (C_i - A_i \cosh^2 \tau)T_i, = 0, \tag{52}$$

where $i = z, c, l$ and C_i is another separation constant connected with angular part of the Laplace operator $Y_i(\theta, \varphi)$. There are only the following solutions of the Eq. (52) which satisfy the condition (48)

$$\begin{aligned} T_z &= \cosh \tau \\ T_c &= \cosh \tau \\ T_l &= \sinh \tau, \end{aligned} \tag{53}$$

with the following separation constants for the electrical part $A_z = A_c = 3$; $C = 2$ and for the magnetic part $A_l = 3$; $C = 0$. Thus the magnetic part of the system of equations do not depends from the angular components, i.e. $Y_l(\theta, \varphi) = 0$, while for electrical part $Y_z(\theta, \varphi), Y_c(\theta, \varphi)$ we have the following linear system of equations

$$\frac{\triangle(\theta, \varphi)Y_i}{Y_i} = -2, \tag{54}$$

where now $i = z, c$. There are only two solutions of the Eq. (54) which satisfy the condition (48)

$$Y_z = \cos \theta \tag{55}$$
$$Y_c = \sin \theta \exp(i\varphi), \tag{56}$$

Using the relation between the separation constants A_i and the real number α we have

$$\alpha^2 = 4 \; ; \alpha = \pm 2 \; ; \gamma = 8. \tag{57}$$

Finally, we can write the exact soliton solution of the system of nonlinear equations (42) representing propagation of electromagnetic wave in nonlinear vacuum

$$\begin{aligned} E_z(r, \tau, \theta) &= \frac{sech(ln(r^{\pm 2}))}{r} \cosh \tau \cos \theta \\ E_c(r, \tau, \theta, \varphi) &= \frac{sech(ln(r^{\pm 2}))}{r} \cosh \tau \sin \theta \exp(i\varphi) \\ B_l(r, \tau) &= \frac{sech(ln(r^{\pm 2}))}{r} \sinh \tau. \end{aligned} \tag{58}$$

If we rewrite the solution in Cartesian coordinates, it not difficult to be shown that the solution (58) of the system (42) admits finite energy and own angular momentum

$$E_z(r, \tau, \theta) = \frac{2r}{r^4+1} \frac{\hat{r}}{r} \frac{z}{\hat{r}} = \frac{2z}{r^4+1}$$

$$E_c(r, \tau, \theta, \varphi) = \frac{2r}{r^4+1} \frac{\hat{r}}{r} \frac{x+iy}{\hat{r}} = \frac{2(x+iy)}{r^4+1} \qquad (59)$$

$$B_l(r, \tau) = \frac{2r}{r^4+1} \frac{ct}{r} = \frac{2ct}{r^4+1},$$

where $r = \sqrt{x^2+y^2+z^2-c^2t^2}$ and $\hat{r} = \sqrt{x^2+y^2+z^2}$. The intensity profile of the soliton solution can be expressed as

$$I(x,y,z,t) = |E|^2 + |B|^2 = \frac{4r^2+8c^2t^2}{(r^4+1)^2} = \frac{4(x^2+y^2+z^2+c^2t^2)}{\left[(x^2+y^2+z^2-c^2t^2)^2+1 \right]^2}. \qquad (60)$$

The solution presents nonlinear spherical wave. The problem with stable charged localized solutions is still open.

CONCLUSIONS

In this paper the wave equation (3) is solved using one simple form of the Courant-Hilbert *ansatz* (2). In this way we reduce the wave equation to parabolic type diffraction equation for the envelope DE (6). The correcponding initial value problem can be solved for this parabolic equation (6) and different kind of finite energy exact solutions can be found for (6) as well as for the wave equation (3). The solutions spread with velocity of light in amedia by different ways. In the laser optics an Gaussian light bullet, or other kind of amplitude functions A are localized initially near one plane wave of kind of $E(x,y,z,0) = A(x,y,z,0)\exp(ik_0z)$ and present a superposition of plane waves near one main wave-number. When such pulses admit a lot of cycles under the amplitude, they spread mainly in transverse direction, following the Fresnel's law of diffraction, and exact cover the laser pulse dynamics measured in the experiments. The new experimental [16] and theoretical [13] observation for few, single and half-cycle pulses is that at few diffraction lengths their intensity profile takes parabolic or semi-spherical form. Simply, the paraxial optics do not works for few, single and HCP's waves. When the initial pulse is not localized near a plane wave and is spherically symmetric, the solutions of the wave equation (3) spread radially with typical forming of inside and outside wavefronts. These conclusions are based on exact analytical solutions of the wave equation and the amplitude equation, presented in the paper. In both cases the initially localized amplitude functions decreases and the energy distributes over while space for finite time. In nonlinear regime we solve nonlinear wave equation and a system of nonlinear wave equations (42) governing propagation of broad-band optical pulses in nonlinear vacuum. The main difference from the solutions of linear wave (3) and amplitude (6) equations, which spread in space, is that the soliton solution of nonlinear wave equation (36) preserve its spatial and spectral shape in time. Thus, one photon-like propagation

is obtained as exact solution of the corresponding $3D + 1$ nonlinear wave equation Eq. (36). The scalar 3D+1 soliton solution of (36) has Lorentz shape with asymmetric k_z spectrum. The soliton solution (58) of the nonlinear vacuum system of equations (42) is with strong localization than scalar case and admits own angular momentum.

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